

# Delegating Learning\*

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February 14, 2019

## Abstract

Learning is crucial to organizational decision making but often needs to be delegated. We examine a dynamic delegation problem where a principal decides on a project with uncertain profitability. A biased agent, who is initially as uninformed as the principal, privately learns the profitability over time and communicates to the principal. We formulate learning delegation as a dynamic mechanism design problem and characterize the optimal delegation scheme. We show that private learning gives rise to the tradeoff between how much information to acquire and how promptly it is reflected in the decision. We discuss implications on learning delegation for distinct organizations.

**Keywords:** private learning, delegation, delays, deadlines, commitment, cheap talk

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\*We are extremely grateful for comments from Nageeb Ali, Federico Echenique, Asher Wolinsky and participants of various conferences and seminars. We acknowledge financial support from the Institute for Research in Market Imperfections and Public Policy, MIPP, ICM IS130002. Zhang also acknowledges financial support from CONICYT-FONDECYT postdoctoral award (Project 3170783).

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# 1 Introduction

Suppose that two people need to decide whether to invest in a project. If they invest, they could receive a gain or suffer a loss. If they do not invest, they wait, obtain new information, and may invest after a delay. Now suppose that one of them, the principal, prefers to learn, while the other, the agent, prefers to invest. If learning has to be delegated to the agent and the principal cannot observe the learning outcome, can the agent convey it truthfully? If so, what should be the optimal delegation scheme? How does it change over time given what the agent has learned so far? How long should learning take place?

Learning delegation is a common occurrence. For example, suppose the board of directors of a company is deliberating whether to acquire another company. Apart from the financial value of the acquisition, its strategic value—e.g., its impact on the price and competition, the current employees, the bargaining power with suppliers—is also relevant. Since much of this information is hard to observe directly, the board needs to rely on learning by the manager, who has direct access to the parties involved. However, learning takes time, and a career-driven manager may not share the board’s vision. Even if they may agree on an acquisition with a certain value, when its value is uncertain, the manager may prefer to move forward with it immediately rather than to devote time learning its value.

For another example, the Food and Drug Administration (FDA) relies on pharmaceutical companies to develop drugs and test their efficacy. Although the companies are required to submit clinical trial results, the trials themselves cannot be fully monitored, and therefore the results can be manipulated.<sup>1</sup> While the FDA’s mission is to promote public health, the pharmaceutical companies’ main motive is to profit. So even though the pharmaceutical companies may not want to introduce a drug known to be harmful, they may have less incentive than the FDA to spend time learning about the value of

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<sup>1</sup>Several authors have documented frauds in clinical trials (George and Buyse, 2015). Seife (2015) shows that the FDA has found substantive evidence of fraudulent data in biomedical research on humans. See also Seife, “Are your medications safe,” *Slate*, February 9, 2015.

an unknown drug.

We study how a principal should delegate an investment decision to an agent who privately learns about the investment over time. Our analysis extends the traditional static delegation approach (Holmstrom (1984); Melumad and Shibano (1991); Alonso and Matouschek (2008); Amador and Bagwell (2013); Grenadier, Malenko, and Malenko (2016); Guo (2016)) to allow for evolving private information. We formulate delegation as a dynamic mechanism design problem and characterize its solution. Our results shed light on how organizations could incentivize learning and informative communication with an evolving delegation scheme and why distinct organizations should implement distinct learning delegation procedures.

In our model, a principal and an agent face a project that never expires. The project's quality, which can be good or bad, is initially uncertain. Players share the same belief about the project's quality. The principal needs to decide when, if ever, to invest in the project. The project generates a signal whose arrival time is random. As long as no investment has happened, the agent privately observes the signal, or the absence thereof, without cost. Hence, investing and learning are two sides of the same coin in that as long as investment has not happened, learning continues. A good project generates a good signal while a bad project generates a bad signal; therefore, the arrival of a signal reveals project quality. Depending on which type of signal arrives more frequently, the agent's belief about project quality evolves when no signal has arrived. At each point in time before investment happens, the agent sends a cheap talk message about the information learned so far to the principal. The principal commits to a delegation rule that specifies, for each point in time and each possible message history at that time, whether to invest or not. Once the investment happens, the game ends.

No one receives a payoff if no investment has happened. Once the investment happens, each player receives a time-discounted payoff determined by project quality and the player's identity. A good investment brings gains to both players, while a bad investment brings losses to both players. Consequently, each player will want to invest if they are optimistic enough about

project quality and will want to wait for more information otherwise. However, the players' gain-loss ratios are different, which leads to different attitudes towards learning. For the agent, the gain-loss ratio is sufficiently high that the common initial belief is also sufficiently high, so he prefers to invest immediately. On the contrary, the gain-loss ratio for the principal is not sufficiently high, so she initially wants to learn and invest only when a good signal arrives. The challenge for the principal is then to incentivize the agent to tell the truth when he has not received any signal, while trying to invest as soon as possible after a good signal arrives.

First, we note that if a good signal triggers immediate investment, the agent would like to pretend to be informed when in fact no signal has arrived, and no learning would take place. Therefore, investment must follow a good signal with delay. To see how the delay should evolve over time to incentivize learning, we need to understand the driving forces behind learning. Learning benefits the agent because if a bad signal arrives, he would then learn that the project is bad and avoid the loss from investing. On the other hand, learning costs the agent in that it takes time. Suppose that no signal has arrived and the agent is still optimistic enough to prefer to invest right away. At this point, the cost of learning outweighs the benefit. To encourage learning, the principal needs to decrease the cost by making investment respond faster to the good signal throughout time. Suppose that the principal wants to encourage the agent to learn for one more day. If a claim of good signal leads the principal to invest immediately, then the cost of learning is one day's delay. However, if a good signal that arrives today leads to investment 5 days later while a good signal that arrives tomorrow leads to investment 4.5 days after tomorrow, the cost of learning is only a half-day's delay. Delays in investment that decrease in the arrival times of the good signal allow the principal to balance the cost and benefit of learning for the agent, hence his truthful revelation.

It is natural to think that if no good signal has arrived, investment should not happen. This is not always the case. First, note that the longer learning takes, the more delay the principal needs to incentivize it. Suppose that at some time  $T$ , even if no signal has arrived, learning stops and investment

happens. Since no incentives for learning is required from  $T$  on, the delay decreases gradually to 0 at  $T$ . If instead the principal decides to incentivize learning after  $T$ , the delay at  $T$  must be positive. Accordingly, delays of investment for claims of the good signal at each time before  $T$  must also be increased. Therefore, the longer learning takes place, the better information the principal receives, the more accurate but less prompt her decision to invest is. If no investment happens unless a good signal arrives, learning could take place for an arbitrarily long period of time. Consequently, the decision to invest is 100% accurate because the principal only invests if she is completely sure that the project is good. However the downside is that she has to provide incentives for learning for a long time. The resulting long delays in investment when a good signal arrives can therefore be a prohibitive cost for the principal. Therefore, due to the tradeoff between the amount of information acquired and how effectively it is used, the principal may find it optimal to commit to investing with a deadline even if no good signal has been claimed.

The optimal duration of learning that balances the tradeoff depends on the players' preferences for learning. Given that the principal prefers to wait while the agent prefers to invest initially, if the players' gain-loss ratios are sufficiently high, maintaining incentives for learning becomes very costly for the principal, and therefore the duration of learning is short. On the other hand, when the players' gain-loss ratios are sufficiently low, it is in the principal's interest to maintain a longer learning phase. Our results speak to how distinct organizations should use distinct protocols to delegate learning. In the FDA example, losses from approving a damaging drug are substantial. This maps to our model when the principal has a low gain-loss ratio and therefore is strongly inclined to learn. The optimal course of action for the FDA is then to be prudent by establishing long revision processes. Not only do they ensure that a damaging drug will never get approved, the ensuing long delays in approval also guarantee truthful revelation from pharmaceutical companies. If a manager's career concern is strong and he has a high gain-loss ratio, the optimal action for the board when it comes to acquisition decisions is to set short learning phases and then acquire as long as no negative news has arrived.

We show the robustness of our results to natural variations of the environment. First, we explore a situation in which the agent incurs an unverifiable cost when he learns. The unverifiable learning cost makes the incentive problem more severe. We show how a contract with delays and deadlines can motivate the agent to dynamically acquire information and to report his signal truthfully. Moreover, apart from the baseline model where the frequency of the good signal is higher than that of the bad signal, we also discuss an alternative model where the opposite is true. That is, no news is either no news or good news. We show that even when the agent becomes more eager to invest as time transpires, it is still optimal for the principal to set a decreasing delay and a deadline. Finally, we complement our results by showing how transfers can completely solve the problem of private learning. Indeed, by using transfers, the principal can motivate the agent to learn as well as extract the whole surplus.

Our paper contributes to the delegation literature initiated by Holmstrom (1984) and extended by Melumad and Shibano (1991), Alonso and Matouschek (2008), Armstrong and Vickers (2010), Amador and Bagwell (2013), and Ambrus and Egorov (2017), among others. As in all these papers, in our model the principal may grant flexibility to the agent so that he can use his information, but granting too much flexibility may open up room for opportunistic behavior. However, these papers study static models and do not address the issue of how to provide incentives to an agent with evolving private information.<sup>2</sup> In particular, our work emphasizes how the dynamic provision of incentives determines how information is used and for how long learning takes place.

Grenadier, Malenko, and Malenko (2016) and Guo (2016) explore delegation models in a dynamic context. In Grenadier, Malenko, and Malenko (2016), a timing decision needs to be made and an agent who is informed at time 0 communicates with the principal throughout time. Whereas Grenadier,

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<sup>2</sup>While in our model the principal dynamically screens the agent's information, we depart from the growing dynamic mechanism design literature (Pavan, Segal, and Toikka, 2014; Bergemann and Valimaki, forthcoming; Madsen, 2018) by assuming transfers are infeasible. Our model features symmetric information at time 0 and therefore transfers allow the principal to achieve the first best. See Section 5.3 for details.

Malenko, and Malenko (2016) explore how the value of commitment for the principal depends on the sign of the agent’s bias, we take commitment for granted but explore how to delegate with evolving private information. As Grenadier, Malenko, and Malenko (2016) point out, their full commitment case is similar to standard static delegation problems and, as a result, interval delegation is optimal. In Guo (2016), the principal delegates the decision to experiment over time to an agent who has private information about its profitability at time 0.<sup>3</sup> Once experimentation starts, however, all signals are public. A comparison between our paper and Guo (2016) highlights the differences between private and public learning, which have important implications for the design of incentive schemes. In her model, since signals are public, once a good signal arrives, the risky project is publicly known to be optimal and is fully implemented. In our model, however, investment decisions commonly known to be optimal are nonetheless delayed. This is the principal’s response to the problem of providing incentives to an agent with evolving private information.

Our paper is also related to the study of optimal delegation decisions when information acquisition is endogenous. In Aghion and Tirole (1997), delegation makes the returns to learning higher for the agent, but it can be costly to the principal when she has also become informed herself. In Deimen and Szalay (forthcoming), the precision of the information acquired by the agent is public and can be used to signal the credibility of his advice. Consequently, less information is acquired for the agent’s sake, but his advice becomes more influential in the decision. Szalay (2005) also studies the optimal delegation set when information acquisition is endogenous and the principal can commit. He shows that to incentivize the agent, compromise actions may need to be removed from the delegation set. In all these models, information acquisition is a one-time decision, therefore the tradeoff between extracting information and using information efficiently is different from ours. In contrast, Lewis and Ottaviani (2008) study a setting where the agent searches for the best alternative

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<sup>3</sup>Guo (2016) focuses on the full commitment case, but she also shows that the sign of the agent’s bias determines the value of commitment.

over time, and find the optimal contract for the principal, who cannot observe the search effort. In the monitored search case, the principal can observe the outcome of search. In the delegation case, the principal cannot observe the search outcome and has to rely on the agent’s report over time. They study when it is possible to replicate monitored search through delegation. In both cases, monetary transfers are used, which we rule out.

Frankel (2016), Li, Matouschek, and Powell (2017), Lipnowski and Ramos (2017), Guo and Hörner (2018), and Chen (2018) study repeated delegation models in which parties face a stream of decisions. In these models, incentives can be provided by linking the different decisions. In contrast, we study situations in which a single, irreversible decision is to be made and therefore linking decisions is infeasible.<sup>4</sup>

Finally, our work is related to dynamic persuasion models, particularly, McClellan (2017), Henry and Ottaviani (forthcoming), and Orlov, Skrzypacz, and Zryumov (forthcoming). These papers explore how to design approval rules when learning is costly, signals are public, and incentives are misaligned ex-post. In contrast, we mainly focus on the case where learning is costless, signals are private, and incentives are misaligned ex-ante. In Section 5.1, we allow costly learning.<sup>5</sup>

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 formalizes the dynamic delegation problem. Section 4 presents our main results. Section 5 discusses some extensions. Section 6 concludes.

## 2 The Model

We consider an infinite-horizon continuous-time game played by a principal and an agent. There is an initially unknown state  $\theta \in \{0, 1\}$ . We call  $\theta = 1$  the good state and  $\theta = 0$  the bad state. At time 0, the agent and the principal are symmetrically uninformed about the state  $\theta$ , with  $\mathbb{P}[\theta = 1] = p_0$  being the

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<sup>4</sup>Another difference is that, with the exception of Guo and Hörner (2018), the repeated delegation literature has focused on serially uncorrelated incomplete information.

<sup>5</sup>It is not possible to extract and use any information in a model with private learning in which incentives are misaligned ex-post (and not just ex-ante as in our baseline model).



initial prior.

The agent privately learns about the state without cost.<sup>6</sup> A signal is generated according to an exponential distribution with arrival rate  $\lambda^\theta$ , which depends on  $\theta$ . Specifically, conditional on  $\theta$ , over an interval  $[t, t + dt]$ , a signal  $s_t = \theta$  is realized with probability  $\lambda^\theta dt$ . The arrival of the signal is privately observed by the agent. Thus, the arrival of a signal perfectly reveals the state to the agent. We say that the agent is uninformed if he has not observed a signal. The agent's private history up to period  $t$  is denoted  $h^t$ . We use  $\emptyset$  to denote the history with no signal.

The private belief process  $p_t = \mathbb{P}[\theta = 1 \mid h^t]$  is formed according to the initial prior  $p_0$  and the agent's private history  $h^t$  up to period  $t$ . The law of motion for the agent's private belief  $p_t$  can be derived as follows. If a signal  $s_t = 1$  arrives during the interval  $[t, t + dt)$ , the belief jumps to 1; if a signal  $s_t = 0$  arrives during the interval, the belief jumps to 0. If no signal arrives, Bayes's rule can be used to deduce that the posterior at the end of  $t + dt$  is

$$p_t + dp_t = \frac{p_t(1 - \lambda^1 dt)}{(1 - p_t)(1 - \lambda^0 dt) + p_t(1 - \lambda^1 dt)}.$$

That is, when no signal arrives, the evolution of the belief is governed by the differential equation<sup>7</sup>

$$\frac{dp_t}{dt} = -(\lambda^1 - \lambda^0)p_t(1 - p_t).$$

We assume that  $\lambda^0 < \lambda^1$  and thus no news is bad news. In other words, the belief decreases in the absence of a signal. We show that our results extend to the case  $\lambda^0 \geq \lambda^1$  in Section 5.2.

The principal chooses  $y_t \in \{0, 1\}$  at each  $t \geq 0$ .  $y_t = 1$  means to invest and  $y_t = 0$  means not to invest. The decision to invest is irreversible: if  $y_t = 1$  for some  $t$ , then  $y_\tau = 1$  for all  $\tau > t$  and the interaction ends.

Players' preferences over investment coincide conditional on  $\theta$ . During each

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<sup>6</sup>In Section 5, we extend our model and results to incorporate costly learning.

<sup>7</sup>See Liptser and Shiryaev (2013) for details.

interval  $[t, t+dt)$  for which  $y_t = 0$ , both players receive zero payoff. Conditional on  $\theta$ , if the principal invests at time  $t$ , she gets total discounted payoffs equal to

$$e^{-rt}V \text{ if } \theta = 1 \text{ and } e^{-rt}(-\nu) \text{ if } \theta = 0,$$

whereas the agent gets total discounted payoffs equal to

$$e^{-rt}W \text{ if } \theta = 1 \text{ and } e^{-rt}(-\omega) \text{ if } \theta = 0,$$

where  $r > 0$  is the common discount rate, and  $V$ ,  $\nu$ ,  $W$ , and  $\omega$  are strictly positive.

To state our assumption on the conflict of interest, it is useful to describe the one-person benchmark. Suppose that the agent not only perfectly observes the arrival of the signal but also has the right to invest. The optimal policy for the agent is characterized by a cutoff  $p^* := \frac{(\lambda^1+r)\omega}{rW+(\lambda^1+r)\omega}$  (Keller, Rady, and Cripps, 2005). The agent finds it optimal to invest given the current belief  $p$  if and only if he is optimistic enough about the state; that is,  $p \geq p^*$ . Intuitively, the optimal policy must be a cutoff policy because if the uninformed agent does not find it attractive to invest at  $t$ , then neither does the uninformed agent at  $t+dt$  who is more pessimistic about the value of the investment than at  $t$ . Similarly, suppose that the principal not only controls decisions but also observes the signal. Given the current belief  $p$ , the principal would find it optimal to invest iff  $p \geq q^* = \frac{(\lambda^1+r)\omega}{rW+(\lambda^1+r)\omega}$ .

We can now state the assumption on the conflict of interest, which is maintained throughout the paper.

**Assumption 1**  $p^* < p_0 < q^*$ .

This assumption implies that at  $t = 0$ , the agent wants to invest immediately whereas the principal wants to invest only after observing a good signal. An equivalent formulation for Assumption 1 is

$$\frac{W}{\omega} \frac{r}{\lambda^1 + r} \frac{p_0}{1 - p_0} > 1 > \frac{V}{\nu} \frac{r}{\lambda^1 + r} \frac{p_0}{1 - p_0}.$$

In other words, the gain-loss ratio for the agent,  $W/\omega$ , is high enough that he

would like to invest at time 0, but the gain-loss ratio for the principal,  $V/\nu$ , is sufficiently low that she would like to invest only after observing a good signal. Note that when Assumption 1 does not hold, the principal can easily align the agent's incentives.<sup>8</sup>

Since  $\lambda^1 > \lambda^0$ , as time goes on and no signal is received, the agent gets more pessimistic. At some point, the agent would prefer to wait and invest only after observing the good signal. Let  $t^*$  be the time at which the agent becomes indifferent between investing and waiting for a good signal. Formally, for  $\lambda^1 > \lambda^0$ ,

$$t^* = \frac{1}{\lambda^1 - \lambda^0} \ln \left( \frac{p_0}{1 - p_0} \frac{W}{\omega} \frac{r}{(\lambda^1 + r)} \right).$$

Figure 1 illustrates  $(p_t)_{t \geq 0}$  and  $t^*$ . For  $t < t^*$ , the principal's and the agent's interests are not aligned when no signal has arrived. For  $t > t^*$ , the principal's and the agent's interests coincide for all private histories. We can thus interpret  $t^*$  as a measure of how long it takes for the incentives to be aligned.  $t^*$  increases as the agent becomes more willing to invest without any information (i.e. when  $W/\omega$  becomes larger) and as the absence of signal becomes less informative (i.e. when  $\lambda^1 - \lambda^0$  becomes smaller so that learning becomes slower).

### 3 The Dynamic Delegation Problem

We set up the principal's problem of eliciting the agent's evolving private information to maximize her expected profits. Following the delegation literature (Holmstrom, 1984), we focus on incentive provision through the design of control rights in the absence of transfers. To do this when learning is private, we formulate a dynamic mechanism design problem with commitment. At each  $t \in [0, \infty)$  the agent sends a costless message  $m_t \in \{0, 1, \emptyset\}$  given the private history  $h^t$ . The principal commits to an action  $y_t \in \{0, 1\}$  as a function of the

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<sup>8</sup>To see this, note that if  $q^* < p_0$ , then the principal would like to invest at  $t = 0$  and would not need the agent. If  $p_0 < p^*$  and  $p_0 < q^*$ , both the principal and the agent would like to invest only after observing a good signal. In this case, both parties' preferences are perfectly aligned throughout the game and the first best can be achieved even without commitment.

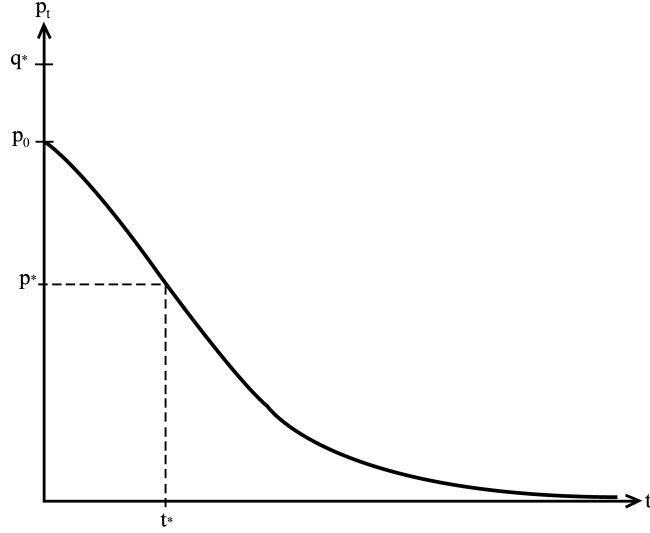


Figure 1: Evolution of  $p_t$  when no signal arrives. Preferences are aligned after time  $t^*$ . Parameter values:  $\lambda^1 = 5$ ,  $\lambda^0 = 4.8$ ,  $r = 0.03$ ,  $p^0 = 0.8$ ,  $w = 10$ ,  $W = 1800$ ,  $t^* = 7.28$ .

message history up to  $t$ ,  $m^t \equiv \{m_\tau\}_{0 \leq \tau < t}$ .

A contract is a tuple  $\langle T, \tau \rangle$ , with  $T \in \mathfrak{R}_+ \cup \{\infty\}$  and  $\tau: [0, T] \rightarrow \mathfrak{R}_+$  if  $T < \infty$  while  $\tau: \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$  if  $T = \infty$ . We use  $\text{dom}(\tau)$  to denote the domain of  $\tau$ . If the agent has reported  $m_t = \emptyset$  for all  $t \in \text{dom}(\tau)$ , the principal invests at time  $T$ . The function  $\tau$  is the time at which the investment is made when the agent reports that he has received the good signal at  $t$  ( $m_t = 1$ ). The principal never invests after the agent reports a bad signal. In Appendix A, we define the general class of contracts and show that our restriction to contracts of the form  $\langle T, \tau \rangle$  is without loss of generality.

We now describe the feasibility and incentive constraints. Since time is irreversible,  $\tau(t) \geq t$  for all  $t \in \text{dom}(\tau)$ . To ensure the agent truthfully reveals when he is informed that the state is good at  $t$  instead of delaying the report, it must be that  $\tau(t)$  is non-decreasing. Otherwise, take  $\tau(t_1) > \tau(t_2)$  with  $t_1 < t_2$  and note that the agent who receives the good signal at  $t_1$  could wait and report the good signal at  $t_2 > t_1$ . The principal also needs to ensure that the informed agent at  $t$  reveals truthfully instead of pretending to be uninformed during the

rest of the game. Formally,  $\tau(t) \leq T$  for all  $t \in \text{dom}(\tau)$ .

A key incentive constraint is to ensure the uninformed agent at  $t$  does not want to claim that he is informed and has received a good signal. To ensure truthful revelation of the uninformed agent at  $t$ ,  $\langle T, \tau \rangle$  must satisfy

$$\int_t^T p_t \lambda^1 \frac{e^{-\lambda^1 s}}{e^{-\lambda^1 t}} e^{-r\tau(s)} W ds + \left( p_t \frac{e^{-\lambda^1 T}}{e^{-\lambda^1 t}} e^{-rT} W + (1 - p_t) \frac{e^{-\lambda^0 T}}{e^{-\lambda^0 t}} e^{-rT} (-\omega) \right) \geq \max \left\{ e^{-r\tau(t)} (-\omega + p_t(W + \omega)), 0 \right\}$$

for all  $t \in \text{dom}(\tau)$ . Note that the agent can always claim that the state is bad and ensure a payoff equal to 0. The right-hand side is the maximum between 0 and the expected payoff of an uninformed agent at  $t$  (who has belief  $p_t$ ) if he claims the state is good and induces investment at  $\tau(t)$ . The left-hand side is the agent's expected payoff if he claims to be uninformed and his continuation policy is to report truthfully. In this case, he could receive the good signal at  $s < T$  and get the payoff  $e^{-r\tau(s)} W$  with conditional probability  $p_t \lambda^1 e^{-\lambda^1(s-t)} ds$ , or receive no signal before  $T$  and induce an uninformed investment decision at  $T$ .<sup>9</sup>

The dynamic delegation problem can be formulated as:

$$\max_{T \in \mathfrak{R}_+ \cup \{\infty\}, \tau(\cdot)} \int_0^T p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau(s)} V ds + \left( p_0 e^{-\lambda^1 T} e^{-rT} V + (1 - p_0) e^{-\lambda^0 T} e^{-rT} (-\nu) \right) \quad (1)$$

subject to

$$\tau(t) \geq t \quad \forall t \in \text{dom}(\tau) \quad (2)$$

$$\tau \text{ is non-decreasing} \quad (3)$$

$$\tau(t) \leq T \quad \forall t \in \text{dom}(\tau) \quad (4)$$

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<sup>9</sup>This incentive constraint could be considered insufficient as the agent could find it optimal to be truthful in some interval  $[t, t + \epsilon]$  and lie after  $t + \epsilon$ . As we show in Appendix A, this is not the case.

$$\int_t^T p_t \lambda^1 \frac{e^{-\lambda^1 s}}{e^{-\lambda^1 t}} e^{-r\tau(s)} W ds + \left( p_t \frac{e^{-\lambda^1 T}}{e^{-\lambda^1 t}} e^{-rT} W + (1 - p_t) \frac{e^{-\lambda^0 T}}{e^{-\lambda^0 t}} e^{-rT} (-\omega) \right) \geq \max \left\{ e^{-r\tau(t)} (-\omega + p_t(W + \omega)), 0 \right\} \quad t \in \text{dom}(\tau). \quad (5)$$

This problem maximizes the principal's expected payoffs (1) over all contracts subject to the feasibility constraint (2) and the dynamic incentive constraints (3)-(5). The dynamic incentive constraints ensure that at any private history, the agent has incentives to truthfully reveal his information. Since private information changes over time, our dynamic delegation problem contrasts with most related problems in the literature that study either static models (Holmstrom, 1984; Melumad and Shibano, 1991; Alonso and Matouschek, 2008; Amador and Bagwell, 2013) or dynamic models in which information asymmetry is present at the beginning of the relationship (Guo, 2016; Grenadier, Malenko, and Malenko, 2016).

## 4 Analysis

In this section, we characterize the solution to the dynamic delegation problem.

### 4.1 Delays

This subsection characterizes the delay with which an investment commonly known to be profitable is implemented. The proofs are relegated to Appendix B. Our first result shows that optimal investments are delayed in any contract that satisfies the dynamic incentive constraints.

**Lemma 1** *Let  $\langle T, \tau \rangle$  satisfy (2) and (5). Then,  $\tau(t) > t$ , for all  $t < \min\{t^*, T\}$ .*

Conditional on the project being revealed profitable at  $t < \min\{t^*, T\}$ , the implementation time is inefficient (from both the principal's and the agent's perspectives). This distortion arises precisely due to the fact that learning is

private: if the implementation time were not distorted and  $\tau(t) = t$  for some  $t < \min\{t^*, T\}$ , the uninformed agent at  $t$  would claim he learned that the state is good in order to induce immediate investment.

In order to solve our dynamic delegation problem, it will be useful to find a solution  $\tau$  to (1) keeping  $T \in \mathfrak{R}_+ \cup \{\infty\}$  fixed. Solving the dynamic delegation problem for a fixed  $T$  is analytically useful and allows us to illustrate the tradeoffs involved when delegating to an agent who privately learns over time. The dynamic delegation problem keeping  $T$  fixed can be analyzed by finding solutions to the following *relaxed problem* (6). It is obtained by ignoring constraints (3)-(4) and by imposing the feasibility constraint (2) and the dynamic incentive constraint (5) over subsets of  $\text{dom}(\tau)$ .

$$\max_{\tau(\cdot)} \int_0^T p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau(s)} V ds + \left( p_0 e^{-\lambda^1 T} e^{-rT} V + (1 - p_0) e^{-\lambda^0 T} e^{-rT} (-\nu) \right) \quad (6)$$

subject to

$$\tau(t) \geq t \quad \forall t \geq \min\{t^*, T\}, \quad (7)$$

$$\begin{aligned} & \int_t^T p_t \lambda^1 \frac{e^{-\lambda^1 s}}{e^{-\lambda^1 t}} e^{-r\tau(s)} W ds + \left( p_t \frac{e^{-\lambda^1 T}}{e^{-\lambda^1 t}} e^{-rT} W + (1 - p_t) \frac{e^{-\lambda^0 T}}{e^{-\lambda^0 t}} e^{-rT} (-\omega) \right) \\ & \geq \max \left\{ e^{-r\tau(t)} (-\omega + p_t(W + \omega)), 0 \right\} \quad \forall t \leq \min\{t^*, T\}. \end{aligned} \quad (8)$$

The following result establishes a necessary and sufficient optimality condition for the relaxed problem (6).

**Lemma 2** *Let  $\tau$  satisfy (7) and (8).*

- (a) *Suppose  $T \leq t^*$ . Then,  $\tau$  solves the relaxed problem iff (8) binds for almost every  $t \in [0, T]$ .*
- (b) *Suppose  $T > t^*$ . Then,  $\tau$  solves the relaxed problem (6) iff (7) binds for almost every  $t \geq t^*$  and (8) binds for almost every  $t \leq t^*$ .*

In the optimal solution to the relaxed problem, the uninformed agent is indifferent between truthful revelation and claiming to know that the state is good for almost all  $t \leq \min\{t^*, T\}$ . To see this, suppose that  $\tau$  is optimal and there is a set  $A \subseteq [0, \min\{t^*, T\}]$  of positive measure such that for any  $t' \in A$ , the uninformed agent strictly prefers to reveal the truth. The principal could construct a new function  $\tau'$  that coincides with  $\tau$  outside of  $A$  but is slightly smaller than  $\tau$  inside  $A$ .  $\tau'$  results in higher expected payoffs for the principal than  $\tau$ , and it satisfies (7) and (8). Thus,  $\tau$  cannot be optimal. Moreover, Lemma 2 also shows that for  $t > \min\{t^*, T\}$ , there is no need to distort investment. Since after  $t^*$  the incentives are aligned, delaying investments only makes it harder to provide incentives before  $t^*$ .

We now further explore an important consequence of the binding incentive constraint (8) over  $[0, \min\{t^*, T\})$ .

**Lemma 3** *Fix  $T$  and  $\tau(\cdot)$  such that (8) binds for all  $t < \min\{t^*, T\}$ . Then, the derivative of  $\tau$  with respect to  $t$  is given by*

$$\dot{\tau}(t) = \left(\frac{\lambda^0}{r}\right) \frac{\omega}{W \frac{p_t}{1-p_t} - \omega}$$

*for all  $t < \min\{t^*, T\}$ . In particular, over  $t < \min\{t^*, T\}$ ,  $\tau$  is strictly increasing and convex, and its slope is strictly less than 1.*

This lemma characterizes the slope of a timing policy  $\tau$  when (8) is binding. It can be intuitively derived as follows. Since (8) is binding everywhere in  $[0, \min\{t^*, T\})$ , the uninformed agent at  $t$  is indifferent between claiming he has received the good signal and truth-telling for all  $t' \geq t$ . The expected payoff the agent gets from truth-telling for  $t' \geq t$  can be decomposed into the current and continuation payoffs. Current payoffs are 0 as by declaring truthfully no investment is made at  $t$ . For continuation payoffs, note that since the incentive constraint (8) is also binding at  $t + dt$ , the uninformed agent at  $t + dt$  gets the same expected payoff from truth-telling for all  $t' \geq t + dt$  and from pretending to have observed the good signal at  $t + dt$ . Combining these two remarks, the payoff the uninformed agent gets at  $t$  from being truthful for



all  $t' \geq t$  is the same as what he gets from truth-telling at  $t$  and lying at  $t + dt$ . As a result, the uninformed agent at  $t$  is indifferent between (i) claiming to have observed the good signal at  $t$  (Lie at  $t$ ), and (ii) being truthful at  $t$  but lying at  $t + dt$  if he is still uninformed (Lie at  $t + dt$ ). Table 1 shows the agent's payoffs from both policies for all possible outcomes.

Outcomes	$s^t = 1$	$s^t = 0$	$s^t = \emptyset, \theta = 1$	$s^t = \emptyset, \theta = 0$
Lie at $t$	$e^{-r\tau(t)}W$	$e^{-r\tau(t)}(-\omega)$	$e^{-r\tau(t)}W$	$e^{-r\tau(t)}(-\omega)$
Lie at $t + dt$	$e^{-r\tau(t+dt)}W$	0	$e^{-r\tau(t+dt)}W$	$e^{-r\tau(t+dt)}(-\omega)$
Probabilities	$p_t\lambda^1 dt$	$(1 - p_t)\lambda^0 dt$	$p_t(1 - \lambda^1 dt)$	$(1 - p_t)(1 - \lambda^0 dt)$

Table 1: Payoffs from two different policies. Under the first policy (Lie at  $t$ ), the uninformed agent claims that the state is good at  $t$ . Under the second policy (Lie at  $t + dt$ ), the uninformed agent claim to be uninformed at  $t$  but lies at  $t + dt$  if he remains uninformed.

Since the expected payoffs from both policies coincide,

$$e^{-r\tau(t)}\left(p_t W + (1 - p_t)(-\omega)\right) = e^{-r\tau(t+dt)}\left(p_t W + (1 - p_t)(1 - \lambda^0 dt)(-\omega) + 0 \cdot (1 - p_t)\lambda^0 dt\right).$$

Equivalently,

$$(1 - p_t)\lambda^0 dt \omega e^{-r\tau(t)} = (e^{-r\tau(t)} - e^{-r\tau(t+dt)})\left(p_t W + (1 - p_t)(1 - \lambda^0 dt)(-\omega)\right).$$

Rearranging terms, dividing by  $dt$  and taking  $dt \rightarrow 0$ , we deduce that

$$(1 - p_t)\lambda^0 \omega = r\dot{\tau}(t)\left(p_t W + (1 - p_t)(-\omega)\right), \quad (9)$$

which provides the characterization in Lemma 3.

Equation (9) illustrates how  $\tau$  balances the costs and benefits of learning for the agent. The left hand side in (9) is the benefit from learning as the agent could avoid investment when the project is bad. The right hand side in (9) is the cost of learning as when no signal arrives the investment is just delayed. An important implication from this characterization is that  $\dot{\tau}(t) < 1$  and thus

the delay with which investment decisions are made,  $\tau(t) - t$ , is decreasing in  $t$ . Intuitively, to motivate the agent to learn, the agent's cost of learning has to be lower than that in the single-player benchmark for the agent and therefore the principal sets  $\dot{\tau}(t) < 1$ . In Section 5.2, we show that this feature of decreasing delays is robust when  $\lambda^0 > \lambda^1$ .

## 4.2 Optimal Dynamic Delegation

This subsection characterizes the solutions to the optimal delegation problem and establishes the tradeoff between the amount of information acquired and how effectively it is used.

We first find a solution  $\tau^T$  to the relaxed problem when  $T \leq t^*$ . We impose (8) binding everywhere in  $[0, T]$ . By Lemma 3, (8) binding in  $[0, T]$  gives

$$\dot{\tau}^T(t) = \left(\frac{\lambda^0}{r}\right) \frac{\omega}{W \frac{p_t}{1-p_t} - \omega}, \quad t < T. \quad (10)$$

(8) binding at  $T$  gives

$$\tau^T(T) = T. \quad (11)$$

These two together give us

$$\tau^T(t) = T - \frac{\lambda^0}{r} \int_t^T \frac{1}{\frac{W}{\omega} \frac{p_s}{1-p_s} - 1} ds, \quad t \leq T. \quad (12)$$

Figure 2 illustrates the solution.

Since  $\tau^T(\cdot)$  satisfies the conditions in Lemma 2, it solves the relaxed problem. We now verify that it actually solves the original dynamic delegation problem (1) for a given  $T$ . First note that  $\tau^T$  satisfies (2). Indeed,  $\tau^T(t) = \tau^T(T) - \int_t^T \dot{\tau}^T(s) ds$  and, since  $\tau^T(T) = T$  and  $\dot{\tau}^T(t) < 1$ ,  $\tau^T(t) \geq \tau^T(T) - (T - t) = t$  for all  $t \in [0, T]$ . Second,  $\tau^T$  satisfies (5) because it holds with equality over  $t \in [0, T]$ . Finally, since  $\tau^T(t)$  is increasing over  $[0, T]$  and  $\tau^T(T) = T$ ,  $\tau^T$  also satisfies the incentive constraints (3)-(4). As a result,  $\tau^T$  indeed solves the dynamic delegation problem (1) for a given  $T$ . As can be seen, the incentive constraint for the uninformed agent is the key to

pinning down the optimal contract when  $T \leq t^*$ .

The following result provides a key insight for solving for the optimal  $T < t^*$ .

**Proposition 1** *Let  $t < T < \hat{T} < t^*$ . Then,  $\tau^T(t) < \tau^{\hat{T}}(t)$ .*

Figure 3 illustrates Proposition 1. Increasing the deadline is beneficial for the principal in that more information is acquired and thus investment in the bad state is less likely to happen. Proposition 1 shows that more learning imposes a nontrivial incentive cost on the principal because when  $T$  increases,  $\tau^T(t)$  must increase too. This means that when  $T$  increases, investments are delayed more when the good signal is received.

Formally, Proposition 1 follows immediately from Equation (12). To better understand Proposition 1, take  $t < T < \hat{T}$  and assume for the moment that  $t$  is close to  $T$ . When the uninformed agent at  $t$  faces the contract  $\langle T, \tau^T \rangle$ , he knows that by declaring truthfully, the investment will be made at  $T$  (unless a bad signal is received in the meanwhile). Now, when the uninformed agent at  $t$  faces the contract  $\langle \hat{T}, \tau^{\hat{T}} \rangle$ , the earliest time at which the investment could be made is  $\tau^{\hat{T}}(T) > T$ . As a result, the expected continuation payoff that the uninformed agent gets at  $t$  by being truthful is lower when he faces  $\langle \hat{T}, \tau^{\hat{T}} \rangle$  than when he faces  $\langle T, \tau^T \rangle$ . Therefore, to provide incentives for truthful revelation at  $t$ , contract  $\langle \hat{T}, \tau^{\hat{T}} \rangle$  must punish the agent even more when he claims a good signal. In other words,  $\tau^{\hat{T}}(t) > \tau^T(t)$ . This intuition can be iteratively applied backwards to render this property for all  $t < T$ .

We solve the relaxed problem given  $T > t^*$  by imposing (8) binding everywhere in  $[0, t^*)$  and (7) binding everywhere in  $[t^*, T]$ . By Lemma 3, (8) binding in  $[0, t^*)$  implies

$$\dot{\tau}^T(t) = \left(\frac{\lambda^0}{r}\right) \frac{\omega}{W \frac{p_t}{1-p_t} - \omega} \quad t \in [0, t^*).$$

Combined with (7) binding for  $t \geq t^*$ , we have

$$\tau^T(t) = \begin{cases} t^* - \frac{\lambda^0}{r} \int_t^{t^*} \frac{1}{\omega \frac{p_s}{1-p_s} - 1} ds & t \leq t^*, \\ t & t > t^*. \end{cases}$$

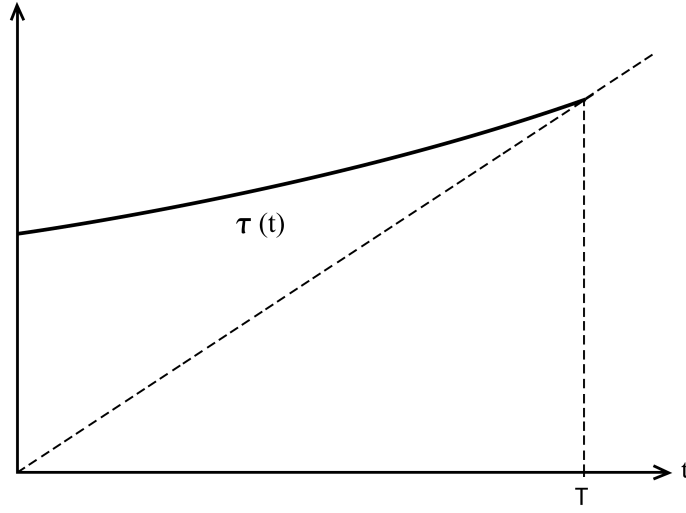


Figure 2: The dark line shows the time at which the investment is made as a function of the time at which the good signal is received. For  $t < T$ , the investment decision is delayed and the delay,  $\tau^T(t) - t$ , is decreasing. Parameter values:  $\lambda^1 = 5$ ,  $\lambda^0 = 4.8$ ,  $r = 0.03$ ,  $p^0 = 0.8$ ,  $v = 120$ ,  $w = 10$ ,  $V = 480$ ,  $W = 1800$ ,  $T = 2.4$ .

Lastly, to make sure that  $\tau^T$  satisfies (8) at  $t^*$  and therefore solves the relaxed problem, we need  $T$  to be infinity. To see this, notice that at  $t^*$ , by being truthful that he has not received a signal, the agent receives the payoff from the policy “invest as soon as a good signal arrives before  $T$  and invest at  $T$  if no signal arrives before  $T$ ,” which is weakly less preferred to the policy “invest as soon as a good signal arrives and do not invest if no signal arrives.” Since at  $t^*$  the agent is indifferent between the latter policy and the policy “invest right away,” we need  $T = \infty$  to ensure incentive compatibility. Therefore a solution to the relaxed problem is

$$\tau^\infty(t) = \begin{cases} t^* - \frac{\lambda^0}{r} \int_t^{t^*} \frac{1}{\frac{W}{\omega} \frac{p_s}{1-p_s} - 1} ds & t \leq t^*, \\ t & t > t^*. \end{cases}$$

Since  $\tau^\infty$  is increasing and (5) is satisfied everywhere in  $[0, \infty)$ , it solves the

original dynamic delegation problem (1) given that  $T > t^*$ .

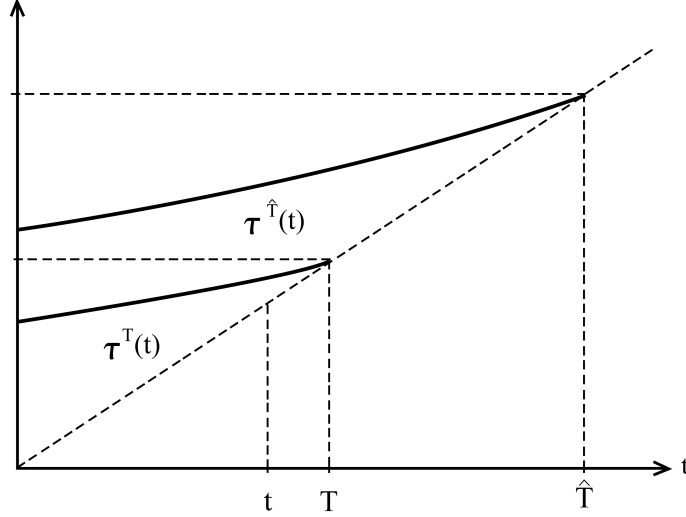


Figure 3: For  $t \leq T < \hat{T}$ ,  $\tau^T(t) < \tau^{\hat{T}}(t)$ . Parameter values:  $\lambda^1 = 5$ ,  $\lambda^0 = 4.8$ ,  $r = 0.03$ ,  $p^0 = 0.8$ ,  $v = 120$ ,  $w = 10$ ,  $V = 480$ ,  $W = 1800$ ,  $T = 2.4$ ,  $\hat{T} = 4.3$ .

The following theorem summarizes our characterization.

**Theorem 1** *The optimal contract takes one of the following two forms:*

- (a) *There is a deadline  $T < t^*$ . If a good signal arrives before  $T$ , investment happens with a delay. If no signal arrives before  $T$ , investment happens at  $T$ .*
- (b) *If a good signal arrives before  $t^*$ , investment happens with a delay. If a good signal arrives after  $t^*$ , investment happens with no delay.*

To find the optimal contract  $\langle T^*, \tau^{T^*} \rangle$ , it suffices to compare the optimal solution when  $T \in [0, t^*]$  to the case in which  $T = \infty$ . It is thus enough to compare the expected payoff for the principal from the optimal  $\tau^T$  when  $T \leq t^*$  to that from  $\tau^\infty$ .

The optimal contract can be implemented by setting time-dependent delegation sets illustrated in Figure 4. At any  $t < \min\{t^*, T^*\}$ , the agent is

allowed to commit to invest in  $[\tau^{T^*}(t), \infty)$  or just wait and commit later. For  $t \geq \min\{t^*, T^*\}$ , the agent is granted full freedom.

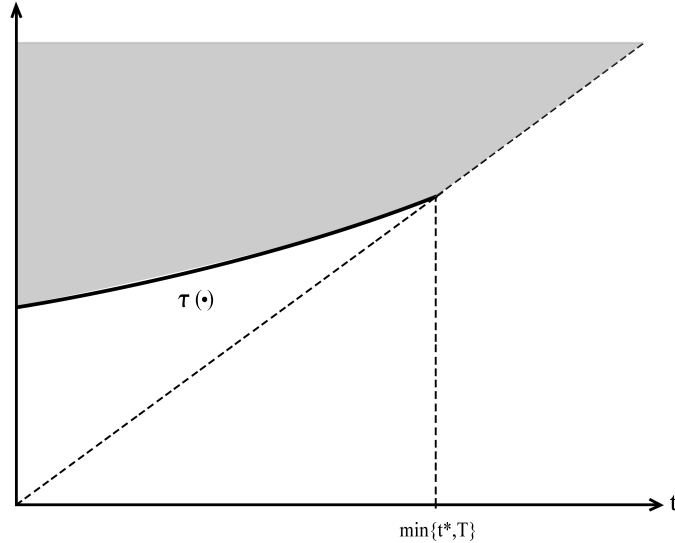


Figure 4: For  $t < \min\{t^*, T^*\}$ , the agent can choose to invest at any  $t' \in [\tau(t), \infty)$ . For  $t \geq \min\{t^*, T^*\}$ , the agent can invest at any  $t' > t$ . Parameter values:  $\lambda^1 = 5$ ,  $\lambda^0 = 4.8$ ,  $r = 0.03$ ,  $p^0 = 0.8$ ,  $v = 120$ ,  $w = 10$ ,  $V = 480$ ,  $W = 1800$ ,  $T = 4.3$ ,  $t^* = 7.2$ .

### 4.3 Comparative Statics

We now derive some comparative statics results. These results assume that parameters satisfy Assumption 1, that is,  $\frac{W}{\omega} \frac{r}{\lambda^1+r} \frac{p_0}{1-p_0} > 1 > \frac{V}{\nu} \frac{r}{\lambda^1+r} \frac{p_0}{1-p_0}$ .

#### Proposition 2

- (a) Fix all parameters except  $W$  and  $\omega$ . There exist cutoffs  $0 < \underline{\kappa} < \bar{\kappa}$  such that for all  $W/\omega < \underline{\kappa}$ , the optimal contract sets no deadline, whereas for  $W/\omega > \bar{\kappa}$  the optimal contract sets a deadline  $T^* < t^*$ .
- (b) Fix all parameters except  $V$  and  $\nu$ . There exist cutoffs  $0 < \underline{\eta} < \bar{\eta}$  such that for all  $V/\nu < \underline{\eta}$ , the optimal contract sets no deadline, whereas for  $V/\nu > \bar{\eta}$  the optimal contract sets a deadline  $T^* < t^*$ .

The proof is relegated to Appendix C. Part (a) shows that when  $W/\omega$  is sufficiently small, it is optimal to invest only after the principal has perfectly learned that  $\theta = 1$ . In this case,  $t^*$  is small, so the incentives will become aligned rapidly, and there is no need to significantly delay investments for  $t < t^*$ . In contrast, when  $W/\omega$  is large,  $t^*$  is large, and the conflict of interest is severe. So decisions to invest need to be significantly distorted for  $t < \min\{T^*, t^*\}$ . To economize on distortions early in the game, the principal commits to invest at a deadline  $T^* < t^*$  even when this means learning stops early.

Part (b) characterizes the solutions as we vary the principal's payoffs  $V$  and  $\nu$ . When  $V/\nu$  is small,  $V$  is small compared to  $\nu$  and it is relatively costly for the principal to invest when the state is bad. To avoid the costs of a failure, the principal prefers to perfectly learn the state even when this entails significant delays for  $t < t^*$ . In contrast, when  $V/\nu$  is large, the cost of a failure is small, and the principal fixes a deadline  $T^* < t^*$  that stops learning and reduces distortions for  $t < T^*$ .

Proposition 2 sheds light on how different organizations should provide incentives for learning. For example, the FDA incurs significant costs when approving bad drugs (so that  $\nu$  is large). Our results suggest that the FDA should set lengthy revision processes to ensure pharmaceutical companies learn the value of the drugs even if this entails substantial delays between the drugs' discovery and the FDA's final approval. In contrast, the board of a company that is contemplating a partially reversible acquisition (so  $\nu$  is small) or that cannot align the manager's career incentives (so  $W/\omega$  is large) should set a deadline  $T^* < t^*$  that facilitates truthful communication even at the possible cost of an incorrect decision.

## 5 Extensions

### 5.1 Costly Learning Effort

We extend the model to incorporate learning with a costly effort. To generate the signal whose arrival time follows an exponential distribution at any time  $t$ , the agent needs to incur a cost  $cdt$  per unit of time. The agent could also decide not to incur the cost, in which case he receives no information. In the baseline model in Section 2, we assume that  $c = 0$ . We now assume that  $c > 0$ . From the principal's perspective, the agent may lie not only about the signal he has observed, but also about his learning effort. This subsection shows how delays and deadlines can induce both truthful revelation and costly learning when  $c > 0$ .

We maintain Assumption 1. This means that the agent has even stronger incentives to invest at time  $t = 0$  due to the fact that now in order to obtain information he needs to delay the investment as well as make a costly effort. Thus, there is a conflict of interest in that at  $t = 0$ , the principal would like to wait for information, but the agent would like to invest immediately.<sup>10</sup> To simplify the exposition, we restrict attention to contracts with  $T$  sufficiently short that the optimal decision for the uninformed agent at  $T$  is to invest immediately.

A contract is a tuple  $\langle T, \tau \rangle$  that suggests an effort of learning at any  $t < T$ . If the agent reports  $m_t = \emptyset$  for all  $t \in \text{dom}(\tau)$ , the principal invests at  $T$  so  $y_T = 1$ . Similar to the baseline model, if the agent reports that he has received a good signal at  $t$ , then the principal invests at  $\tau(t) \geq t$ .

Fix a contract  $\langle T, \tau \rangle$  and define  $u^{\langle T, \tau \rangle}(t, S)$  as the total expected utility for the agent who is uninformed at  $t$ , learns in  $[0, S]$ , and reports truthfully at any

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<sup>10</sup>Using standard arguments, the optimization problem faced by the agent when he has decision rights can be formulated as a dynamic programming problem. This formulation results in the following optimal policy for the agent: there exists cutoffs  $\bar{p} > \underline{p}$  such that the agent invests if  $p_t \geq \bar{p}$ , does not invest but incurs the learning cost if  $\bar{p} > p_t \geq \underline{p}$ , and does not invest or learn if  $p_t < \underline{p}$ .



given  $t' \in [0, S]$ :

$$\begin{aligned}
u^{\langle T, \tau \rangle}(t, S) &= \int_t^S p_t \lambda^1 \frac{e^{-\lambda^1 s}}{e^{-\lambda^1 t}} \left( e^{-r\tau(s)} W - c \frac{1 - e^{-rs}}{r} \right) ds \\
&\quad + \int_t^S (1 - p_t) \lambda^0 \frac{e^{-\lambda^0 s}}{e^{-\lambda^0 t}} (-c) \frac{1 - e^{-rs}}{r} ds + p_t \frac{e^{-\lambda^1 S}}{e^{-\lambda^1 t}} e^{-rT} W \\
&\quad + (1 - p_t) \frac{e^{-\lambda^0 S}}{e^{-\lambda^0 t}} e^{-rT} (-\omega) - c \frac{1 - e^{-rS}}{r} \left( p_t \frac{e^{-\lambda^1 S}}{e^{-\lambda^1 t}} + (1 - p_t) \frac{e^{-\lambda^0 S}}{e^{-\lambda^0 t}} \right).
\end{aligned}$$

We now introduce the incentive constraints. The first incentive constraint ensures that the agent finds it optimal to make the costly effort at all times  $t \in \text{dom}(\tau)$ . In other words,

$$T \in \arg \max_{S \in \text{dom}(\tau)} u^{\langle T, \tau \rangle}(t, S) \quad (13)$$

for all  $t \in \text{dom}(\tau)$ . The second incentive constraint ensures that the agent finds it optimal to report truthfully that he is uninformed. Formally,

$$u^{\langle T, \tau \rangle}(t, T) \geq p_t e^{-r\tau(t)} W + (1 - p_t) e^{-r\tau(t)} (-\omega) - c \frac{1 - e^{-rt}}{r} \quad (14)$$

for all  $t \in \text{dom}(\tau)$ . The *dynamic delegation problem* is therefore formulated by maximizing (1) over all contracts  $\langle T, \tau \rangle$  subject to (2)-(3)-(4)-(13)-(14).

The following result simplifies the analysis.

**Lemma 4** *There exists  $\bar{c} > 0$  such that for all  $c < \bar{c}$ ,  $\frac{\partial}{\partial S} u^{\langle T, \tau \rangle}(t, S) > 0$ .*

When investment occurs only at time  $T$  or when a good signal has been received, the agent finds it optimal to learn at a cost if the cost is below a threshold. While the agent would find it attractive to invest immediately, the fact that he has no incentives to pretend to be informed when he is not implies that he strictly values learning in order to rule out a type II error.

An important implication from this observation is that for  $c < \bar{c}$ , we can solve the delegation problem by ignoring (13). Indeed,  $u^{\langle T, \tau \rangle}(t, S)$  is increasing in  $S$  and thus, at any  $t$ , the best choice for the agent is to set  $S = T$ . Now, similar to Lemma 1 and Lemma 2, equation (14) implies that  $\tau(t) > t$  and,

fixing  $T$ , (14) binds for  $t < T$  for any optimal solution. Given  $T$ , the solution  $\tau^T$  to the dynamic delegation problem satisfies

$$\tau(T) = T, \quad \dot{\tau}(t) = \frac{(1 - p_t)\lambda^0\omega - ce^{r(\tau(t)-t)}}{p_tW + (1 - p_t)(-\omega)} \quad \forall t \leq T. \quad (15)$$

From Peano's theorem (Coddington and Levinson, 1955), the ordinary differential equation (15) has a unique solution  $\tau^T$ . Finding a close form solution for  $\tau^T$  does not seem feasible. The following result shows that a key comparative statics result from our baseline model extends to the current setup.

**Lemma 5** *Let  $t < T < \hat{T}$ . Then,  $\tau^T(t) < \tau^{\hat{T}}(t)$ .*

This result states that if the principal wants to incentivize a longer learning phase, then she must distort investment decisions more. Thus, the main features of the optimal contract in our baseline model extend to the case of costly learning.

## 5.2 No News Is No News and No News Is Good News

In the absence of a signal, the evolution of the agent's belief  $(p_t)_{t \geq 0}$  satisfies

$$\frac{dp_t}{dt} = -(\lambda^1 - \lambda^0)p_t(1 - p_t).$$

We first consider the case when  $\lambda^0 = \lambda^1$ ; that is,  $\frac{dp_t}{dt} = 0$ . In this case, the agent's belief remains constant if no signal has arrived, and jumps to 0 or 1 at the first signal. Therefore the uninformed agent is never indifferent between investing and waiting to invest after a good signal and we define  $t^* = \infty$ .

The single-player problem is solved identically as that in Section 2. The dynamic delegation problem and the relaxed problem are set up in the same way. When solving the relaxed problem, since  $t^* = \infty$ , we only need to consider the  $T \leq t^*$  case. Since it is infeasible to set  $T = \infty$  for any combination of parameters that satisfy Assumption 1, the optimal contract always features a deadline  $T$  and the corresponding contract  $\tau^T$  is solved for in the same way as in Section 4.2.

When  $\lambda^0 > \lambda^1$ , the agent's belief drifts up as times goes on. Suppose that the agent observes the signal and decides whether to invest at each point in time. Following arguments similar to those in Section 2, it is relatively simple to show that there exists  $p^*$  such that the agent invests iff  $p_t \geq p^*$ . Analogously, there exists  $q^*$  such that the principal would make the decision iff  $p_t \geq q^*$ .<sup>11</sup> We assume that  $p^* < p_0 < q^*$ . This means that at time 0, the agent would like to invest whereas the principal would like to wait for information. In contrast to Section 2, the assumption  $\lambda^0 > \lambda^1$  now implies that there exists  $t^*$  such that if no signal has been received, the principal would like to invest at any  $t > t^*$ . In particular, for  $t > t^*$ , the principal's and the agent's preferences surely coincide as both would like to invest. This implies that there will always be a deadline  $T \leq t^*$ .

We find the contract  $\langle T, \tau \rangle$  that solves (1) under constraints (2)-(3)-(4)-(5). All these constraints remain relevant in this setup as they capture feasibility and truth-telling incentives that need to be provided regardless of the direction followed by the belief path.

The solution method is similar to Section 4. We now sketch and discuss the main steps.

**Lemma 6** *Let  $\langle T, \tau \rangle$  satisfy (2) and (5). Then,  $\tau(t) > t$ , for all  $t \leq T$ .*

This result is similar to Lemma 1. The main difference is that now the uninformed agent prefers to invest for all  $t \in \text{dom}(\tau)$  and, as a result, all the investment times need to be distorted.

We also solve the dynamic delegation problem for fixed  $T$  and, as in Section 4, it will be convenient to formulate the following relaxed problem:

$$\max_{\tau(\cdot)} \int_0^T p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau(s)} V ds + \left( p_0 e^{-\lambda^1 T} e^{-rT} V + (1-p_0) e^{-\lambda^0 T} e^{-rT} (-\nu) \right) \quad (16)$$

subject to

$$\tau(T) \geq T \quad (17)$$

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<sup>11</sup>Note that the thresholds  $p^*$  and  $q^*$  in this subsection do not coincide with the thresholds derived in Section 2.

$$\begin{aligned}
& \int_t^T p_t \lambda^1 \frac{e^{-\lambda^1 s}}{e^{-\lambda^1 t}} e^{-r\tau(s)} W ds + \left( p_t \frac{e^{-\lambda^1 T}}{e^{-\lambda^1 t}} e^{-rT} W + (1 - p_t) \frac{e^{-\lambda^0 T}}{e^{-\lambda^0 t}} e^{-rT} (-\omega) \right) \\
& \geq \max \left\{ e^{-r\tau(t)} (-\omega + p_t(W + \omega)), 0 \right\} \quad \forall t \leq T. \tag{18}
\end{aligned}$$

**Lemma 7** *Let  $\tau^*$  satisfy (17)-(18). Then,  $\tau^*$  solves the relaxed problem (16) iff (18) binds for almost every  $t \in [0, T]$ .*

This lemma is similar to Lemma 2. Intuitively, if the constraint were slack, the principal could slightly reduce the investment time and improve her expected payoffs.

A solution to the relaxed problem is then found by imposing (18) binding over  $[0, T]$ . Since  $-\omega + p_t(W + \omega) > 0$  for all  $t \geq 0$ , (18) binding at  $T$  implies  $\tau(T) = T$ . Using Lemma 3, we can solve for the binding constraint (18) by simply solving the system

$$\tau(T) = T, \quad \dot{\tau}(t) = \left( \frac{\lambda^0}{r} \right) \frac{\omega}{W \frac{p_t}{1-p_t} - \omega} \quad t < T.$$

The solution  $\tau^T$  to this system is given by (12). This function is concave and its slope is less than 1. As it satisfies all the constraints of the dynamic delegation problem,  $\tau^T$  actually solves the dynamic delegation problem for fixed  $T$ . As  $T$  increases, so does  $\tau^T(t)$  and thus the principal needs to distort more investment decisions. The optimal  $T$  is chosen as follows. Over  $T \geq t^*$  the principal should optimally set  $T = t^*$  since for all  $t > t^*$ , the principal is optimistic enough to invest without any news. Over  $T < t^*$ , the solution solves the tradeoff characterized in Proposition 2.

### 5.3 Transfers

In this subsection, we show that transfers can align the agent's incentives. A *contract with transfers* is a tuple  $\langle T, \tau, F, q \rangle$  where  $T \in \mathfrak{R}_+ \cup \{\infty\}$  and  $\tau(\cdot)$  are as in Section 3, while  $F \in \mathfrak{R}$  is a fixed amount paid from the agent to the principal at  $t = 0$  and  $q \in \mathfrak{R}$  is a transfer from the agent to the principal

whenever a decision is made. A contract with transfers allows for payments at the beginning of the relationship and whenever a decision is made and it rules out schemes in which transfers depend on time.

Consider the problem of maximizing the sum of the principal's and agent's payoffs. This problem is also a stopping time problem in which the investment is made at  $t = 0$  iff  $\frac{V+W}{\nu+\omega} \frac{r}{\lambda^1+r} \frac{p_0}{1-p_0} > 1$ . Assume that in the policy that maximizes the sum of payoffs, investment occurs only after a good signal is observed. This policy results in total payoffs equal to  $p_0(V+W) \frac{\lambda^1}{\lambda^1+r}$ . We claim that the principal can achieve this first-best payoff by designing a contract with transfers.

To see this, set  $q > 0$  such that  $\frac{W-q}{\omega+q} \frac{r}{\lambda^1+r} \frac{p_0}{1-p_0} = 1$ . This means that the principal charges a tax  $q$  to the investment decision such that the agent is indifferent between investing at  $t = 0$  and investing after a good signal. The principal also sets  $T = \infty$  and  $\tau(t) = t$ . The agent facing this contract with transfers has incentives to truthfully reveal his information. Finally, the principal charges  $F = p_0(W - q) \frac{\lambda^1}{\lambda^1+r}$  at the beginning of the relationship. By offering  $\langle T, \tau, F, q \rangle$ , the principal gets expected payoffs equal to

$$F + p_0(V + q) \frac{\lambda^1}{\lambda^1 + r} = p_0(V + W) \frac{\lambda^1}{\lambda^1 + r}.$$

We have therefore proven the following result.

**Proposition 3** *The principal can achieve first best payoffs by using a contract with transfers.*

This result is similar to others in contract theory showing that a principal can implement an efficient allocation and extract the whole surplus using unrestricted transfers (Bolton and Dewatripont, 2005). The key driver behind all these results is the fact that the contract is signed under symmetric information and that payments allow for arbitrary utility transfers.

## 6 Concluding Remarks

This paper studies a dynamic delegation model in which learning is private. Evolving private information shapes the optimal contract in distinctive ways. Indeed, we show that to ensure truthful revelation from the agent, the principal needs to delay investments commonly known to be optimal. As time goes on, the principal grants more flexibility to the agent and, eventually, the agent is free to make any decision. Our analysis uncovers a new tradeoff between how much information is acquired in the relationship and how efficiently new information is used. In sum, our analysis brings out a number of new economic features arising in delegation models with evolving private information.

Our model is stylized. The learning process is assumed to be Poisson, investment is irreversible, and the agent has little freedom to decide how to learn.<sup>12</sup> The model could also be extended to allow for money burning.<sup>13</sup> In that model, the principal may induce truthful reporting by asking the agent to burn money instead of delaying decisions. Future work should address these extensions.

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<sup>12</sup>At the other extreme, the agent could decide any experiment that reveals information about the state.

<sup>13</sup>This means that the agent can spend resources that have no value for the principal.

## Appendix (For Online Publication)

### A Formulation of Mechanism Design Problem

First we define a contract. To do this, we first need some terminology. A *public history at  $t$*  is  $h_t^p := \{(m_\tau, y_\tau)\}_{0 \leq \tau < t} \in H_t^p$ . It contains the sequence of messages and investment decisions strictly before  $t$ . A *private history at  $t$*  is  $h_t^\alpha := \{(m_\tau, y_\tau)\}_{0 \leq \tau < t}, \{h_\tau\}_{0 \leq \tau \leq t}\} \in H_t^\alpha$ . It contains the sequence of signal, messages and investment decisions strictly before  $t$  as well as the signal observation at  $t$ . The message space at time  $t$ ,  $M_t : H_t^p \rightarrow 2^{\{0,1,\emptyset\}}$ , is defined as follows:  $\forall t$ ,

$$M_t(h_t^p) = \begin{cases} \{m_\tau\}, & \text{if } \exists \tau < t \text{ s.t. } m_\tau \neq \emptyset \text{ or } y_\tau = 1 \\ \{0, 1, \emptyset\}, & \text{otherwise.} \end{cases}$$

We use  $m^t$  to denote the message sequence up to and including  $t$ :  $m^t = \{m_\tau\}_{0 \leq \tau \leq t}$ . Through abusing of notation, we say that  $m^t = \emptyset$  if  $m_\tau = \emptyset$  for all  $0 \leq \tau \leq t$ .

A contract  $\Gamma$  is a function mapping  $\left\{ \{m_\tau\}_{0 \leq \tau \leq t}, \{y_\tau\}_{0 \leq \tau < t} \right\}$  to  $y_t \in \{0, 1\}$  with the following irreversibility property: for any  $t$ , if  $y_\tau = 1$  for some  $0 \leq \tau < t$ , then  $y_t = 1$ . From now on, we keep in mind this property and omit the dependence of  $y_t$  on  $\{y_\tau\}_{0 \leq \tau < t}$  and simply write  $y_t = y(m^t)$ .

Our next goal is to simplify the principal's problem. To do this, we first show that any contract can be represented by three components: a "deadline"  $\bar{T} \in \mathfrak{R}_+ \cup \{\infty\}$ , a function  $\tau_0(\cdot)$  which maps the arriving time of the first 0-message to an investment time  $\mathfrak{R}_+ \cup \{\infty\}$ , and a function  $\tau_1(\cdot)$  which maps the arriving time of the first 1-message to an investment time  $\mathfrak{R}_+ \cup \{\infty\}$ .

Let us define  $\bar{T} := \inf\{t : m^t = \emptyset, y(m^t) = 1\}$ . It follows that for any  $t$  such that  $m^t = \emptyset$ , if  $t < \bar{T}$ , then  $y(m^t) = 0$ ; otherwise  $y(m^t) = 1$ . In other words,  $\bar{T}$  pins down the principal's action for an empty message history of any length. Now let us consider  $m^t \neq \emptyset$ . We define  $\gamma(m^t) := \min\{\tau : m_\tau \neq \emptyset\}$ , the time that a non-empty message history jumps from  $\emptyset$  to 0 or 1. Then

by the definition of  $M_t$ , any  $m^t \neq \emptyset$  is completely characterized by  $\gamma(m^t)$ , the value of  $m_t \in \{0, 1\}$  and the value of  $t$ . For each  $x \geq 0$ , let us define  $\tau_0(x) = \inf\{t : y(m^t) = 1, m_t = 0, \gamma(m^t) = x\}$ . Therefore, for each  $m^t$  for which  $m_t = 0$  and  $t < \tau_0(\gamma(m^t))$ , we have  $y(m^t) = 0$ ; for each  $m^t$  for which  $m_t = 0$  and  $t \geq \tau_0(\gamma(m^t))$ ,  $y(m^t) = 1$ . Similarly,  $\tau_1(\cdot) := \inf\{t : y(m^t) = 1, m_t = 1, \gamma(m^t) = x\}$  pins down  $y(m^t)$  for all  $m^t \neq \emptyset$  and  $m_t = 1$ . Therefore  $\tau_0(\cdot)$  and  $\tau_1(\cdot)$  pin down the principal's action at any non-empty message history. Note that since the infimum of an empty set is  $+\infty$ , we allow the case that  $\bar{T} = \infty$ , or  $\tau_i(x) = \infty$  for  $i = 0, 1$  for some  $x$ .

Right now the domains of  $\tau_0(\cdot)$  and  $\tau_1(\cdot)$  are both  $[0, \infty)$ . We argue that it suffices to restrict them to  $[0, \bar{T}]$  whenever  $\bar{T} < \infty$ . In other words, it is redundant to define the investment time for a history  $m^t \neq \emptyset$  for which  $\gamma(m^t) > \bar{T}$ . The argument is simple: if for some  $m^t$  we have  $\gamma(m^t) > \bar{T}$ , then it must be the case that  $m^{\bar{T}} = \emptyset$  and  $y(m^{\bar{T}}) = y(m^t) = 1$ . We sum up our discussion in the following proposition:

**Proposition 4** *A contract belongs to one of the two following groups:*

1.  $\bar{T} < \infty$ ,  $\tau_0, \tau_1 : [0, \bar{T}] \rightarrow [0, \infty]$ ,
2.  $\bar{T} = \infty$ ,  $\tau_0, \tau_1 : [0, \bar{T}) \rightarrow [0, \infty]$ .

Now we have demonstrated that a contract consists of three components  $\bar{T}$ ,  $\tau_0(\cdot)$  and  $\tau_1(\cdot)$ . We are now ready to state the principal's objective function:

$$\int_0^{\bar{T}} \left[ p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau_1(s)} V + (1 - p_0) \lambda^0 e^{-\lambda^0 s} e^{-r\tau_0(s)} (-\nu) \right] ds \\ + \left[ p_0 e^{-\lambda^1 \bar{T}} + (1 - p_0) e^{-\lambda^0 \bar{T}} \right] e^{-r\bar{T}} \left[ -\nu + p_{\bar{T}} (V + \nu) \right]$$

For the constraints faced by the principal, first note that the principal's actions must be feasible, therefore  $\tau_i(x) \geq x$  for all  $x$ . For the incentive compatibility constraints of the agent, we require that at any on- or off-path history, the agent prefers to tell the truth from then on. Hence we discuss the possible histories  $h_t^\alpha$  faced by the agent at which  $M_t$  is not a singleton:



1. Suppose that  $h^t$  contains signal 1. Then choosing  $m_t = 1$  is preferred by the agent to:

- (a) choosing  $m_t = 0$ :  $e^{-r\tau_1(t)}W \geq e^{-r\tau_0(t)}W$ ;
- (b) choosing  $m_t = \emptyset$  and  $m_s = 1$  for some  $s > t$ :  $e^{-r\tau_1(t)}W \geq e^{-r\tau_1(s)}W$ ,  
 $\forall s > t$ ;
- (c) choosing  $m_t = \emptyset$  and  $m_s = 0$  for some  $s > t$ :  $e^{-r\tau_1(t)}W \geq e^{-r\tau_0(s)}W$ ,  
 $\forall s > t$ ;
- (d) choosing  $m_s = \emptyset$  for all  $s \geq t$ .

That is,

- (a)  $\tau_1(t) \leq \tau_0(t)$ ,  $\forall t$ ;
- (b)  $\tau_1(t) \leq \tau_1(s)$ ,  $\forall s > t$ ;
- (c)  $\tau_1(t) \leq \tau_0(s)$ ,  $\forall s > t$ ;
- (d)  $\tau_1(t) \leq \bar{T}$ ,  $\forall t$ .

2. Suppose that  $h^t$  contains signal 0. Then choosing  $m_t = 0$  is preferred by the agent to:

- (a) choosing  $m_t = 1$ :  $e^{-r\tau_0(t)}(-\omega) \geq e^{-r\tau_1(t)}(-\omega)$ ;
- (b) choosing  $m_t = \emptyset$  and  $m_s = 0$  for some  $s > t$ :  $e^{-r\tau_0(t)}(-\omega) \geq e^{-r\tau_0(s)}(-\omega)$ ,  $\forall s > t$ ;
- (c) choosing  $m_t = \emptyset$  and  $m_s = 1$  for some  $s > t$ :  $e^{-r\tau_0(t)}(-\omega) \geq e^{-r\tau_1(s)}(-\omega)$ ,  $\forall s > t$ ;
- (d) choosing  $m_s = \emptyset$  for all  $s \geq t$ .

That is,

- (a)  $\tau_0(t) \geq \tau_1(t)$ ,  $\forall t$ ;
- (b)  $\tau_0(t) \geq \tau_0(s)$ ,  $\forall s > t$ ;
- (c)  $\tau_0(t) \geq \tau_1(s)$ ,  $\forall s > t$ ;

(d)  $\tau_0(t) \geq \bar{T}, \forall t$ .

3. Suppose that  $h^t = \emptyset$ . Then truth-telling forever from now on maximizes the agent's expected payoff. That is, the agents expected payoff from using the truth-telling strategy given that  $h^t = \emptyset$

$$V(t) = \left( \int_t^{\bar{T}} e^{-\lambda^1 s - r\tau_1(s)} ds \right) p_t \lambda^1 e^{(r+\lambda^1)t} W \quad (19)$$

$$+ \left( \int_t^{\bar{T}} e^{-\lambda^0 s - r\tau_0(s)} ds \right) (1 - p_t) \lambda^0 e^{(r+\lambda^0)t} (-\omega) \quad (20)$$

$$+ p_t e^{(r+\lambda^1)(t-\bar{T})} W + (1 - p_t) e^{(r+\lambda^0)(t-\bar{T})} (-\omega) \quad (21)$$

satisfies

$$U(t) = \max \left\{ e^{-r[\tau_1(t)-t]} \left[ -\omega + p_t(W + \omega) \right], e^{-r[\tau_0(t)-t]} \left[ -\omega + p_t(W + \omega) \right], \right.$$

$$\left. e^{-rdt} p_t \lambda^1 dt e^{-r[\tau_1(t+dt)-t-dt]} W + e^{-rdt} (1 - p_t) \lambda^0 dt e^{-r[\tau_0(t+dt)-t-dt]} (-\omega) \right. \\ \left. + e^{-rdt} \left[ 1 - p_t \lambda^1 dt - (1 - p_t) \lambda^0 dt \right] U(t + dt) \right\}.$$

The first and second term denote the agent's expected payoff if he chooses  $m_t = 1$  and  $m_t = 0$ , respectively. Both actions essentially end the game and there is no need to specify future actions. The last term denotes the agent's expected payoff if he chooses  $m_t = \emptyset$  and the optimal action at  $t + dt$ . The first component is the agent's payoff if he gets a 1-signal during  $(t, t + dt)$ . The IC conditions in 1 ensures that the optimal action is to choose  $m_{t+dt} = 1$  in this case, which leads to an investment time  $\tau_1(t + dt)$ . The second component is the agent's payoff if he gets a 0-signal during  $(t, t + dt)$ . The third component is the agent's payoff if he receives no signal during  $(t, t + dt)$ .

The next lemma simplifies the incentive condition at  $h^t = \emptyset$ .

**Lemma 8** *Suppose that a contract  $\langle \bar{T}, \tau_0, \tau_1 \rangle$  satisfies IC at any history  $h_t^\alpha$  for which  $h^t \neq \emptyset$ . Moreover, suppose that at  $h_t^\alpha$  for which  $h^t = \emptyset$ , the following holds:*

$$e^{-rt}V(t) \geq \max \left\{ e^{-r\tau_0(t)} \left[ p_t W + (1-p_t)(-\omega) \right], e^{-r\tau_1(t)} \left[ p_t W + (1-p_t)(-\omega) \right] \right\}.$$

*Then, the strategy of truth-telling at every history also maximizes the agent's expected payoff at any  $h_t^\alpha$  for which  $h^t = \emptyset$ .*

**Proof.** Fix an arbitrary  $h_t^\alpha$  for which  $M_t(h_t^\alpha) = \{0, 1, \emptyset\}$  and  $h^t = \emptyset$ . Let  $\sigma^*$  denote the strategy of truth-telling at every history, whenever doing so is possible. Let  $\sigma$  denote an alternative strategy such that either  $\sigma(h_t^\alpha) \neq \sigma^*(h_t^\alpha)$ , or there exists a concatenation history of  $h_t^\alpha, h_s^\alpha$ , such that  $s > t$  and  $\sigma(h_s^\alpha) \neq \sigma^*(h_s^\alpha)$ .

If  $\sigma(h_t^\alpha) \neq \emptyset$ , then by inequality 8,  $\sigma^*$  renders higher payoff than  $\sigma$ .

If  $\sigma(h_t^\alpha) = \emptyset$ , then take a concatenation history of  $h_t^\alpha$  for which  $\sigma^*(h_s^\alpha) \neq \sigma(h_s^\alpha)$ . Note that at  $h_s^\alpha$ ,  $m_\tau = \emptyset$  for all  $\tau < s$ ; otherwise  $M_s(h_s^\alpha)$  is a singleton. Moreover,  $y_\tau = 0$  for all  $\tau < s$ ; otherwise a decision is already made at  $\bar{T}$  and  $M_s(h_s^\alpha)$  is again a singleton. Therefore the agent's cumulative payoff during  $[0, s)$  equals 0 for both  $\sigma^*$  and  $\sigma$ . Now, if  $h^s \neq \emptyset$ ,  $\sigma^*$  renders higher payoff since the contract is incentive compatible at such a history. If  $h^s = \emptyset$ , then  $\sigma^*(h_s^\alpha) = \emptyset$  while  $\sigma(h_s^\alpha) \in \{0, 1\}$ . By inequality 8,  $\sigma^*$  still renders higher payoff. We have thus shown that  $\sigma^*$  renders higher payoff at any future (on- or off-path) history  $h_s^\alpha$  for which  $\sigma^*$  and  $\sigma$  differ. Therefore  $\sigma^*$  renders higher expected payoff than  $\sigma$  at the information set  $h_t^\alpha$ . ■

Now we will characterize the optimal contract. First we notice that any incentive compatible optimal contract  $\langle \bar{T}^*, \tau_0^*, \tau_1^* \rangle$  must have  $\tau_0^* = \infty$  almost surely.

**Proposition 5** *Given an incentive compatible optimal contract  $\langle \bar{T}^*, \tau_0^*, \tau_1^* \rangle$ , let us define  $A := \{t : \tau_0^*(t) < \infty\}$ . Then  $A$  has measure 0.*

**Proof.** First we notice that if  $\bar{T}^* = \infty$ , then IC requires that  $\tau_0^*(t) = \infty$  for all  $t$ . So for the rest of the proof let us assume that  $\bar{T}^* < \infty$ . By way of contradiction, suppose that  $A$  has positive measure. Then, the part of the principal's payoff involving  $\tau_0^*(\cdot)$  can be rewritten as:

$$\begin{aligned}
& \int_0^{\bar{T}^*} e^{-r\tau_0^*(s)}(-\nu)(1-p_0)\lambda^0 e^{-\lambda^0 s} ds \\
&= \int_A e^{-r\tau_0^*(s)}(-\nu)(1-p_0)\lambda^0 e^{-\lambda^0 s} ds + \int_{[0, \bar{T}^*] \setminus A} e^{-r\tau_0^*(s)}(-\nu)(1-p_0)\lambda^0 e^{-\lambda^0 s} ds \\
&= \int_A e^{-r\tau_0^*(s)}(-\nu)(1-p_0)\lambda^0 e^{-\lambda^0 s} ds \\
&< 0
\end{aligned}$$

Let  $\bar{t}$  be such that  $p_{\bar{t}}W + (1-p_{\bar{t}})(-\omega) = 0$ . We will propose an alternative contract depending on whether  $\bar{T}^*$  is greater or smaller than  $\bar{t}$ .

Case 1.  $\bar{T}^* \leq \bar{t}$

Consider an alternative contract  $\langle \bar{T}^*, \tilde{\tau}_0, \tau_1^* \rangle$ , where

$$\tilde{\tau}_0(s) = \infty, \forall s$$

Since the principal's payoff involving  $\tilde{\tau}_0$  is 0, this contract strictly increases the principal's payoff. Now we will show that  $\langle \bar{T}^*, \tilde{\tau}_0, \tau_1^* \rangle$  is incentive compatible, contradicting to  $\langle \bar{T}^*, \tau_0^*, \tau_1^* \rangle$  being a solution.

It is obvious that the IC conditions when  $h^t \neq \emptyset$  (i.e. cases 1 and 2) are still satisfied for the new contract. For the case when  $h^t = \emptyset$ , notice that under the new contract, the agent's payoff from truth-telling forever from time  $t$  on is

$$\begin{aligned}
& \left[ p_t e^{-\lambda^1(\bar{T}^*-t)} + (1-p_t)e^{-\lambda^0(\bar{T}^*-t)} \right] e^{-rT^*} \left[ p_{\bar{T}^*}W + (1-p_{\bar{T}^*})(-\omega) \right] \\
& + \int_t^{\bar{T}^*} (1-p_t)\lambda^0 e^{-\lambda^0(s-t)} e^{-r\tau_0^*(s)} \cdot 0 ds + \int_t^{\bar{T}^*} p_t \lambda^1 e^{-\lambda^1(s-t)} e^{-r\tau_1^*(s)} W ds \\
& \geq \left[ p_t e^{-\lambda^1(\bar{T}^*-t)} + (1-p_t)e^{-\lambda^0(\bar{T}^*-t)} \right] e^{-rT^*} \left[ p_{\bar{T}^*}W + (1-p_{\bar{T}^*})(-\omega) \right] \\
& + \int_t^{\bar{T}^*} (1-p_t)\lambda^0 e^{-\lambda^0(s-t)} e^{-r\tau_0^*(s)} (-\omega) ds + \int_t^{\bar{T}^*} p_t \lambda^1 e^{-\lambda^1(s-t)} e^{-r\tau_1^*(s)} W ds
\end{aligned}$$

$$\begin{aligned} &\geq \left[ p_t W + (1 - p_t)(-\omega) \right] e^{-r\tau_1^*(t)} \\ &\geq 0 \end{aligned}$$

The second inequality follows from the the fact that the old contract satisfies IC. Therefore truth-telling forever from  $t$  on is preferred to lying that  $m_t = 1$ . The last inequality follows because for any  $t \leq \bar{T}^* \leq \bar{t}$ ,  $p_t W + (1 - p_t)(-\omega) \geq 0$ . Therefore truth-telling forever from  $t$  on is preferred to lying that  $m_t = 0$ . We have just established that under the new contract, truth-telling forever from  $t$  on is preferred to lying at  $t$ . By Lemma 8, this ensures that the new contract is incentive compatible.

Case 2.  $\bar{T}^* > \bar{t}$ .

Since  $p_t$  decreases in  $t$ ,  $p_{\bar{T}^*} W + (1 - p_{\bar{T}^*})(-\omega) < 0$ . Consider an alternative contract  $\langle \tilde{T}, \tilde{\tau}_0, \tilde{\tau}_1 \rangle$ , where

$$\tilde{T} = \infty, \tilde{\tau}_0(s) = \infty, \forall s$$

and

$$\tilde{\tau}_1(t) = \begin{cases} \tau_1^*(t) & \text{if } t \leq \bar{T}^* \\ t & \text{otherwise.} \end{cases}$$

The principal's payoff under this contract equals

$$\begin{aligned} &\int_0^{\bar{T}^*} (1 - p_0) \lambda^0 e^{-\lambda^0 s} \cdot 0 \, ds + \int_0^{\bar{T}^*} p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau_1^*(s)} W \, ds \\ &\quad + \left[ p_0 e^{-\lambda^1 \bar{T}^*} + (1 - p_0) e^{-\lambda^0 \bar{T}^*} \right] \cdot \int_{\bar{T}^*}^{\infty} p_{\bar{T}^*} \lambda^1 e^{-\lambda^1 (s - \bar{T}^*)} e^{-rs} W \, ds \\ &> \int_0^{\bar{T}^*} (1 - p_0) \lambda^0 e^{-\lambda^0 s} e^{-r\tau_0^*(s)} \cdot (-\omega) \, ds + \int_0^{\bar{T}^*} p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau_1^*(s)} W \, ds \\ &\quad + \left[ p_0 e^{-\lambda^1 \bar{T}^*} + (1 - p_0) e^{-\lambda^0 \bar{T}^*} \right] \cdot e^{-r\bar{T}^*} \left[ p_{\bar{T}^*} W + (1 - p_{\bar{T}^*})(-\omega) \right]. \end{aligned}$$

The inequality follows because

$$\int_{\bar{T}^*}^{\infty} p_{\bar{T}^*} \lambda^1 e^{-\lambda^1 (s - \bar{T}^*)} e^{-rs} W \, ds > 0 > e^{-r\bar{T}^*} \left[ p_{\bar{T}^*} W + (1 - p_{\bar{T}^*})(-\omega) \right]$$

and  $A$  has positive measure by assumption. Now we show the new contract is incentive compatible. First, it is easy to see that at  $h^t \neq \emptyset$  and  $t \leq \bar{T}^*$ , IC are satisfied. Second, at any  $t > \bar{T}^* > t^*$ , the interests of the principal and the agent are aligned. Therefore the first-best action is incentive compatible. Lastly, at any  $h^t = \emptyset$  and  $t \leq \bar{T}^*$ , the agent's payoff if he is truth-telling since then on equals

$$\int_t^{\bar{T}^*} p_t \lambda^1 e^{-\lambda^1(s-t)} e^{-r\tau_1^*(s)} W ds + \int_{\bar{T}^*}^{\infty} p_{\bar{T}^*} \lambda^1 e^{-\lambda^1(s-\bar{T}^*)} e^{-rs} W ds \geq 0.$$

Therefore the payoff of truth-telling is greater than the payoff of lying that  $m_t = 0$ .

On the other hand, the payoff of lying that  $m_t = 1$  is

$$\begin{aligned} & \left[ p_t W + (1 - p_t)(-\omega) \right] e^{-r\tau_1^*(t)} \\ & \leq \left[ p_t e^{-\lambda^1(\bar{T}^*-t)} + (1 - p_t) e^{-\lambda^0(\bar{T}^*-t)} \right] e^{-r\bar{T}^*} \left[ p_{\bar{T}^*} W + (1 - p_{\bar{T}^*})(-\omega) \right] \\ & \quad + \int_t^{\bar{T}^*} (1 - p_t) \lambda^0 e^{-\lambda^0(s-t)} e^{-r\tau_0^*(s)} (-\omega) ds + \int_t^{\bar{T}^*} p_t \lambda^1 e^{-\lambda^1(s-t)} e^{-r\tau_1^*(s)} W ds \\ & \leq \int_t^{\bar{T}^*} p_t \lambda^1 e^{-\lambda^1(s-t)} e^{-r\tau_1^*(s)} W ds \\ & \leq \int_t^{\bar{T}^*} p_t \lambda^1 e^{-\lambda^1(s-t)} e^{-r\tau_1^*(s)} W ds + \int_{\bar{T}^*}^{\infty} p_{\bar{T}^*} \lambda^1 e^{-\lambda^1(s-t)} e^{-rs} W ds, \end{aligned}$$

which is the agent's payoff of truth-telling. The first inequality follows because the contract  $\langle \bar{T}^*, \tau_0^*, \tau_1^* \rangle$  is incentive compatible. The second inequality follows because  $p_{\bar{T}^*} W + (1 - p_{\bar{T}^*})(-\omega) < 0$ . Applying Lemma 8 again, we know that the new contract is incentive compatible. ■

It is easy to argue then that

**Proposition 6** *Given an incentive compatible optimal contract  $\langle \bar{T}^*, \tau_0^*, \tau_1^* \rangle$  for which  $\bar{T}^* < \infty$ ,*

$$\tau_0^*(t) = \infty, \quad \forall t < \bar{T}^*.$$

**Proof.** Suppose that  $\tau_0^*(t) < \infty$  for some  $t < \bar{T}^*$ . Then  $h_t^\alpha$  which includes a

0-signal, the agent could deviate to  $m_\tau = \emptyset$  for all  $t \leq \tau < s$  and  $m_s = 0$  for some  $s > t$ . By Proposition 5, such  $s$  must exist. ■

We have shown that  $\tau_0^*(t) = \infty$  for all  $t$  with the possible exception of  $\tau_0^*(\bar{T}^*)$  when  $\bar{T}^* < \infty$ . We set  $\tau_0(\bar{T}^*) = \bar{T}^*$  automatically whenever  $\bar{T}^* < \infty$ . This makes sure that for any  $\bar{T}^* < \infty$ , truth-telling is incentive-compatible at  $\bar{T}^*$  and at  $t < \bar{T}^*$  and  $h^t \neq \emptyset$ .

Now we are ready to rewrite the principal's constrained maximization problem as follows:

$$\max_{\bar{T} \in \mathfrak{R}_+ \cup \{\infty\}, \tau(\cdot)} \int_0^{\bar{T}} p\lambda^1 e^{-\lambda^1 s} e^{-r\tau(s)} V ds + \left[ p e^{-\lambda^1 \bar{T}} + (1-p) e^{-\lambda^0 \bar{T}} \right] e^{-r\bar{T}} \left[ -\nu + p_{\bar{T}}(V + \nu) \right] \quad (22)$$

subject to

$$\tau(t) \geq t \quad \forall t \in [0, \bar{T}]; \quad (23)$$

$$\tau(s) \geq \tau(t) \quad \forall s \geq t; \quad (24)$$

$$\tau(t) \leq \bar{T} \quad \forall t \in [0, \bar{T}]; \quad (25)$$

$$\begin{aligned} & \int_t^{\bar{T}} p_t \lambda^1 \frac{e^{-\lambda^1 s}}{e^{-\lambda^1 t}} e^{-r\tau(s)} W ds + \left[ p_t \frac{e^{-\lambda^1 \bar{T}}}{e^{-\lambda^1 t}} e^{-r\bar{T}} W + (1 - p_t) \frac{e^{-\lambda^0 \bar{T}}}{e^{-\lambda^0 t}} e^{-r\bar{T}} (-\omega) \right] \\ & \geq \max \left\{ e^{-r\tau(t)} \left[ -\omega + p_t(W + \omega) \right], 0 \right\}, \forall t \in [0, \bar{T}]. \end{aligned}$$

## B Proofs for Section 4.1

**Proof of Lemma 1.** We prove that  $\tau(t) > t$  for  $t < \min\{t^*, T\}$ . For simplicity, take  $t = 0$ . By contradiction, assume that  $\tau(0) = 0$ . The left hand

side of (5) can be written as

$$\begin{aligned} & \int_0^T \left( p_0 \lambda^1 \exp(-\lambda^1 s) \exp(-r\tau(s)) W \right) ds + \left( p_0 \exp(-\lambda^1 T) e^{-rT} W \right. \\ & \quad \left. + (1 - p_0) \exp(-\lambda^0 T) e^{-rT} (-\omega) \right) \leq \int_0^T \left( \lambda^1 \exp(-\lambda^1 s) \exp(-rs) p_0 W \right) ds \\ & \quad + \left( p_0 \exp(-\lambda^1 T) e^{-rT} W + (1 - p_0) \exp(-\lambda^0 T) e^{-rT} (-\omega) \right) \end{aligned}$$

The inequality follows since  $\tau(s) \geq s$ . The term on the right hand side of the inequality above is the expected payoff that the agent would get following the policy of investing if any good signal is revealed before  $T$  and investing at  $T$  if no signal is revealed before  $T$ . Since  $p_0 > p^*$ , this policy must result in strictly lower payoffs than the expected payoff from investing at  $t = 0$ . So,

$$\begin{aligned} & \int_0^T \left( \lambda^1 \exp(-\lambda^1 s) \exp(-rs) p_0 W \right) ds + \left( p_0 \exp(-\lambda^1 T) e^{-rT} W \right. \\ & \quad \left. + (1 - p_0) \exp(-\lambda^0 T) e^{-rT} (-\omega) \right) < p_0 W + (1 - p_0)(-\omega). \end{aligned}$$

Combining these inequalities we deduce that (5) is violated at  $t = 0$  when  $\tau(0) = 0$ . It follows that  $\tau(0) > 0$ . ■

**Proof of Lemma 2.** Let  $\tau^*$  solve the relaxed problem. By way of contradiction, assume that for some  $A \subseteq [0, \min\{t^*, T\})$  with positive Lebesgue measure, and for all  $t \in A$ , the constraint (8) is slack. For  $t \in \text{dom}(\tau)$ , define

$$\varphi_t = \int_t^T p_t \lambda^1 \frac{e^{-\lambda^1 s}}{e^{-\lambda^1 t}} e^{-r\tau^*(s)} W ds + \left( p_t \frac{e^{-\lambda^1 T}}{e^{-\lambda^1 t}} e^{-rT} W + (1 - p_t) \frac{e^{-\lambda^0 T}}{e^{-\lambda^0 t}} e^{-rT} (-\omega) \right)$$

Now, define  $\tau'$  as follows. For  $t \notin A$ ,  $\tau'(t) = \tau^*(t)$ , while for  $t \in A$ ,

$$e^{-r\tau'(t)} (-\omega + p_t(W + \omega)) = \varphi_t.$$

For  $t \in A$ ,  $e^{-r\tau'(t)} > e^{-r\tau^*(t)}$ . Therefore,  $\tau'(t) \leq \tau^*(t)$  for all  $t \in [0, \min\{t^*, T\}]$ , with strict inequality for  $t \in A$ . We claim that  $\tau'$  is feasible. To see this, note



that for all  $t \in [0, \min\{t^*, T\}]$

$$\begin{aligned}
& \int_t^T p_t \lambda^1 \frac{e^{-\lambda^1 s}}{e^{-\lambda^1 t}} e^{-r\tau'(s)} W ds + \left( p_t \frac{e^{-\lambda^1 T}}{e^{-\lambda^1 t}} e^{-rT} W + (1 - p_t) \frac{e^{-\lambda^0 T}}{e^{-\lambda^0 t}} e^{-rT} (-\omega) \right) \\
& \geq \int_t^T p_t \lambda^1 \frac{e^{-\lambda^1 s}}{e^{-\lambda^1 t}} e^{-r\tau^*(s)} W ds + \left( p_t \frac{e^{-\lambda^1 T}}{e^{-\lambda^1 t}} e^{-rT} W + (1 - p_t) \frac{e^{-\lambda^0 T}}{e^{-\lambda^0 t}} e^{-rT} (-\omega) \right) \\
& = \varphi_t \\
& \geq e^{-r\tau'(t)} (-\omega + p_t(W + \omega)) \\
& \geq \max\{0, e^{-r\tau'(t)} (-\omega + p_t(W + \omega))\}.
\end{aligned}$$

The first inequality follows since  $\tau'$  is below  $\tau^*$  and the equality is by definition of  $\varphi_t$ . The second inequality follows with equality when  $t \in A$  (by definition of  $\tau'$ ) and for  $t \notin A$  follows since  $\tau'$  and  $\tau^*$  coincide and  $\tau^*$  satisfies 8. The third inequality follows since  $t < t^*$ . It follows that  $\tau'$  satisfies (7)-(8) and results in higher expected payoffs than  $\tau^*$ . This contradicts the optimality of  $\tau^*$  for the relaxed problem.

We now argue that when  $T > t^*$  (case b in the statement of the Proposition),  $\tau^*(t) = t$  for almost every  $t \in [t^*, T]$ . Otherwise, there is a set  $A \subseteq [t^*, T]$  of positive measure such that for all  $t \in A$ ,  $\tau^*(t) > t$ . Construct  $\tau'$  that coincides with  $\tau^*$  outside  $A$ , but  $\tau'(t) = t$  for  $t \in A$ . It is clear that  $\tau'$  satisfies (8) since for  $t < t^*$ ,  $\tau'$  does not change the payoff from lying but increased the payoff from truth-telling. It follows that  $\tau'$  is feasible for the relaxed problem and results in higher expected payoffs for the principal than  $\tau^*$ . This is a contradiction.

Now, to prove the converse, we assume that  $T > t^*$ . The proof of the converse when  $T \leq t^*$  is analogous. Take  $\tau^*$  such that (7)-(8) bind almost everywhere. Take  $\tau'$  that solves the relaxed problem (6). From the first part of this proof,  $\tau'$  and  $\tau^*$  coincide for almost every  $t \in [t^*, T]$ . The previous step also shows that  $\tau'$  is such that (8) binds for almost every  $t \in [0, \min\{t^*, T\}]$ . Define

$$u(t) = \int_t^T e^{-\lambda^1 s} (e^{-r\tau^*(s)} - e^{-r\tau'(s)}) ds$$

for  $t \in [0, \min\{t^*, T\}]$ . Note that  $u(t)$  is absolutely continuous and its deriva-

tive is defined almost everywhere and equals  $-e^{-\lambda^1 t}(e^{-r\tau^*(t)} - e^{-r\tau'(t)})$ . Now, using the fact that the constraint binds almost everywhere for both  $\tau'$  and  $\tau^*$ , we deduce that for almost every  $t \in [0, \min\{t^*, T\}]$ ,

$$-p_t \lambda^1 W u(t) = u'(t) \left( -\omega + p_t(W + \omega) \right)$$

and  $u(\min\{t^*, T\}) = 0$ . It follows that for almost every  $t \in [0, \min\{t^*, T\}]$ ,  $\frac{d}{dt} \left( u(t) e^{\int_0^t H(s) ds} \right) = 0$  where  $H$  is a continuous function. Since  $u(\min\{t^*, T\}) = 0$ ,  $u(t) = 0$  for all  $t \in [0, \min\{t^*, T\}]$ . In particular,  $0 = u'(t) = -e^{-\lambda^1 t}(e^{-r\tau^*(t)} - e^{-r\tau'(t)})$  almost everywhere and therefore  $\tau'$  and  $\tau^*$  coincide for almost every  $t \in [0, \min\{t^*, T\}]$ . Since  $\tau^*$  satisfies (7)-(8),  $\tau^*$  solves the relaxed problem. ■

**Proof of Lemma 3.** Since (5) is binding for all  $t \in [0, \min\{t^*, T\}]$ ,

$$\begin{aligned} \int_t^T \lambda^1 e^{-\lambda^1 s} e^{-r\tau(s)} W ds + \left( e^{-\lambda^1 T} e^{-rT} W + \frac{1-p_t}{p_t} e^{(\lambda^0 - \lambda^1)t} e^{-\lambda^0 T} e^{-rT} (-\omega) \right) \\ = e^{-\lambda^1 t} e^{-r\tau(t)} \left( W + \frac{1-p_t}{p_t} (-\omega) \right) \end{aligned}$$

where we use the fact that  $t \leq t^*$ . Since the right hand side of this equation and  $p_t$  are differentiable, so is  $\tau$ . Taking derivatives and using the fact that  $\frac{d}{dt} \left( \frac{1-p_t}{p_t} \right) = (\lambda^1 - \lambda^0) \frac{1-p_t}{p_t}$ , we deduce that

$$\begin{aligned} -\lambda^1 e^{-\lambda^1 t} e^{-r\tau(t)} W = -(\lambda^1 + r\dot{\tau}(t)) e^{-\lambda^1 t - r\tau(t)} \left( W + \frac{1-p_t}{p_t} (-\omega) \right) \\ + e^{-\lambda^1 t - r\tau(t)} (\lambda^1 - \lambda^0) \frac{1-p_t}{p_t} (-\omega). \end{aligned}$$

Solving for  $\dot{\tau}(t)$ , we deduce that

$$\dot{\tau}(t) = \left( \frac{\lambda^0}{r} \right) \frac{\omega}{W \frac{p_t}{1-p_t} - \omega}.$$

The slope of  $\tau$  is nonnegative. To see that  $\tau$  is convex, note that  $p_t/(1-p_t)$  is non-increasing and thus  $\dot{\tau}$  is non-decreasing. To see that  $\dot{\tau}$  is less than 1, note

that

$$\dot{\tau} < 1 \quad \text{iff} \quad 1 < \frac{W}{\omega} \frac{r}{\lambda^0 + r} \frac{p_t}{1 - p_t}.$$

To verify this last property, note that investing at  $t$  results in higher expected payoffs for the agent than learning at  $t$  and investing at  $t + dt$  unless the bad state is revealed. That is,

$$p_t W + (1 - p_t)(-\omega) \geq (1 - p_t)(1 - \lambda^0 dt)e^{-rdt}(-\omega) + p_t e^{-rdt}W.$$

Reordering terms and taking  $dt \rightarrow 0$ , we deduce that  $1 < \frac{W}{\omega} \frac{r}{\lambda^0 + r} \frac{p_t}{1 - p_t}$ . ■

## C Proofs for Section 4.3

**Proof of Proposition 2.** We first prove part (a). Note that the principal's expected payoff from setting  $T = \infty$  equals

$$\varphi\left(\frac{W}{\omega}\right) = \int_0^{t^*} p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau_{\frac{W}{\omega}}^{t^*}(s)} V ds + \int_{t^*}^{\infty} p_0 \lambda^1 e^{-\lambda^1 s} e^{-rs} V ds$$

where  $\tau_{\frac{W}{\omega}}^X(t) = X - \left(\frac{\lambda^0}{r}\right) \int_t^X \frac{1}{\frac{W}{\omega} \frac{p_s}{1 - p_s} - 1} ds$  for all  $X \geq t$ . We claim that for all  $\epsilon > 0$ , there exists  $L$  such that for all  $\frac{W}{\omega} > L$ ,  $\varphi\left(\frac{W}{\omega}\right) < \epsilon$ .

First, notice that since  $t^* \rightarrow \infty$  as  $\frac{W}{\omega} \rightarrow \infty$ , there exists  $L_1$  such that for all  $\frac{W}{\omega} > L_1$ ,

$$\int_{t^*}^{\infty} p_0 \lambda^1 e^{-\lambda^1 s} e^{-rs} V ds < \epsilon/2.$$

Now we show that there exists  $L_2$  such that for all  $\frac{W}{\omega} > L_2$ ,

$$\int_0^{t^*} p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau_{\frac{W}{\omega}}^{t^*}(s)} V ds < \frac{\epsilon}{2}.$$

To show this, we first show that for any  $\delta$ , there exists  $L_3$  such that for all  $\frac{W}{\omega} > L_3$ ,

$$\frac{1}{\frac{W}{\omega} \frac{p_s}{1 - p_s} - 1} < \delta, \forall s \in [0, t^*].$$

Since  $\frac{p_s}{1-p_s}$  decreases in  $s$ , suffices to show

$$\frac{1}{\frac{W}{\omega} \frac{p_0}{1-p_0} - 1} < \delta.$$

This is done by letting  $L_3 = \frac{\delta+1}{\delta} \frac{2(1-p_0)}{p_0}$ .

Given this, we now show that for any  $\eta$ , there exists  $L_4$  such that  $\frac{W}{\omega} > L_4$  implies

$$e^{-r\tau(s)} < \eta, \forall s \in [0, t^*].$$

To show this, first we notice that  $\tau(s)$  increases in  $s$ , so it suffices to show that there exists  $L_4$  such that  $\frac{W}{\omega} > L_4$  implies

$$e^{-r \left[ t^* - \frac{\lambda^0}{r} \int_0^{t^*} \frac{1}{\frac{W}{\omega} \frac{p_s}{1-p_s} - 1} ds \right]} < \eta.$$

In other words,

$$t^* - \frac{\lambda^0}{r} \int_0^{t^*} \frac{1}{\frac{W}{\omega} \frac{p_s}{1-p_s} - 1} ds > \frac{\ln \eta}{-r}.$$

Given what we showed in the previous step, we can find  $L_3$  such that

$$\frac{1}{\frac{W}{\omega} \frac{p_s}{1-p_s} - 1} < \frac{r}{2\lambda^0}, \forall s \in [0, t^*].$$

Therefore

$$\begin{aligned} t^* - \frac{\lambda^0}{r} \int_0^{t^*} \frac{1}{\frac{W}{\omega} \frac{p_s}{1-p_s} - 1} ds &> t^* - \frac{\lambda^0}{r} \int_0^{t^*} \frac{r}{2\lambda^0} ds \\ &= t^* - \frac{\lambda^0}{r} \frac{r}{2\lambda^0} t^* \\ &= \frac{t^*}{2} \rightarrow \infty \end{aligned}$$

as  $\frac{W}{\omega} \rightarrow \infty$ . We have therefore shown that there exists  $L_4$  such that  $\frac{W}{\omega} > L_4$  implies

$$e^{-r\tau(s)} < \eta, \forall s \in [0, t^*].$$

Now find  $L_4$  such that

$$e^{-r\tau(s)} < \frac{\epsilon}{4Vp_0}, \forall s \in [0, t^*].$$

Therefore

$$\begin{aligned} \int_0^{t^*} p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau(s)} V ds &< \int_0^{t^*} p_0 \lambda^1 e^{-\lambda^1 s} \frac{\epsilon}{4Vp_0} V ds \\ &= p_0 \lambda^1 \frac{\epsilon}{4Vp_0} V \int_0^{t^*} e^{-\lambda^1 s} ds \\ &= p_0 \lambda^1 \frac{\epsilon}{4Vp_0} V \cdot \frac{1}{\lambda^1} (1 - e^{-\lambda^1 t^*}) \\ &= p_0 \frac{\epsilon}{4Vp_0} V (1 - e^{-\lambda^1 t^*}) \\ &= \frac{\epsilon}{4} (1 - e^{-\lambda^1 t^*}) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Therefore, for  $\frac{W}{\omega} > L_2 := \max\{L_3, L_4\}$ , we have

$$\int_0^{t^*} p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau_{\frac{W}{\omega}}^*(s)} V ds < \frac{\epsilon}{2}.$$

Lastly, letting  $L := \max\{L_1, L_2\}$ , we then have

$$\varphi\left(\frac{W}{\omega}\right) < \epsilon$$

for  $\frac{W}{\omega} > L$ .

Now, note that by setting an optimal deadline  $T \in [0, t^*]$ , the principal's payoff equals

$$\Phi\left(\frac{W}{\omega}\right) = \max_{T \in [0, t^*]} \Phi\left(\frac{W}{\omega}, T\right)$$

where

$$\Phi\left(\frac{W}{\omega}, T\right) = \int_0^T p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau_{\frac{W}{\omega}}^*(s)} V ds + \left( p_0 e^{-\lambda^1 T} e^{-rT} V + (1-p_0) e^{-\lambda^0 T} e^{-rT} (-\nu) \right)$$

Note that

$$\begin{aligned}\Phi\left(\frac{W}{\omega}, T\right) &> p_0 e^{-rT} \left(1 - e^{-\lambda^1 T}\right) V + \left(p_0 e^{-\lambda^1 T} e^{-rT} V + (1 - p_0) e^{-\lambda^0 T} e^{-rT} (-\nu)\right) \\ &= e^{-rT} \left(p_0 V + (1 - p_0) e^{-\lambda^0 T} (-\nu)\right)\end{aligned}$$

Fix any  $T$  such that the expression above is strictly positive and equals  $\eta > 0$ . Let  $\epsilon = \eta/2$  and take  $W/\omega > L$  such that  $\varphi(W/\omega) < \epsilon = \eta/2$  and  $T < t^*$ . In particular,

$$\Phi\left(\frac{W}{\omega}\right) \geq \eta > \eta/2 \geq \varphi\left(\frac{W}{\omega}\right)$$

which proves that there exists some  $\bar{\kappa}$  such that for all  $\frac{W}{\omega} > \bar{\kappa}$ ,  $T \in [0, t^*]$  results in higher payoffs than  $T = \infty$ .

To complete the proof of part (a), note that as  $W/\omega$  goes to  $x$  where  $x \frac{r}{\lambda^1 + r} \frac{p_0}{1 - p_0} = 1$ ,  $t^* \rightarrow 0$ . In particular,

$$\varphi\left(\frac{W}{\omega}\right) \rightarrow \int_0^\infty p_0 \lambda^1 e^{-\lambda^1 s} e^{-rs} V ds = p_0 V \frac{\lambda^1}{\lambda^1 + r}$$

whereas

$$\Phi\left(\frac{W}{\omega}\right) \rightarrow p_0 V + (1 - p_0)(-\nu).$$

Since  $p_0 V \frac{\lambda^1}{\lambda^1 + r} > p_0 V + (1 - p_0)(-\nu)$ , there exists  $\underline{\kappa}$  such that for all  $W/\omega < \underline{\kappa}$ ,  $\varphi(W/\omega) > \Phi(W/\omega)$ .

To prove part (b), we normalize the principal's expected payoffs by  $\nu$  and write

$$\varphi\left(\frac{V}{\nu}\right) = \int_0^{t^*} p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau \frac{W}{\omega}(s)} \frac{V}{\nu} ds + \int_{t^*}^\infty p_0 \lambda^1 e^{-\lambda^1 s} e^{-rs} \frac{V}{\nu} ds$$

for the principal's payoff when  $T = \infty$  and

$$\Phi\left(\frac{V}{\nu}, T\right) = \int_0^T p_0 \lambda^1 e^{-\lambda^1 s} e^{-r\tau \frac{W}{\omega}(s)} \frac{V}{\nu} ds + \left(p_0 e^{-\lambda^1 T} e^{-rT} \frac{V}{\nu} + (1 - p_0) e^{-\lambda^0 T} e^{-rT} (-1)\right)$$

for the principal payoff when setting  $T < t^*$ . Note that as  $V/\nu \rightarrow 0$ ,

$$\varphi\left(\frac{V}{\nu}\right) \rightarrow 0, \quad \Phi\left(\frac{V}{\nu}, T\right) \rightarrow -(1-p_0)e^{-\lambda^0 T} e^{-rT}$$

Since  $\Phi$  is continuous in  $(\frac{V}{\nu}, T)$ , there exists  $\underline{\eta} > 0$  such that for all  $V/\nu < \underline{\eta}$ ,

$$\varphi\left(\frac{V}{\nu}\right) > \max_{T \in [0, t^*]} \Phi\left(\frac{V}{\nu}, T\right)$$

and thus it is optimal for the principal to set  $T = \infty$ .

To complete part (b), define  $y$  such that  $1 = y \frac{r}{\lambda^1 + r} \frac{p_0}{1-p_0}$ . By definition,

$$\varphi(y) < \max_{T \in [0, t^*]} \Phi(y, T)$$

where the maximum on the right is attained at  $T = 0$ . By continuity, there exists  $\bar{\eta} < y$  such that for all  $V/\nu > \bar{\eta}$ ,

$$\varphi\left(\frac{V}{\nu}\right) < \max_{T \in [0, t^*]} \Phi\left(\frac{V}{\nu}, T\right)$$

and the principal sets a deadline  $T < t^*$ . ■

## D Proofs for Section 5

**Proof of Lemma 4.** Note that

$$\begin{aligned} \frac{\partial}{\partial S} \left( u^{(T, \tau)}(t, S) \right) &= (1-p_t)e^{\lambda^0 t} \left\{ \frac{p_t}{1-p_t} e^{(\lambda^1 - \lambda^0)t} \lambda^1 e^{-\lambda^1 S} (e^{-r\tau(S)} - e^{-rT}) \right. \\ &\quad \left. + \lambda^0 \omega e^{-\lambda^0 S - rT} - \left( \frac{p_t}{1-p_t} e^{(\lambda^1 - \lambda^0)t} e^{-\lambda^1 S} + e^{-\lambda^0 S} \right) e^{-rS} c \right\} \\ &\geq (1-p_t)e^{\lambda^0 t} \left\{ \lambda^0 \omega e^{-\lambda^0 T} e^{-rT} - \left( \frac{p_t}{1-p_t} e^{(\lambda^1 - \lambda^0)t} + 1 \right) c \right\} \end{aligned}$$

where the inequality follows since  $\tau(S) \leq T$  for all  $S \leq T$ . The result follows by taking

$$\bar{c} = \min_{t \leq T} \frac{\lambda^0 \omega e^{-\lambda^0 T} e^{-rT}}{\frac{p_t}{1-p_t} e^{(\lambda^1 - \lambda^0)t} + 1}.$$

■

**Proof of Lemma 5.** Since the slope of  $\tau^{\hat{T}}$  is less than 1,  $\tau^T(T) = T$ , and  $\tau^{\hat{T}}(\hat{T}) = \hat{T}$ , it follows that  $\tau^T(T) < \tau^{\hat{T}}(T)$ . Moreover, from (15),  $\dot{\tau}^T(T) > \dot{\tau}^{\hat{T}}(T)$  and thus there exists an interval  $[a_1, T]$  such that  $\tau^T(t) < \tau^{\hat{T}}(t)$  for all  $t \in [a_1, T]$ . In particular,  $\tau^T(a_1) < \tau^{\hat{T}}(a_1)$  and, applying the same argument, there exists an interval  $[a_2, a_1]$  such that  $\tau^T(t) < \tau^{\hat{T}}(t)$  for all  $t \in [a_2, a_1]$ . All times  $t \in ]0, T]$  will be covered by this procedure and thus  $\tau^T(t) < \tau^{\hat{T}}(t)$ . The argument for  $t = 0$  follows by continuity. ■

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