Multidimensional Apportionment through Discrepancy Theory

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Abstract

Deciding how to allocate the seats of a house of representatives is one of the most fundamental problems in the political organization of societies, and has been widely studied over already two centuries. The idea of proportionality is at the core of most approaches to tackle this problem, and this notion is captured by the divisor methods, such as the Jefferson/D'Hondt method. In a seminal work, Balinski and Demange extended the single-dimensional idea of divisor methods to the setting in which the seat allocation is simultaneously determined by two dimensions, and proposed the so-called *biproportional apportionment* method. The method, currently used in several electoral systems, is however limited to two dimensions and the question of extending it is considered to be an important problem both theoretically and in practice. In this work we initiate the study of multidimensional proportional apportionment. We first formalize a notion of multidimensional proportionality that naturally extends that of Balinski and Demange. By means of analyzing an appropriate integer linear program we are able to prove that, in contrast to the two-dimensional case, the existence of multidimensional proportional apportionments is not guaranteed and deciding its existence is NP-complete. Interestingly, our main result asserts that it is possible to find approximate multidimensional proportional apportionments that deviate from the marginals by a small amount. The proof arises through the lens of discrepancy theory, mainly inspired by the celebrated Beck-Fiala Theorem. We finally evaluate our approach by using the data from the recent 2021 Chilean Constitutional Convention election.

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1 Introduction

A cornerstone of modern democracies is the division of the political organization, generally including a house of representatives with the goal of reflecting the needs of different segments across the population. In the apportionment problem, the purpose is to allocate the total number of seats in a house of representatives, and how to solve this problem is something that has been discussed and studied extensively in modern history. A natural goal that is at the core of many apportionment systems is the idea of *proportionality*. That is, a party receives an amount of seats that is proportional to the number of votes that the party garnered in the election. Since in general the seats are not divisible, it is necessary to properly formalize the notion of proportionality in an integral setting. The divisor methods provide an answer to this problem, based on appropriately scaling the votes and rounding the result in order to meet the house size. These methods are widely used at national and regional levels in many democracies around the world. The two most common versions are, by far, the Jefferson/D'Hondt method proposed by Thomas Jefferson in 1792, and the Webster/Sainte-Laguë method first proposed by Daniel Webster in 1832. While both methods correspond to divisor methods, the latter rounds to the nearest integer while the former takes integer part [8].

In their seminal work, Balinski and Demange extended the notion of proportionality and divisor methods to the case in which the apportionment is ruled by two dimensions, studying this extension from an axiomatic and algorithmic point of view [4, 3]. In this setting, an instance is given by an integral matrix (of votes) where the rows typically represent the political parties and the columns represent the regions. We are also given a list of strictly positive integers, called *marginals*, specifying the row and column sums for any feasible biproportional apportionment. Thus the *row* marginals account for the number of seats that have to be allocated to the corresponding party[1], and the *column* marginals correspond to the number of seats a given district should get.[2] The goal is to find a matrix (of seats) satisfying the marginals and keeping proportionality with respect to the votes simultaneously in both dimensions. This notion is captured by a set of *multipliers*, one for each row and column. The biproportional apportionment method is currently used in elections of several cantons in Switzerland.[3]

A distinctive feature of the biproportional method is that the existence (and essential uniqueness) is guaranteed [4], under very natural conditions. However, by design, the biproportional method is limited to the case of apportionments ruled by two dimensions. Departing from the two-dimensional case is not only a challenging mathematical question, but also a relevant practical problem. Indeed, as modern societies become more complex, representation of dimensions beyond political affiliation and geography is increasingly demanded. For instance, New Zealand's parliament includes ethnic representation while the recently elected 2021 Chilean Constitutional Convention includes gender balance as a constraint. Another example, mentioned by Demange [18], is the proposed division of three types of "Constituent People" (Bosniacs, Croats and Others) in the Parliament of the Federation of Bosnia and Herzegovina, which led Demange to coin the multidimensional proportional apportionment as a challenging question.

In this paper, we initiate the study of multidimensional proportional apportionment, and establish that if we allow small deviations from the prescribed marginals then existence is again guaranteed. As an illustration, in the case of three dimensions, say political, regional and gender,

[1] Generally, these marginals correspond to the proportion of seats each party should get, given its proportion on the number of votes, by applying some divisor method. Sometimes, however, a certain minimum proportion of the votes is imposed in order to obtain seats in the house.

[2] Often proportional to the population of the district.

[3] These include Zurich, Aargau, Schaffhausen, Nidwalden, Zug, Schwyz and Valais.

our main theorem states that there exists a 3-dimensional proportional apportionment that deviates by at most one on each of the prescribed marginals.

1.1 Our Contribution

One of the key technical ingredients used to study the biproportional apportionment corresponds to a linear program, inspired by the closely related *matrix scaling* problem [19, 20, 29]. Following this approach, we introduce an integer linear program to analyze the multidimensional setting and provide structural results by studying its linear relaxation. Specifically, we prove that the existence of a multidimensional proportional apportionment is fully characterized by the fact that the linear relaxation of this integer LP admits an integer optimal solution. We can then use this technique to establish that in general multidimensional proportional apportionments may fail to exist. This result is established by extending the network flow approach [30, 19] and conducting a careful primal-dual analysis. Furthermore, we use this approach to show that determining the existence of a proportional apportionment in the multidimensional setting (dimension 3 and higher) is NP-complete. This is in sharp contrast with the two-dimensional case, in which this decision problem is polynomially solvable [3, 4]. These results can be found in Section 3, and the Appendix 6 contain their proofs.

Given that multidimensional proportional apportionments may fail to exist (and are in general hard to compute) we study what happens when we allow for small violations in the marginals. Specifically, we consider the question of whether we can obtain an apportionment satisfying the proportionality condition by allowing to violate the marginals by a certain amount, and whether this can be done efficiently (polynomial in the house size). Our main result provides a positive answer to this question, and prove that if $(u_1, \dots, u_d) \in \mathbb{N}^d$ are the target maximum violations in each dimension,[4] a d-dimensional proportional apportionment exists so long as $\sum_{\ell=1}^{d} 1/(u_\ell + 2) \leq 1$.[5] In dimension 2, with $u_1 = u_2 = 0$ this recovers the existence result of Balinski and Demange [4], while in dimension 3 a violation of one seat in each dimension is enough to guarantee existence. The main new technical ingredient is to follow the lens of discrepancy theory, mainly the celebrated Beck-Fiala Theorem [13]. Furthermore, we provide a finer analysis on the deviations allowed by our algorithm, going beyond the typical ℓ_∞ analysis of deviations in discrepancy rounding of linear programs. To this end, we design an algorithm for an appropriate discrepancy problem in hypergraphs that might be of independent interest. Starting from an optimal solution of our base linear relaxation for the multidimensional problem, we run our discrepancy algorithm to get an apportionment with the desired deviations, and by using the structure of our linear program we show that proportionality is satisfied. These results can be found in Section 4, and the Appendix 7 contain their proofs.

Finally, in Section 5, we test our method for finding a 3-dimensional proportional apportionment in the context of the 2021 Chilean Constitutional Convention election. This election sought to elect a convention achieving proportionality across three dimensions: political, geographical and gender. We observe that our method leads to an apportionment fulfilling the prescribed marginals and achieving exact gender parity. We also conclude that our method is significantly more representative than the one used. Finally, by simulating small random perturbations to the votes, we conclude that our approach is more robust in that these perturbations translate into only small changes in the house configuration.

[4] That is, in each dimension $\ell \in \{1, \dots, d\}$ we allow the marginals to be additively violated by at most u_ℓ.

[5] The result actually requires a mild additional assumption that, for instance, is satisfied if the original vote matrix does not contain zeros.

1.2 Literature Overview

Divisor Methods. There is a long and rich body of literature for the apportionment problem and the divisor methods, intersecting different areas such as operations research, computer science and political science. For a formal treatment of the theory and a historical survey, we refer to the book of Balinski and Young [9] and to the recent book by Pukelsheim [27]. For a deeper treatment of social choice and new methods, we also refer to the book and article by Baliski and Laraki [6, 7].

Biproportionality and Matrix Scaling. After Balinski and Demange first develop the biproportional method [3, 4], some variants of it have been later proposed by Balinski [2, 5]. Rote and Zachariasen [29] and Gaffke and Pukelsheim [19, 20] provided a unified view of biproportionality by using network flow formulations. Pukelsheim et al. also provided a wider view of network flow methods and their usage for electoral systems [28]. The matrix scaling problem has been studied extensively in the optimization, statistics, algorithms and machine learning communities and we refer to the survey by Idel for an extensive treatment of this problem [22]. Particularly relevant is the work by Sinkhorn [32] and subsequent complexity and algorithmic results by Sinkhorn and Knopp [33], Rothblum and Schneider [30] and Nemirovski and Rothblum [26]. Kalantari et al. [23] analyzed an algorithm for matrix scaling introduced by Balinski and Demange [3] and very recently there have been several works on developing faster algorithms for matrix scaling and improved analysis of existing methods [16, 1, 17].

Discrepancy Theory. In the classic discrepancy minimization problem, there is a fractional vector x with entries in $[0, 1]$ satisfying $Ax = b$ for some binary matrix A and an integer vector b, and the goal is to round x to get an integral vector $\tilde{x}$ with entries in $\{0, 1\}$ in a way such that the maximum deviation $\|Ax - A\tilde{x}\|_\infty$ is as small as possible. In a celebrated result, Beck and Fiala [13] provided an algorithm to perform such rounding while achieving a maximum deviation of at most the maximum ℓ_1 norm of a column in A. This result was later improved for certain regimes [14, 15, 31]. Also remarkable are the recent works by Bansal et al. [10, 12, 11], Lovett and Meka [25] and Rothvoss [31] that provide algorithmic results for different discrepancy problems involving ℓ_∞ and ℓ_2 violations.

2 Preliminaries

In the classic apportionment problem, the input is given by a pair $(\mathcal{P}, H)$ where $\mathcal{P}$ is a vector in $\mathbb{N}^n$ containing the votes obtained by each party $i \in \{1, \ldots, n\}$ and H is the *house size*, representing the total number of seats to allocate. The goal is to decide how many of the H seats should be given to each party. A feasible solution to this problem is formally described by a vector x such that $\sum_{i=1}^n x_i = H$ and x_i is a nonnegative integer for every $i \in \{1, \ldots, n\}$. The value x_i is the number of seats to be allocated for party $i \in \{1, \ldots, n\}$. Clearly, a feasible solution always exists, but the challenge is to allocate the H seats *proportionally* to the votes. Naturally, seats cannot be divided fractionally, and therefore proportionality in this context needs to be defined appropriately. This paradigm is captured by a family of broadly used methods called *divisor methods*, that we formally describe in what follows.

2.1 Signpost Sequences, Rounding Rules and Divisor Methods

Following the formalization introduced by Balinski and Young [9], a *signpost sequence* is a function, defined over the nonnegative integers, $s : \mathbb{N} \to \mathbb{R}_+$ satisfying $s(0) = 0$, $s(q) \in [q - 1, q]$ for every strictly positive integer q and the following *left-right disjunction* property: (i) If $s(p) = p - 1$ for some $p \geq 2$, then $s(q) < q$ for every $q \geq 1$, and (ii) if $s(q) = q$ for some $q \geq 1$, then $s(p) > p - 1$ for every $p \geq 2$. In particular, any signpost sequence is strictly increasing over the strictly positive integers. To every signpost sequence s we can associate a *rounding rule* $[\![\cdot]\!]_s$ as follows: $[\![0]\!]_s = \{0\}$, $[\![t]\!]_s = \{q\}$ when $t \in (s(q), s(q+1))$ and $[\![t]\!]_s = \{q-1, q\}$ when $t = s(q) > 0$. Especially relevant are the signpost sequences of the form $s(q) = q - \Delta$ for every strictly positive integer q and some $\Delta \in [0, 1]$, since they capture the usual rounding operations. These signpost sequences are called *stationary*. To mention a few, $\Delta = 0$ coincides with the classic downward rounding when t is fractional, $\Delta = 1$ coincides with the upward rounding when t is fractional and $\Delta = 1/2$ coincides with the standard rounding when $t - 1/2$ is fractional.

The divisor method associated to a signpost sequence s works as follows: Given a pair $(\mathcal{P}, H)$, compute a vector $x \in \mathbb{N}^n$ with $\sum_{i=1}^n x_i = H$ for which there is a strictly positive value λ, called *multiplier*, such that $x_i \in [\![\lambda \mathcal{P}_i]\!]_s$ for every $i \in \{1, \dots, n\}$. For every signpost sequence and every pair $(\mathcal{P}, H)$ the divisor method is guaranteed to provide a solution [9]. In the context of voting, the classic Jefferson/D'Hondt method corresponds to the divisor method associated to the stationary signpost sequence with $\Delta = 0$. Other classic methods are the one by Adams, corresponding to the divisor method associated to the stationary signpost sequence with $\Delta = 1$, and the method by Webster/Sainte-Laguë, corresponding to the divisor method associated to the stationary signpost sequence with $\Delta = 1/2$. For an extensive treatment of the theory of divisor methods and their historical aspects, we refer to the book by Balinski and Young [9] and the one by Pukelsheim [27].

2.2 Multidimensional Apportionment

In the *d-dimensional apportionment problem*, the input is given by a tuple $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ described as follows. For each $\ell \in \{1, \dots, d\}$ we have a set N_ℓ such that $\mathcal{N} = \{N_1, \dots, N_d\}$ and $\mathcal{V}$ is a vector with nonnegative integer entries in the Cartesian product $\prod_{\ell=1}^d N_\ell$. For each $\ell \in \{1, \dots, d\}$ and each $v \in N_\ell$ there are integer values m_v^- and $m_v^+ > 0$ called *lower and upper marginals*, respectively, and H is a strictly positive integer value such that $\sum_{v \in N_\ell} m_v^- \leq H \leq \sum_{v \in N_\ell} m_v^+$ for every $\ell \in \{1, \dots, d\}$. In the context of a political election, the values $\mathcal{V}_e$ represent the number of votes garnered by the tuple e and H represents the house size. For an instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ we denote by $E(\mathcal{V})$ the subset of tuples $e \in \prod_{\ell=1}^d N_\ell$ such that $\mathcal{V}_e > 0$.

For $d = 2$, Balinski and Demange extended the classic notion of proportionality captured by the divisor methods [3, 4]. In this case, the sets N_1 and N_2 may represent, for example, the set of parties and the set of districts in a political election, respectively. We now generalize this approach to arbitrary dimension in the natural way, described as follows. Given a d-dimensional instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ and a signpost sequence s, we say that $x \in \mathbb{N}^{E(\mathcal{V})}$ is a *d-dimensional proportional apportionment* if there exists a strictly positive value μ, and for every $\ell \in \{1, \dots, d\}$ and every $v \in N_\ell$ there exists a strictly positive value λ_v, called *multiplier*, such that the following holds:

$$m_v^- \leq \sum_{e \in E(\mathcal{V}): e_\ell = v} x_e \leq m_v^+ \quad \text{for every } \ell \in \{1, \dots, d\} \text{ and every } v \in N_\ell, \tag{1}$$

$$\sum_{e \in E(\mathcal{V})} x_e = H, \tag{2}$$

$$s(x_e) \leq \mathcal{V}_e \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell} \leq s(x_e + 1) \quad \text{for every } e \in E(\mathcal{V}), \tag{3}$$

and furthermore, we have the following conditions regarding the values of the multipliers for every $\ell \in \{1,\ldots,d\}$ and every $v \in N_\ell$,

$$\text{If } \lambda_v > 1, \text{ then we have } \sum_{e \in E(\mathcal{V}): e_\ell = v} x_e = m_v^-, \tag{4}$$

$$\text{If } \lambda_v < 1, \text{ then we have } \sum_{e \in E(\mathcal{V}): e_\ell = v} x_e = m_v^+. \tag{5}$$

We denote by $\mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ the set of triplets (x, μ, λ) where x is integral, μ is strictly positive, λ is strictly positive in every entry and the triplet satisfies conditions (1)-(5). Note that for the case $s(1) = 0$, any x satisfying (3) must be strictly positive in every entry. Observe that given the strict positivity of the values μ and λ_v, condition (3) is equivalent to $[\![\mathcal{V}_e \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell}]\!]_s = x_e$, thus it captures the idea of proportionality. We remark that for $d = 2$ this corresponds to the proportionality notion of Balinski and Demange, in the sense that their definition of a biproportional apportionment is equivalent to our definition of a 2-dimensional proportional apportionment.

3 A Linear Programming Approach

In this section, we introduce an integer linear program inspired by transportation and matrix scaling problems. We prove a correspondence between the integer optimal solutions of its linear relaxation and the multidimensional proportional apportionments. Using this characterization, we show the inexistence of proportional apportionments for some instances of the problem and the computational hardness of deciding the existence of such apportionments.

3.1 An Integer Linear Program Inspired by Matrix Scaling

We follow a related network flow approach introduced by Rote and Zachariasen [29] for matrix scaling and used by Gaffke and Pukelsheim [20, 19] to model biproportional apportionments. Our integer linear program to study the d-dimensional apportionment problem is constructed as follows. Consider a d-dimensional instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ and a signpost sequence s. For each $e \in E(\mathcal{V})$ and each $t \in \{1,\ldots,H\}$ we have a binary variable y_e^t and its cost in the objective function is given by $\log(s(t)/\mathcal{V}_e)$ if $s(t) > 0$ and zero otherwise.

$$\text{minimize} \quad \sum_{e \in E(\mathcal{V})} \sum_{\substack{t \in \{1,\ldots,H\}: \\ s(t) > 0}} y_e^t \log\left(\frac{s(t)}{\mathcal{V}_e}\right) \tag{6}$$

$$\text{subject to} \quad \sum_{t=1}^{H} y_e^t = x_e \qquad \text{for every } e \in E(\mathcal{V}), \tag{7}$$

$$\sum_{e \in E(\mathcal{V})} x_e = H, \tag{8}$$

$$\sum_{e \in E(\mathcal{V}): e_\ell = v} x_e \geq m_v^- \qquad \text{for every } \ell \in \{1,\ldots,d\} \text{ and every } v \in N_\ell, \tag{9}$$

$$\sum_{e \in E(\mathcal{V}): e_\ell = v} x_e \leq m_v^+ \qquad \text{for every } \ell \in \{1,\ldots,d\} \text{ and every } v \in N_\ell, \tag{10}$$

$$y_e^1 \geq \lfloor 1 - s(1) \rfloor \qquad \text{for every } e \in E(\mathcal{V}), \tag{11}$$

$$y_e^t \in \{0,1\} \qquad \text{for every } e \in E(\mathcal{V}) \text{ and every } t \in \{1,\ldots,H\}. \tag{12}$$

The variable x_e represents the total number of seats to be allocated in the apportionment for the tuple e and constraint (7) takes care of aggregating the seats in these variables. Constraint (8) ensures to respect the house size and constraints (9) and (10) enforces every feasible solution to satisfy the marginals. Finally, constraint (11) ensures $x_e \geq 1$ if $s(1) = 0$. We remark that this integer linear program can be equivalently written by omitting the variables y at the price of having a nonlinear convex objective.

3.2 Characterizing Optimal Solutions of the Linear Relaxation

When $d = 2$, the above problem is as hard as a transportation problem in a bipartite network, and in consequence, one can recover an optimal solution of this problem by relaxing integrality and solving the linear relaxation. Furthermore, it can be shown when $d = 2$ that any extreme point defines a proportional apportionment, where multipliers are obtained by computing the exponential of the corresponding dual solution [29, 19, 27]. Therefore, in the general d-dimensional setting, the first natural question that we address is the following: Can we characterize the set of proportional apportionments in terms of the set of optimal solutions of the linear relaxation of (6)-(12)?

By duality, we know that for any feasible solution (x, y) of the linear relaxation of (6)-(12), (x, y) is optimal if and only if there exists a dual solution $(\mathcal{U}, \Lambda^-, \Lambda^+, \beta)$ such that the following conditions hold for every $e \in E(\mathcal{V})$ and every $t \in \{1, \ldots, H\}$:

$$\mathcal{U} + \sum_{\ell=1}^{d} (\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) + \beta^t_e \leq \log\left(\frac{s(t)}{\mathcal{V}_e}\right), \qquad \text{if } s(t) > 0, \tag{13}$$

$$\mathcal{U} + \sum_{\ell=1}^{d} (\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) + \beta^t_e \leq 0, \qquad \text{if } s(t) = 0, \tag{14}$$

$$y^t_e \left[\mathcal{U} + \sum_{\ell=1}^{d} (\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) + \beta^t_e - \log\left(\frac{s(t)}{\mathcal{V}_e}\right)\right] = 0, \qquad \text{if } s(t) > 0, \tag{15}$$

$$y^t_e \left[\mathcal{U} + \sum_{\ell=1}^{d} (\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) + \beta^t_e\right] = 0, \qquad \text{if } s(t) = 0, \tag{16}$$

$$\beta^t_e (y^t_e - 1) = 0, \tag{17}$$

$$\beta^t_e \leq 0, \qquad \text{if } s(t) > 0, \tag{18}$$

and such that the following conditions hold for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$:

$$\Lambda^-_v \left(\sum_{e \in E(\mathcal{V}): e_\ell = v} \sum_{t=1}^{H} y^t_e - m^-_v\right) = 0, \tag{19}$$

$$\Lambda^+_v \left(\sum_{e \in E(\mathcal{V}): e_\ell = v} \sum_{t=1}^{H} y^t_e - m^+_v\right) = 0, \tag{20}$$

$$\Lambda^-_v, \ -\Lambda^+_v \geq 0, \tag{21}$$

where $\mathcal{U}$ is the dual variable associated to the constraint (8), Λ^-_v is the dual variable associated to the constraint (9) for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$, Λ^+_v is the dual variable associated to the constraint (10) for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$, and β^t_e is the dual variable associated to the upper bound of one on the value of y^t_e for every $e \in E(\mathcal{V})$ and every $t \in \{1, \ldots, H\}$. We refer to a tuple $(x, y, \mathcal{U}, \Lambda^-, \Lambda^+, \beta)$ satisfying (13)-(21), with (x, y) feasible for the linear relaxation of (6)-(12), as an optimal primal-dual pair for this linear relaxation. Now we answer the initial question of this subsection. The following result asserts that there exists a d-dimensional proportional apportionment if and only if the linear relaxation of (6)-(12) admits an integer optimal solution.

Theorem 1. *Let $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ be an instance of the d-dimensional apportionment problem, let s be a signpost sequence and let $x \in \mathbb{N}^{E(\mathcal{V})}$. Then, there exist μ and λ such that $(x, \mu, \lambda) \in \mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ if and only if there exists y such that (x, y) is an optimal solution for the linear relaxation of (6)-(12).*

This theorem provides a natural way of studying the existence of d-dimensional proportional apportionments for a given instance: solving the program (6)-(12) and its linear relaxation and comparing the objective values. If they coincide, any optimal solution of the integer linear program defines a proportional apportionment; otherwise, there is no such apportionment. We use this procedure in the following subsection to show that there are instances that do not admit proportional apportionments, and in 3.4 we prove that, unless $\mathsf{P} = \mathsf{NP}$, there is no efficient algorithm to decide whether this is the case for a given instance.

3.3 Nonexistence of 3-dimensional Proportional Apportionments

We recall that the existence of d-dimensional proportional apportionments when $d \in \{1, 2\}$ is completely understood and necessary and sufficient conditions are provided in general [4, 3]. In particular, when the apportionment instance is 2-dimensional and $\mathcal{V}$ is strictly positive, there is always a proportional apportionment. This follows from the fact that when $d = 2$ the linear relaxation of (6)-(12) is integral, as a consequence of total unimodularity, and the feasibility of this program is guaranteed by $\mathcal{V}$ strictly positive. For practical purposes, this is relevant since it guarantees the existence of proportional apportionments for a fairly natural setting. Then, the following question raises naturally: Given a d-dimensional instance with $d \geq 3$ and $\mathcal{V}$ strictly positive, can we always find a proportional apportionment? We answer this question in the negative when s belongs to the relevant family of stationary signpost sequences.

Theorem 2. *There exists an instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ of the 3-dimensional apportionment problem, with $\mathcal{V}$ strictly positive in each of its entries, such that for every stationary signpost sequence s we have $\mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H) = \emptyset$.*

This result is proved in a 3-dimensional instance where $|N_\ell| = 2$ for each $\ell \in \{1, 2, 3\}$, and $\mathcal{V}_e > 0$ for every $e \in N_1 \times N_2 \times N_3$. This shows that even for very small instances with $\mathcal{V}$ strictly positive, the existence is not guaranteed when $d = 3$. We remark that this contrasts the 2-dimensional case where the existence is always guaranteed when $\mathcal{V}$ is strictly positive.

3.4 Complexity of the Multidimensional Apportionment Problem

So far, we know that a proportional apportionment for a given instance is always optimal for the linear relaxation of (6)-(12), and moreover, it defines an extreme point of its feasible region. The natural question is how to determine the existence of an integer extreme point of this optimal region or to assure that no such a point exists. More formally, consider the following decision problem: Given an instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ of the d-dimensional apportionment problem and a signpost sequence s, decide if $\mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ is empty or not. Within this subsection, we refer to this decision problem as the *(d, s)-proportional apportionment problem*. We remark that the work of Balinski and Demange shows that the $(2, s)$-proportional apportionment problem can be solved in polynomial time for every signpost sequence s [3, 4]. Then, the natural question that arises is the following: What is the complexity of the (d, s)-proportional apportionment problem when $d \geq 3$? The following is our main result in this line.

Theorem 3. *For every signpost sequence s and every $d \geq 3$, the (d, s)-proportional apportionment problem is* NP*-complete.*

We prove this theorem by a hardness reduction from the perfect matching problem in d-partite hypergraphs. Recall that $G = (P, F)$ is a *d-partite hypergraph* if the set of vertices P can be partitioned into d disjoint sets $P_1, \ldots, P_d$, and every hyperedge $f \in F$ intersects each of the parts exactly once, that is, $|f \cap P_\ell| = 1$ for every $f \in F$ and every $\ell \in \{1, \ldots, d\}$. A 2-partite hypergraph is just a bipartite graph. We say that $F' \subseteq F$ is a *perfect matching* of G if for every $v \in P$ we have $|\{f \in F' : v \in f\}| = 1$. The problem of determining if a d-partite hypergraph contains a perfect matching is NP-complete even when $|P_1| = |P_2| = \cdots = |P_d|$ and $d = 3$, which is known as the 3-dimensional matching problem [24].

As a remark, if one considers the case where $|N_\ell|$ is constant for every $\ell \in \{1, \ldots, d\}$, there exists a polynomial algorithm for the (d, s)-proportional apportionment problem: One can enumerate every possible base defining an extreme point of the linear relaxation of (6)-(12), and therefore we can check if there exists an optimal integer extreme point. Theorem 1 guarantees the correctness of this algorithm.

4 An LP Rounding Algorithm for Multidimensional Apportionment

In the previous section, we have addressed the multidimensional apportionment problem from an existence and complexity point of view. In particular, we have seen that there exist d-dimensional instances for which it is not possible to simultaneously satisfy conditions (1)-(5). Therefore, in this chapter we address the following question: Is it possible to compute a vector that satisfies condition (3) and such that the violation in the other conditions is under control? We provide a positive answer to this question, summarized in the following theorem.

Theorem 4. *Let $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ be an instance of the d-dimensional apportionment problem and let s be a signpost sequence such that the linear relaxation of (6)-(12) is feasible. Let $u_1, \ldots, u_d$ be nonnegative integers such that $\sum_{\ell=1}^d 1/(u_\ell + 2) \leq 1$. Then, there exists an integral vector $X \in \mathbb{N}^{E(\mathcal{V})}$ such that the following holds:*

(i) $m_v^- - u_\ell \leq \sum_{e \in E(\mathcal{V}): e_\ell = v} X_e \leq m_v^+ + u_\ell$ *for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$.*

(ii) There exists $\mu > 0$ and a vector λ with strictly positive entries such that:

  (1) $s(X_e) \leq \mathcal{V}_e \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell} \leq s(X_e + 1)$ *for every $e \in E(\mathcal{V})$.*

  (2) For every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$, if $\lambda_v > 1$ then $|\sum_{e \in E(\mathcal{V}): e_\ell = v} X_e - m_v^-| \leq u_\ell$.

  (3) For every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$, if $\lambda_v < 1$ then $|\sum_{e \in E(\mathcal{V}): e_\ell = v} X_e - m_v^+| \leq u_\ell$.

Furthermore, X can be found in time polynomial in $|E(\mathcal{V})|$, $\sum_{\ell=1}^d |N_\ell|$ and H.

We have seen in Section 3.3 that there exist instances for the d-dimensional apportionment problem with $d \geq 3$ for which there are no integral optimal solutions for the linear relaxation of (6)-(12). In contrast, we show that it is possible to round an optimal fractional solution of this linear relaxation in a way that the obtained integral vector satisfies the proportionality condition (3) and at the same time the violation in the marginals condition (1), as well as the multipliers conditions

(4)-(5), is under control. To maintain these violations under control we study in Subsection 4.1 a particular discrepancy problem in hypergraphs, inspired by the work of Beck and Fiala for the *discrepancy minimization* problem [13]. We present the algorithm necessary for Theorem 4 and a brief analysis of this result in Subsection 4.2. We remark that the feasibility of the linear relaxation of (6)-(12) is guaranteed under mild assumptions, for instance, that every entry of $\mathcal{V}$ is strictly positive.

Note that for the case $d = 2$, Theorem 4 is valid for $u_1 = u_2 = 0$, therefore it implies the existence of 2-dimensional proportional apportionments whenever the linear relaxation of (6)-(12) is feasible, i.e. whenever there exists a vector in $\mathbb{R}_+^{E(\mathcal{V})}$ respecting the marginals and preserving the zeros of $\mathcal{V}$, with the additional condition that $\mathcal{V}_e > 0$ implies a strictly positive entry if $s(1) = 0$. This is one of the main results of Balinski and Demange [4], so Theorem 4 can be seen as a generalization of their existence result for the case of arbitrary dimension.

4.1 A Discrepancy Problem in d-partite Hypergraphs

Consider a d-partite hypergraph G with vertex partition $\{P_1, \dots, P_d\}$ and hyperedges E, and let $x \in [0,1]^E$ be such that $\sum_{e\in\delta(v)} x_e$ is integral for every vertex v of G, where $\delta(v)$ is the set of hyperedges containing v. We are also given d nonnegative integer values $u_1, \dots, u_d$ and the goal is to round x into an integral vector in $\{0,1\}^E$ in a way such that the deviation from $\sum_{e\in\delta(v)} x_e$ on every vertex $v \in P_\ell$ is at most u_ℓ for each $\ell \in \{1, \dots, d\}$. Naturally, when $u_1 = \cdots = u_d$ we fall in the classic discrepancy minimization approach. In contrast, we are interested in providing a fine upper bound on the deviation throughout the different parts of the hypergraph. The following is our main result in this line.

Theorem 5. *Let G be a d-partite hypergraph with vertex partition $\{P_1, \dots, P_d\}$ and hyperedges E. Let $x \in [0,1]^E$ be such that $\sum_{e\in\delta(v)} x_e$ is integral for every vertex v of G and let $u_1, \dots, u_d$ be nonnegative integers such that $\sum_{\ell=1}^d 1/(u_\ell + 2) \le 1$. Then, there exists $z \in \{0,1\}^E$ such that for every $\ell \in \{1, \dots, d\}$ and every $v \in P_\ell$ it holds $|\sum_{e\in\delta(v)} (z_e - x_e)| \le u_\ell$, and $z_e = x_e$ when x_e is integer. Furthermore, z can be computed in time polynomial in $|E|$ and $\sum_{\ell=1}^d |P_\ell|$.*

We present an iterative rounding algorithm inspired by the classic discrepancy minimization result by Beck and Fiala [13] that computes a solution z satisfying the conditions guaranteed by Theorem 5. To present this procedure, we introduce a simple linear program that will be used during its execution. Given a vector $Y \in [0,1]^F$ with $F \subseteq E$, a subset of edges $\mathcal{E} \subseteq F$ and a subset of vertices $Q_\ell \subseteq P_\ell$ for each $\ell \in \{1, \dots, d\}$, we consider the following linear program with variables y_e for each $e \in \mathcal{E}$:

$$\sum_{e\in\delta(v)\cap\mathcal{E}} y_e = \sum_{e\in\delta(v)\cap\mathcal{E}} Y_e \quad \text{for every } v \in \bigcup_{\ell=1}^d Q_\ell, \tag{22}$$

$$0 \le y_e \le 1 \quad \text{for every } e \in \mathcal{E}. \tag{23}$$

We denote by $\mathcal{K}(Y, \mathcal{E}, Q)$ the polytope of feasible solutions for this linear program.

Algorithm 1 iteratively solves a linear program in the form (22)-(23). The condition in the loop guarantees that the algorithm makes progress in fixing at least one new variable into a binary value. Once the loop condition is not satisfied, the algorithm rounds up or down the rest of the fractional variables and its output satisfies the properties guaranteed by Theorem 5.

Algorithm 1 Iterative Rounding Algorithm

Input: A d-partite hypergraph G with vertex partition $\{P_1, \dots, P_d\}$ and hyperedges E, a vector $x \in [0,1]^E$ and nonnegative integer values $u_1, \dots, u_d$.
Output: Binary vector $z \in \{0,1\}^E$.

```
Initialize $y^0 \leftarrow x$
Let $\mathcal{E}^0 = \{e \in E : y_e^0$ is fractional$\}$
For each $\ell \in \{1, \dots, d\}$, let $Q_\ell^0 = \{v \in P_\ell : |\delta(v) \cap \mathcal{E}^0| \geq u_\ell + 2\}$
Let $z_e = y_e^0$ for every $e \notin \mathcal{E}^0$.
Initialize $t \leftarrow 0$
while there exists $\ell \in \{1, \dots, d\}$ such that $Q_\ell^t \neq \emptyset$ do
    Compute an extreme point $y^{t+1}$ of $\mathcal{K}(y^t, \mathcal{E}^t, Q^t)$
    Let $\mathcal{E}^{t+1} = \{e \in E : y_e^{t+1}$ is fractional$\}$
    For each $\ell \in \{1, \dots, d\}$, let $Q_\ell^{t+1} = \{v \in P_\ell : |\delta(v) \cap \mathcal{E}^{t+1}| \geq u_\ell + 2\}$
    Let $z_e = y_e^{t+1}$ for every $e \in \mathcal{E}^t \setminus \mathcal{E}^{t+1}$. Update $t \leftarrow t + 1$
Let $T$ be the value of $t$ that did not satisfy the loop condition
Let $z_e \in \{\lfloor y_e^T \rfloor, \lceil y_e^T \rceil\}$ for every $e \in \mathcal{E}^T$
Return $z$.
```

It is worth mentioning two observations. First, note that the last step of the algorithm allows to round the remaining fractional entries as desired. Maybe the most natural way in order to minimize deviations is to choose the nearest integer, although it does not allow to improve the bound in the worst case. The second observation is that the bound can be slightly improved for particular cases, when at least one of the parts in the vertex partition is small. In particular, the sufficient condition over integers $u_1, \dots, u_d$ in Theorem 5 can be replaced by $\sum_{\ell=1}^{d} \min\{\lfloor q/(u_\ell + 2) \rfloor, |P_\ell|\} < q$ for every strictly positive integer q. We remark that this is not an improvement over Theorem 5, since the inequality over integers $u_1, \dots, u_d$ becomes strict in the new condition.

4.2 Rounding an Optimal Solution of the Linear Relaxation

To obtain the result stated in Theorem 4, we present our Algorithm 2 which is based on two key steps. In the first step, we solve the linear relaxation of (6)-(12) to get a solution. If this solution is integral then this is the output of the algorithm. Otherwise, we use this solution to feed our Algorithm 1 to get an integral vector. Let α be such that $\alpha(e) = \{e_1, \dots, e_d\}$ for each $e \in E(\mathcal{V})$. The function α captures the natural representation of the (ordered) tuples in $E(\mathcal{V})$ as (unordered) sets. We use this representation to go from the apportionment setting to the hypergraph representation. Algorithm 2 describes the detailed procedure.

Theorem 4 allows constructing vectors that satisfy (3) with fixed maximum marginal deviations in some relevant cases, in particular, when d is fixed. For example, when $d = 3$, Algorithm 2 can be ran using any vector u in $\{(0,1,4), (0,2,2), (1,1,1)\}$. These vectors and their permutations actually constitute the Pareto Frontier of deviations satisfying the sufficient condition in the theorem, which is particularly relevant in the case of elections, since it allows the incorporation of a new feature on top of the classic party and district dimension. We also remark that when $d \geq 3$, Algorithm 2 can be ran using $u = (d-2, \dots, d-2)$. When one dimension has only two options (e.g. gender) and say this is the first dimension, following the logic of the discussion after Theorem 5 one can achieve even better bounds, like $u = (0,1,3)$ when $d = 3$ and $u = (0,2,3,4)$ when $d = 4$ (instead of $u = (0,2,3,18)$ which is part of the Pareto Frontier in the general case).

Algorithm 2 Apportionment Rounding Algorithm

Input: A d-dimensional instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ and $u_1, \ldots, u_d$ with $\sum_{\ell=1}^{d} 1/(u_\ell + 2) \leq 1$.
Output: An integral vector $X \in \mathbb{N}^{E(\mathcal{V})}$.

1: Let $(x^\star, y^\star)$ be an optimal solution of the linear relaxation of (6)-(12) in instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$
2: If $x^\star$ is integral **return** $X \leftarrow x^\star$
3: If $x^\star$ is fractional, consider the d-partite hypergraph G with vertex partition $N_1, \ldots, N_d$ and hyperedges $\alpha(E(\mathcal{V}))$. Run Algorithm 1 over the hypergraph G, the fractional vector w defined as $w_{\alpha(e)} = x_e^\star - \lfloor x_e^\star \rfloor$ for every $e \in E(\mathcal{V})$ and the values $u_1, \ldots, u_d$, and let $z \in \{0,1\}^{\alpha(E(\mathcal{V}))}$ be its output.
4: **Return** $X_e = \lfloor x_e^\star \rfloor + z_{\alpha(e)}$ for every $e \in E(\mathcal{V})$.

Finally, we remark that the bounds obtained cannot be strictly improved, in the sense that if we denote $f(u) = \sum_{\ell=1}^{d} 1/(u_\ell + 2)$, there is no function $g < f$ such that $g(u) \leq 1$ ensures the existence of a multidimensional proportional apportionment with violation at most u_ℓ in each dimension $\ell \in \{1, \ldots, d\}$. In particular, we prove that in the case $d = 3$ it is not possible to ensure a maximum deviation given by $u = (0, 0, K)$ for any constant K, and we extend this impossibility to the case of higher dimension by induction. We leave as an open question whether there are other vectors u with $f(u) > 1$, for instance $u = (0, 1, 1)$, defining deviations that are reachable for every instance of the problem, i.e., whether our sufficient condition for u defining feasible deviations is also necessary.

5 Results from the Chilean Constitutional Convention

In this last section, we test our method for the case of three dimensions, namely political lists, districts, and gender, and the signpost sequence $s(q) = q$ for every positive integer q. List marginals are calculated according to the votes obtained by each list through a single-dimensional Jefferson/D'Hondt method, district marginals are predefined by law and gender marginals ensure parity. We refer to this as the three-proportional method (TPM) in the following. The testing ground is provided by the recent election of the Chilean Constitutional Convention (May 15-16, 2021), and the basis of the comparison is given by the Constitutional Convention method (CCM). Chile's electoral map is divided into 28 electoral districts with a specified number of seats to be allocated in each district. In total 155 seats were to be allocated, 17 of which were reserved for ethnic minority groups, so that 138 seats were allocated to the 28 districts. Our comparison only considers these non-ethnic seats. We also mention that each voter votes for at most one candidate of his/her district.

In the recent Chilean Constitutional Convention election a total of 70 lists, including over 1300 candidates, competed for these 138 seats. Three of these lists correspond to well-established political alliances. The XP list represented the right-wing parties, including not only the traditional parties *Renovación Nacional* and *Unión Demócrata Independiente*, but also the newer centrist *Evopoli* and the extreme right *Partido Republicano*. The YB list represented the center-left parties that have mostly governed Chile in the last three decades, including the *Democracia Cristiana* and the *Partido Socialista*. The third list is the YQ list and corresponds to the left-wing parties such as the *Partido Comunista* and a number of much newer parties. Additionally, there were two important politically independent players in the election that arose as conglomerates encompassing different lists (but that did not compete in any district). These correspond to what we denote by LP (for *Lista del*

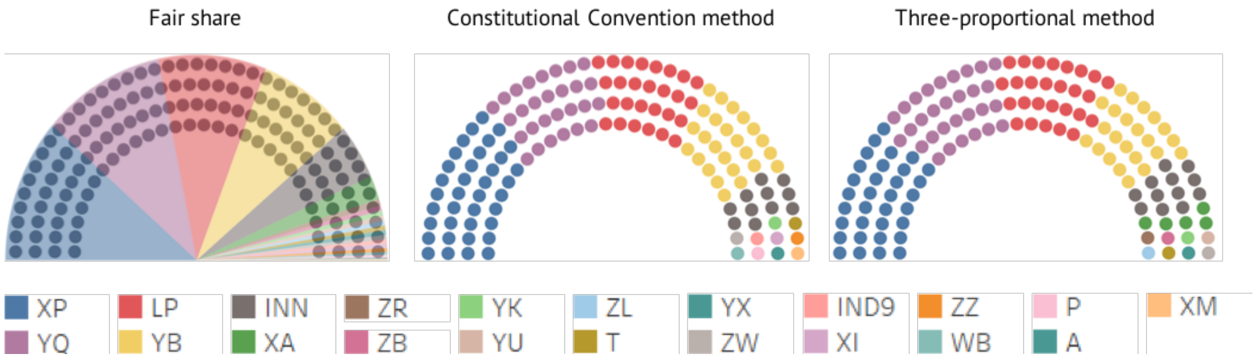

Figure 1: Political distribution by method and fair share.

Pueblo) and INN (for *Independientes No Neutrales*).[6] [7] By observing the outcome of both TPM and CCM, we have that among the 70 lists and 28 independent candidates, only 20 lists and one independent candidate[8] obtain enough votes to be elected in either system. For ease of exposition, when presenting the results we omit the votes of the other lists and independent candidates, none of which obtained more than 0.51% of the votes and jointly represent less than the 10% of them. Note that the results are not affected by this modification.

In what follows we compare the CCM results with what would have happened if TPM was in place. To this end let us first describe CCM, which works as follows. In the first step, the seats of each district are divided between the lists and independent candidates according to the single-dimensional Jefferson/D'Hondt method, using the votes obtained by all the candidates of each list. Then, the seats assigned to each list are divided between its sublists (usually political parties) through the same method and provisionally assigned to the candidates of these parties with more individual votes. If at this point the set of elected candidates achieve gender balance (meaning the same number of men and women if the number of seats of the district is even and at most one more man/woman if it is odd), the seats are assigned to these candidates. Otherwise, the following procedure is repeated until the gender balance condition is satisfied: Pick the provisionally elected candidate of the over-represented gender with the lowest number of votes, and assign in his/her place the provisionally non-elected candidate of the other gender and his/her same party (or list, in case the former is not possible) with the highest number of individual votes.

Gender Balance. Both methods achieve this property: CCM leads to a house composed of 70 men and 68 women, while an absolute gender balance of 69 men and 69 women is achieved by TPM.

Political Balance. As a quality measure of an apportionment, we consider the deviation of the political distribution from the perfectly fair distribution, a.k.a. *fair share* in the literature, which assigns to each list the (possibly fractional) number of seats that corresponds to the proportion of the house that the votes obtained represent. Figure 1 shows the proportion obtained by each list with the 138 seats as a background, in order to give a graphical idea of the fair share, and

[6]This association is standard as reported, for instance, by https://2021.decidechile.cl/#/ev/2021. Full election data can be found in https://pv.servelelecciones.cl/.

[7]When presenting the results for the remaining lists we use the election codes.

[8]This independent candidate is denoted as IND9 because of the number of the district where he participated.

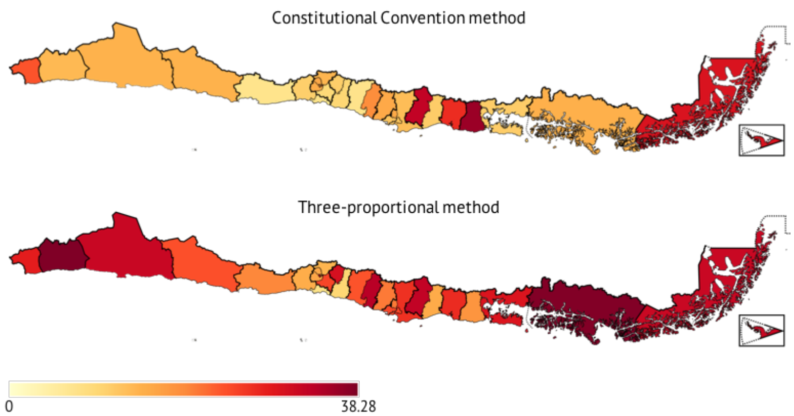

Figure 2: Standard deviation of political distribution with respect to the fair share by method and district.

the political distribution of the Constitutional Convention under both methods.[9] It is observed that TPM generates a political distribution much closer to the fair share than the Constitutional Convention method, with a smaller overrepresentation of the most voted list and an assignment of seats to the 14 top-voted lists. It is particularly relevant to remark that CCM does not assign any seat to list XA, which is the sixth most voted list with almost 4% of the votes, while TPM allocates 5 seats to this list.

The notion of closeness or dispersion with respect to the fair share can be formalized through the Euclidean distance between an apportionment and the fair share or, in other words, the standard deviation of an apportionment with respect to the fair share. This notion is easily extended to the apportionment of a single district as well, comparing the political distribution of the seats assigned in the district with the fair distribution according to the votes.[10] Figure 2 shows the standard deviation of the apportionments obtained with each method by district.[11] Naturally, CCM is locally closer to the fair share than TPM, which is essentially a property of the design since CCM achieves local proportionality. However, when summing up the results by district, the local errors generated by CCM start to add up and the distortion with respect to the fair share increases. On the other hand, TPM is designed to achieve global proportionality so that the national results are much closer to the fair share of the vote. Indeed the standard deviation of the apportionment obtained with TPM with respect to the fair share is 2.49, and with CCM this value is 6.44. There is, therefore, a trade-off between local and global political representation.

Robustness. Another criterion we use to compare CCM and TPM is their robustness to small perturbations in the votes. To evaluate this aspect we conduct $n = 1000$ simulations, and in each we multiply the votes obtained by each candidate by a normally distributed value with mean one and standard deviation 0.05. We then compute the distribution of the number of seats transferred from one list to any other on each simulation starting from the original apportionment. Denoting the seats obtained by each list $\ell \in L$ in the original apportionment as y_ℓ, the seats obtained by each list $\ell \in L$ in simulation $i \in \{1, \ldots, n\}$ as y^i_ℓ and the variable of interest as T^i, this variable is

[9]The data used for this figure is contained in Table 1 of Appendix 8.
[10]District data can be found in Tables 2 and 3 of Appendix 8.
[11]The data used for this figure is contained in Table 4 of Appendix 8.

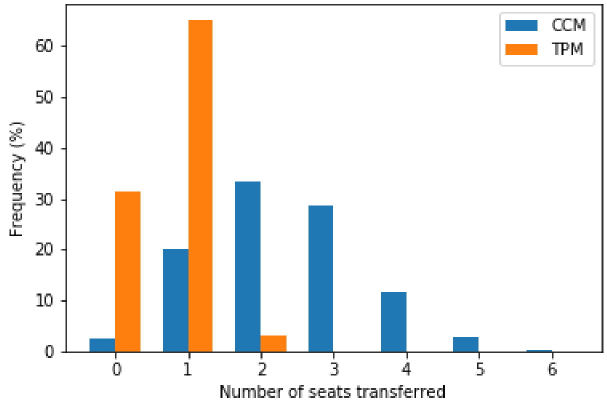

Figure 3: Distribution of the seats transferred between lists under perturbations of the votes.

given by[12]

$$T^i = \frac{1}{2} \sum_{\ell \in L} |y_\ell - y^i_\ell|.$$

Figure 3 plots the distribution of this variable under each method. Since the three-proportional method assigns to each list a number of seats determined by its total votes, instead of the votes by district, this method generates a more robust result in the face of changes in the votes. Indeed, the average number of seats transferred under this method is 0.72, and this value increases to 2.37 under the Constitutional Convention method.

The value of a vote. As a final observation, we remark that when using TPM each vote is equally valuable in favor of the chosen list, because its seats, as it was pointed, depend only on its total number of votes. In the Constitutional Convention election, comparing the number of seats assigned to each district and the people who voted in each of them, the votes of some people were 5.93 times more valuable than the votes of people living in a different district.[13] Incorporating geographic division, gender, and possibly other criteria as additional dimensions instead of making separate elections for each clearly allows getting closer to the well-known principle of *one person, one vote*.

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[12] Note that since seat transfers are counted twice in the summation, we divide the expression by 2.
[13] And this ratio becomes higher than 1000 when considering the seats reserved to ethnic groups.

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6 Appendix: Proofs of Section 3

6.1 Proof of Theorem 1

To prove Theorem 1 we first state the following lemma about the structure of an optimal primal-dual pair for the linear relaxation of (6)-(12).

Lemma 1. *Let $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ be an instance of d-dimensional apportionment problem, let s be a signpost sequence and let $(x, y, \mathcal{U}, \Lambda^-, \Lambda^+, \beta)$ be an optimal primal-dual pair for the linear relaxation of (6)-(12). For every $e \in E(\mathcal{V})$, define the value $t_e = \max\{t \in \{1, \ldots, H\} : y_e^t > 0\}$ when $x_e > 0$ and $t_e = 0$ when $x_e = 0$. Then, the following holds:*

(i) For every $e \in E(\mathcal{V})$, we have that $t_e = \lceil x_e \rceil$. In particular, $y_e^t = 1$ for every $t < t_e$ when $x_e > 1$.

(ii) For every $e \in E(\mathcal{V})$, we have that $s(t_e) \leq \mathcal{V}_e \cdot \exp(\mathcal{U}) \cdot \prod_{\ell=1}^d \exp(\Lambda_{e_\ell}^- + \Lambda_{e_\ell}^+) \leq s(t_e + 1)$.

Proof. When $x_e = 0$, part (i) follows directly, so in the following we suppose $x_e > 0$. Clearly, for every $e \in E(\mathcal{V})$ we have that $t_e \geq \lceil x_e \rceil$, otherwise constraint (7) would be violated. Suppose that $e \in E(\mathcal{V})$ is such that $t_e > \lceil x_e \rceil$. Then, there exists $t > \lfloor x_e \rfloor$ such that y_e^t is strictly positive, and consider the solution $(\bar{x}, \bar{y})$ obtained as follows: We define $\bar{y}_e^t = 1$ for every $t < \lfloor x_e \rfloor + 1$, $\bar{y}_e^p = x_e - \lfloor x_e \rfloor$ for $p = \lfloor x_e \rfloor + 1$ and $\bar{y}_e^t = 0$ otherwise; and $\bar{x}_e = \sum_{t=1}^H \bar{y}_e^t$ for every $e \in E(\mathcal{V})$. The solution $(\bar{x}, \bar{y})$ is feasible for the linear relaxation of (6)-(12) since $\bar{x}_e = \sum_{t=1}^H \bar{y}_e^t = (\lfloor x_e \rfloor + 1) - 1 + x_e - \lfloor x_e \rfloor = x_e$. Since the signpost sequence is strictly increasing in $\{1, \ldots, H\}$, for every $e \in E(\mathcal{V})$ the function $\log(s(t)/\mathcal{V}_e)$ is strictly increasing as a function of $t \in \{1, \ldots, H\}$. Therefore, $\sum_{t=1}^H \bar{y}_e^t \log(s(t)/\mathcal{V}_e) < \sum_{t=1}^H y_e^t \log(s(t)/\mathcal{V}_e)$, but this contradicts the optimality of (x, y). That concludes the proof for (i).

Now we prove (ii). We know that for every $t > t_e$ it holds $y_e^t = 0$, so complementary slackness condition (17) implies that $\beta_e^t = 0$. Note that $t > t_e$ guarantees $s(t) > 0$, and therefore condition (13) implies that $\mathcal{U} + \sum_{\ell=1}^d (\Lambda_{e_\ell}^- + \Lambda_{e_\ell}^+) \leq \log(s(t)/\mathcal{V}_e)$, which is verified in particular for $t = t_e + 1$. Therefore, $\mathcal{V}_e \cdot \exp(\mathcal{U}) \cdot \prod_{\ell=1}^d \exp(\Lambda_{e_\ell}^- + \Lambda_{e_\ell}^+) \leq s(t_e + 1)$. On the other hand, when $t = t_e$ with $s(t_e) > 0$, the complementary slackness condition (15) implies that $\mathcal{U} + \sum_{\ell=1}^d (\Lambda_{e_\ell}^- + \Lambda_{e_\ell}^+) + \beta_e^{t_e} - \log(s(t_e)/\mathcal{V}_e) = 0$, since $y_e^{t_e} > 0$. By (18) we have that $\beta_e^{t_e} \leq 0$ and therefore we conclude that $\mathcal{U} + \sum_{\ell=1}^d (\Lambda_{e_\ell}^- + \Lambda_{e_\ell}^+) \geq \log(s(t_e)/\mathcal{V}_e)$, implying $s(t_e) \leq \mathcal{V}_e \cdot \exp(\mathcal{U}) \cdot \prod_{\ell=1}^d \exp(\Lambda_{e_\ell}^- + \Lambda_{e_\ell}^+)$. The case $s(t_e) = 0$ (which can only happen for $t_e \in \{0, 1\}$) is straightforward, since by definition $\mathcal{V}_e > 0$ for every $e \in E(\mathcal{V})$. That concludes (ii). ☐

Now we are ready to prove Theorem 1

Proof of Theorem 1. Let $(x, \mu, \lambda) \in \mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$, and recall that x is integral and λ, μ are strictly positive. To prove that there exists a binary vector y such that (x, y) is an optimal solution of (6)-(12) we use duality, and more specifically, we define a dual solution $(\mathcal{U}, \Lambda^-, \Lambda^+, \beta)$ such that $(x, y, \mathcal{U}, \Lambda^-, \Lambda^+, \beta)$ is an optimal primal-dual pair for the linear relaxation of (6)-(12).

Let $\mathcal{U} = \log(\mu)$ and for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$ we define Λ_v^-, Λ_v^+ as follows. We let $\Lambda_v^- = \log(\lambda_v)$ and $\Lambda_v^+ = 0$ in the following cases: (a) if $\sum_{e \in E(\mathcal{V}): e_\ell = v} x_e = m_v^- = m_v^+$ and $\lambda_v \geq 1$, or (b) if $\sum_{e \in E(\mathcal{V}): e_\ell = v} x_e = m_v^- \neq m_v^+$. We let $\Lambda_v^- = 0$ and $\Lambda_v^+ = \log(\lambda_v)$ in the following cases: (a) if $\sum_{e \in E(\mathcal{V}): e_\ell = v} x_e = m_v^- = m_v^+$ and $\lambda_v < 1$, or (b) if $\sum_{e \in E(\mathcal{V}): e_\ell = v} x_e = m_v^+ \neq m_v^-$. In any other case, we have $\sum_{e \in E(\mathcal{V}): e_\ell = v} x_e \notin \{m_v^-, m_v^+\}$, so conditions (4)-(5) imply that $\lambda_v = 1$, and hence we define $\Lambda_v^- = \Lambda_v^+ = \log(\lambda_v) = 0$. Note that we defined Λ^- and Λ^+ in a way such that for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$ we have $\Lambda_v^- + \Lambda_v^+ = \log(\lambda_v)$. For each $e \in E(\mathcal{V})$ we define y_e^t and

β_e^t in the following way: For each $t > x_e$ we define $y_e^t = 0$ and $\beta_e^t = 0$, and for each $t \leq x_e$ we define $y_e^t = 1$ and $\beta_e^t = \log(s(t)/\mathcal{V}_e) - \mathcal{U} - \sum_{\ell=1}^d (\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell})$.

Suppose first that the signpost sequence is such that $s(1) > 0$. By construction, the solution is feasible for the linear relaxation of (6)-(12) and it satisfies conditions (15), (17), (19) and (20). To check condition (21), note that if $\Lambda^-_v < 0$ for some v, the only possibility from the definition of this variable is that $\lambda_v < 1$ and $\sum_{e\in E(\mathcal{V}):e_\ell = v} x_e = m^-_v \neq m^+_v$, but this cannot happen because of condition (5). For Λ^+_v it is analogous. To check condition (18) for $t \leq x_e$, observe that

$$\beta_e^t = \log\left(\frac{s(t)}{\mathcal{V}_e}\right) - \mathcal{U} - \sum_{\ell=1}^{d}(\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) \leq \log\left(\frac{s(x_e)}{\mathcal{V}_e}\right) - \mathcal{U} - \sum_{\ell=1}^{d}(\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell})$$
$$= \log\left(\frac{s(x_e)}{\mathcal{V}_e \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell}}\right) \leq 0,$$

where the first inequality comes from the monotonicity of the logarithm and the signpost sequence s, the equality comes from the definition of $\mathcal{U}, \Lambda^-$ and Λ^+ and the last inequality comes from the fact that (x, μ, λ) satisfies (3) and therefore $s(x_e) \leq \mathcal{V}_e \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell}$. To check that (13) holds, observe that since $\mathcal{V}_e \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell} \leq s(x_e + 1)$ from condition (3), we have that

$$\mathcal{U} + \sum_{\ell=1}^{d} (\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) = \log\left(\mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell}\right) \leq \log\left(\frac{s(x_e+1)}{\mathcal{V}_e}\right).$$

If $t \geq x_e + 1$, since $\beta_e^t = 0$ we conclude that

$$\mathcal{U} + \sum_{\ell=1}^{d} (\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) + \beta_e^t = \mathcal{U} + \sum_{\ell=1}^{d} (\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) \leq \log\left(\frac{s(x_e+1)}{\mathcal{V}_e}\right) \leq \log\left(\frac{s(t)}{\mathcal{V}_e}\right),$$

where the last inequality comes from the monotonicity of the logarithm and the signpost sequence s. Otherwise, when $t \leq x_e$, by the definition of β_e^t it follows directly that $\mathcal{U} + \sum_{\ell=1}^d (\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) + \beta_e^t = \log(s(t)/\mathcal{V}_e)$. We conclude that $(x, y, \mathcal{U}, \Lambda^-, \Lambda^+, \beta)$ is an optimal primal-dual pair for the linear relaxation of (6)-(12). When $s(1) = 0$, we define y and β in the same way as before except when $t = 1$, case for which we consider $\beta_e^1 = -\mathcal{U} - \sum_{\ell=1}^d (\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell})$ for every $e \in E(\mathcal{V})$. Once again, the solution is feasible for the linear relaxation of (6)-(12) and moreover, as we know that for $t \geq 2$ we have $s(t) > 0$, the previous analysis is still valid for every $e \in E(\mathcal{V})$ and $t \geq 2$. For $t = 1$, we know that $y_e^1 = 1$ for every $e \in E(\mathcal{V})$, so (17) follows directly. Fixing β_e^1 as mentioned, we conclude that $(x, y, \mathcal{U}, \Lambda^-, \Lambda^+, \beta)$ also verifies the remaining conditions (14) and (16). As the proof of conditions (19)-(21) for the case $s(1) > 0$ remain valid, $(x, y, \mathcal{U}, \Lambda^-, \Lambda^+, \beta)$ is an optimal primal-dual pair for the linear relaxation of (6)-(12). By duality, we conclude that (x, y) is an optimal solution for the linear relaxation of (6)-(12).

For the converse, let (x, y) be an integral optimal solution for the linear relaxation of (6)-(12), with a corresponding dual optimal solution $(\mathcal{U}, \Lambda^-, \Lambda^+, \beta)$. Let $\mu = \exp(\mathcal{U})$ and for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$ let $\lambda_v = \exp(\Lambda^-_v + \Lambda^+_v)$. We verify in what follows that (x, μ, λ) satisfies (1)-(5). Observe that condition (1) is directly satisfied since (x, y) satisfies constraints (9)-(10) in the program. Condition (2) also follows directly from constraint (8). To check condition (4), let $\ell \in \{1, \ldots, d\}$ and $v \in N_\ell$ be such that $\lambda_v > 1$. From the definition of λ, this is equivalent to $\Lambda^-_v + \Lambda^+_v > 0$, and since $\Lambda^+_v \leq 0$ from condition (21), this implies $\Lambda^-_v > 0$. Therefore, condition (19) allows to conclude $\sum_{e\in E(\mathcal{V}):e_\ell = v} x_e = m^-_v$. Condition (5) follows analogously. It remains to show that (x, μ, λ) also verifies (3). From Lemma 1 (i) and the integrality of x, we know that for every $e \in E(\mathcal{V})$ we have that $t_e = x_e$ and therefore, by Lemma 1 (ii) we have that $s(x_e) \le \mathcal{V}_e \cdot \exp(\mathcal{U}) \cdot \prod_{\ell=1}^{d} \exp(\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) \le s(x_e + 1)$, which by the definition of μ and λ is equivalent to $s(x_e) \le \mathcal{V}_e \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell} \le s(x_e + 1)$. We conclude that $(x, \mu, \lambda) \in \mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$. This concludes the theorem. □

6.2 Proof of Theorem 2

Recall that s is a stationary signpost sequence if $s(q) = q - \Delta$ for some $\Delta \in [0, 1]$ and every strictly positive integer q. Consider the 3-dimensional instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ defined as follows: $N_1 = \{p_1, p_2\}$, $N_2 = \{r_1, r_2\}$ and $N_3 = \{g_1, g_2\}$. In order to represent vectors with entries in $N_1 \times N_2 \times N_3$ we use two matrices, the first one for g_1 and the second one for g_2, which rows represent N_1 (the first row p_1 and the second p_2) and which columns represent N_2 (the first column r_1 and the second r_2). The matrix representation of $\mathcal{V}$ is given by

$$\begin{pmatrix} 81 & 40 \\ 50 & 54 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 66 & 53 \\ 81 & 48 \end{pmatrix}.$$

The house size is equal to $H = 10$ and the marginals m^- and m^+ are all equal to five. We denote by $\mathcal{T}$ this instance. Consider (X, Y) where the matrix representation of X is given by

$$\begin{pmatrix} 1.5 & 1 \\ 1 & 1.5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1.5 \\ 1.5 & 1 \end{pmatrix},$$

and Y is defined as in Lemma 1, in particular $Y_e^1 = 1$ for every $e \in E(\mathcal{V})$, $Y_e^2 = 0.5$ for every $e \in E(\mathcal{V})$ such that $X_e = 1.5$ and $Y_e^t = 0$ in any other case. By construction, (X, Y) is a feasible solution of the linear relaxation of (6)-(12) for every signpost sequence. Consider the solution $(x^\star, y^\star)$ where the matrix representation of $x^\star$ is given by

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and $y^\star$ is defined as in Lemma 1, meaning $y_e^{\star 1} = 1$ for every $e \in E(\mathcal{V})$, $y_e^{\star 2} = 1$ for $e \in E(\mathcal{V})$ such that $x_e^\star = 2$ and $y_e^{\star t} = 0$ in any other case.

Claim 1. *For every $\Delta \in [0, 1]$, $(x^\star, y^\star)$ is the optimal solution of the integer program (6)-(12).*

We show how to prove the theorem using Claim 1. For every $\Delta \in [0, 1]$ we have that the difference of the objective values for the solutions $(x^\star, y^\star)$ and (X, Y) is equal to

$$\log\left(\frac{2-\Delta}{81}\right) + \log\left(\frac{2-\Delta}{48}\right) - \frac{1}{2}\left[\log\left(\frac{2-\Delta}{81}\right) + \log\left(\frac{2-\Delta}{54}\right) + \log\left(\frac{2-\Delta}{53}\right) + \log\left(\frac{2-\Delta}{81}\right)\right]$$
$$= \frac{1}{2}\log\left(\frac{53 \cdot 54}{48^2}\right) > 0,$$

and therefore $(x^\star, y^\star)$ is not an optimal solution for the linear relaxation of (6)-(12). Since $(x^\star, y^\star)$ is the optimal integer solution of (6)-(12), any other optimal solution has the same objective value, and therefore we conclude that there is no integral solution that is optimal for the linear relaxation. By Theorem 1 we conclude that $\mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H) = \emptyset$.

To prove Claim 1 we consider two cases. Suppose first that $\Delta \in [0,1)$. In this case, the program (6)-(12) is given by

$$\begin{aligned}
&\text{minimize} \quad \Phi(y,\Delta) && (24)\\
&\text{subject to} \quad \sum_{t=1}^{10} y_e^t = x_e \qquad \text{for every } e \in N_1 \times N_2 \times N_3, && (25)\\
&\sum_{e \in N_1\times N_2\times N_3 : e_\ell = v} x_e = 5 \qquad \text{for every } \ell \in \{1,2,3\} \text{ and every } v \in N_\ell, && (26)\\
&y_e^t \in \{0,1\} \qquad \text{for every } e \in N_1 \times N_2 \times N_3 \text{ and } t \in \{1,\dots,10\}, && (27)
\end{aligned}$$

where the function $\Phi(y,\Delta)$ is given by

$$\Phi(y,\Delta) = \sum_{e \in N_1\times N_2\times N_3} \sum_{t=1}^{10} y_e^t \log\left(\frac{t-\Delta}{\mathcal{V}_e}\right).$$

In particular, Φ is strictly decreasing in its second variable $\Delta \in [0,1)$, and $\Phi(y,\Delta) \to -\infty$ when $\Delta \to 1$ if there exists $e \in N_1 \times N_2 \times N_3$ such that $y_e^1 > 0$. For the instance $\mathcal{T}$ there are 104 integer vectors x satisfying (26), hence using Lemma 1 (i) it can be checked computationally that for $\Delta^\star = 0.99995$ and $\varepsilon = 5\cdot 10^{-7}$ the following holds: for every $\Delta \in \{\varepsilon, 2\varepsilon, \dots, \Delta^\star - \varepsilon, \Delta^\star\}$ and every integer solution (x,y) satisfying (25)-(27), we have that $\Phi(y,\Delta) > \Phi(y^\star, \Delta - \varepsilon)$ and $\Phi(y,0) > \Phi(y^\star,0)$. We show in what follows that this implies $\Phi(y,\Delta) > \Phi(y^\star,\Delta)$ for every $\Delta \in [0,\Delta^\star]$. Let $\Delta \in [0,\Delta^\star]$. If $\Delta = 0$, by hypothesis we have $\Phi(y,\Delta) > \Phi(y^\star,\Delta)$, so we are done in this case. Otherwise, if $(k-1)\varepsilon < \Delta \le k\varepsilon$ for some $k \in \{1,\dots,\Delta^\star/\varepsilon\}$, we have that $\Phi(y,\Delta) \ge \Phi(y,k\varepsilon) > \Phi(y^\star,(k-1)\varepsilon) > \Phi(y^\star,\Delta)$, where the first and last inequalities come from the fact that Φ is strictly decreasing in its second variable and the second inequality comes from the mentioned property.

It remains to show the inequality for $\Delta \in (\Delta^\star, 1)$. We first observe that only four of the integer vectors x satisfying (26) are strictly positive in every entry, including $x^\star$. Among integer solutions (x,y) satisfying (25)-(27) with strictly positive x, is easy to check algebraically and using Lemma 1 (i) that for every $\Delta \in [0,1]$ $(x^\star, y^\star)$ is the one with smallest objective value. Thus, it only remains to show this for $\Delta \in (\Delta^\star, 1)$ and feasible solutions (x,y) where x has at least one component equal to zero, i.e. where $|\text{supp}(x)| = |\{e \in N_1 \times N_2 \times N_3 : x_e \neq 0\}| < 8$. Note that from Lemma 1 (i) and the integrality of x and $x^\star$

$$\Phi(y,\Delta) - \Phi(y^\star,\Delta) \ge \sum_{e\in N_1\times N_2\times N_3 : x_e > x_e^\star} \sum_{t=x_e^\star+1}^{x_e} \log\left(\frac{t-\Delta}{\mathcal{V}_e}\right) - \sum_{e\in N_1\times N_2\times N_3 : x_e^\star > x_e} \sum_{t=x_e+1}^{x_e^\star} \log\left(\frac{t-\Delta}{\mathcal{V}_e}\right)$$

and denote the first term on the righthand side as $\mathcal{S}_1(x,x^\star,\Delta)$ and the substracted term as $\mathcal{S}_2(x,x^\star,\Delta)$. Suppose first that $|\text{supp}(x)| = 7$. In this case, the fact that $\sum_{e\in N_1\times N_2\times N_3} x_e = 10$ and the marginals constraint (26) implies that there are exactly three entries $e \in \text{supp}(x)$ such that $x_e = 2$. As all the entries of $x^\star$ are at least one, in this case we can bound $\mathcal{S}_1(x,x^\star,\Delta)$ as

$$\mathcal{S}_1(x,x^\star,\Delta) \ge \min_{\substack{\Delta\in[0,1],\\ e\in N_1\times N_2\times N_3}} 3\log\left(\frac{2-\Delta}{\mathcal{V}_e}\right) \ge 3\log\left(\frac{1}{81}\right),$$

where the first inequality follows from the fact that the logarithm is negative since $2 - \Delta < \mathcal{V}_e$ for every $\Delta \in [0,1]$ and every $e \in N_1 \times N_2 \times N_3$ and the second one from replacing the minimizers.

Similarly, we can bound $\mathcal{S}_2(x, x^\star, \Delta)$ as follows, using that one entry of x is 0:

$$\mathcal{S}_2(x, x^\star, \Delta) \leq \max_{e \in N_1 \times N_2 \times N_3} \log\left(\frac{1-\Delta}{\mathcal{V}_e}\right) = \log\left(\frac{1-\Delta}{40}\right),$$

where once again, the first inequality uses the negativity of logarithms and the second comes from replacing e by the one with the smallest value of $\mathcal{V}_e$. Writing all together, we have that for every feasible solution (x, y) it holds that

$$\Phi(y, \Delta) - \Phi(y^\star, \Delta) \geq 3\log\left(\frac{1}{81}\right) - \log\left(\frac{1-\Delta}{40}\right) = \log\left(\frac{40}{81^3 \cdot (1-\Delta)}\right).$$

Thus, imposing $81^3 \cdot (1-\Delta) < 40$ or equivalently $\Delta > 1 - 40/81^3 \approx 0.999925$ we conclude $\Phi(y, \Delta) > \Phi(y^\star, \Delta)$.

Now consider the case $|\text{supp}(x)| \leq 6$. In this case, we achieve a lower bound for $\mathcal{S}_1(x, x^\star, \Delta)$ computing $\mathcal{S}_1(x, x^\star, 1)$ for each feasible solution with two or more entries equal to zero. The minimum is achieved in the solution x with entries (p_2, r_2, g_1) and (p_1, r_1, g_2) equal to 5 and all the others equal to 0, and the value of $\mathcal{S}_1(x, x^\star, 1)$ is approximately -26.36. For $\mathcal{S}_2(x, x^\star, \Delta)$, we bound in the same way as before, but using the fact that the entries with zeros are at least two:

$$\mathcal{S}_2(x, x^\star, \Delta) \leq 2 \max_{e \in N_1 \times N_2 \times N_3} \log\left(\frac{1-\Delta}{\mathcal{V}_e}\right) = 2\log\left(\frac{1-\Delta}{40}\right).$$

Using these two bounds, we can bound the difference of the objective values as

$$\Phi(y, \Delta) - \Phi(y^\star, \Delta) > -26.5 - 2\log\left(\frac{1-\Delta}{40}\right).$$

So a sufficient condition for our result is $2\log((1-\Delta)/40) \leq -26.5$, which is equivalent to $\Delta \geq 1 - 40\exp(-13.25) \approx 0.99993$. Thus, recalling that $\Delta^\star = 0.99995$, in particular we have that $\Phi(y, \Delta) > \Phi(y^\star, \Delta)$ for every $\Delta \in (\Delta^\star, 1)$. Hence, we have proved the optimality of $(x^\star, y^\star)$ for every $\Delta < 1$. When $\Delta = 1$ the claim follows by a computational certification of the optimality of $(x^\star, y^\star)$ for the integer program (6)-(12). □

6.3 Proof of Theorem 3

Consider a d-partite hypergraph $G = (P, F)$ with vertices partition $P_1, \ldots, P_d$ such that $|P_1| = \cdots = |P_d| = c$. We define two instances for the (d, s)-proportional apportionment problem as follows. Let $\mathcal{P} = \{P_1, \ldots, P_d\}$ and let $\mathcal{V}(G)$ be such that for every $e \in \prod_{\ell=1}^{d} P_\ell$ is defined as follows: $\mathcal{V}_e(G) = 1$ when $\{e_1, \ldots, e_d\} \in F$, and $\mathcal{V}_e(G) = 0$ otherwise. For every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell$, let $m_v^1 = 1$ and $m_v^2 = |\delta(v)| + 1$, where $\delta(v)$ is the set of hyperedges containing v. Consider the apportionment instance $\mathcal{T}_1(G) = (\mathcal{P}, \mathcal{V}(G), m^1, m^1, c)$ and the instance $\mathcal{T}_2(G) = (\mathcal{P}, \mathcal{V}(G), m^2, m^2, |F| + c)$. The following lemma establishes the correspondence between the (d, s)-proportional apportionment problem and the perfect matching problem in d-partite hypergraphs.

Lemma 2. *Let $G = (P, F)$ be a d-partite graph with vertices partition $P_1, \ldots, P_d$ and such that $|P_1| = \cdots = |P_d| = c$. Then, for every signpost sequence s, the following holds:*

(a) When $s(1) > 0$, we have that G has a perfect matching if and only if $\mathcal{A}_s(\mathcal{T}_1(G)) \neq \emptyset$.

(b) When $s(1) = 0$, we have that G has a perfect matching if and only if $\mathcal{A}_s(\mathcal{T}_2(G)) \neq \emptyset$.

Proof. Let $G = (P, F)$ be a d-partite hypergraph as described in the statement of the lemma. Recall that $E(\mathcal{V}(G))$ corresponds to the subset of tuples in the product $\prod_{\ell=1}^{d} P_\ell$ where $\mathcal{V}(G)$ is strictly positive, and therefore, $E(\mathcal{V}(G)) = \{e \in \prod_{\ell=1}^{d} P_\ell : \{e_1, \ldots, e_d\} \in F\}$, that is, exactly the tuples in the product that are in correspondence with the edges of G. Within the proof, the following function will be useful. Let $\alpha : E(\mathcal{V}(G)) \to F$ be such that $\alpha(e) = \{e_1, \ldots, e_d\}$. The function α is a bijection, and it captures the natural representation of the (ordered) tuples in $\prod_{\ell=1}^{d} P_\ell$ as (unordered) hyperedges in G.

We first consider the case $s(1) > 0$ and suppose that G has a perfect matching $F' \subseteq F$. Consider the integral vector x with entries in $E(\mathcal{V}(G))$ defined as follows: $x_e = 1$ when $\alpha(e) \in F'$ and $x_e = 0$ otherwise. Let $\mu = 1$ and for every $v \in P$, let $\lambda_v = s(1)^{1/d}$. We check now that $(x, \mu, \lambda) \in \mathcal{A}_s(\mathcal{T}_1(G))$. Since F' is a perfect matching, for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell$ we have that

$$\sum_{e \in E(\mathcal{V}(G)): e_\ell = v} x_e = |\delta(v) \cap F'| = 1 = m_v^1,$$

and therefore conditions (1), (2), (4) and (5) are satisfied. For every $e \in E(\mathcal{V}(G))$, we have that $\mathcal{V}_e(G) \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell} = 1 \cdot (s(1)^{1/d})^d = s(1)$. When $x_e = 1$, from the monotonicity of the signpost sequence s we therefore have that $s(x_e) = s(1) = \mathcal{V}_e(G) \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell} \leq s(2) = s(x_e + 1)$, and when $x_e = 0$, we have that $s(x_e) = s(0) \leq \mathcal{V}_e(G) \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell} = s(1) = s(x_e + 1)$. We conclude that condition (3) is satisfied and therefore $(x, \mu, \lambda) \in \mathcal{A}_s(\mathcal{T}_1(G))$. Conversely, suppose now that there exists a pair $(\bar{x}, \bar{\lambda}) \in \mathcal{A}_s(\mathcal{T}_1(G))$. The marginals condition (1) implies that $\sum_{e \in E(\mathcal{V}(G)): e_\ell = v} \bar{x}_e = 1$ for every $v \in P$, and therefore for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell$ there exists a unique $e \in E(\mathcal{V}(G))$ where $e_\ell = v$ and such that $\bar{x}_e = 1$. Then, the subset $\{\alpha(e) : \bar{x}_e = 1\}$ is a perfect matching. This concludes the proof of (a).

Consider now the case where $s(1) = 0$, and suppose that G has a perfect matching $F' \subseteq F$. Consider an integral vector x with entries in $E(\mathcal{V}(G))$ defined as follows: $x_e = 2$ when $\alpha(e) \in F'$ and $x_e = 1$ otherwise. Let $\mu = 1$ and for every $v \in P$, let $\lambda_v = s(2)^{1/d}$. We check now that $(x, \mu, \lambda) \in \mathcal{A}_s(\mathcal{T}_2(G))$. Since F' is a perfect matching, for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell$ we have that

$$\sum_{e \in E(\mathcal{V}(G)): e_\ell = v} x_e = |\delta(v)| - 1 + 2 = |\delta(v)| + 1 = m_v^2,$$

and therefore conditions (1), (2), (4) and (5) are satisfied. For every $e \in E(\mathcal{V}(G))$, we have that $\mathcal{V}_e(G) \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell} = 1 \cdot (s(2)^{1/d})^d = s(2)$. From the monotonicity of the signpost sequence s, when $x_e = 2$ we have that $s(x_e) = s(2) = \mathcal{V}_e(G) \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell} \leq s(3) = s(x_e + 1)$, and when $x_e = 1$ we have that $s(x_e) = s(1) \leq \mathcal{V}_e(G) \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell} = s(2) = s(x_e + 1)$. We conclude that condition (3) is satisfied and therefore $(x, \mu, \lambda) \in \mathcal{A}_s(\mathcal{T}_2(G))$. Conversely, suppose now that there exists a pair $(\bar{x}, \bar{\lambda}) \in \mathcal{A}_s(\mathcal{T}_2(G))$. The marginals condition (1) implies that $\sum_{e \in E(\mathcal{V}(G)): e_\ell = v} \bar{x}_e = |\delta(v)| + 1$ for every $v \in P$. Since $s(1) = 0$, we have that $\bar{x}_e \geq 1$ for every $e \in E(\mathcal{V}(G))$, and therefore for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell$ there exists a unique $e \in E(\mathcal{V}(G))$ where $e_\ell = v$ and such that $\bar{x}_e = 2$. Therefore, the subset $\{\alpha(e) : \bar{x}_e = 2\}$ is a perfect matching of G. This concludes the proof of (b). □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. The problem is in NP since for a triplet (x, μ, λ) checking conditions (1)-(5) can be done in polynomial time. Given a signpost sequence s, by Lemma 2 we have that the perfect matching problem in d-partite hypergraphs where each part is the same size, is reducible to the (d, s)-proportional apportionment problem, since for every such hypergraph G the instances $\mathcal{T}_1(G)$

and $\mathcal{T}_2(G)$ can be computed in polynomial time. Since the perfect matching problem in d-partite hypergraphs where each part has equal size is NP-complete for every $d \geq 3$ [24, 21], we conclude that for every $d \geq 3$ and every signpost sequence s the (d, s)-proportional apportionment problem is NP-complete. □

7 Appendix: Proofs of Section 4

7.1 Proof of Theorem 5

In the following lemmas, we summarize the key properties of our Algorithm 1.

Lemma 3. *Let G be a d-partite hypergraph with vertex partition $\{P_1, \dots, P_d\}$ and hyperedges E, let $u_1, \dots, u_d$ be nonnegative integers and let $x \in [0,1]^E$. When $\sum_{\ell=1}^{d} 1/(u_\ell + 2) \leq 1$, we have that Algorithm 1 terminates. Furthermore, we have $T \leq |E|$, and for every $t \in \{0, \dots, T-1\}$ we have that $\mathcal{E}^{t+1}$ is strictly contained in $\mathcal{E}^t$.*

Lemma 4. *Let G be a d-partite hypergraph with vertex partition $\{P_1, \dots, P_d\}$ and hyperedges E, let $u_1, \dots, u_d$ be nonnegative integers and let $x \in [0,1]^E$. Suppose that T as defined in Algorithm 1 is finite and let z be the output of Algorithm 1 . Then, for every $\ell \in \{1, \dots, d\}$ and every $v \in P_\ell$ there exists $t(v) \in \{0, 1, \dots, T\}$ such that $|\delta(v) \cap \mathcal{E}^t| \geq u_\ell + 2$ for every $t < t(v)$, when $t(v) > 0$, and $|\delta(v) \cap \mathcal{E}^t| \leq u_\ell + 1$ for every $t \geq t(v)$. Furthermore, for every $t \leq t(v)$ we have that*

$$\sum_{e \in \delta(v)} x_e = \sum_{e \in \delta(v) \cap \mathcal{E}^t} y_e^t + \sum_{e \in \delta(v) \setminus \mathcal{E}^t} z_e.$$

Now we show how to prove Theorem 5 using these lemmas. In the rest of this appendix, we prove Lemma 3 and Lemma 4.

Proof of Theorem 5. We show in what follows that Algorithm 1 computes a solution z that satisfies the conditions guaranteed by Theorem 5. By Lemma 3 we have that the algorithm terminates and $T \leq |E|$. Furthermore, $\mathcal{E}^{t+1}$ is strictly contained in $\mathcal{E}^t$ for every $t \in \{0, \dots, T-1\}$. In the initialization we have that $z_e = x_e \in \{0,1\}$ for every $e \in E \setminus \mathcal{E}^0$, which in particular implies that if x_e is integer, then $z_e = x_e$. For every $t \in \{0, \dots, T-1\}$ we have $z_e = y_e^{t+1} \in \{0,1\}$ for each $e \in \mathcal{E}^t \setminus \mathcal{E}^{t+1}$, and since we have that $E \setminus \mathcal{E}^T$ is equal to the union of $E \setminus \mathcal{E}^0$ and $\bigcup_{t=0}^{T-1} (\mathcal{E}^t \setminus \mathcal{E}^{t+1})$, where all these unions are disjoint, we conclude that $z_e \in \{0,1\}$ for every $e \notin \mathcal{E}^T$. Since $z_e \in \{\lfloor y_e^T \rfloor, \lceil y_e^T \rceil\}$ for every $e \in \mathcal{E}^T$, we conclude that $z \in \{0,1\}^E$. By Lemma 4, for every $\ell \in \{1, \dots, d\}$ and every $v \in P_\ell$ there exists $t(v) \in \{0, 1, \dots, T\}$ such that $|\delta(v) \cap \mathcal{E}^t| \geq u_\ell + 2$ for every $t < t(v)$, when $t(v) > 0$, and $|\delta(v) \cap \mathcal{E}^t| \leq u_\ell + 1$ for every $t \geq t(v)$. Furthermore,

$$\begin{aligned}\sum_{e \in \delta(v)} x_e &= \sum_{e \in \delta(v) \cap \mathcal{E}^{t(v)}} y_e^{t(v)} + \sum_{e \in \delta(v) \setminus \mathcal{E}^{t(v)}} z_e \\ &= \sum_{e \in \delta(v) \cap \mathcal{E}^{t(v)}} y_e^{t(v)} - \sum_{e \in \delta(v) \cap \mathcal{E}^{t(v)}} z_e + \sum_{e \in \delta(v)} z_e.\end{aligned}$$

We have that $0 < y_e^{t(v)} < 1$ for every $e \in \delta(v) \cap \mathcal{E}^{t(v)}$, and therefore the above equality implies that

$$\Big| \sum_{e \in \delta(v)} (x_e - z_e) \Big| \leq \sum_{e \in \delta(v) \cap \mathcal{E}^{t(v)}} |y_e^{t(v)} - z_e| < |\delta(v) \cap \mathcal{E}^{t(v)}| \leq u_\ell + 1.$$

Since for every vertex v of G we have that $\sum_{e\in\delta(v)} x_e$ is integral and $\sum_{e\in\delta(v)} z_e$ is integral as well, we conclude that $|\sum_{e\in\delta(v)}(x_e - z_e)| \leq u_\ell$ for every $\ell \in \{1,\ldots,d\}$ and every $v \in P_\ell$. Finally, as the algorithm executes at most $|E|$ iterations, and in each iteration it solves a linear program with at most $|E|$ variables and at most $2|E| + \sum_{\ell=1}^d |P_\ell|$ constraints, we conclude that z can be computed efficiently. □

The following two propositions will be used to prove Lemma 3.

Proposition 1. *Consider a tuple $(Y,\mathcal{E},Q)$ and suppose that $|\bigcup_{\ell=1}^d Q_\ell| < |\mathcal{E}|$. Then, for every extreme point $y \in \mathcal{K}(Y,\mathcal{E},Q)$, there exists at least one hyperedge $e \in \mathcal{E}$ such that $y_e \in \{0,1\}$.*

Proof. Observe that the total number of variables in the linear program is $|\mathcal{E}|$ while the number of equality constraints in (22) is equal to $|\bigcup_{\ell=1}^d Q_\ell|$. In particular, since $|\bigcup_{\ell=1}^d Q_\ell| < |\mathcal{E}|$ we conclude that the number of variables in the linear program is strictly larger than the number of equality constraints in (22). Therefore, any extreme point y of $\mathcal{K}(Y,\mathcal{E},Q)$ must satisfy at least one inequality constraint in (23) with equality, so at least one entry of y is in $\{0,1\}$. □

Proposition 2. *Let $w_1,\ldots,w_d$ be integer numbers such that $w_\ell \geq 2$ for every $\ell \in \{1,\ldots,d\}$. Then, the following holds:*

(i) $q \geq \sum_{\ell=1}^d \lfloor q/w_\ell \rfloor$ for every strictly positive q if and only if $\sum_{\ell=1}^d 1/w_\ell \leq 1$.

(ii) $q > \sum_{\ell=1}^d \lfloor q/w_\ell \rfloor$ for every strictly positive q if and only if $\sum_{\ell=1}^d 1/w_\ell < 1$.

(iii) Suppose that $q \geq \sum_{\ell=1}^d \lfloor q/w_\ell \rfloor$ for every strictly positive q and let $\bar{q}$ be a strictly positive integer such that $\bar{q} = \sum_{\ell=1}^d \lfloor \bar{q}/w_\ell \rfloor$. Then, $\bar{q}$ is a common multiplier of $w_1,\ldots,w_d$.

Proof. Consider the function $\phi_d(q,w) = q - \sum_{\ell=1}^d \lfloor q/w_\ell \rfloor$. First observe that for any strictly positive q we have that $\phi_d(q,w) \geq q - \sum_{\ell=1}^d q/w_\ell = q(1-\sum_{\ell=1}^d 1/w_\ell)$. When $\sum_{\ell=1}^d 1/w_\ell \leq 1$ we have that $\phi_d(q,w) \geq 0$. Consider $\hat{q} = \prod_{\ell=1}^d w_\ell$. In particular, we have that $\lfloor \hat{q}/w_\ell \rfloor = \hat{q}/w_\ell$ for every $\ell \in \{1,\ldots,d\}$ and therefore $\phi_d(\hat{q},w) = \hat{q}(1-\sum_{\ell=1}^d 1/w_\ell)$. Then, when $\phi_d(\hat{q},w) \geq 0$ we have that $\sum_{\ell=1}^d 1/w_\ell \leq 1$. That proves (i). Property (ii) follows from the same argument by using strict inequalities. For the third property, if $\bar{q}$ is not a common multiplier of $w_1,\ldots,w_d$, then we necessarily have that $\lfloor \bar{q}/w_\ell \rfloor < \bar{q}/w_\ell$ for some $\ell \in \{1,\ldots,d\}$. Therefore,

$$\phi_d(\bar{q},w) = \bar{q} - \sum_{\ell=1}^d \left\lfloor \frac{\bar{q}}{w_\ell} \right\rfloor > \bar{q} - \sum_{\ell=1}^d \frac{\bar{q}}{w_\ell} = \bar{q}\left(1 - \sum_{\ell=1}^d \frac{1}{w_\ell}\right).$$

By property (i) we have that $\sum_{\ell=1}^d 1/w_\ell \leq 1$ and therefore the last term in the above inequality is nonnegative, which contradicts the fact that $\phi_d(\bar{q},w) = 0$. That concludes (iii). □

Now we are ready to prove Lemma 3 and Lemma 4.

Proof of Lemma 3. In what follows we show how to conclude the lemma by using the following claim: Given an integer t, if there exists $\ell \in \{1,\ldots,d\}$ for which $Q_\ell^t \neq \emptyset$ and if $\sum_{\ell=1}^d 1/(u_\ell+2) \leq 1$, we have that every extreme point of $\mathcal{K}(y^t,\mathcal{E}^t,Q^t)$ has at least one integral entry. If the claim is true, as $\mathcal{K}(y^t,\mathcal{E}^t,Q^t)$ is nonempty since y^t is always a feasible solution for the linear program (22)-(23), then $\mathcal{E}^{t+1}$ is strictly contained in $\mathcal{E}^t$, so as long as the loop condition is satisfied, we have that $|\mathcal{E}^t|$ is strictly decreasing in t. We conclude that T is finite and furthermore $T \leq |E|$. We now show how to prove the claim. Note that at iteration t, for every $\ell \in \{1,\ldots,d\}$ we have $|Q_\ell^t| \leq \lfloor |\mathcal{E}^t|/(u_\ell+2) \rfloor$. If this is not the case, it would mean that $\sum_{v\in Q_\ell^t} |\delta(v)\cap\mathcal{E}^t| \geq (u_\ell+2)\cdot(1+\lfloor|\mathcal{E}^t|/(u_\ell+2)\rfloor) > |\mathcal{E}^t|$, which is a contradiction since each hyperedge in $\mathcal{E}^t$ contains at most one vertex in Q^t_ℓ, hence the term in the left is a sum of cardinalities of disjoint subsets of $\mathcal{E}^t$. Therefore, we have that

$$\left|\bigcup_{\ell=1}^{d} Q^t_\ell\right| \leq \sum_{\ell=1}^{d} |Q^t_\ell| \leq \sum_{\ell=1}^{d} \left\lfloor \frac{|\mathcal{E}^t|}{u_\ell + 2} \right\rfloor \leq |\mathcal{E}^t|,$$

where the last inequality comes from Proposition 2 (i). From Proposition 2 (ii), we know that if $\sum_{\ell=1}^{d} 1/(u_\ell+2) < 1$, then we have $\sum_{\ell=1}^{d}\lfloor|\mathcal{E}^t|/(u_\ell+2)\rfloor < |\mathcal{E}^t|$, and therefore we have $|\bigcup_{\ell=1}^{d} Q^t_\ell| < |\mathcal{E}^t|$. By Proposition 1 we conclude that every extreme point of $\mathcal{K}(y^t, \mathcal{E}^t, Q^t)$ has at least one integral entry, so we are done in this case. In the following suppose that $\sum_{\ell=1}^{d} 1/(u_\ell+2) = 1$. From Proposition 2 (i) we know that $\sum_{\ell=1}^{d}\lfloor|\mathcal{E}^t|/(u_\ell+2)\rfloor \leq |\mathcal{E}^t|$, and from Proposition 2 (iii) the equality is only possible when $|\mathcal{E}^t|$ is a common multiple of the values $u_1+2, \dots, u_d+2$. In what follows, we suppose the latter holds and we consider two cases.

Case 1. Suppose first that for every $\ell \in \{1,\dots,d\}$ and every $v \in P_\ell$ we have $|\delta(v) \cap \mathcal{E}^t| \in \{0, u_\ell+2\}$. Then, in this case the polytope $\mathcal{K}(y^t, \mathcal{E}^t, Q^t)$ is obtained from the linear program

$$\sum_{e\in\delta(v)\cap\mathcal{E}^t} z_e = \sum_{e\in\delta(v)\cap\mathcal{E}^t} y^t_e \quad \text{for every } \ell \in \{1,\dots,d\} \text{ and every } v \in P_\ell \text{ s.t. } |\delta(v)\cap\mathcal{E}^t| = u_\ell + 2,$$
$$0 \leq z_e \leq 1 \qquad \text{for every } e \in \mathcal{E}^t.$$

Since G is a d-partite hypergraph, we have that for every $\ell \in \{1,\dots,d\}$ the set $\mathcal{E}^t$ is equal to the union of the sets $\delta(v)\cap\mathcal{E}^t$ over $v \in P_\ell$. Furthermore, for every $\ell \in \{1,\dots,d\}$ and every $v \in P_\ell \setminus Q^t_\ell$ we have $|\delta(v)\cap\mathcal{E}^t| = 0$, and therefore we conclude that $\mathcal{E}^t$ is equal to the union of the sets $\delta(v)\cap\mathcal{E}^t$ over $v \in Q^t_\ell$. Then, taking $\ell \in \{1,\dots,d\}$ and summing over $v \in Q^t_\ell$ we have that

$$\sum_{v\in Q^t_\ell}\sum_{e\in\delta(v)\cap\mathcal{E}^t} z_e = \sum_{v\in Q^t_\ell}\sum_{e\in\delta(v)\cap\mathcal{E}^t} y^t_e,$$

which implies $\sum_{e\in\mathcal{E}^t} z_e = \sum_{e\in\mathcal{E}^t} y^t_e$, and this last equality does not depend on ℓ. Then, in the linear program defining $\mathcal{K}(y^t, \mathcal{E}^t, Q^t)$, the constraints given by two different sets Q^t_ℓ and $Q^t_{\hat\ell}$, with $\ell \neq \hat\ell$, are linearly dependent. Since any extreme point of $\mathcal{K}(y^t, \mathcal{E}^t, Q^t)$ is obtained by a linear system of $|\mathcal{E}^t|$ linearly independent constraints, we necessarily have that at least one of them is from the set of constraints (23), so it has at least one integral entry.

Case 2. Suppose there exists a value $\hat\ell \in \{1,\dots,d\}$ and $\hat v \in P_{\hat\ell}$ such that $|\delta(\hat v)\cap\mathcal{E}^t| \notin \{0, u_{\hat\ell}+2\}$. If $1 \leq |\delta(\hat v)\cap\mathcal{E}^t| < u_{\hat\ell}+2$, observe that $\hat v$ does not induce a constraint in the linear program defining $\mathcal{K}(y^t, \mathcal{E}^t, Q^t)$. On the other hand, we have $|\bigcup_{v\in P_{\hat\ell}: v\neq\hat v}(\delta(v)\cap\mathcal{E}^t)| \leq |\mathcal{E}^t| - |\delta(\hat v)\cap\mathcal{E}^t|$. Thus, we have that

$$|Q^t_{\hat\ell}| \leq \left\lfloor \frac{|\mathcal{E}^t| - |\delta(\hat v)\cap\mathcal{E}^t|}{u_{\hat\ell}+2} \right\rfloor = \left\lfloor \frac{|\mathcal{E}^t|}{u_{\hat\ell}+2} - \frac{|\delta(\hat v)\cap\mathcal{E}^t|}{u_{\hat\ell}+2} \right\rfloor \leq \frac{|\mathcal{E}^t|}{u_{\hat\ell}+2} - 1,$$

where the last inequality comes from the fact that $|\mathcal{E}^t|/(u_{\hat\ell}+2)$ is an integer and $|\delta(\hat v)\cap\mathcal{E}^t|$ is strictly positive and strictly less than $u_{\hat\ell}+2$. Suppose now that $|\delta(\hat v)\cap\mathcal{E}^t| > u_{\hat\ell}+2$. We have $|\bigcup_{v\in P_{\hat\ell}: v\neq\hat v}(\delta(v)\cap\mathcal{E}^t)| \leq |\mathcal{E}^t| - |\delta(\hat v)\cap\mathcal{E}^t|$, and therefore

$$|Q^t_{\hat\ell}| \leq 1 + \left\lfloor \frac{|\mathcal{E}^t| - |\delta(\hat v)\cap\mathcal{E}^t|}{u_{\hat\ell}+2} \right\rfloor = 1 + \left\lfloor \frac{|\mathcal{E}^t|}{u_{\hat\ell}+2} - \frac{|\delta(\hat v)\cap\mathcal{E}^t|}{u_{\hat\ell}+2} \right\rfloor \leq 1 + \frac{|\mathcal{E}^t|}{u_{\hat\ell}+2} - 2 = \frac{|\mathcal{E}^t|}{u_{\hat\ell}+2} - 1,$$

where we used that $|\mathcal{E}^t|/(u_{\hat{\ell}}+2)$ is an integer and that $|\delta(\hat{v})\cap\mathcal{E}^t| > u_{\hat{\ell}}+2$. Therefore, in both situations we have that

$$\left|\bigcup_{\ell=1}^{d} Q_\ell^t\right| \le \sum_{\ell\neq\hat{\ell}} |Q_\ell^t| + |Q_{\hat{\ell}}^t| \le \sum_{\ell\neq\hat{\ell}} \left\lfloor \frac{|\mathcal{E}^t|}{u_\ell+2}\right\rfloor + \frac{|\mathcal{E}^t|}{u_{\hat{\ell}}+2} - 1$$
$$= \sum_{\ell\neq\hat{\ell}} \frac{|\mathcal{E}^t|}{u_\ell+2} + \frac{|\mathcal{E}^t|}{u_{\hat{\ell}}+2} - 1 = \sum_{\ell=1}^{d} \frac{|\mathcal{E}^t|}{u_\ell+2} - 1 = |\mathcal{E}^t| - 1,$$

where the first equality comes from the fact that $|\mathcal{E}^t|$ is a common multiple of $u_1+2,\ldots,u_d+2$, and the last equality holds since $\sum_{\ell=1}^d 1/(u_\ell+2) = 1$. Once again, this implies $|\bigcup_{\ell=1}^d Q_\ell^t| < |\mathcal{E}^t|$, hence by Proposition 1 we conclude that every extreme point of $\mathcal{K}(y^t,\mathcal{E}^t,Q^t)$ has at least one integral entry. This concludes the proof of the claim and the proof of the lemma. □

Proof of Lemma 4. We first observe that for every $t\in\{0,\ldots,T-1\}$ it holds that $\mathcal{E}^{t+1}\subseteq\mathcal{E}^t$, and therefore for every $\ell\in\{1,\ldots,d\}$ and every $v\in P_\ell$ we have that $\delta(v)\cap\mathcal{E}^{t+1}\subseteq\delta(v)\cap\mathcal{E}^t$. Since Algorithm 1 terminates, we conclude that for every $\ell\in\{1,\ldots,d\}$ and every $v\in P_\ell$ either $|\delta(v)\cap\mathcal{E}^0|\le u_\ell+1$, and in this case for every $t\ge t(v)=0$ it holds $|\delta(v)\cap\mathcal{E}^t|\le u_\ell+1$, or there exists $t(v)\in\{1,\ldots,T\}$ such that $|\delta(v)\cap\mathcal{E}^t|\ge u_\ell+2$ for every $t<t(v)$ and $|\delta(v)\cap\mathcal{E}^t|\le u_\ell+1$ for every $t\ge t(v)$. Consider now a vertex v in G. If $t(v)=0$, by definition we have that $y_e^0 = x_e$ for every $e\in\delta(v)\cap\mathcal{E}^0$ and $z_e=x_e$ for every $e\in\delta(v)\setminus\mathcal{E}^0$, so the identity follows directly. Otherwise, we proceed by induction. For the base case observe that by definition we have that $y_e^0=x_e$ for every $e\in\delta(v)\cap\mathcal{E}^0$ and $z_e=x_e$ for every $e\in\delta(v)\setminus\mathcal{E}^0$. Given a value $t\le t(v)-1$, since $\mathcal{E}^{t+1}\subseteq\mathcal{E}^t$ we have that $\delta(v)\cap\mathcal{E}^{t+1}=(\delta(v)\cap\mathcal{E}^t)\setminus(\delta(v)\cap(\mathcal{E}^t\setminus\mathcal{E}^{t+1}))$ and $\delta(v)\setminus\mathcal{E}^{t+1}=(\delta(v)\setminus\mathcal{E}^t)\cup(\delta(v)\cap(\mathcal{E}^t\setminus\mathcal{E}^{t+1}))$. Therefore, we have that

$$\sum_{e\in\delta(v)\cap\mathcal{E}^{t+1}} y_e^{t+1} + \sum_{e\in\delta(v)\setminus\mathcal{E}^{t+1}} z_e$$
$$= \sum_{e\in\delta(v)\cap\mathcal{E}^{t}} y_e^{t+1} - \sum_{e\in\delta(v)\cap(\mathcal{E}^t\setminus\mathcal{E}^{t+1})} y_e^{t+1} + \sum_{e\in\delta(v)\setminus\mathcal{E}^{t}} z_e + \sum_{e\in\delta(v)\cap(\mathcal{E}^t\setminus\mathcal{E}^{t+1})} z_e.$$

We have that by construction in Algorithm 1 it holds that $z_e = y_e^{t+1}$ for every $e\in\mathcal{E}^t\setminus\mathcal{E}^{t+1}$, and therefore

$$\sum_{e\in\delta(v)\cap\mathcal{E}^{t+1}} y_e^{t+1} + \sum_{e\in\delta(v)\setminus\mathcal{E}^{t+1}} z_e = \sum_{e\in\delta(v)\cap\mathcal{E}^{t}} y_e^{t+1} + \sum_{e\in\delta(v)\setminus\mathcal{E}^{t}} z_e$$
$$= \sum_{e\in\delta(v)\cap\mathcal{E}^{t}} y_e^{t} + \sum_{e\in\delta(v)\setminus\mathcal{E}^{t}} z_e = \sum_{e\in\delta(v)} x_e,$$

where the second equality holds since $y^{t+1}\in\mathcal{K}(y^t,\mathcal{E}^t,Q^t)$ and the third equality holds by the inductive hypothesis. This concludes the proof of the lemma. □

We finish by showing the alternative sufficient condition over integers $u_1,\ldots,u_d$ in Theorem 5 pointed at the end of Subsection 4.1. As mentioned, this condition can be replaced by $\sum_{\ell=1}^d \min\{\lfloor q/(u_\ell+2)\rfloor, |P_\ell|\} < q$ for every strictly positive integer q. In fact, given an integer t, if there exists $\ell\in\{1,\ldots,d\}$ for which $Q_\ell^t\neq\emptyset$, this inequality implies that $|\bigcup_{\ell=1}^d Q_\ell^t| \le \sum_{\ell=1}^d |Q_\ell^t| \le \sum_{\ell=1}^d \min\{\lfloor|\mathcal{E}^t|/(u_\ell+2)\rfloor, |P_\ell|\} < |\mathcal{E}^t|$, where the second inequality comes from the fact that for every $\ell\in\{1,\ldots,d\}$, $|Q_\ell^t|$ cannot be higher than $|P_\ell|$. Since this implies $|\mathcal{E}^t| > |\bigcup_{\ell=1}^d Q_\ell^t|$ for every t such that there exists $\ell\in\{1,\ldots,d\}$ for which $Q_\ell^t\neq\emptyset$, we conclude the same result of Lemma 3, namely that Algorithm 1 terminates after at most $|E|$ steps, and for every $t \in \{0, \dots, T-1\}$ we have that $\mathcal{E}^{t+1}$ is strictly contained in $\mathcal{E}^t$. Since Lemma 4 and the analysis on the proof of Theorem 5 under Lemmas 3 and 4 remain valid under this new condition, we conclude the result.

7.2 Proof of Theorem 4

We first provide one of the main ingredients to analyze Algorithm 2. The next lemma states that the result of any rounding, down or up, of the fractional entries of an optimal solution of the linear relaxation of (6)-(12), satisfies the proportionality condition (3).

Lemma 5. *Let $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ be an instance of the d-dimensional apportionment problem, let s be a signpost sequence and let $(x, y, \mathcal{U}, \Lambda, \beta)$ be an optimal primal-dual pair for the linear relaxation of (6)-(12). Suppose that $\bar{x}$ is an integral vector such that $\bar{x}_e \in \{\lfloor x_e \rfloor, \lceil x_e \rceil\}$ for every $e \in E(\mathcal{V})$. Then, for every $e \in E(\mathcal{V})$ we have that $s(\bar{x}_e) \le \mathcal{V}_e \cdot \exp(\mathcal{U}) \cdot \prod_{\ell=1}^d \exp(\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) \le s(\bar{x}_e + 1)$.*

Proof. For every $e \in E(\mathcal{V})$, let $t_e = \max\{t \in \{1, \dots, H\} : y_e^t > 0\}$ when $x_e > 0$ and $t_e = 0$ when $x_e = 0$. By Lemma 1 (ii) we have that for every $e \in E(\mathcal{V})$, $s(t_e) \le \mathcal{V}_e \cdot \exp(\mathcal{U}) \cdot \prod_{\ell=1}^d \exp(\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) \le s(t_e + 1)$, and by Lemma 1 (i) we have that $t_e = \lceil x_e \rceil$. Let $e \in E(\mathcal{V})$. If x_e is integral, then necessarily $\bar{x}_e = x_e = t_e$, so we are done in this case. Now suppose that x_e is fractional. In particular, we have $s(t_e) > 0$, since $x_e > 0$ and if $s(1) = 0$ the fractionality of x_e implies $x_e > 1$. If we apply condition (15) with $t = t_e$, we have that

$$\mathcal{U} + \sum_{\ell=1}^{d} (\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) + \beta_e^{t_e} - \log\left(\frac{s(t_e)}{\mathcal{V}_e}\right) = 0,$$

since $y_e^{t_e}$ is strictly positive. By exponentiating and rearranging terms we get that

$$\mathcal{V}_e \cdot \exp(\mathcal{U}) \cdot \prod_{\ell=1}^{d} \exp(\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) = s(t_e) \cdot \exp(-\beta_e^{t_e}).$$

Since $0 < y_e^{t_e} < 1$, by the complementary slackness condition (17) we have that $\beta_e^{t_e} = 0$, and therefore we conclude that $\mathcal{V}_e \cdot \exp(\mathcal{U}) \cdot \prod_{\ell=1}^d \exp(\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) = s(t_e) = s(\lceil x_e \rceil)$. When $\bar{x}_e = \lceil x_e \rceil$ we have that $s(\bar{x}_e) = s(\lceil x_e \rceil) = \mathcal{V}_e \cdot \exp(\mathcal{U}) \cdot \prod_{\ell=1}^d \exp(\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) \le s(\lceil x_e \rceil + 1) = s(\bar{x}_e + 1)$, where the inequality holds by the monotonicity of s. Similarly, when $\bar{x}_e = \lfloor x_e \rfloor$ we have that $s(\bar{x}_e + 1) = s(\lfloor x_e \rfloor + 1) = s(\lceil x_e \rceil) = \mathcal{V}_e \cdot \exp(\mathcal{U}) \cdot \prod_{\ell=1}^d \exp(\Lambda^-_{e_\ell} + \Lambda^+_{e_\ell}) \ge s(\lfloor x_e \rfloor) = s(\bar{x}_e)$, and again the inequality comes from the monotonicity of the signpost sequence. □

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Let $(x^\star, y^\star, \mathcal{U}, \Lambda^-, \Lambda^+, \beta)$ be an optimal primal-dual pair for the linear relaxation of (6)-(12) and define $\mu = \exp(\mathcal{U})$ and $\lambda_v = \exp(\Lambda^-_v + \Lambda^+_v)$ for every $\ell \in \{1, \dots, d\}$ and every $v \in N_\ell$. If the vector $x^\star$ is integral, by Theorem 1 it holds that $(x^\star, \mu, \lambda) \in \mathcal{A}_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$. That is, the triplet $(x^\star, \mu, \lambda)$ satisfies the marginal condition (1) and the proportionality condition (3), as well as conditions (4) and (5), and as $x^\star$ is integral we are done in this case.

Otherwise, suppose that $x^\star$ is fractional. Consider the d-partite hypergraph G with vertex partition $N_1, \dots, N_d$ and hyperedges $\alpha(E(\mathcal{V}))$ and let z be the output of Algorithm 1 over the graph G and the fractional vector $w \in [0,1]^{\alpha(E(\mathcal{V}))}$, defined as $w_{\alpha(e)} = x^\star_e - \lfloor x^\star_e \rfloor$ for every $e \in E(\mathcal{V})$. By Theorem 5, we have $z \in \{0,1\}^{\alpha(E(\mathcal{V}))}$ and $|\sum_{e \in \delta(v)} (z_e - w_e)| \le u_\ell$ for every $\ell \in \{1, \dots, d\}$ and every $v \in N_\ell$, where $\delta(v)$ is the set of hyperedges incident to v in hypergraph G. This is equivalent to $|\sum_{e\in E(\mathcal{V}):e_\ell=v}(z_{\alpha(e)}-x^\star_e+\lfloor x^\star_e\rfloor)|\le u_\ell$ for every $\ell\in\{1,\dots,d\}$ and every $v\in N_\ell$. For every $e\in E(\mathcal{V})$ let $X_e=z_{\alpha(e)}+\lfloor x^\star_e\rfloor$. We therefore have that for every $\ell\in\{1,\dots,d\}$ and every $v\in N_\ell$ it holds that

$$\sum_{e\in E(\mathcal{V}):e_\ell=v} X_e - m^-_v \ge \sum_{e\in E(\mathcal{V}):e_\ell=v}(z_{\alpha(e)}-x^\star_e+\lfloor x^\star_e\rfloor)\ge -u_\ell,$$

where the first inequality comes from the definition of X and since $x^\star$ satisfies the marginals condition (9). If $\lambda_v>1$, from condition (21) we must have $\Lambda^-_v>0$, so complementary slackness condition (19) implies $\sum_{e\in E(\mathcal{V}):e_\ell=v}x^\star_e=m^-_v$. Therefore, in this case we have $|\sum_{e\in E(\mathcal{V}):e_\ell=v}X_e-m^-_v|=|\sum_{e\in E(\mathcal{V}):e_\ell=v}(z_{\alpha(e)}-x^\star_e+\lfloor x^\star_e\rfloor)|\le u_\ell$. Similarly, for every $\ell\in\{1,\dots,d\}$ and every $v\in N_\ell$ we have that

$$\sum_{e\in E(\mathcal{V}):e_\ell=v} X_e - m^+_v \le \sum_{e\in E(\mathcal{V}):e_\ell=v}(z_{\alpha(e)}-x^\star_e+\lfloor x^\star_e\rfloor)\le u_\ell,$$

where the first inequality comes from the definition of X and since $x^\star$ satisfies the marginals condition (10). If $\lambda_v<1$, from condition (21) we must have $\Lambda^+_v<0$, so complementary slackness condition (20) implies $\sum_{e\in E(\mathcal{V}):e_\ell=v}x^\star_e=m^+_v$. Therefore, in this case we have $|\sum_{e\in E(\mathcal{V}):e_\ell=v}X_e-m^+_v|=|\sum_{e\in E(\mathcal{V}):e_\ell=v}(z_{\alpha(e)}-x^\star_e+\lfloor x^\star_e\rfloor)|\le u_\ell$. Finally, observe that $X_e\in\{\lfloor x^\star_e\rfloor,\lceil x^\star_e\rceil\}$ for every $e\in E(\mathcal{V})$ and therefore by Lemma 5 we have that $s(X_e)\le\mathcal{V}_e\cdot\mu\cdot\prod_{\ell=1}^d\lambda_{e_\ell}\le s(X_e+1)$ for every $e\in E(\mathcal{V})$. The complexity follows since the algorithm first solves a linear program with $O(H|E(\mathcal{V})|)$ variables and $O(H|E(\mathcal{V})|+\sum_{\ell=1}^d|N_\ell|)$ constraints, and Algorithm 1 runs in time polynomial in $O(|E(\mathcal{V})|+\sum_{\ell=1}^d|N_\ell|)$ from Theorem 5. □

7.3 Tightness of the Bounds for Deviation from the Marginals

For ease of writing, we first define a u-approximate d-dimensional proportional apportionment as an apportionment that satisfies the proportionality condition and deviates a maximum fixed amount from the marginals, as described in Theorem 4. Formally, let $(\mathcal{N},\mathcal{V},m^-,m^+,H)$ be an instance of the d-dimensional apportionment problem, let s be a signpost sequence, let $u\in\mathbb{N}^d$ and let $X\in\mathbb{N}^{E(\mathcal{V})}$. We say that X is a *u-approximate d-dimensional proportional apportionment* for this instance if the following conditions hold:

(i) $m^-_v-u_\ell\le\sum_{e\in E(\mathcal{V}):e_\ell=v}X_e\le m^+_v+u_\ell$ for every $\ell\in\{1,\dots,d\}$ and every $v\in N_\ell$.

(ii) There exists $\mu>0$ and a vector λ with strictly positive entries such that:

  (1) $s(X_e)\le\mathcal{V}_e\cdot\mu\cdot\prod_{\ell=1}^d\lambda_{e_\ell}\le s(X_e+1)$ for every $e\in E(\mathcal{V})$.

  (2) For every $\ell\in\{1,\dots,d\}$ and every $v\in N_\ell$, if $\lambda_v>1$ then $\left|\sum_{e\in E(\mathcal{V}):e_\ell=v}X_e-m^-_v\right|\le u_\ell$.

  (3) For every $\ell\in\{1,\dots,d\}$ and every $v\in N_\ell$, if $\lambda_v<1$ then $\left|\sum_{e\in E(\mathcal{V}):e_\ell=v}X_e-m^+_v\right|\le u_\ell$.

The following theorem formally establishes the notion of tightness mentioned in Subsection 4.2.

Theorem 6. *Let $d\in\mathbb{N}$ with $d\ge3$ and let $f_d:\mathbb{N}^d\to\mathbb{R}_+$ be defined as $f_d(u)=\sum_{\ell=1}^d\frac{1}{u_\ell+2}$. Let $h_d:\mathbb{N}^d\to\mathbb{R}_+$ be a continuous fuction such that $h_d<f_d$ and the limit $\lim_{u_d\to\infty}h_d(u)$ exists and is strictly less than $\lim_{u_d\to\infty}f_d(u)$. Then, there exists a vector $u\in\mathbb{N}^d$ satisfying $h_d(u)\le1$ for which the following holds: There exists an instance $(\mathcal{N},\mathcal{V},m^-,m^+,H)$ of the d-dimensional apportionment*

problem and a signpost sequence s for which the linear relaxation of (6)-(12) is feasible, such that there is no u-approximate d-dimensional proportional apportionment for this instance.

In order to prove this result, we first state two lemmas. The first one establishes the nonexistence of $(0, 0, c)$-approximate 3-dimensional proportional apportionments for certain instances and any $c \in \mathbb{N}$, while the second one allows extending this result to a higher dimension.

Lemma 6. *For every nonnegative integer c, there exists an instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ of the 3-dimensional apportionment problem, with $\mathcal{V}$ strictly positive in each entry, and a signpost sequence s for which the linear relaxation of (6)-(12) is feasible, such that there is no $(0, 0, c)$-approximate 3-dimensional proportional apportionment for this instance.*

Lemma 7. *Let $d \in \mathbb{N}$ with $d \geq 3$, let $u \in \mathbb{N}^d$ and suppose that there exists an instance of the d-dimensional apportionment problem and a signpost sequence s for which the linear relaxation of (6)-(12) is feasible, such that there is no u-approximate d-dimensional proportional apportionment for this instance. Define $u_{max} = \max_{\ell \in \{1,\dots,d\}} u_\ell$. Then, there exists an instance of the $(d+1)$-dimensional apportionment problem and a signpost sequence s for which the linear relaxation of (6)-(12) is feasible, such that there is no (u, u_{max})-approximate $(d+1)$-dimensional proportional apportionment for this instance.*

We now show how to prove Theorem 6 using these lemmas. In the rest of this appendix, we prove Lemma 6 and Lemma 7.

Proof of Theorem 6. We first claim that for every $d \geq 3$ and every $c \in \mathbb{N}$, there exists an instance of the d-dimensional apportionment problem and a signpost sequence s for which the linear relaxation of (6)-(12) is feasible, such that there is no $(0, 0, c, \dots, c)$-approximate d-dimensional proportional apportionment for this instance. This can be seen by induction over $d \geq 3$: the base case is immediate from Lemma 6 and, given that this is true for a certain $d \geq 3$, Lemma 7 allows to conclude the result for the $(d+1)$-dimensional case.

In order to prove the theorem, let $d \geq 3$ and a continuous function $h_d : \mathbb{N}^d \to \mathbb{R}_+$ satisfying $h_d < f_d$ and $\lim_{u_d \to \infty} h_d(u) < \lim_{u_d \to \infty} f_d(u)$. In particular, we have that for every $c \in \mathbb{N}$ it holds $h_d(0, 0, c, \dots, c) < f_d(0, 0, c, \dots, c) = 1 + (d-2)/(c+2)$, where the last expression tends to 1 as c tends to infinity. Hence, $\lim_{c \to \infty} h_d(0, 0, c, \dots, c) < 1$ and since h_d is continuous, there exists $\bar{c} \in \mathbb{N}$ such that $h_d(0, 0, \bar{c}, \dots, \bar{c}) \leq 1$. Since the result of the previous paragraph holds in particular for the d-dimensional case and taking $c = \bar{c}$, the theorem follows. □

Proof of Lemma 6. Let c be a nonnegative integer and consider the instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ of the 3-dimensional apportionment problem defined as follows: $\mathcal{N} = \{N_1, N_2, N_3\}$ with $N_1 = \{p_1, \dots, p_{2(c+1)}\}, N_2 = \{r_1, \dots, r_{2(c+1)}\}$ and $N_3 = \{g_1, g_2\}$. The vote matrix $\mathcal{V}$ is given by[14]

$$\mathcal{V}_{p_i r_j g_1} = \begin{cases} M & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}, \qquad \mathcal{V}_{p_i r_j g_2} = \begin{cases} M & \text{if } i = j+1 \\ 1 & \text{otherwise} \end{cases}$$

for every $i, j \in \{1, \dots, 2(c+1)\}$, where $M > 1$ and we denote $p_{2(c+1)+1} := p_1$, $r_{2(c+1)+1} := r_1$. Note that $E(\mathcal{V}) = N_1 \times N_2 \times N_3$ The marginals are $m^-_v = m^+_v = 1$ for every $v \in N_1 \cup N_2$ and $m^-_{g_1} = m^+_{g_1} = m^-_{g_2} = m^+_{g_2} = c + 1$, and the house size is $H = 2(c+1)$. We also consider any signpost sequence s with $s(1) > 0$.

[14]Note that we write $p_i r_j g_k$ instead of (p_i, r_j, g_k) for subindices to make the notation easier.

The linear relaxation of (6)-(12) is feasible for this instance. In fact, defining

$$E' = \{e \in E(\mathcal{V}) : \mathcal{V}_e = M\} = \bigcup_{i=1}^{2(c+1)} \{(p_i, r_i, g_1), (p_{i+1}, r_i, g_2)\},$$

the pair (x, y) given by $x_e = 1/2$ for every $e \in E'$, $x_e = 0$ for every $e \in E(\mathcal{V}) \setminus E'$, $y_e^1 = 1/2$ for every $e \in E'$, $y_e^1 = 0$ for every $e \in E(\mathcal{V}) \setminus E'$ and $y_e^t = 0$ for every $e \in E(\mathcal{V})$ and $t \in \{2, \dots, H\}$ verifies constraints (7)-(11) and $y_e^t \in [0,1]$ for every $e \in E(\mathcal{V})$ and every $t \in \{1, \dots, H\}$. However, we will show that there is no vector satisfying the definition of $(0,0,c)$-approximate 3-dimensional proportional apportionment for this instance.

Suppose that $X \in \mathbb{N}^{E(\mathcal{V})}$ is a $(0,0,c)$-approximate 3-dimensional proportional apportionment for this instance. This means that $\sum_{e \in E(\mathcal{V}): e_\ell = v} X_e = 1$ for every $\ell \in \{1,2\}$ and every $v \in N_\ell$, $|\sum_{e \in E(\mathcal{V}): e_3 = v} X_e - (c+1)| \le c$ for every $v \in N_3$ and there exist values $\mu > 0$, $\lambda_v > 0$ for every $v \in N_1 \cup N_2 \cup N_3$ such that $s(X_e) \le \mathcal{V}_e \cdot \mu \cdot \prod_{\ell=1}^{3} \lambda_{e_\ell} \le s(X_e + 1)$ for every $e \in E(\mathcal{V})$. In the following we use these conditions to obtain a contradiction and conclude that such vector X cannot exist, distinguishing the case $\lambda_{g_1} \neq \lambda_{g_2}$ and the case $\lambda_{g_1} = \lambda_{g_2}$.

If $\lambda_{g_1} \neq \lambda_{g_2}$, we can assume without loss of generality that $\lambda_{g_1} > \lambda_{g_2}$ (if $\lambda_{g_1} < \lambda_{g_2}$ the proof is analogous). Let $j_1 \in \arg\max_{j \in \{1,\dots,2(c+1)\}} \lambda_{r_j}$. Note that $\mathcal{V}_{p_{j_1} r_{j_1} g_1} = M \ge \mathcal{V}_{p_{j_1} r_j g_k}$ for every $j \in \{1, \dots, 2(c+1)\}$ and every $k \in \{1,2\}$, with strict inequality if $j \neq j_1$ and $k = 1$. Using that $\lambda_{g_1} > \lambda_{g_2}$, we conclude that

$$\mathcal{V}_{p_{j_1} r_{j_1} g_1} \cdot \mu \cdot \lambda_{p_{j_1}} \cdot \lambda_{r_{j_1}} \cdot \lambda_{g_1} > \mathcal{V}_{p_{j_1} r_j g_k} \cdot \mu \cdot \lambda_{p_{j_1}} \cdot \lambda_{r_j} \cdot \lambda_{g_k} \qquad \text{for every } (j,k) \in \{1, \dots, 2(c+1)\} \times \{1,2\} \text{ with } (j,k) \notin \{(j_1, 1)\}.$$

Therefore, from the proportionality condition and the monotonicity of s we obtain that $X_{p_{j_1} r_{j_1} g_1} \ge X_{p_{j_1} r_j g_k}$ for every $(j,k) \in \{1, \dots, 2(c+1)\} \times \{1,2\}$ with $(j,k) \notin \{(j_1,1)\}$, and since we know that $\sum_{j=1}^{2(c+1)} \sum_{k=1}^{2} X_{p_{j_1} r_j g_k} = 1$ from the marginals condition and X is integral, we conclude that $X_{p_{j_1} r_{j_1} g_1} = 1$. Since the marginals condition also guarantees $\sum_{i=1}^{2(c+1)} \sum_{k=1}^{2} X_{p_i r_{j_1} g_k} = 1$, we can delete p_{j_1} and r_{j_1} an iterate. Specifically, we now let $j_2 \in \arg\max_{j \in \{1,\dots,2(c+1)\} \setminus \{j_1\}} \lambda_{r_j}$ and obtain that

$$\mathcal{V}_{p_{j_2} r_{j_2} g_1} \cdot \mu \cdot \lambda_{p_{j_2}} \cdot \lambda_{r_{j_2}} \cdot \lambda_{g_1} > \mathcal{V}_{p_{j_2} r_j g_k} \cdot \mu \cdot \lambda_{p_{j_2}} \cdot \lambda_{r_j} \cdot \lambda_{g_k} \qquad \text{for every } (j,k) \in \{1, \dots, 2(c+1)\} \times \{1,2\} \text{ with } (j,k) \notin \{(j_1,1), (j_2,1)\},$$

so $X_{p_{j_2} r_{j_2} g_1} \ge X_{p_{j_2} r_j g_k}$ for every $(j,k) \in \{1, \dots, 2(c+1)\} \times \{1,2\}$ with $(j,k) \notin \{(j_1,1), (j_2,1)\}$. Since $\sum_{j=1}^{2(c+1)} \sum_{k=1}^{2} X_{p_{j_2} r_j g_k} = 1$, we conclude as before that $X_{p_{j_2} r_{j_2} g_1} = 1$. Repeating this process allows to conclude that X is given by

$$X_{p_i r_j g_k} = \begin{cases} 1 & \text{if } i = j \text{ and } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

and therefore $\sum_{i=1}^{2(c+1)} \sum_{j=1}^{2(c+1)} X_{p_i r_j g_1} = 2(c+1)$ and $\sum_{i=1}^{2(c+1)} \sum_{j=1}^{2(c+1)} X_{p_i r_j g_2} = 0$, so for each $v \in \{g_1, g_2\}$ we have $|\sum_{e \in E(\mathcal{V}): e_3 = v} X_e - (c+1)| = c + 1 > c$, violating one of the conditions we have imposed over X and therefore obtaining a contradiction.

We now consider the case $\lambda_{g_1} = \lambda_{g_2}$ and denote this term as λ_g. We first show that there exists $e \in E(\mathcal{V}) \setminus E'$ such that $X_e = 1$. To see this, note that if $X_e = 0$ for every $e \in E(\mathcal{V}) \setminus E'$, in order to satisfy the marginals condition without deviations for N_1 and N_2 we must have $x_{p_i r_i g_1} + x_{p_{i+1} r_i g_2} = 1$ and $x_{p_{i+1} r_{i+1} g_1} + x_{p_{i+1} r_i g_2} = 1$ for every $i \in \{1, \dots, 2(c+1)\}$, implying $x_{p_i r_i g_1} = x_{p_{i+1} r_{i+1} g_1}$ for every

$i \in \{1, \ldots, 2(c+1)\}$ and therefore either $x_{p_i r_i g_1} = 1$ for every $i \in \{1, \ldots, 2(c+1)\}$ or $x_{p_i r_i g_1} = 0$ for every $i \in \{1, \ldots, 2(c+1)\}$. In any case, this implies $|\sum_{e \in E(\mathcal{V}): e_3 = g_1} X_e - (c+1)| = c+1 > c$, violating once again a condition we have imposed over X. We conclude that there exists $\bar{e} \in E(\mathcal{V}) \setminus E'$ such that $X_{\bar{e}} = 1$, and without loss of generality we assume that $\bar{e} = (p_{i_0}, r_{i_1}, g_2)$ with $i_0, i_1 \in \{1, \ldots, 2(c+1)\}$ and $i_0 \neq i_1 + 1$. In particular, since $\mathcal{V}_{\bar{e}} = 1$ we have that

$$s(1) \leq \mu \cdot \lambda_{p_{i_0}} \cdot \lambda_{r_{i_1}} \cdot \lambda_g \leq s(2). \tag{28}$$

If $i_0 = i_1$, since $\sum_{j=1}^{2(c+1)} \sum_{k=1}^{2} X_{p_{i_0} r_j g_k} = 1$ from the marginals condition, we must have $X_{p_{i_0} r_{i_0} g_1} = 0$ and therefore $s(0) \leq \mathcal{V}_{p_{i_0} r_{i_0} g_1} \cdot \mu \cdot \lambda_{p_{i_0}} \cdot \lambda_{r_{i_0}} \cdot \lambda_g \leq s(1)$. Nevertheless, $\mathcal{V}_{p_{i_0} r_{i_0} g_1} = M$, so using (28) with $i_0 = i_1$ we obtain

$$s(1) \leq \mu \cdot \lambda_{p_{i_0}} \cdot \lambda_{r_{i_0}} \cdot \lambda_g < M \cdot \mu \cdot \lambda_{p_{i_0}} \cdot \lambda_{r_{i_0}} \cdot \lambda_g \leq s(1),$$

which is a contradiction.

If $i_0 \neq i_1$, since $\sum_{j=1}^{2(c+1)} \sum_{k=1}^{2} X_{p_{i_1} r_j g_k} = \sum_{i=1}^{2(c+1)} \sum_{k=1}^{2} X_{p_i r_{i_1} g_k} = 1$ there exists $i_2 \in \{1, \ldots, 2(c+1)\} \setminus \{i_1\}$ such that $X_{p_{i_1} r_{i_2} g_k} = 1$ for some $k \in \{0, 1\}$. Since $i_2 \neq i_1$ and $2(c+1)$ is a finite value, we can repeat this argument and conclude that there exists a sequence of pairs $(i_0, i_1), (i_1, i_2), \ldots, (i_{q-1}, i_q), (i_q, i_0)$, with $q \geq 1$, such that the following holds:

(i) $i_j \in \{1, \ldots, 2(c+1)\}$ for every $j \in \{0, \ldots, q\}$.

(ii) If $j, j' \in \{0, \ldots, q\}$ with $j \neq j'$, then $i_j \neq i_{j'}$.

(iii) For every $j \in \{0, \ldots, q\}$ there exists $k \in \{1, 2\}$ such that $X_{p_{i_j} r_{i_{j+1}} g_k} = 1$ (where we denote $r_{i_{q+1}} := r_{i_0}$).

From the proportionality condition we have $s(1) \leq \mathcal{V}_{p_{i_j} r_{i_{j+1}} g_k} \cdot \mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g \leq M \cdot \mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g$ for every $j \in \{1, \ldots, q\}$ and some $k \in \{1, 2\}$, and therefore

$$\prod_{j=1}^{q} (\mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g) \geq \left(\frac{s(1)}{M} \right)^q. \tag{29}$$

On the other hand, the marginals condition applied to $p_{i_0}, \ldots, p_{i_q}$ and $r_{i_0}, \ldots, r_{i_q}$ implies $X_{p_{i_j} r_{i_j} g_1} = 0$ for every $j \in \{0, \ldots, q\}$. Using the proportionality condition, this implies $s(1) \geq \mathcal{V}_{p_{i_j} r_{i_j} g_1} \cdot \mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_j}} \cdot \lambda_g = M \cdot \mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_j}} \cdot \lambda_g$ for every $j \in \{0, \ldots, q\}$, and therefore

$$\prod_{j=0}^{q} (\mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_j}} \cdot \lambda_g) \leq \left(\frac{s(1)}{M} \right)^{q+1}. \tag{30}$$

Putting (28), (29) and (30) together, we obtain

$$\prod_{j=0}^{q} (\mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_j}} \cdot \lambda_g) \leq \left(\frac{s(1)}{M} \right)^{q+1} < \frac{(s(1))^{q+1}}{M^q} \leq \prod_{j=0}^{q} (\mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g).$$

But the first and the last terms are equal, so this is a contradiction. □

Proof of Lemma 7. Let $d \in \mathbb{N}$ with $d \geq 3$, let $u \in \mathbb{N}^d$, let $(\{\bar{N}_1, \ldots, \bar{N}_d\}, \bar{\mathcal{V}}, \bar{m}^-, \bar{m}^+, \bar{H})$ be an instance of the d-dimensional apportionment problem and let $\bar{s}$ be a signpost sequence for which the linear relaxation of (6)-(12) is feasible (denote as $\bar{x}$ any feasible solution for it) and such that there is no u-approximate d-dimensional proportional apportionment for this instance. Let $\hat{\ell} \in \operatorname{argmax}_{\ell \in \{1,\ldots,d\}} u_\ell$ and denote $N_{\hat{\ell}} = \{v_1, \ldots, v_{|N_{\hat{\ell}}|}\}$. Consider the $(d+1)$-dimensional instance $(\mathcal{N}, \mathcal{V}, m^-, m^+, \bar{H})$ defined as follows. $\mathcal{N} = \{\bar{N}_1, \ldots, \bar{N}_d, \{w_1, \ldots, w_{|N_{\hat{\ell}}|}\}\}$, $\mathcal{V}_e = \bar{\mathcal{V}}_{e_1 \ldots e_d}$ if $e_{\hat{\ell}} = v_i$ and $e_{d+1} = w_i$ for some $i \in \{1, \ldots, |N_{\hat{\ell}}|\}$ and $\mathcal{V}_e = 0$ otherwise, $m^-_v = \bar{m}^-_v$ and $m^+_v = \bar{m}^+_v$ for every $v \in \{N_1, \ldots, N_d\}$, $m^-_{w_i} = \bar{m}^-_{v_i}$ and $m^+_{w_i} = \bar{m}^+_{v_i}$ for every $i \in \{1, \ldots, |N_{\hat{\ell}}|\}$. Note that

$$E(\mathcal{V}) = \left\{ e \in \prod_{\ell=1}^{d} N_\ell \times \{w_1, \ldots, w_{|N_{\hat{\ell}}|}\} \;\middle|\; (\bar{\mathcal{V}}_{e_1 \ldots e_d} > 0) \wedge (\exists\, i \in \{1, \ldots, |N_{\hat{\ell}}|\} : e_{\hat{\ell}} = v_i \wedge e_{d+1} = w_i) \right\},$$

so clearly the linear relaxation of (6)-(12) for this instance and the signpost sequence $\bar{s}$ is feasible, since the vector x defined as $x_e = \bar{x}_{e_1 \ldots e_d}$ if $e \in E(\mathcal{V})$ and $x_e = 0$ otherwise is a feasible solution for it. Furthermore, there cannot exist a $(u, u_{\hat{\ell}})$-approximate $(d+1)$-dimensional proportional apportionment for this instance and the signpost sequence $\bar{s}$. If it was the case, denoting as $X \in \mathbb{N}^{E(\mathcal{V})}$ such vector and μ, λ the multipliers for which the definition of $(u, u_{\hat{\ell}})$-approximate $(d+1)$-dimensional proportional apportionment is verified, we claim that $\bar{X} \in \mathbb{N}^{E(\bar{\mathcal{V}})}$ defined as $\bar{X}_e = X_{e_1 \ldots e_d w_i}$ for every $e \in E(\bar{\mathcal{V}})$, where $i \in \{1, \ldots, |N_{\hat{\ell}}|\}$ is such that $e_{\hat{\ell}} = v_i$, is a u-approximate d-dimensional proportional apportionment for the instance $(\{\bar{N}_1, \ldots, \bar{N}_d\}, \bar{\mathcal{V}}, \bar{m}^-, \bar{m}^+, \bar{H})$ with signpost sequence $\bar{s}$, which contradicts the hypothesis and allows to conclude. We prove the claim in what follows.

In order to check that $\bar{X}$ satisfies the conditions in the definition of a u-approximate d-dimensional proportional apportionment for the instance $(\{\bar{N}_1, \ldots, \bar{N}_d\}, \bar{\mathcal{V}}, \bar{m}^-, \bar{m}^+, \bar{H})$ with signpost sequence $\bar{s}$, we define the multipliers as $\bar{\mu} = \mu$, $\bar{\lambda}_v = \lambda_v$ for every $\ell \in \{1, \ldots, d\} \setminus \{\hat{\ell}\}$ and every $v \in N_\ell$, and $\bar{\lambda}_{v_i} = \lambda_{v_i} \cdot \lambda_{w_i}$ for every $i \in \{1, \ldots, |N_{\hat{\ell}}|\}$. To check the conditions, first consider $\ell \in \{1, \ldots, d\} \setminus \{\hat{\ell}\}$ and $v \in N_\ell$, and note that

$$\sum_{\substack{e \in E(\bar{\mathcal{V}}): \\ e_\ell = v}} \bar{X}_e = \sum_{i=1}^{|N_{\hat{\ell}}|} \sum_{\substack{e \in E(\mathcal{V}): e_\ell = v, \\ e_{\hat{\ell}} = v_i, e_{d+1} = w_i}} X_e = \sum_{i=1}^{|N_{\hat{\ell}}|} \sum_{\substack{e \in E(\mathcal{V}): e_\ell = v, \\ e_{\hat{\ell}} = v_i}} X_e = \sum_{\substack{e \in E(\mathcal{V}): \\ e_\ell = v}} X_e,$$

where the first equality comes from the definition of $\bar{X}$ and the second one from the fact that for every $e \in E(\mathcal{V})$ we have that there exists $i \in \{1, \ldots, |N_{\hat{\ell}}|\}$ for which $e_{\hat{\ell}} = v_i$ and $e_{d+1} = w_i$. Similarly, for any $i \in \{1, \ldots, |N_{\hat{\ell}}|\}$ the same analysis leads to

$$\sum_{\substack{e \in E(\bar{\mathcal{V}}): \\ e_{\hat{\ell}} = v_i}} \bar{X}_e = \sum_{\substack{e \in E(\mathcal{V}): e_{\hat{\ell}} = v_i, \\ e_{d+1} = w_i}} X_e = \sum_{\substack{e \in E(\mathcal{V}): \\ e_{\hat{\ell}} = v_i}} X_e = \sum_{\substack{e \in E(\mathcal{V}): \\ e_{d+1} = w_i}} X_e.$$

Since the marginals $\bar{m}^-_v$ and $\bar{m}^+_v$ are equal to m^-_v and m^+_v, respectively, for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_\ell$, condition (i) in the definition of a u-approximate d-dimensional proportional apportionment follows directly, as well as conditions (ii) (2) and (ii) (3) for every $\ell \in \{1, \ldots, d\} \setminus \{\hat{\ell}\}$ and every $v \in N_\ell$, because the corresponding multipliers $\bar{\lambda}_v$ are equal to λ_v. To see condition (ii) (2) for $v \in N_{\hat{\ell}}$, let $i \in \{1, \ldots, |N_{\hat{\ell}}|\}$ and note that if $\bar{\lambda}_{v_i} = \lambda_{v_i} \lambda_{w_i} > 1$, then necessarily $\lambda_{v_i} > 1$ or $\lambda_{w_i} > 1$. Since X is a $(u, u_{\hat{\ell}})$-approximate $(d+1)$-dimensional proportional apportionment, we must have either $|\sum_{e \in E(\mathcal{V}): e_{\hat{\ell}} = v_i} X_e - m^-_{v_i}| \leq u_{\hat{\ell}}$ or $|\sum_{e \in E(\mathcal{V}): e_{d+1} = w_i} X_e - m^-_{w_i}| \leq u_{d+1}$, but we know that $\sum_{e \in E(\bar{\mathcal{V}}): e_{\hat{\ell}} = v_i} \bar{X}_e = \sum_{e \in E(\mathcal{V}): e_{\hat{\ell}} = v_i} X_e = \sum_{e \in E(\mathcal{V}): e_{d+1} = w_i} X_e$, $\bar{m}^-_{v_i} = m^-_{v_i} = m^-_{w_i}$ and $u_{\hat{\ell}} = u_{d+1}$, so we conclude that $|\sum_{e \in E(\bar{\mathcal{V}}):e_{\hat{\ell}}=v_i} \bar{X}_e - \bar{m}_{v_i}^-| \le u_{\hat{\ell}}$. The proof of condition (ii) (3) for $v \in N_{\hat{\ell}}$ is completely analogous. Finally, to see condition (ii) (1), let $e \in E(\bar{\mathcal{V}})$, define $i \in \{1, \ldots, |N_{\hat{\ell}}|\}$ such that $e_{\hat{\ell}} = v_i$ and note that

$$\bar{\mathcal{V}}_e \cdot \bar{\mu} \cdot \prod_{\ell=1}^{d} \bar{\lambda}_{e_\ell} = \mathcal{V}_{e_1 \ldots e_d w_i} \cdot \mu \cdot \lambda_{w_i} \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell}.$$

From condition (ii) (1) applied to X with multipliers μ and λ, we know that the last term lies in the interval $[s(X_{e_1 \ldots e_d w_i}), s(X_{e_1 \ldots e_d w_i} + 1)] = [s(\bar{X}_e), s(\bar{X}_e + 1)]$, so we conclude that $\bar{X}$, with multipliers $\bar{\mu}$ and $\bar{\lambda}$, satisfies this condition as well. This concludes the proof of the claim and the proof of the lemma. □

8 Appendix: Results from the Chilean Constitutional Convention

In this appendix we present some tables with detailed results of the methods compared in Section 5. Table 1 shows the percentage of votes obtained by each of the lists that obtains at least one seat in one or both of the evaluated methods, as well as the number and percentage of seats obtained by each of these lists according to each system.

List	Votes (%)	Seats CCM	Seats CCM (%)	Seats TPM	Seats TPM (%)
XP	22.79	37	26.81	33	23.91
YQ	20.77	28	20.29	30	21.74
LP	18.28	27	19.57	27	19.57
YB	16.02	25	18.12	23	16.67
INN	8.73	11	7.97	12	8.7
XA	3.78	0	0	5	3.62
ZR	1.02	0	0	1	0.72
ZB	0.82	0	0	1	0.72
YK	0.81	1	0.72	1	0.72
YU	0.81	0	0	1	0.72
ZL	0.76	0	0	1	0.72
T	0.74	1	0.72	1	0.72
YX	0.73	0	0	1	0.72
ZW	0.70	1	0.72	1	0.72
IND9	0.68	1	0.72	0	0
XI	0.65	1	0.72	0	0
ZZ	0.61	1	0.72	0	0
WB	0.46	1	0.72	0	0
P	0.44	1	0.72	0	0
A	0.22	1	0.72	0	0
XM	0.20	1	0.72	0	0

Table 1: Elected candidates by list.

Tables 2 and 3 show the same information as the previous table (omitting the percentage of seats), but in this case by district. For each district, the corresponding table displays only the lists that obtained at least one seat in that district in either system.

District 1 (3 seats)			
List	**Votes (%)**	**CCM**	**TPM**
INN	18.51	0	1
XP	22.44	1	0
YB	27.55	1	1
YQ	25.66	1	1

District 2 (3 seats)			
List	**Votes (%)**	**CCM**	**TPM**
A	20.73	1	0
XP	28.61	1	1
YQ	34.84	1	2

District 3 (4 seats)			
List	**Votes (%)**	**CCM**	**TPM**
LP	20.43	1	1
XP	16.73	1	1
YQ	18.02	1	1
ZR	8.40	0	1
ZZ	24.76	1	0

District 4 (4 seats)			
List	**Votes (%)**	**CCM**	**TPM**
INN	19.34	1	1
LP	23.65	1	1
XA	8.19	0	1
YB	15.43	1	0
YQ	21.03	1	1

District 5 (6 seats)			
List	**Votes (%)**	**CCM**	**TPM**
LP	28.96	2	2
WB	11.76	1	0
XP	17.34	1	1
YB	18.18	1	1
YQ	18.56	1	2

District 6 (8 seats)			
List	**Votes (%)**	**CCM**	**TPM**
INN	10.40	1	1
LP	18.49	2	1
XP	18.38	1	2
YB	16.38	1	2
YK	14.27	1	1
YQ	20.65	2	1

District 7 (7 seats)			
List	**Votes (%)**	**CCM**	**TPM**
LP	22.81	2	2
XP	24.13	2	2
YB	15.53	1	1
YQ	26.90	2	2

District 8 (7 seats)			
List	**Votes (%)**	**CCM**	**TPM**
LP	20.41	2	2
XP	16.22	1	1
YB	10.47	1	0
YQ	35.56	3	3
YU	9.63	0	1

District 9 (6 seats)			
List	**Votes (%)**	**CCM**	**TPM**
IND9	11.75	1	0
INN	9.89	0	1
LP	24.94	2	2
XA	4.60	0	1
XP	12.43	1	0
YB	11.41	1	0
YQ	21.39	1	2

District 10 (7 seats)			
List	**Votes (%)**	**CCM**	**TPM**
INN	12.55	1	1
LP	11.14	1	0
XP	23.04	2	2
YB	14.32	1	1
YQ	24.42	2	2
ZL	9.79	0	1

District 11 (6 seats)			
List	**Votes (%)**	**CCM**	**TPM**
XP	51.16	4	4
YB	15.46	1	1
YQ	15.03	1	1

District 12 (6 seats)			
List	**Votes (%)**	**CCM**	**TPM**
INN	20.98	2	2
LP	19.46	1	1
T	11.00	1	1
XP	18.18	1	1
YQ	17.97	1	1

District 13 (4 seats)			
List	**Votes (%)**	**CCM**	**TPM**
LP	30.39	2	2
YB	25.31	1	1
YQ	18.86	1	1

District 14 (5 seats)			
List	**Votes (%)**	**CCM**	**TPM**
INN	12.20	1	1
LP	20.69	1	1
XP	17.06	1	0
YB	19.92	1	1
YQ	24.15	1	2

District 15 (5 seats)			
List	**Votes (%)**	**CCM**	**TPM**
LP	22.52	1	2
P	14.67	1	0
XP	18.97	1	1
YB	16.29	1	1
YQ	13.80	1	0
ZB	5.81	0	1

District 16 (4 seats)			
List	**Votes (%)**	**CCM**	**TPM**
LP	18.87	1	1
XP	29.99	1	1
YB	19.11	1	1
YQ	24.78	1	1

District 17 (7 seats)			
List	**Votes (%)**	**CCM**	**TPM**
LP	13.34	1	1
XI	15.50	1	0
XP	26.57	2	3
YB	14.49	1	1
YQ	21.77	2	2

District 18 (4 seats)			
List	**Votes (%)**	**CCM**	**TPM**
LP	32.18	2	2
XP	30.80	1	2
YB	21.30	1	0

Table 2: Votes and elected candidates by list in districts 1 to 18.

District 19 (5 seats)			
List	**Votes (%)**	**CCM**	**TPM**
INN	13.90	1	1
LP	22.22	1	2
XP	26.79	2	1
YB	15.06	1	1

District 20 (7 seats)			
List	**Votes (%)**	**CCM**	**TPM**
INN	12.95	1	1
LP	12.53	1	0
XA	8.64	0	2
XP	20.60	2	1
YB	15.42	1	1
YQ	16.06	1	1
ZW	12.78	1	1

District 21 (4 seats)			
List	**Votes (%)**	**CCM**	**TPM**
INN	19.42	1	1
XP	23.28	1	1
YB	19.41	1	0
YQ	19.58	1	1
YX	8.73	0	1

District 22 (3 seats)			
List	**Votes (%)**	**CCM**	**TPM**
XP	34.74	2	2
YB	25.71	1	1

District 23 (6 seats)			
List	**Votes (%)**	**CCM**	**TPM**
INN	13.19	1	1
LP	18.76	1	1
XA	6.45	0	1
XP	26.52	2	1
YB	15.18	1	1
YQ	12.70	1	1

District 24 (4 seats)			
List	**Votes (%)**	**CCM**	**TPM**
LP	14.73	0	1
XP	23.67	1	1
YB	29.78	2	2
YQ	14.97	1	0

District 25 (3 seats)			
List	**Votes (%)**	**CCM**	**TPM**
XP	36.67	2	1
YB	24.20	1	1
YQ	18.03	0	1

District 26 (4 seats)			
List	**Votes (%)**	**CCM**	**TPM**
INN	14.63	1	0
LP	21.55	1	1
XP	22.75	1	1
YB	24.50	1	2

District 27 (3 seats)			
List	**Votes (%)**	**CCM**	**TPM**
XP	29.51	1	1
YB	34.13	1	2
YQ	19.22	1	0

District 28 (3 seats)			
List	**Votes (%)**	**CCM**	**TPM**
LP	18.95	1	1
XM	20.25	1	0
XP	19.84	1	1
YQ	18.45	0	1

Table 3: Votes and elected candidates by list in districts 19 to 28.

Using these results, we calculate the standard deviation of each apportionment with respect to the fair share. These values by district and for the whole country are shown in the following table.

District	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
CCM	23.52	13.54	14.44	14.43	7.04	10.89	7.57	15.2	16.78	12.52
TPM	28.56	38.28	32.04	23.88	19.45	15.1	7.57	15.96	28.76	13.97

District	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
CCM	15.65	14.01	20.55	9.34	11.04	9.86	7.25	19.1	15.52	12.24
TPM	15.65	14.01	20.55	24.57	30.45	9.86	23.51	33.77	20.59	24.55

District	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**Country**
CCM	13.07	32.83	11.07	26.98	36.17	11.17	14.64	30	**6.44**
TPM	26.55	32.83	15.36	27.20	18.13	29.68	37.99	31.94	**2.49**

Table 4: Standard deviation of the list distribution with respect to the fair share.