

The Design of Durable Goods

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The use of a durable good is limited by both its physical life and usable life. For example, an electric-car battery can last for five years (physical life) or 100,000 miles (usable life), whichever comes first. We propose a framework for examining how a profit-maximizing firm might choose the usable life, physical life, and selling price of a durable good. The proposed framework considers differences in usage rates and product valuations by consumers and allows for the effects of technological constraints and product obsolescence on a product's usable and physical lives. Our main result characterizes a relationship between optimal price, cost elasticities, and opportunity costs associated with relaxing upper bounds on usable and physical lives. We describe conditions under which either usable life or physical life, or both, obtains its maximum possible values; examine why a firm might devote effort to relaxing nonbinding constraints on usable life or physical life; consider when price cuts might be accompanied with product improvements; and examine how a firm might be able to cross-subsidize product improvements.

Key words: product life; product design; technology development; durable goods; pricing

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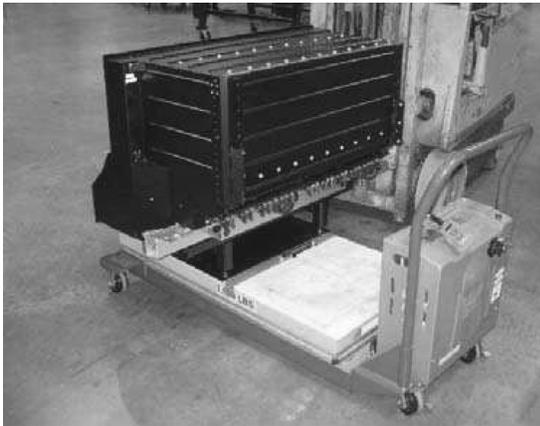
1. Introduction

The life of a durable good can end in one of two ways—through use or through obsolescence or deterioration that is at least partly independent of use. We use the term *usable life* to refer to the maximum amount of use a consumer can obtain from a product, and we use the term *physical life* to refer to the time before a product becomes unusable for reasons other than use, such as physical deterioration resulting from exposure to the environment, leakage of fluids and charges, or technological obsolescence. We examine how a firm might select the usable life, physical life, and price of a durable good. To the best of our knowledge, the distinction between usable life and physical life, questions relating to how a firm chooses such properties, and the proposed framework for addressing these questions are new to the marketing literature.

To illustrate the difference between usable life and physical life, consider the lithium-ion battery in Tesla Motors' all-electric roadster (Figure 1), which is expected to maintain good driving performance for about 100,000 miles or five years, whichever comes first. That is, the battery has a physical life of five years and a usable life of 100,000 miles. Similarly, some DVD burners can last six to eight years (phys-

ical life) or 5,000 to 7,000 DVD copies (usable life), whichever comes first. As in these examples, physical life is measured in time units, typically years for durable goods; usable life is measured in quantity units, such as the number of miles or disk copies. Other examples of usable life are the number of impulses emitted by a cardiac pacemaker, the number of heartbeats before a cardiovascular stent fractures or deforms as a result of the cumulative effect of biomechanical forces, and the number of pages an inkjet printer can print over its life.

For some products, such as medical devices, regulatory agencies specify minimum values for the usable life and physical life. Additionally, a firm has to consider consumer preferences and usage rates, design and material choices, alternative technologies and their limitations, and the cost of making a product when selecting the usable life and physical life of a product. To illustrate, consider the effect of the state of technology on product design. Firms have been working for many years not only to develop alternative sources of energy but also to design storage devices that are simultaneously able to hold a large amount of charge; are efficient, lightweight, inexpensive, and long-lasting; and are convenient to carry and use. Television manufacturers continue to make

Figure 1 Lithium-Ion Battery Pack for the Tesla Roadster

Source. Used with permission of Tesla Motors.

efforts to improve image quality, increase usable life, reduce size and weight, and cut the manufacturing and assembly costs of plasma and LCD sets. However, at any given point in time, technology limits what can be made and the cost at which it can be made. Both these considerations place constraints on a product's usable life and physical life, and it is useful to know which of these constraints can be more easily relaxed and what effects this might have on the design and pricing of a product.

The purpose of this paper is to present a framework and model for examining how a firm might choose usable life, physical life, and price for a durable good. The proposed model restricts consumers to purchasing at most one unit of the product. It only considers a monopolist seller (but §5 discusses the potential impact of competition, product lines, and repurchases by consumers). The single-purchase assumption is reasonable for many durable goods, including implantable medical devices like pacemakers and cardiovascular stents, rechargeable battery packs for electric cars, household washers and dryers, vacuum cleaners, and home heating and air-conditioning units. It is also common to a number of marketing models, including diffusion models (e.g., Bass 1969, Mahajan et al. 1990), and conjoint-based models for optimal product and product-line design (e.g., Green and Krieger 1985, Kohli and Krishnamurti 1987, Belloni et al. 2008). The monopoly assumption is more reasonable for new, innovative products. It is also appropriate for firms that can operate as monopolies for long periods of time because they have patent protection, possess proprietary technologies, or sell to consumers with high switching costs.

The present model considers the effects of the following four factors on a product's usable life, physical life, and price.

(i) *Usage rate*: How much use a consumer obtains from a product can depend on his or her (average)

rate of product use. If the usage rate is low, a product can come to the end of its physical life, forcing a consumer to abandon a product that still has remaining usable life. Above some usage-rate threshold, usable life ends before physical life, and these consumers can obtain full product use. Differences in usage rates therefore play an important role in the design of a durable good. If all consumers had the same usage rate, an optimally designed product would simultaneously reach the end of both its usable and physical lives.

(ii) *Utility and willingness to pay*: Utility is obtained by consumers from using, not simply owning, such products as air conditioners, washing machines, car tires, medicated stents, and cardiac pacemakers. How much consumers are willing to pay depends on how much value they place on a marginal unit of use. We allow consumers to differ in their marginal valuations and allow these to increase, decrease, or remain constant with the amount of use. This, in turn, implies that a firm cannot choose a price by equating marginal cost and marginal valuation, because the latter differs across consumers. Observe that the marginal valuations can increase for some products (e.g., implantable medical devices, the replacement of which carries physical risks), decrease for others (e.g., cars and car tires, which can become less reliable with use), and remain constant for still other products (e.g., air-conditioning and heating units, which can function reliably, if less efficiently, over long periods of time).

(iii) *Upper bounds on usable life or physical life*: All products have finite usable life and physical life. The upper bounds on their values depend on at least two factors. The first is product obsolescence, which imposes an upper bound on physical life and which sometimes is a result of a deliberate strategy by a firm to create replacement demand for its product. The maximum use any consumer can obtain over this time period is the associated upper bound on the usable life of the product. The second consideration is technology, which often becomes a constraint in providing a higher usable life. The constraining effect of technology on usable life is especially pronounced in the early years of product development and has been witnessed for products such as plasma television screens, pacemakers, cardiovascular stents, electric-car batteries, and fuel cells. The situation in which there are no constraints on usable and physical lives can be easily represented as a special case of our model.

(iv) *Production cost*: In many cases, marginal cost can increase with the usable life and/or physical life. Such an increase might be especially sharp closer to the upper bounds on the usable and physical lives. When these bounds are reached, the marginal cost effectively becomes infinitely high. In the present

model, the variable cost of making a product can be any arbitrary increasing function of usable life and physical life.

A notable feature of the present analysis is that it makes fairly weak assumptions about heterogeneities in usage rates and product valuations by consumers and about the cost function. We allow any suitable statistical distributions to capture the heterogeneities in consumer usage rates and product valuations; and as already noted, the only assumption we make concerning cost is that it increases with usable life and physical life. Obviously, these specifications are too general to obtain a specific optimal solution. Instead, our main result characterizes a relationship between optimal price, cost elasticities, and opportunity costs of relaxing the upper bounds on usable life and physical life. We describe conditions under which either usable life or physical life, or both, can take its maximum possible values. We examine why a firm might devote research and development effort to relaxing the upper bound on usable (physical) life, even when it does not want to increase the usable (physical) life and when it might be able to both cut price and increase the usable life or physical life we also examine how the firm might be able to separate the manner in which it improves the product from the way in which it invests to reduce production cost.

The rest of this paper is organized as follows. Section 2 gives a brief overview of related research. Section 3 describes the proposed model and considers questions related to optimal product design and pricing. It also examines when and how a firm should invest in technologies that reduce the manufacturing cost or relax the constraints on the maximum usable life or physical life of a product. Section 4 discusses the results. Section 5 concludes the paper.

2. Background

To our knowledge, product life has not been studied as an aspect of product design in marketing. However, a stream of research in economics has examined the closely related question of how a monopolist might choose the durability of a product. Early research by Kleiman and Ophir (1966), Levhari and Srinivasan (1969), and Schmalensee (1970) suggested that a monopolist would choose a level of durability that was not socially optimal. Later, Swan (1970, 1977) and Sieper and Swan (1973) showed that, in many cases, a monopolist would offer a socially optimal level of durability but would also supply a smaller quantity of the product than might be available in a perfectly competitive market. Further research suggested that this result is robust if some of Swan's assumptions were to be relaxed (see Barro 1972, Schmalensee 1974, Su 1975, Auernheimer and Saving 1977, Schmalensee 1979). Subsequently, Muller

and Peles (1988, 1990) showed that a monopolist should optimally choose a decreasing level of durability over time.

A related stream of research in marketing examines the role of durability on competition in secondhand markets (e.g., Levinthal and Purohit 1989; Desai and Purohit 1998, 1999; Desai et al. 2007) and the timing of new product introductions (e.g., Moorthy and Png 1992, Thölke et al. 2001, Sankaranarayanan 2007). This research considers durability to be a unidimensional construct with greater durability translating into longer product life.

The present research departs from the above-mentioned work in at least four ways. First, it distinguishes between the usable life and the physical life of a product, defining them as distinct but related aspects of durability. An important consequence of this distinction is the ability to separate users into those who use the product to the end of its usable (but not physical) life and those who use the product to the end of its physical (but not usable) life. Second, economics-based research considers a stock of durable goods as providing a fungible supply of an underlying service—a quantity of electricity or the distance that can be traveled using the number of cars on the market, for example. Demand for the service depends in a continuous manner on its market price. In contrast, the present model assumes that the product is purchased in discrete units by consumers, that each unit of the product offers an indivisible “quantity bundle” of the underlying service that cannot be distributed in a continuous manner across consumers, and that part of the product may, in one way or another, ultimately remain unused. Third, we explicitly allow differences in consumer usage rates and willingness to pay, which in turn determine the demand function for the product. Finally, unlike models that examine issues concerning time inconsistency and secondhand purchases, we only consider a static model that allows at most a single purchase by a consumer. Modeling repeat purchases is important, but this is more natural in a dynamic model, which is beyond the scope of this paper (allowing consumers to buy multiple units in a cross-sectional model would imply that consumers purchase several units of a durable good at one time, an assumption that is difficult to justify for such products as medical devices, electric-car batteries, and home washers and dryers).

3. Model

We consider a firm that makes a durable good and wishes to select the profit-maximizing usable life (x), physical life (y), and price (p) for its product. We

define usable life in quantity units (e.g., the number of charge–recharge cycles provided by an average lithium-ion battery) and physical life in time units (e.g., months). For simplicity, we assume that at the time of purchase, all consumers believe that the product has the same usable life and physical life. These values might be disclosed by the seller and correspond to values obtained from product testing, sometimes under regulatory supervision (for example, by the U.S. Food and Drug Administration). We first assume that the firm can separately and independently choose the usable life and physical life of a product. In §3.4, we examine how the results change when there is linear interdependence between the usable life and physical life.

We denote the maximum usable life of the product by x_{\max} and the maximum physical life by y_{\max} . These upper bounds can depend on such factors as the nature of the technology, the materials used, the product architecture and design, and the state of technological knowledge at the time of product development. For some products, these bounds are so weak that they impose no practical constraints on product design; for example, some furnaces and refrigerators can last 50 years. However, for other products, including energy storage and medical devices, these bounds act as significant constraints and can impel firms to make investments in research and development to increase a product's usable life and/or physical life.

Product obsolescence is a separate factor affecting how long consumers use a product. For example, if a firm introduces new versions of a product every year and if most consumers do not use it for more than three or four years, the product is, for all practical purposes, obsolete in five years. We reflect the effect of product obsolescence in the value of y_{\max} . That is, we define the upper bound on the physical life as the time after which a product is either obsolete or has degraded to the point that it no longer functions.

We assume that consumers obtain utility from using, not just owning, the product. The amount of use, in turn, depends on the (average) rate at which a consumer uses the product and the marginal value he or she associates with a unit of use. If usage rate is high enough, a consumer will obtain use equal to the usable life of the product; otherwise, he or she will obtain as much use as is possible over the physical life of the product. Heterogeneities in usage rates and marginal valuations determine the distribution of utilities across consumers and thus the aggregate demand, which we use to determine the profit-maximizing values for usable life, physical life, and price.¹ We now describe the model in detail.

¹ A revenue-maximizing solution is obtained by setting the cost equal to zero. If there are no constraints on the usable life or

3.1. Consumer Model

Consider a consumer with average usage rate θ . Let q denote the amount of use the consumer obtains from the product. Then $q = \theta y$ if the consumer uses the product for the entire duration of its physical life. In addition, $q \leq x$, because, by definition, the usable life is the upper bound on the amount of use available from the product. Thus,

$$q = \min\{\theta y, x\}.$$

Consumers who obtain $q = \theta y < x$ units of use therefore have average usage rates $\theta < x/y$; those who obtain $q = x$ units of use have average usage rates $\theta \geq x/y$. A consumer buys the product only if

$$u = v(q) - p > 0,$$

where u denotes utility, v is the value associated with q units of use, and p is the price of the product. In the rest of this paper, we assume that

$$v(q) = kq^\gamma, \quad \gamma, k > 0,$$

where $\gamma = (\partial v / \partial q)(q/v)$ is the elasticity of product valuation with respect to the quantity of use. We refer to k as the unit valuation. Consumers can have different product valuations for two reasons. First, they can obtain different amounts of use if their usage rates are below the threshold x/y . Second, they can associate a different value, $v(q)$, with the same amount of use. We model these two sources of heterogeneity by allowing the usage rate θ and unit valuation k to vary according to independent, continuous density functions $f(\theta)$ and $g(k)$, defined over suitable (but otherwise arbitrary) ranges $[\theta_{\min}, \theta_{\max}]$ and $[k_{\min}, k_{\max}]$, respectively. For simplicity, we assume that the value of γ is the same across consumers. We observe that $\theta_{\min} > 0$ because consumers would not buy a product they never intend to use, and $\theta_{\max} < \infty$ because there are physical limits to the rate of product use. For example, everyone who buys an electric car presumably intends to use it at least once, and no one can drive the car for more than 24 hours a day. In reality, the lower and upper bounds are likely much tighter. Similarly, $k_{\min} > 0$, $k_{\max} < \infty$, so that a consumer's marginal value for incremental use is positive, but bounded. Figure 2 shows a family of utility functions with $\gamma = 1/2$.

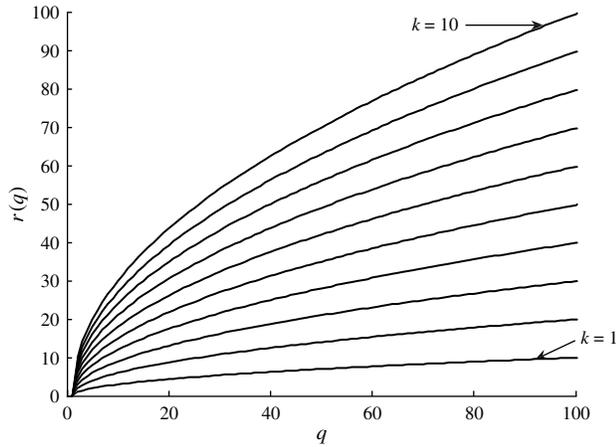
3.2. Demand Function

We assume that each consumer buys at most one unit of the product. Let D denote the demand for the product. Then

$$0 \leq D \leq \int_{\theta_{\min}}^{\theta_{\max}} \int_{k_{\min}}^{k_{\max}} f(\theta)g(k) d\theta dk = 1.$$

physical life, the solution is obtained by relaxing these constraints (or equivalently, setting the relevant Lagrange multipliers to zero).

Figure 2 $v(q) = kq^{1/2}$, Where $k \in [1, \dots, 10]$



We decompose demand into two parts, $D = D_1 + D_2$, where D_1 is the demand from buyers with usage rates $\theta < x/y$, who obtain $q = \theta y$ ($< x$) units of use; and D_2 is the demand from buyers with usage rates $\theta \geq x/y$, who obtain $q = x$ ($\leq \theta y$) units of use.

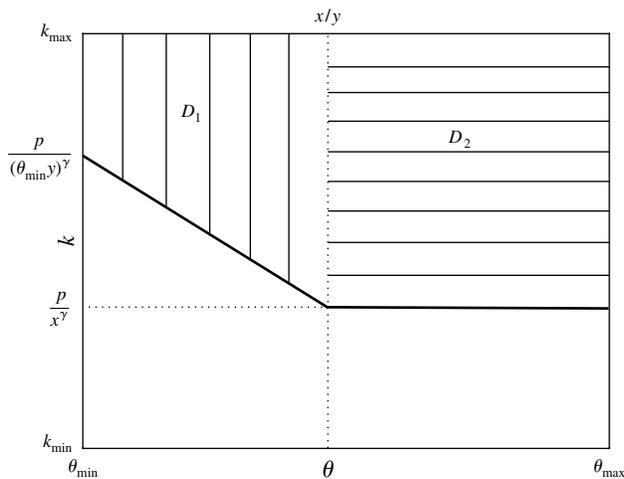
Figure 3 shows how D_1 and D_2 depend on the usable life, physical life, price, usage rate, unit valuation, and elasticity of product valuation with respect to the quantity of use. Usage rate is plotted on the horizontal axis, with a critical break at $\theta = x/y$. Unit valuation is plotted on the vertical axis, with a critical break at p/x^γ . Consumers with usage rates $\theta < x/y$ buy the product if

$$u = kq^\gamma - p = k(\theta y)^\gamma - p > 0,$$

which is equivalent to the condition $k > p/(\theta y)^\gamma$. These buyers are associated with the region above the sloping line on the left-hand side of Figure 3. The demand from these buyers is

$$D_1 = \int_{\theta_{\min}}^{x/y} \int_{p/(\theta y)^\gamma}^{k_{\max}} f(\theta)g(k) d\theta dk$$

Figure 3 Demand from Two Types of Buyers



$$= \int_{\theta_{\min}}^{x/y} \left\{ 1 - G\left(\frac{p}{(\theta y)^\gamma}\right) \right\} f(\theta) d\theta,$$

where $G(k)$ is the cumulative density function associated with $g(k)$. Similarly, consumers with usage rates $\theta \geq x/y$ buy the product if

$$u = kq^\gamma - p = kx^\gamma - p > 0,$$

which is equivalent to the condition $k > p/x^\gamma$. These buyers are associated with the region above the flat line on the right-hand side of Figure 3. The demand from these buyers is

$$D_2 = \int_{x/y}^{\theta_{\max}} \int_{p/x^\gamma}^{k_{\max}} f(\theta)g(k) d\theta dk$$

$$= \left\{ 1 - G\left(\frac{p}{x^\gamma}\right) \right\} \left\{ 1 - F\left(\frac{x}{y}\right) \right\},$$

where $F(\theta)$ is the cumulative density function associated with $f(\theta)$.

Observe that $D_1 = 0$ and $D_2 = D$ when $x/y = \theta_{\min}$. As x/y increases, D_1 increases, D_2 decreases, more consumers use the product to the end of its physical life (obtaining only partial use of the product), and the firm has greater ability to discriminate among these buyers on the basis of their usage rates. When $x/y = \theta_{\max}$, product use varies by usage rate for all consumers, and $D_1 = D$, $D_2 = 0$. Also, note that when x/y is equal to θ_{\min} or θ_{\max} , the firm only needs to choose either usable life or physical life; the other variable takes a value determined by the minimum or maximum usage rate.

3.3. Product Design and Pricing

Let $c \equiv c(x, y)$ denote the cost of producing a unit of the product with usable life x and physical life y . We assume increasing marginal costs ($\partial c/\partial x$, $\partial c/\partial y > 0$), but we make no assumption about how these change with usable life or physical life. For example, it is possible that the marginal cost increases slowly until the usable life or physical life reaches an upper bound that is due to planned product obsolescence. Alternatively, the marginal cost might increase at a high enough rate that the upper bounds on the usable life or physical life are not binding. Let

$$e_x = \frac{\partial c}{\partial x} \cdot \frac{x}{c}, \quad e_y = \frac{\partial c}{\partial y} \cdot \frac{y}{c}$$

denote the cost elasticities with respect to x and y , respectively. As the marginal costs are positive, $e_x, e_y > 0$, for all values of x and y .

The optimal values for usable life, physical life, and price are obtained by solving the following maximization problem in which usable life and physical life

are constrained to lie below their maximum possible values:

$$\begin{aligned} \max_{x, y, p} \quad & \pi = (p - c)D \\ \text{subject to} \quad & x \leq x_{\max}, \quad y \leq y_{\max}. \end{aligned}$$

Consider the Lagrangian

$$L = (p - c)D - \lambda_x(x - x_{\max}) - \lambda_y(y - y_{\max}),$$

where $\lambda_x, \lambda_y \geq 0$ are the shadow prices, which reflect the marginal change in profit for a unit increase in x_{\max} and y_{\max} , respectively. Let the maximum value of L be obtained for $x = x^*, y = y^*, p = p^*$. Let e_{x^*} and e_{y^*} denote the associated cost elasticities. We define

$$\tau_{x^*} = \frac{\lambda_{x^*}}{D^*} \cdot \frac{x^*}{c^*}, \quad \tau_{y^*} = \frac{\lambda_{y^*}}{D^*} \cdot \frac{y^*}{c^*},$$

where λ_{x^*}/D^* (λ_{y^*}/D^*) is the marginal opportunity cost of producing a unit of the product with optimal usable (physical) life and c^*/x^* (c^*/y^*) is the associated cost per unit of the usable (physical) life. Thus, τ_{x^*} and τ_{y^*} are the ratios of the two costs. For brevity, we refer to τ_{x^*} and τ_{y^*} as the *opportunity ratios* in the rest of this paper. We begin with the following theorem (all proofs are in Appendix A).

THEOREM 1. *The optimal values of the usable life, x^* , the physical life, y^* , and the price, p^* , satisfy the condition*

$$\frac{p^*}{c^*} = \frac{e_{x^*} + e_{y^*} + \tau_{x^*} + \tau_{y^*}}{\gamma},$$

where c^* is the cost at the optimal values x^* and y^* .

Theorem 1 implies that $p^* > c^*$ only if $e_{x^*} + e_{y^*} + \tau_{x^*} + \tau_{y^*} > \gamma$. That is, the firm will make the product only if the sum of cost elasticities and opportunity ratios exceeds the elasticity of product valuation with respect to quantity of use. Given usable and physical lives, a market in which γ has a smaller value will attract fewer buyers. Theorem 1 implies that a firm in such a market will have a higher price and earn a higher profit margin.

Theorem 1 is also useful to understanding the effect of marginal cost on product design and pricing. A firm facing a higher factor–cost elasticity may either charge a higher price and/or make a product with lower usable life or physical life. Suppose it charges a higher price. Theorem 1 implies that the firm should then increase both unit price and unit margin, both of which have the effect of reducing demand. Alternatively, suppose the firm makes a product with lower usable life or physical life, which it sells at a lower price. Theorem 1 implies that the price reduction should then be small enough to ensure that the firm earns a higher unit margin.

Evidently, $\lambda_{x^*} = \tau_{x^*} = 0$, if $x^* < x_{\max}$; and $\lambda_{y^*} = \tau_{y^*} = 0$, if $y^* < y_{\max}$. Thus, if the optimal usable life and physical life are both below their maximum possible values, $p^*/c^* = (e_{x^*} + e_{y^*})/\gamma$, and from Theorem 1, the firm will make the product only if $e_{x^*} + e_{y^*} > \gamma$, so that price exceeds cost. Is it possible that $e_{x^*} + e_{y^*} < \gamma$ for a profit-maximizing solution with $x^* < x_{\max}$, $y^* < y_{\max}$? We show that it is not. Suppose such a solution did exist, so that $p^* < c^*$ and $\pi^* < 0$. Given a cost function, c , it is always possible to find another cost function, c' , such that $c' \geq c$ and $e_x + e_y > \gamma$ for all $x, y > 0$. An example of such a cost function is $c' = Ax^\alpha y^\beta$ with suitably large constant A (>0) and cost elasticities $e'_x = \alpha$, $e'_y = \beta$ with $\alpha + \beta > \gamma$. Then at any interior optima, say, (x', y', p') , we have $p' = (e'_x + e'_y)c' > \gamma c'$, which implies $\pi' > 0$. But $c' \geq c$ for all $x, y > 0$, so that if the cost associated with the solution (x', y', p') were c instead of c' , the profit, say, π , would be no smaller than π' . In addition, $\pi \leq \pi^*$ because π^* is the maximum profit associated with the cost function c . It follows that $0 < \pi' \leq \pi \leq \pi^*$, a contradiction of the assumption $\pi^* < 0$. We thus conclude that there can be no interior profit-maximizing solution when $e_x + e_y < \gamma$. It follows that there must instead be a profit-maximizing solution for which the usable life and/or the physical life takes the maximum possible value.

COROLLARY 1. *If $e_{x^*} + e_{y^*} \leq \gamma$, then the firm will make the product only if (i) $x^* = x_{\max}$ and/or $y^* = y_{\max}$, and (ii) $x^* \leq \min\{x_{\max}, \theta_{\max} y_{\max}\}$ and $y^* \leq \min\{y_{\max}, x_{\max}/\theta_{\min}\}$.*

If $x^* = x_{\max}$ and/or $y^* = y_{\max}$, then $\tau_{x^*} > 0$ and/or $\tau_{y^*} > 0$, and the firm can profitably sell the product when $e_{x^*} + e_{y^*} + \tau_{x^*} + \tau_{y^*} > \gamma$. Thus, the firm increases its price and unit margin in response to a constraint that limits its choice of the product's usable life or physical life. The other alternative—to cut price (and profit margin) and attract buyers—is not as profitable. Both technological constraints and high cost elasticities are particularly pronounced in the early stages of a new product's life, and these factors can contribute to its price being set at a substantial multiple of its cost.

Consider $x^* = x_{\max}$, $y^* = y_{\max}$. Then the optimal price is obtained by solving the following problem:

$$\max_p \pi = (p - c^*)D,$$

where $D = D(p)$ is a function only of price. This problem has a well-known solution:

$$\frac{p^*}{c^*} = \frac{\eta_{p^*}}{1 + \eta_{p^*}},$$

where $\eta_p = (dD/dp)(p/D)$ is the price elasticity of demand. Profit decreases with η_{p^*} and is positive only

if $\eta_{p^*} < -1$. It follows from Corollary 1 that the optimality condition relating the price, cost elasticities, and opportunity costs can be written as

$$\frac{p^*}{c^*} = \frac{e_{x^*} + e_{y^*} + \tau_{x^*} + \tau_{y^*}}{\gamma} = \frac{\eta_{p^*}}{1 + \eta_{p^*}}.$$

Now consider $y^* = y_{\max}$, $x^* < x_{\max}$. Then a similar analysis gives the optimality condition:

$$\frac{p^*}{c^*} = \frac{e_{x^*} + e_{y^*} + \tau_{y^*}}{\gamma} = -\frac{e_{x^*} \eta_{p^*}}{\eta_{x^*}} = \frac{\eta_{x^*} + e_{x^*}}{\eta_{x^*}},$$

where $\eta_x = (dD/dx)(x/D)$ is the elasticity of demand with respect to the usable life. Thus, the unit markup

$$\frac{p^* - c^*}{c^*} = \frac{e_{x^*}}{\eta_{x^*}}$$

is equal to the ratio of the cost elasticity and price elasticity with respect to the usable life. Similarly, if $x^* = x_{\max}$, $y^* < y_{\max}$, then the optimality condition is

$$\frac{p^*}{c^*} = \frac{e_{x^*} + e_{y^*} + \tau_{x^*}}{\gamma} = -\frac{e_{y^*} \eta_{p^*}}{\eta_{y^*}} = \frac{\eta_{y^*} + e_{y^*}}{\eta_{y^*}},$$

where $\eta_y = (dD/dy)(y/D)$ is the elasticity of demand with respect to the physical life. The unit markup

$$\frac{p^* - c^*}{c^*} = \frac{e_{y^*}}{\eta_{y^*}}$$

is equal to the ratio of the cost elasticity and price elasticity with respect to the physical life.

3.4. Interdependent Usable Life and Physical Life

So far, we have made the simplifying assumption that the firm can select the usable life and physical life independently. In many cases, the use of certain materials or designs can affect both the usable life and physical life at the same time. We now consider such partial dependence between the two. In particular, suppose

$$x = X + r_x Y \quad \text{and} \quad y = Y + r_y X.$$

We interpret these expressions as follows. The firm can choose a usable life X and a physical life Y . However, the choice of Y changes the usable life available to customers by an amount that is proportional to the value of Y . Similarly, the choice of X changes the physical life by an amount proportional to the value of X . We note that this formulation is equivalent to assuming a linear relationship between x and y , because we can rewrite the above expression for y in the form $Y = y - r_y X$, substitute for Y $x = X + r_x Y$, and obtain the relation $x = (1 - r_x r_y)X + r_x y$. Symmetrically, $y = (1 - r_x r_y)Y + r_y x$.

If $r_x = r_y = 0$, then there is no interdependence between X and Y , and we obtain the results of the

previous sections. If $r_x > 0$, then designing a product with a longer physical life also increases its usable life (based, for example, on using more durable materials or components). If $r_x < 0$, a longer physical life reduces the usable life. Similar interpretations apply for r_y . In this formulation of the problem, the firm selects values of X and Y , and those choices determine, in part, usable life (x) and physical life (y). We show in Appendix B that the optimal solution in this case is for the firm to choose values of X and Y that satisfy the same conditions as in Theorem 1 and Corollary 1, with X replacing x and Y replacing y in their statements in obvious ways.

An alternative approach to modeling the interdependence between usable life and physical life is to consider a situation in which both of these are functions of a common underlying factor, say, z . To consider the simplest such case, suppose x and y are independent linear functions of z : $x = r_1 z$, $y = r_2 z$, $r_1, r_2 > 0$. Then $y = (r_2/r_1)x = rx$. We can then show that, analogous to Theorem 1, $p^* = (e_{x^*} + \tau_{x^*})$ and $y^* = rx^*$. The firm thus loses the freedom to separately choose the usable life and the physical life of a product, and its profit will be lower if technological interdependence forces the firm to choose a higher or lower level of the usable life or the physical life than it might do otherwise. Underdesign can lead to lower demand, and overdesign can lead to higher costs and higher price.

4. Implications

4.1. Factors Affecting Price-to-Cost Ratio

In general, higher cost elasticities, higher opportunity costs, and lower values of the elasticity of the product valuation with respect to use increase the price-to-cost ratio and the profit margin. To illustrate, consider a multiplicative (Cobb–Douglas) cost function, $c = Ax^\alpha y^\beta$, with cost elasticities of $e_x = \alpha$, $e_y = \beta$, where $\alpha, \beta > 0$. Suppose cost is a convex function of x and y . Then $\alpha, \beta \geq 1$ and, since $\tau_{x^*}, \tau_{y^*} \geq 0$, Theorem 1 implies

$$p^* = \frac{(e_{x^*} + e_{y^*} + \tau_{x^*} + \tau_{y^*})c^*}{\gamma} \geq \frac{2c^*}{\gamma}.$$

Thus, the optimal price is at least twice the cost if the utility obtained by consumers increases at a decreasing rate with product use ($\gamma < 1$). Also, as x_{\max} and y_{\max} take larger values, the opportunity costs (and the opportunity ratios, τ_{x^*} and τ_{y^*}) can become smaller. It follows from Theorem 1 that the price multiple at which a product is sold can decrease as technological and design improvements allow a firm to relax constraints and make products with higher usable life

or physical life. Indeed, changes in design and production processes can yield two types of cost reduction: (i) lower total factor costs and (ii) lower marginal costs (which, in turn, implies lower cost elasticities). For example, if $c = Ax^\alpha y^\beta$, then smaller values of α and β reflect lower marginal costs and a lower value of A reflects a lower total factor cost, which can be due to a more efficient use of resources, improvements in production processes, economies of scale, and experience effects. Both lead to lower costs and thus a lower price. However, Theorem 1 implies that a change in the total factor cost will not affect the optimal price-to-cost ratio or the percent profit margin. In contrast, lowering cost elasticities can reduce both the cost and the price-to-cost ratio (which implies that the optimal price will decline at a faster rate than the cost). The extent of these reductions is moderated by the value of γ —the larger its value, the smaller the effect of reductions in the cost elasticities on the price-to-cost ratio.

4.2. Relaxing the Upper Bound on Usable Life or Physical Life

When should a firm make investments to increase the value of x_{\max} (y_{\max})? The obvious answer is when $x^* = x_{\max}$ ($y^* = y_{\max}$). However, following Corollary 1, there are at least two conditions. The first occurs when $y^* = x_{\max}/\theta_{\min}$, that is, when the physical life corresponds to the maximum duration of time any consumer can use a product with a usable life of x_{\max} . Recall that $x_{\max}/\theta_{\min} > x_{\max}/\theta$ for all $\theta > \theta_{\min}$, where θ is the (exogenous) usage rate for a consumer. The only way to profitably increase the physical life is to increase x_{\max} , even if $x^* < x_{\max}$. The second situation occurs when $x^* = \theta_{\max} y_{\max}$, the maximum number of uses a consumer can obtain from a product that has a physical life of y_{\max} . Again, note that $\theta_{\max} y_{\max} > \theta y_{\max}$ for each $\theta < \theta_{\max}$. The only way to profitably increase the usable life is to increase y_{\max} , even if $y^* < y_{\max}$.

4.3. Pricing Product Improvements

We examine below if and when the firm should increase the usable life or physical life of the product and simultaneously lower its price ($p_2^* < p_1^*$).

Consider a firm for which the cost elasticities and/or the opportunity ratios have a lower value at time $t = 2$ than at time $t = 1$. Let c_t^* denote the cost, let $e_{x_t^*}$ and $e_{y_t^*}$ denote the cost elasticities, and let $\tau_{x_t^*}$ and $\tau_{y_t^*}$ denote the opportunity ratios at time t . Because the cost elasticities and/or the opportunity ratios are lower at time $t = 2$,

$$\frac{e_{x_1^*} + e_{y_1^*} + \tau_{x_1^*} + \tau_{y_1^*}}{e_{x_2^*} + e_{y_2^*} + \tau_{x_2^*} + \tau_{y_2^*}} = K > 1.$$

From Theorem 1,

$$p_t^* = \frac{e_{x_t^*} + e_{y_t^*} + \tau_{x_t^*} + \tau_{y_t^*}}{\gamma}, \quad t = 1, 2.$$

Thus, $p_2 < p_1$ if

$$\frac{c_2^*}{c_1^*} < \frac{e_{x_1^*} + e_{y_1^*} + \tau_{x_1^*} + \tau_{y_1^*}}{e_{x_2^*} + e_{y_2^*} + \tau_{x_2^*} + \tau_{y_2^*}} = K.$$

That is, $c_2^* < Kc_1^*$ is a sufficient condition for the firm to lower the price at time $t = 2$. Since $K > 1$, the cost of making the product at time $t = 2$ may be higher and yet the firm will find it optimal to cut the price. The value of K increases with the magnitude of the reduction in the cost elasticities and opportunity ratios; as K increases, so does the upper limit on the cost c_2^* at which a price reduction is optimal for the firm. In other words, a larger reduction in cost elasticities and opportunity ratios allows the firm to not only absorb a larger cost increase but also to simultaneously lower the price. Note that reductions in cost elasticities can lead to (i) increases in the values of x_t^* and y_t^* , so that the product becomes more attractive to consumers; and (ii) under certain conditions, they also allow the firm to reduce the price and increase demand.

4.4. Cross-Subsidizing Product Improvements

Sometimes a firm might find it cheaper or easier to lower the marginal cost for (say) physical life. On the other hand, it might prefer to increase the usable life of the product. Is it possible for the firm to cross-subsidize product development so that the cost savings from lowering the marginal cost of usable life can be used to increase the physical life of the product without raising price?

We illustrate that it is. Consider the cost function $c_t = A_t x^{\alpha_t} y^{\beta_t}$. Let $A_2 = A_1$, so that there is no reduction in total factor cost. Also, let $\tau_{x_t} = \tau_{y_t} = 0$, $t = 1, 2$, so that both usable life and physical life are below their upper bounds at $t = 1$ and $t = 2$. Following the development in §4.3, $p_2^* < p_1^*$ if

$$\frac{x_2^{\alpha_2}}{x_1^{\alpha_1}} \cdot \frac{y_2^{\beta_2}}{y_1^{\beta_1}} < K.$$

Thus, $\beta_1 > \beta_2$ (i.e., a lower cost elasticity for the physical life) allows the firm to increase the usable life and reduce the price of the product even if the cost elasticity for the usable life is unchanged ($\alpha_2 = \alpha_1$). Similarly, a decrease in the marginal cost for the usable life can be used to increase the physical life. Put another way, investment in cost reduction for the usable (physical) life does not necessarily have to translate into a higher usable (physical) life for a product. Technical (and investment) considerations can determine where the firm should cut costs and market factors can be used to decide whether product improvements should extend the usable life or the physical life of the product. Such cross-subsidies can be useful when the firm wants to lower the price and/or improve the product and finds it easier or cheaper to reduce the

marginal cost of one factor (physical life or usable life). For example, suppose it is easier to reduce the marginal cost for the physical life of a lithium-ion battery than for its usable life. Doing so can allow the firm to lower the selling price for a battery that has a higher usable life and an unchanged physical life.

5. Summary and Future Research

The present paper is a first step in developing models that consider the optimal design of durable goods that have distinct usable and physical lives. It proposes a framework for examining how a firm should choose the usable life or physical life and the selling price of a product. The framework allows consideration of the following four factors: (i) the marginal cost of a product's usable life and physical life, (ii) the limitations of technology that at any given time place an upper bound on the usable life or physical life, (iii) the heterogeneity in consumer usage rates, and (iv) the heterogeneity in the value obtained from product use by consumers. We allow arbitrary cost functions and statistical distributions to characterize the heterogeneity in consumer usage rates and product valuations. We find that a firm's decision to make a product with the greatest possible usable life or physical life depends not only on technological feasibility and cost but also on the distribution of consumers' usage rates and utilities and their marginal valuations of product use. The price-to-cost ratio (and thus the percent profit margin) (i) increases with cost elasticities and the opportunity cost of relaxing the technological constraints, and (ii) decreases with the marginal valuation of product use by consumers. If this marginal valuation is nonincreasing in the amount of product use (i.e., if $\gamma \leq 1$), then the optimal price is at least twice the cost of the product for a cost function that has constant cost elasticities.

Lower cost elasticities and opportunity costs can make it optimal for the firm to lower prices and margins and simultaneously increase the usable life or physical life of the product. Sometimes a longer usable life can be obtained by lowering the marginal cost for the physical life, and vice versa. Put another way, how a firm cuts cost can be separated from how it improves a product. In some instances, consumers cannot avail themselves of a longer physical life unless the upper bound on the usable life is relaxed. In such a situation, the firm may find it profitable to increase the upper bound on the usable life but not the usable life of the product itself. A parallel situation can occur when the firm finds it profitable to relax the upper bound on the physical life but not the physical life of the product itself.

The present research could be further developed to study the effects of ownership on product value,

uncertainty in failure time, purchases of multiple units over time, competition, and the evolution of product design over time. It can also be used to examine how a firm might select usable life and physical life for a line of products. We briefly comment on each of these aspects.

Failed products can sometimes have ownership value for consumers. For example, one might keep a toy for emotional reasons, even if it no longer works. Such residual value, which might even increase with product use, can potentially allow the firm to charge a higher price for the product.²

It is also useful to examine the effect of relaxing the present assumption that a product has a deterministic (predictable) usable life and physical life. Uncertainty in this context can lead to lower willingness to pay and thus to lower price or higher expected usable life or physical life of the product. The more skewed the failure distribution is to the left (i.e., the higher the probability of a shorter product life), the greater the possibility that the firm will charge a lower price and may increase the target level of the usable life or physical life depending on the cost of such improvement. Additionally, the firm might be able to make an investment that reduces the probability of product failure before the end of its usable life or physical life. How much it should invest in these efforts, and when, can be a useful question to examine in future research.

Competition can have two effects. On the one hand, it can increase the pressure on a firm to increase the usable life or physical life of a product and lower its cost. On the other hand, it can create incentives for firms to differentiate their products by focusing on different dimensions—one firm, for example, could choose to make a product that has a long physical life but is targeted to consumers with low usage rates while another could make a product with a short physical life that targets consumers with high usage rates. Further differentiation between consumers with high and low product valuations may also be feasible for firms. Additionally, competition can increase the incentives for firms to make investments that reduce costs and relax the technological barriers that limit the usable and physical lives of products.

Modeling multiple purchases over time requires the development of a dynamic model. In such a model, the timing of purchases would depend on the usage rates of consumers; brand choice would depend on alternatives available at a given time. Dynamic models could also be used to examine product-launch strategies over time and to study the effects of planned product obsolescence on consumer

²We are grateful to the editor for bringing this point to our attention.

purchases, product design, and pricing of durable goods.

Finally, it might be useful to examine how a firm might select the usable life and physical life across a product line. Multiple products might allow price discrimination and reduce the amount of usable (physical) life left unused by consumers with low (high) usage rates.

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Appendix A. Proofs of Theorems

THEOREM 1. *The optimal values of the usable life x^* , the physical life y^* , and the price p^* , satisfy the condition*

$$\frac{p^*}{c^*} = \frac{e_{x^*} + e_{y^*} + \tau_{x^*} + \tau_{y^*}}{\gamma},$$

where c^* is the cost at the optimal values x^* and y^* .

PROOF. A profit-maximizing firm selects x , y , and p to solve

$$\begin{aligned} \max_{x, y, p} \quad & \pi = (p - c)D \\ \text{subject to} \quad & x \leq x_{\max}, \quad y \leq y_{\max}, \end{aligned}$$

where p is the unit price, $c \equiv c(x, y)$ is the cost of producing a product with usable life x and physical life y , x_{\max} and y_{\max} correspond to the maximum possible values of the usable life and physical life, respectively, and D is the demand given by the expression

$$D = \left\{ 1 - G\left(\frac{p}{x^\gamma}\right) \right\} \left\{ 1 - F\left(\frac{x}{y}\right) \right\} + \int_{\theta_{\min}}^{x/y} \left\{ 1 - G\left(\frac{p}{(\theta y)^\gamma}\right) \right\} f(\theta) d\theta.$$

Let

$$L = (p - c)D - \lambda_x(x - x_{\max}) - \lambda_y(y - y_{\max}),$$

where $\lambda_x, \lambda_y \geq 0$ are Lagrange multipliers. We use the Leibniz integral rule to obtain the following first-order conditions:

$$\begin{aligned} \frac{\partial L}{\partial x} &= \left[\frac{\gamma p}{x^{\gamma+1}} g\left(\frac{p}{x^\gamma}\right) \left\{ 1 - F\left(\frac{x}{y}\right) \right\} \right] (p - c) - D \frac{\partial c}{\partial x} - \lambda_x = 0; \\ \frac{\partial L}{\partial y} &= \left[\int_{\theta_{\min}}^{x/y} \frac{\gamma p}{y(\theta y)^\gamma} g\left(\frac{p}{(\theta y)^\gamma}\right) f(\theta) d\theta \right] (p - c) - D \frac{\partial c}{\partial y} - \lambda_y = 0; \\ \frac{\partial L}{\partial p} &= \left[\frac{1}{x^\gamma} g\left(\frac{p}{x^\gamma}\right) \left\{ 1 - F\left(\frac{x}{y}\right) \right\} \right. \\ &\quad \left. + \int_{\theta_{\min}}^{x/y} \frac{1}{(\theta y)^\gamma} g\left(\frac{p}{(\theta y)^\gamma}\right) f(\theta) d\theta \right] (p - c) - D = 0. \end{aligned}$$

Let

$$V = \frac{1}{x^\gamma} g\left(\frac{p}{x^\gamma}\right) \left\{ 1 - F\left(\frac{x}{y}\right) \right\} (p - c)$$

and

$$W = (p - c) \int_{\theta_{\min}}^{x/y} \frac{1}{(\theta y)^\gamma} g\left(\frac{p}{(\theta y)^\gamma}\right) f(\theta) d\theta.$$

Then we can express the preceding first-order conditions as

$$\begin{aligned} \frac{\partial L}{\partial x} &= \frac{\gamma p}{x} V - D \frac{\partial c}{\partial x} - \lambda_x = 0; \\ \frac{\partial L}{\partial y} &= \frac{\gamma p}{y} W - D \frac{\partial c}{\partial y} - \lambda_y = 0; \\ \frac{\partial L}{\partial p} &= V + W - D = 0. \end{aligned}$$

The expressions for $\partial L/\partial x$ and $\partial L/\partial y$ give

$$V = \frac{x}{\gamma p} \left(D \frac{\partial c}{\partial x} + \lambda_x \right)$$

and

$$W = \frac{y}{\gamma p} \left(D \frac{\partial c}{\partial y} + \lambda_y \right).$$

Substituting for V and W in the expression for $\partial L/\partial p$ and rearranging terms yields

$$\gamma p = x \left(\frac{\partial c}{\partial x} + \frac{\lambda_x}{D} \right) + y \left(\frac{\partial c}{\partial y} + \frac{\lambda_y}{D} \right).$$

We divide both sides of this expression by c , substitute

$$e_x = \frac{\partial c}{\partial x} \cdot \frac{x}{c}, \quad e_y = \frac{\partial c}{\partial y} \cdot \frac{y}{c}, \quad \tau_x = \frac{\lambda_x}{D} \cdot \frac{x}{c}, \quad \tau_y = \frac{\lambda_y}{D} \cdot \frac{y}{c},$$

and rearrange the terms to obtain

$$\frac{p^*}{c^*} = \frac{e_{x^*} + e_{y^*} + \tau_{x^*} + \tau_{y^*}}{\gamma},$$

where $*$ denotes the optimal values of the relevant terms. \square

COROLLARY 1. *If $e_{x^*} + e_{y^*} \leq \gamma$, then the firm will make the product only if (i) $x^* = x_{\max}$ and/or $y^* = y_{\max}$, and (ii) $x^* \leq \min\{x_{\max}, \theta_{\max} y_{\max}\}$ and $y^* \leq \min\{y_{\max}, x_{\max}/\theta_{\min}\}$.*

PROOF. If $\tau_{x^*} + \tau_{y^*} \leq \gamma - (e_{x^*} + e_{y^*})$, then $p \leq c$, so that the firm has no incentive to make the product. Thus, at least one of τ_{x^*} and τ_{y^*} must be positive; i.e., $x^* = x_{\max}$ and/or $y^* = y_{\max}$. We consider each case below.

Suppose $x = x_{\max}$. Then x_{\max}/θ_{\min} is the maximum amount of time any consumer can use a product, and the firm will not make a product with longer physical life because this increases the cost, but not the demand, for the product. It follows that $y^* \leq \min\{y_{\max}, x_{\max}/\theta_{\min}\}$. If $y^* = y_{\max}$, then $\tau_{y^*} > 0$, and the firm makes the product when $\tau_{x^*} + \tau_{y^*} > \gamma - (e_{x^*} + e_{y^*})$; otherwise, $\tau_{y^*} = 0$, and the firm makes the product only when $\tau_{x^*} > \gamma - (e_{x^*} + e_{y^*})$.

Next, suppose $y = y_{\max}$. Then $\theta_{\max} y_{\max}$ is the maximum amount of use any consumer can obtain from a product, and the firm will not make a product with a greater usable life because this increases the cost, but not the demand, for the product. It follows that $x^* \leq \min\{x_{\max}, \theta_{\max} y_{\max}\}$. If $x^* = x_{\max}$, then $\tau_{x^*} > 0$, and the firm makes the product when $\tau_{x^*} + \tau_{y^*} > \gamma - (e_{x^*} + e_{y^*})$; otherwise, $\tau_{x^*} = 0$, and the firm makes the product only when $\tau_{y^*} > \gamma - (e_{x^*} + e_{y^*})$. \square

Appendix B. Interdependent Usable Life and Physical Life

We consider here the case in which the usable and physical lives of a product are interdependent. Suppose the firm

were to choose the values $x = X$ and $y = Y$ for the usable life and the physical life, respectively. Because of the interdependence, the choice of Y affects the value of x and the choice of X affects the value of y . For simplicity, we assume that these effects are linear:

$$x = X + r_x Y, \quad y = Y + r_y X,$$

where r_x and r_y have unrestricted sign. The special case with $r_x = r_y = 0$ reduces to the case examined in Theorem 1. Here, we show that the result in Theorem 1 is still obtained if we replace x by X and y by Y .

Consider a profit-maximizing firm that selects X , Y , and p , where $X \leq X_{\max}$ and $Y \leq Y_{\max}$. The corresponding optimization problem is

$$\begin{aligned} \max_{X, Y, p} \quad & \pi = (p - c)D \\ \text{subject to} \quad & X \leq X_{\max}, \quad Y \leq Y_{\max}, \end{aligned}$$

where p is the unit price, $c \equiv c(X, Y)$ is the cost of producing a product with usable life X and physical life Y , X_{\max} and Y_{\max} correspond to the maximum possible values for usable life and physical life, respectively, and D is the demand given by the expression

$$\begin{aligned} D = & \int_{\theta_{\min}}^{(X+r_x Y)/(Y+r_y X)} \left\{ 1 - G\left(\frac{p}{(\theta^\gamma(Y+r_y X)^\gamma)}\right) \right\} f(\theta) d\theta \\ & + \int_{(X+r_x Y)/(Y+r_y X)}^{\theta_{\max}} \left\{ 1 - G\left(\frac{p}{(X+r_x Y)^\gamma}\right) \right\} f(\theta) d\theta. \end{aligned}$$

Let

$$L = (p - c)D - \lambda_X(X - X_{\max}) - \lambda_Y(Y - Y_{\max}),$$

where $\lambda_X, \lambda_Y \geq 0$ are Lagrange multipliers. We use the Leibniz integral rule to obtain the following first-order conditions:

$$\begin{aligned} \frac{\partial L}{\partial X} = & (p - c) \left[\int_{\theta_{\min}}^{(X+r_x Y)/(Y+r_y X)} \left\{ \frac{\gamma p r_Y}{\theta^\gamma (Y+r_y X)^{\gamma+1}} \right. \right. \\ & \left. \left. \cdot g\left(\frac{p}{\theta^\gamma (Y+r_y X)^\gamma}\right) \right\} f(\theta) d\theta \right. \\ & \left. + \int_{(X+r_x Y)/(Y+r_y X)}^{\theta_{\max}} \left\{ \frac{\gamma p}{(X+r_x Y)^{\gamma+1}} \right. \right. \\ & \left. \left. \cdot g\left(\frac{p}{(X+r_x Y)^\gamma}\right) \right\} f(\theta) d\theta \right] - D \frac{\partial c}{\partial X} - \lambda_X \\ = & 0. \\ \frac{\partial L}{\partial Y} = & \left[\int_{\theta_{\min}}^{(X+r_x Y)/(Y+r_y X)} \left\{ \frac{\gamma p}{\theta^\gamma (Y+r_y X)^{\gamma+1}} \right. \right. \\ & \left. \left. \cdot g\left(\frac{p}{\theta^\gamma (Y+r_y X)^\gamma}\right) \right\} f(\theta) d\theta \right. \\ & \left. + \int_{(X+r_x Y)/(Y+r_y X)}^{\theta_{\max}} \left\{ \frac{\gamma p r_X}{(X+r_x Y)^{\gamma+1}} \right. \right. \\ & \left. \left. \cdot g\left(\frac{p}{(X+r_x Y)^\gamma}\right) \right\} f(\theta) d\theta \right] (p - c) - D \frac{\partial c}{\partial Y} - \lambda_Y \\ = & 0. \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial p} = & \left[\int_{\theta_{\min}}^{(X+r_x Y)/(Y+r_y X)} \left\{ \frac{1}{\theta^\gamma (Y+r_y X)^\gamma} \right. \right. \\ & \left. \left. \cdot g\left(\frac{p}{\theta^\gamma (Y+r_y X)^\gamma}\right) \right\} f(\theta) d\theta \right. \\ & \left. + \int_{(X+r_x Y)/(Y+r_y X)}^{\theta_{\max}} \left\{ \frac{1}{(X+r_x Y)^\gamma} \right. \right. \\ & \left. \left. \cdot g\left(\frac{p}{(X+r_x Y)^\gamma}\right) \right\} f(\theta) d\theta \right] (p - c) - D \\ = & 0. \end{aligned}$$

Let

$$\begin{aligned} V = & (p - c) \int_{\theta_{\min}}^{(X+r_x Y)/(Y+r_y X)} \left\{ \frac{1}{\theta^\gamma (Y+r_y X)^\gamma} \right. \\ & \left. \cdot g\left(\frac{p}{\theta^\gamma (Y+r_y X)^\gamma}\right) \right\} f(\theta) d\theta, \\ W = & (p - c) \int_{(X+r_x Y)/(Y+r_y X)}^{\theta_{\max}} \left\{ \frac{1}{(X+r_x Y)^\gamma} \right. \\ & \left. \cdot g\left(\frac{p}{(X+r_x Y)^\gamma}\right) \right\} f(\theta) d\theta. \end{aligned}$$

The expressions for $\partial L/\partial X$ and $\partial L/\partial Y$ give

$$V = \frac{X}{\gamma p} \left(D \frac{\partial c}{\partial X} + \lambda_X \right), \quad W = \frac{Y}{\gamma p} \left(D \frac{\partial c}{\partial Y} + \lambda_Y \right).$$

Substituting for V and W in the expression for $\partial L/\partial p$ and rearranging the terms yield

$$\gamma p = x \left(\frac{\partial c}{\partial X} + \frac{\lambda_X}{D} \right) + y \left(\frac{\partial c}{\partial Y} + \frac{\lambda_Y}{D} \right).$$

We divide both sides of this expression by c , substitute

$$e_X = \frac{\partial c}{\partial X} \cdot \frac{X}{c}, \quad e_Y = \frac{\partial c}{\partial Y} \cdot \frac{Y}{c}, \quad \tau_X = \frac{\lambda_X}{D} \cdot \frac{X}{c}, \quad \tau_Y = \frac{\lambda_Y}{D} \cdot \frac{Y}{c},$$

and rearrange the terms to obtain

$$\frac{p^*}{c^*} = \frac{e_{X^*} + e_{Y^*} + \tau_{X^*} + \tau_{Y^*}}{\gamma},$$

where $*$ denotes the optimal values of the relevant terms.

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