On the Explanatory Value of Condition Numbers for Convex Optimization: Theoretical Issues and Computational Experience.

by

Fernando Ordóñez

B.S. Mathematical Engineering, Universidad de Chile, 1997

Submitted to the Sloan School of Management
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Operations Research

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2002

Author ....
Sloan School of Management
July 12, 2002

Certified by ....
Robert M. Freund
Theresa Seley Professor of Operations Research
Thesis Supervisor

Accepted by ....
James B. Orlin
Edward Pennell Brooks Professor of Operations Research,
Co-Director, Operations Research Center

On the Explanatory Value of Condition Numbers for Convex Optimization: Theoretical Issues and Computational Experience.

by

Fernando Ordóñez

Submitted to the Sloan School of Management
on July 12, 2002, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Operations Research

Abstract

The modern theory of condition numbers for convex optimization problems was developed for convex problems in conic format:

$$(CP_d): \qquad z_* := \min_x \{c^t x \mid Ax - b \in C_Y, x \in C_X\} \ .$$

The condition number $C(d)$ for (CP_d) has been shown in theory to provide upper and/or lower bounds on many behavioral and computational characteristics of (CP_d), from sizes of feasible and optimal solutions to the complexity of algorithms for solving (CP_d). However, it is not known to what extent these bounds might be reasonably close to their actual measures of interest. One difficulty in testing the practical relevance of such theoretical bounds is that most practical problems are not presented in conic format. While it is usually easy to transform convex optimization problems into conic format, such transformations are not unique and do not maintain the original data, making this strategy somewhat irrelevant for computational testing of the theory.

The purpose of this thesis is to overcome the obstacles stated above. We introduce an extension of condition number theory to include convex optimization problems not in conic form, and is thus more amenable to computational evaluation. This extension considers problems of the form:

$$(GP_d): \qquad z_* := \min_x \{c^t x \mid Ax - b \in C_Y, x \in P\} \ ,$$

where P is a closed convex set, no longer required to be a cone. We extend many results

of condition number theory to problems of form (GP_d), including bounds on optimal solution sizes, optimal objective function values, interior-point algorithm complexity, etc.

We also test the practical relevance of condition number bounds on quantities of interest for linear optimization problems. We use the NETLIB suite of linear optimization problems as a test-bed for condition number computation and analysis. Our computational results indicate that: (i) most of the NETLIB suite problems have infinite condition number (prior to pre-processing heuristics) (ii) there exists a positive linear relationship between the IPM iterations and $\log C(d)$ for the post-processed problem instances, which accounts for 42% of the variation in IPM iterations, (iii) condition numbers provide fairly tight upper bounds on the sizes of minimum-norm feasible solutions, and (iv) condition numbers provide fairly poor upper bounds on the sizes of optimal solutions and optimal objective function values.

Thesis Supervisor: Robert M. Freund
Title: Theresa Seley Professor of Operations Research

Acknowledgments

This adventure of pursuing graduate studies, which culminates with this work, has been a long but at the same time rewarding and inspiring experience. During the 5 years that lead to this moment a number of people have in many ways influenced me and my life here, and consequently this work. I want to now thank everyone that has helped me finish this thesis.

I want to express my gratitude to professor Robert Freund. His dedication and enthusiasm to research and teaching have been inspiring during my years at MIT to the point that his love for academia influenced my decision to pursue academic life. Rob has also given me valuable advise on life: from changing diapers to running the Bay to Breakers. I am thankful of having the opportunity of meeting such a good friend. I would also like to thank my committee members, Georgia Perakis and Andreas Schulz. Their comments and suggestions on the thesis proposal and final draft were a significant contribution to the current work.

I will always be indebted to my parents, brother and sisters. The family I grew in made me the person I am today, and even now, spread out over different continents, they are constantly present in my thoughts and provide a source of inspiration by their example.

Many friends have helped me during these last five years. Friends in Boston, back in Chile, spread throughout the world and especially at the OR Center. These friends, both near and far, gave me much needed distraction from work and are vital in keeping my sanity and love of life. I owe every one of you a beer. I am especially thankful for all who have in someway have suffered my recently discovered passion for running.

Finally, I want to thank and dedicate this thesis to my new family Andrea and Tatiana, for their love and support; but in particular for their patience with me and their embracing of the adventure that will be our new life.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Objectives and results

The modern theory of condition numbers for convex optimization problems was initially developed by Renegar in [28] for problems in the following conic format:

$$(CP_d) \qquad \begin{array}{rl} z_* := \min & c^t x \\ \text{s.t.} & Ax - b \in C_Y \\ & x \in C_X \ , \end{array} \tag{1.1}$$

where, for concreteness, we consider A to be an $m \times n$ real matrix, $b \in I\!R^m$, $c \in I\!R^n$, and $C_X \subseteq I\!R^n$, $C_Y \subseteq I\!R^m$ are closed convex cones, and the data of the problem is the array $d = (A, b, c)$. We assume that we are given norms $\|x\|$ and $\|y\|$ on $I\!R^n$ and $I\!R^m$, respectively, and let $\|A\|$ denote the usual operator norm; let $\|v\|_*$ denote the dual norm associated with the norm $\|w\|$ on $I\!R^n$ or $I\!R^m$. We define the norm of the data instance $d = (A, b, c)$ by $\|d\| := \max\{\|A\|, \|b\|, \|c\|_*\}$.

We denote by $C(d)$ the condition number of the problem with data d. Roughly speaking $C(d)$ is a scale-invariant reciprocal of the smallest data perturbation Δd which would render the perturbed data instance, $d + \Delta d$, either primal or dual infeasible. The theory of condition numbers for (CP_d) uses the quantity $C(d)$ to bound various behavioral and computational measures of (CP_d).

The condition number $C(d)$ has been shown in theory to be connected to a wide variety of behavioral characteristics of (CP_d) and its dual, including bounds on sizes of feasible solutions, bounds on sizes of optimal solutions, bounds on optimal objective values, bounds on the sizes and aspect ratios of inscribed balls in the feasible region, bounds on the rate of deformation of the feasible region under perturbation, bounds on changes in optimal objective values under perturbation, and numerical bounds related to the linear algebra computations of certain algorithms, see [28], [9], [8], [13], [11], [14], [35], [33], [36], [34], [24], [26]. In the context of interior-point methods for linear and semidefinite optimization, this condition number has also been shown to be connected to various quantities of interest regarding the central trajectory, see [22] and [23]. The connection of these condition numbers to the complexity of algorithms has been developed in [13], [11], [29], [3], and [6], and some of the references contained therein.

Given the theoretical importance of these results, it is natural to ask whether these theoretical results are meaningful for problem instances that one encounters in practice? What are typical values of condition numbers that arise in practice?, and are such problems typically well- or ill-conditioned? These questions motivate the need for computation and analysis of the condition number theory for problems that arise in practice.

Problem (CP_d) covers a very general class of convex problems; in fact any convex optimization problem can be transformed to an equivalent instance of (CP_d). However, such transformations are not necessarily unique and are sometimes rather unnatural given the "natural" data for the problem. This ambiguity makes using the conic format impractical for computation. It is for this reason that we consider the following more general format for convex optimization:

$$
(GP_d) \qquad \begin{array}{rll} z_*(d) = & \min & c^t x \\ & \text{s.t.} & Ax - b \in C_Y \\ & & x \in P, \end{array} \tag{1.2}
$$

where now P, which we call the "ground-set," is allowed to be any closed convex set, possibly unbounded, and possibly lacking an interior. For example, P could be the solution set of box constraints of the form $l \leq x \leq u$ where some components of l and/or u might be unbounded, or P might be the solution of network flow constraints of the form $Nx = g, x \geq 0$. And of course, P might also be a closed convex cone. We refer to problem (GP_d) as the ground-set model (GSM) format.

Inspired by the construction and theory pertaining to condition numbers for conic convex problems, we extend the concepts of condition numbers to the ground-set model format (GP_d) herein, and we extend many condition number results to analogous results for problems in the GSM format. In this thesis we define the condition number for the ground-set model format (GP_d), and we characterize the condition number using the optimal values of certain associated optimization problems. We then use this characterization to bound (i) the size of least-norm feasible solutions of (GP_d), (ii) relative error and optimal solution of (GP_d), (iii) changes in optimal values under perturbation, and (iv) the sizes and distances of solutions from the relative boundary of the feasible region. Finally we use these theoretical results to bound the complexity of an interior-point-method (IPM) algorithm for solving (GP_d).

The GSM format allows us to define the condition number for a convex optimization problem without requiring a transformation to conic form. Using the GSM format, we compute the condition number for linear programs that arise in practice. We analyze the condition numbers for the NETLIB suite of industrial and academic LP problems. We present computational results that indicate that 72% of the NETLIB suite of linear optimization problem instances are ill-conditioned. However, after routine pre-processing by CPLEX 7.1, we find that only 19% of post-processed problem instances in the NETLIB suite are ill-conditioned, and that $\log C(d)$ of the finitely-conditioned post-processed problems is fairly nicely distributed.

Using the condition numbers computed for the NETLIB suite we investigate whether condition number theoretical bounds are reasonably close to the actual measure of interest for linear programming problems that one might encounter in practice. The computational experiments in this thesis concentrate on the performance of state-of-the-art IPM algorithms and the bounds on the minimum-norm feasible solution, optimal solution size, and optimal objective function value.

In the case of modern IPM algorithms for linear optimization, the number of IPM iterations needed to solve a linear optimization instance has been observed to vary from 10 to 100 iterations, over a huge range of problem sizes, see [19], for example. Using the condition-number model for complexity analysis, one can bound the IPM iterations by $O\left(\sqrt{n}\log(C(d)+\cdots)\right)$ for linear optimization in standard form, where the other terms in the bound are of a more technical nature, see [29] for details. (Of course, the IPM algorithms that are used in practice are different from the IPM algorithms that are used in the development of the complexity theory.)

A natural question to ask then is whether the observed variation in the number of IPM iterations (albeit already small) can be accounted for by the condition numbers of the problem instances? In this work we show that the number of IPM iterations needed to solve the problems in the NETLIB suite varies roughly linearly (and monotonically) with $\log C(d)$ of the post-processed problem instances. A simple linear regression model of IPM iterations as the dependent variable and $\log C(d)$ as the independent variable yields a positive linear relationship between IPM iterations and $\log C(d)$ for the post-processed problem instances, significant at the 95% confidence level, with $R^2 = 0.4258$.

Therefore, over 42% of the variation in IPM iterations among the NETLIB suite problems is accounted for by $\log C(d)$ of the problem instances after pre-processing.

Classic condition number theory bounds the norm of the optimal solution, the optimal objective function value and the norm of the minimum feasible solution by condition number quantities. These results, as mentioned above, were extended for the GSM format and the bounds compared to the actual values obtained for the problems in the NETLIB suite.

We find that the condition number bound for the minimum-norm primal feasible solution is small, on average $10^{2.98}$ times the actual value of the minimum-norm primal feasible solution size. The condition number bounds for the optimal solution size and optimal objective function value are larger. On average the corresponding condition number bound is $10^{7.49}$ times the primal optimal solution size, $10^{10.18}$ times the dual optimal solution size, and $10^{6.25}$ times the optimal objective function value. Each average above are corresponds to the geometric mean of the ratio defined by the condition number bound over the actual value.

1.2 Structure of thesis

The structure of this thesis is as follows: In Chapter 2, we present the basis of condition number theory, and we provide the definitions and notation that we use in this work. We also review the theoretical research on condition number theory and show an example that illustrates the need to develop an extension of the condition number theory to include problems not in conic form.

In Chapter 3 we present the ground-set model framework, conditions for strong duality, the characterization of the condition number, and a result providing sufficient conditions for strong duality in terms of the condition number. In Chapter 4 we present a number of geometric bounds for the primal and dual feasible regions using condition numbers. These results prove the existence of reliable solutions in the feasible region, which are bounded by condition-number-related quantities.

We show in Chapter 5 that, for a problem in the GSM format, the condition number bounds the size of feasible and optimal solutions, the size of the objective function, and the size of change in solutions due to perturbations in the data. In Chapter 6 we present a complexity result for problems in the GSM format in terms of the condition number. We describe a standard IPM which solves problem (GP_d) in a number of Newton iterations that is bounded by the logarithm of the condition number of the problem.

The computational results are presented in Chapter 7. We use the NETLIB suite of linear programs for these computational experiments. The results describe the distribution of the condition number for the problems in the NETLIB suite, study the relation of the condition number to the number of iterations a state-of-the-art IPM solver takes to solve the instance, and studies the relation of the condition number to the size of feasible and optimal solutions.

The last chapter in the thesis presents extensions of the current work. We mention the applicability of this procedure to compute condition number for semi-definite programming (SDP), what has been done to relate condition numbers to the homogeneous self-dual model, some limitations of the GSM format, and other future directions of research.

Chapter 2

Research Review

In this chapter we review elements of the foundation of condition number theory. We describe these ideas, first in the context of linear systems of equations (LSE), and then in the context of convex optimization (CO). We also state theoretical results that are relevant to our work in each setting.

We present characterizations of the distance to ill-posedness and condition numbers for CO problems. These characterizations provide a basis for computing these quantities. We provide examples that illustrate how these characterizations stumble into difficulties when trying to compute the distance to ill-posedness and condition numbers for CO problems that arise in practice. In this chapter we also mention previous work on computing condition numbers.

2.1 Condition Numbers for LSE

Research has been conducted on condition numbers in many different mathematical contexts. For example, this notion is present in research involving the solution of LSE, computing eigenvalues of a matrix, and solving linear least squares problems. Lately, condition numbers have been used to study CO problems and the feasibility of conic linear systems.

In all settings, the condition number of the problem is defined using the same three concepts. The condition number depends on the definitions of what is a data instance, what is an ill-posed data instance, and what is the distance to ill-posedness of a data instance.

To illustrate these definitions, we present the condition number in the setting of a LSE.

2.1.1 The LSE

Consider the following finite dimensional LSE. For a given $A \in I\!R^{n \times n}$ and $b \in I\!R^n$, the problem is to find a vector $x \in I\!R^n$ such that

$$Ax = b \ .$$

In this problem, the data is defined as (A, b). The dimension of the problem, n, is fixed. Each pair (A, b) defines a different data instance of the same problem. We refer to (A, b) as a data instance, or just an instance, of the problem.

The set $\mathcal{D}$ is defined as the set containing all possible data instances of the problem. In our LSE example, $\mathcal{D}$ is the set of all possible n by n matrices and n dimensional right hand side vectors:

$$\mathcal{D} = \left\{ (A, b) \mid A \in I\!R^{n \times n}, b \in I\!R^n \right\} \ .$$

2.1.2 Ill-posed instances

The subset of instances, $\mathcal{F} \subset \mathcal{D}$, is defined as the set of consistent instances. Out of all possible instances, we group into $\mathcal{F}$ the instances that are solvable. For the LSE example we have

$$\mathcal{F} = \left\{(A, b) \in I\!R^{n\times n} \times I\!R^{n} \mid b \in \text{range}(A)\right\} .$$

In LSE, we refer to instances in the interior of $\mathcal{F}$ as well-posed instances. These are instances (A, b) for which all instances in a neighborhood around (A, b) also have a solution. We can characterize the interior of $\mathcal{F}$ for LSE as

$$\text{int}\mathcal{F} = \left\{(A, b) \in I\!R^{n\times n} \times I\!R^{n} \mid \det(A) \neq 0\right\} .$$

The set of ill-posed instances, which we denote by $\mathcal{I}$, is the boundary of $\mathcal{F}$, that is instances (A, b) which have infeasible instances arbitrarily close. In this case $\mathcal{F}^C \subset (\text{int}\mathcal{F})^C = \mathcal{I} = \partial\mathcal{F}$, and we have

$$\mathcal{I} = \left\{(A, b) \in I\!R^{n\times n} \times I\!R^{n} \mid \det(A) = 0\right\}.$$

2.1.3 Distance to ill-posedness

The set of all data instances, $\mathcal{D}$, has to be contained in a normed vector space to be able to define a distance to ill-posedness. Let $\| \quad \|$ denote the norm on the vector space containing $\mathcal{D}$. For any instance $d \in \mathcal{D}$ the distance to ill-posedness, $\rho(d)$, is defined as

$$\rho(d) = \text{dist}\,(d, \mathcal{I}) = \inf\left\{\|\Delta d\| \mid d + \Delta d \in \mathcal{I}\right\}.$$

The quantity $\rho(d)$ is the smallest data perturbation, Δd, of instance d that makes the instance $d + \Delta d$ ill-posed.

In the LSE case, the norm of the data is defined in terms of the norms used in $I\!R^n$ and the operator norm in $I\!R^{n \times n}$. The usual definition is $\|d\| = \max\{\|A\|, \|b\|\}$. Representing a change in the data by $\Delta d = (\Delta A, \Delta b)$, we can express the distance to ill-posedness as

$$\begin{aligned} \rho(d) &= \inf\{\|(\Delta d)\| \mid \det(A + \Delta A) = 0\} \\ &= \inf\{\max\{\|\Delta A\|, \|\Delta b\|\} \mid \det(A + \Delta A) = 0\} \\ &= \inf\{\|\Delta A\| \mid \det(A + \Delta A) = 0\}. \end{aligned}$$

Note that the distance to ill-posedness, $\rho(d)$, does not depend on the right hand side, b, for LSE. In this case, the distance to ill-posedness equals the distance of the matrix A to the set of singular matrices. In other words,

$$\rho(d) = \rho(A),$$

where $\rho(A)$ is the distance to the set of singular matrices for a square matrix A.

Consider A non-singular and let the norm on A be

$$\begin{aligned} \|A\| &= \max\left\{\sqrt{\lambda_i} \mid \lambda_i \text{ is an eigenvector of } A^t A\right\} \\ &= \max\{\sigma_i \mid \sigma_i \text{ is a singular value of } A\}. \end{aligned}$$

In this case, the following characterization of $\rho(A)$ is due to Eckart and Young [4]:

$$\rho(A) = \frac{1}{\|A^{-1}\|}.$$

This result is valid for any operator norm on the matrix A. The proof for an arbitrary operator norm is due to Gastinel [15].

2.1.4 Condition Number

The idea of the condition number of a square matrix is present in the classic LSE literature. This condition number is defined as $\kappa(A) = \|A\| \, \|A^{-1}\|$ for non-singular matrices, and $\kappa(A) = \infty$ if the matrix A is singular.

Using the previous characterization of $\rho(A)$, we can rewrite this condition number as

$$\kappa(A) = \|A\| \left\|A^{-1}\right\| = \frac{\|A\|}{\rho(A)} = \frac{\|A\|}{\rho(d)}$$

for a non-singular matrix, and $\kappa(A) = \infty$ for a singular matrix. Therefore, the condition number of a matrix A is the ratio of the size of A to the distance from A to the set of singular matrices.

The definition of a condition number for a general problem follows this idea. For a problem instance d, the condition number is the ratio of the norm of d to the distance from d to the set of ill-posed instances. In other words, the condition number of the data instance d is defined as

$$C(d) = \frac{\|d\|}{\rho(d)}$$

for instances where $\rho(d) > 0$ and $C(d) = \infty$ for ill-posed instances. The condition number is a scaled reciprocal of the distance to ill-posedness.

The quantities $\kappa(A)$ and $C(d)$ are very similar. In fact, in the LSE case, we have

$$\kappa(A) \max\left\{1, \frac{\|b\|}{\|A\|}\right\} = C(d).$$

If $\|b\| \leq \|A\|$, these quantities are the same. If $\|b\| > \|A\|$, we can scale b to obtain an equivalent problem such that $\|b\| = \|A\|$. This idea, and the fact that b is not involved in determining whether the data d is consistent or not, suggest that b is not part of the data for a LSE. If we define the data of a LSE problem as $d = A$, the classic condition number of a matrix is in fact the condition number of the problem.

2.1.5 Results in Condition Number for LSE

The following results illustrate that $\kappa(A)$ is known to bound the solution to the LSE and the size of changes in the solution due to changes in the data.

Proposition 1 *For any instance $d = (A, b)$ of LSE, let x be the solution to the system $Ax = b$. Then*

1. $1 \leq \|A\| \, \|A^{-1}\| = \kappa(A)$.

2. $\|x\| \leq \frac{\|b\|}{\rho(A)} = \kappa(A)\frac{\|b\|}{\|A\|}$.

3. *If x' is the solution of the perturbed system $Ax = b'$, then*

$$\frac{\|x - x'\|}{\|x\|} \leq \kappa(A)\frac{\|b - b'\|}{\|b\|}.$$

4. *If x' is the solution of the perturbed system $A'x = b$, then*

$$\frac{\|x - x'\|}{\|x'\|} \leq \kappa(A)\frac{\|A - A'\|}{\|A\|}.$$

These are classic results in the numerical linear algebra literature. Proofs of these results can be found, for example, in [17], Chapter 7. We can re-express the previous results using $C(d)$ since $\kappa(A) \leq C(d)$.

Some observations on these results follow:

- The condition number satisfies $C(d) \geq 1$. As the problem instance approaches the set of ill-posed instances, the condition number diverges to $+\infty$.

- The norm of any solution to LSE is bounded by the condition number.

- The last two results show that the relative change in the solution is bounded by the condition number times the relative change in the data.

- The relative changes in the solution in items 3 and 4 are defined differently. In item 3 it is the relative change with respect to the original solution, while in item 4 it is with respect to the perturbed solution.

- A similar result exists for a simultaneous change in both A and b.

- All of these results are valid for ill-posed instances, where $\kappa(A) = C(d) = \infty$.

2.2 Condition Numbers for CO

The first analysis of the distance to ill-posedness for convex optimization was done by Renegar in [28]. In this work, he obtained bounds for the norms of solutions and for sensitivity to data perturbation in terms of the distance to ill-posedness, for convex optimization problems in conic form. For simplicity, we present the definition of condition number for finite dimensional CO problems in conic form. We also restrict our exposition to the case of consistent data instances.

2.2.1 Conic convex optimization

The problem is stated as:

$$(CP_d) \qquad \begin{aligned} z_* = \quad \min \quad & c^t x \\ \text{s.t.} \quad & b - Ax \in C_Y, \\ & x \in C_X, \end{aligned} \tag{2.1}$$

where $C_X \subseteq \mathbb{R}^n$ and $C_Y \subseteq \mathbb{R}^m$ are closed convex cones. These cones are considered fixed. The data of the problem is the tuple formed by the matrix $A \in \mathbb{R}^{m \times n}$ and the vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. We refer to a data instance as $d = (A, b, c)$.

We refer to problem (2.1) as (CP_d) for conic primal. The subscript d is used to emphasize the dependency of this problem on the data.

Problem (CP_d) is a very general model, which includes as special cases linear programming (LP) and semidefinite programming (SDP), among others. The choice of cones C_X and C_Y determines the specific type of problem (CP_d) models. For example:

1. An LP problem in standard form can be obtained by setting $C_Y = \{0\}$ and $C_X = \mathbb{R}^n_+$.
2. An SDP problem is obtained by setting $C_Y = \{0\}$ and $C_X = S^n_+$, which is the set of symmetric positive semi-definite $n \times n$ matrices.

The norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are defined on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Based on these norms we define the norm of the data as $\|d\| = \max\{\|A\|, \|b\|_Y, \|c\|_{X^*}\}$, where $\|A\|$ is the appropriate operator norm:

$$\begin{aligned} \|A\| = \; & \max \|Ax\|_Y \\ & \text{s.t. } \; \|x\|_X \leq 1 \; , \end{aligned}$$

and $\|\cdot\|_{X^*}$ is the corresponding norm on the space of linear functionals:

$$\begin{aligned} \|c\|_{X^*} = \; & \max c^t x \\ & \text{s.t. } \; \|x\|_X \leq 1 \; . \end{aligned}$$

For the remainder of the work, we drop the subscripts X and Y on the norms. The norm we refer to will be clear from the context. We still differentiate the original norms from the dual norms using the subscript *.

The primal feasible region, for the data instance d, is denoted by

$$X_d = \{x \in I\!R^n \mid b - Ax \in C_Y, x \in C_X\} .$$

The problem in conic form has a corresponding dual problem:

$$(CD_d) \qquad \begin{array}{rll} z^* = & \max & -b^t y \\ & \text{s.t.} & A^t y + c \in C_X^* \\ & & y \in C_Y^* \ . \end{array} \tag{2.2}$$

Here, C_X^* and C_Y^* denote the (positive) dual cones of C_X an C_Y respectively. The dual cone C^*, for a convex cone $C \in I\!R^n$, is defined by

$$C^* = \left\{y \in I\!R^n \mid x^t y \geq 0 \text{ for all } x \in C\right\} .$$

The dual feasible region, for the data instance d, is denoted by

$$Y_d = \left\{y \in I\!R^m \mid A^t y + c \in C_X^*, y \in C_Y^*\right\} .$$

Consider now the example with cones $C_X = \{0\}$ and $C_Y = I\!R^m$. With this cone selection we obtain the following primal and dual pair of problems:

$$\begin{array}{rllllll} z_* = & \min & c^t x & = \min & c^t 0 & & = 0 \ , \\ & \text{s.t.} & b - Ax \in I\!R^m & \text{s.t.} & b - A0 \in I\!R^m & & \\ & & x \in \{0\} & & & & \end{array}$$

and

$$z^* = \begin{array}{ll} \max & -b^t y \\ \text{s.t.} & A^t y + c \in I\!R^n \\ & y \in \{0\} \end{array} = \begin{array}{ll} \max & -b^t 0 \\ \text{s.t.} & A^t 0 + c \in I\!R^n \end{array} = 0 \ .$$

Since $b \in I\!R^m$ and $c \in I\!R^n$ by the definition of the problem, these problems are feasible for all data $d \in \mathcal{D}$. In this trivial case $\mathcal{D} = \mathcal{F}$ and $\rho(d) = \infty$ for every instance, we will exclude in the remainder of this work this trivial case.

2.2.2 Ill-posed instances and distance to ill-posedness

For conic CO problems, the set of consistent instances, $\mathcal{F}$, is defined by

$$\begin{aligned} \mathcal{F} &= \{d \in \mathcal{D} \mid (CP_d) \text{ and } (CD_d) \text{ are feasible }\} \\ &= \{d \in \mathcal{D} \mid X_d \neq \emptyset \text{ and } Y_d \neq \emptyset\} \ . \end{aligned}$$

In this setting, the ill-posed instances are, again, the boundary of the consistent instances, that is, $\mathcal{I} = \partial\mathcal{F}$. For a consistent instance, $d \in \mathcal{F}$, the distance to ill-posedness, $\rho(d)$, is its distance to the boundary of $\mathcal{F}$. In other words,

$$\rho(d) = \inf\left\{\|\Delta d\| \mid X_{d+\Delta d} = \emptyset \text{ or } Y_{d+\Delta d} = \emptyset\right\} \ .$$

If we use the definitions

$$\mathcal{I}_1 = \{d \in \mathcal{D} \mid X_d = \emptyset\}$$

and

$$\mathcal{I}_2 = \{d \in \mathcal{D} \mid Y_d = \emptyset\} \ ,$$

then $\rho(d)$ becomes

$$\begin{aligned}
\rho(d) &= \inf\left\{\|\Delta d\| \mid X_{d+\Delta d} = \emptyset \text{ or } Y_{d+\Delta d} = \emptyset\right\} \\
&= \inf\left\{\|\Delta d\| \mid d + \Delta d \in \mathcal{I}_1 \cup \mathcal{I}_2\right\} \\
&= \min\left\{ \ \inf\left\{\|\Delta d\| \mid d + \Delta d \in \mathcal{I}_1\right\}, \inf\left\{\|\Delta d\| \mid d + \Delta d \in \mathcal{I}_2\right\} \ \right\} \\
&= \min\left\{ \ \inf\left\{\|\Delta d\| \mid X_{d+\Delta d} = \emptyset\right\}, \inf\left\{\|\Delta d\| \mid Y_{d+\Delta d} = \emptyset\right\} \ \right\}.
\end{aligned}$$

This means that $\rho(d)$ is the size of the smallest data perturbation Δd that makes the primal problem $(CP_{d+\Delta d})$ infeasible or the dual problem $(CD_{d+\Delta d})$ infeasible. Therefore the characterization of $\rho(d)$ for conic CO depends on what is defined as the primal and dual distances to infeasibility. These distances are defined by

$$\rho_P(d) = \inf\left\{\|\Delta d\| \mid X_{d+\Delta d} = \emptyset\right\}$$

and

$$\rho_D(d) = \inf\left\{\|\Delta d\| \mid Y_{d+\Delta d} = \emptyset\right\}.$$

Then, for a consistent instance, $d \in \mathcal{F}$, the distance to ill-posedness is

$$\rho(d) = \min\left\{\rho_P(d), \rho_D(d)\right\}.$$

2.2.3 Condition Number

The condition number, $C(d)$, is defined as the following scale-invariant (by a positive scalar) reciprocal of the distance to ill-posedness:

$$C(d) = \frac{\|d\|}{\rho(d)}$$

for $\rho(d) > 0$ and $C(d) = \infty$ if $\rho(d) = 0$. This definition implies that, as the problem instance is closer to being ill-posed, the condition number approaches infinity.

It is easy to show that when $C_X \neq \{0\}$ or $C_Y \neq I\!R^m$, the instance $(A, b, c) = (0, 0, 0)$ is ill-posed. This fact is used to bound $\rho(d)$ for any instance d by

$$\rho(d) = \inf \{ \|\Delta d\| \mid d + \Delta d \in \mathcal{I} \} \leq \|d\| \, .$$

The above implies that $C(d) \geq 1$. As in the LSE case, the condition number lies between 1 and $+\infty$, being larger for data instances that are closer to being ill-posed.

2.3 Theory of $C(d)$ in conic CO

In this section, we review recent research on $C(d)$ for a conic convex optimization problem. We state the results for the case of consistent instances and finite dimensional CO problems.

These results are categorized into three areas:

1. Results where $C(d)$ provides information on geometric properties of (CP_d).

2. Results that bound the complexity of algorithms for solving (CP_d) in terms of $C(d)$.

3. Results that characterize the distance to ill-posedness as the solution to an optimization problem. These results are the impetus for computing $C(d)$.

2.3.1 Geometric properties

A simple argument that illustrates that the condition number is related to the geometry of the problem is the following:

We know that an ill-posed instance has an infeasible primal problem, dual problem, or both. Therefore, an instance with a large condition number will have either a primal feasible region, or a dual feasible region about to disappear. At least one of these feasible regions is "thin" or "small". On the other hand, a condition number close to one means we have primal and dual feasible regions that do not disappear easily. We can think of these regions as "fat" or "large". This intuitive relationship between geometry and condition number is formalized in various theoretical results.

In [28], Renegar proves that the condition number is related to the size of solutions, the size of optimal solutions, and the sensitivity of solutions to perturbations in the data. This result, which is Theorem 1.1 in [28], is

Theorem 1 *[28] Suppose that $d \in \mathcal{F}$. If d satisfies $\rho_P(d) > 0$, then the following are true:*

1. *There exists $x \in X_d$ such that*

$$\|x\| \leq \frac{\|b\|}{\rho_P(d)}.$$

2. *If $x' \in X_{d+\Delta d}$ where $\Delta d = (0, \Delta b, 0)$ then there exists $x \in X_d$ such that*

$$\|x - x'\| \leq \|\Delta b\| \frac{\max\{1, \|x'\|\}}{\rho_P(d)}.$$

If $\rho(d) > 0$, then

3.
$$\frac{-\|b\|\,\|c\|_*}{\rho_P(d)} \le z_* = z^* \le \frac{\|b\|\,\|c\|_*}{\rho_D(d)}.$$

4. The optimal solution set is not empty, and for every x^* optimal for (CP_d),
$$\|x^*\| \le \frac{\|b\|}{\rho_D(d)} \frac{\|c\|_*}{\rho_P(d)}.$$

This theorem states that for a well-posed problem, there exists a feasible point x such that $\|x\| \le C(d)$. Moreover, that for any optimal solution x^*, we have $\|x^*\| \le C(d)^2$ and $|c^t x^*| \le \|d\| C(d)$. The second point shows how the condition number bounds the change in feasible solutions under a perturbation of the data b.

The following result gives a geometric description of how the feasible region X_d is related to the condition number. For this result each of the cones C_X and C_Y needs to be a regular cone, i.e. a closed convex pointed cone with a non-empty interior. (A pointed cone is a cone that does not contain a line.) The result below also uses the measure of the minimum width of a cone. For a non-empty convex cone $C \neq \{0\}$, the minimum width of cone C is defined by
$$\tau_C = \sup_{\check{x} \in C} \tau(\check{x}, C) = \sup_{\check{x} \in C} \frac{\text{dist}\,(\check{x}, \text{rel}\partial C)}{\|\check{x}\|}, \tag{2.3}$$
where
$$\tau(\check{x}, C) = \frac{\text{dist}\,(\check{x}, \text{rel}\partial C)}{\|\check{x}\|}$$
is the distance from the point $\check{x}$ to the relative boundary of C.

Theorem 2 *[14] Suppose that $d \in \mathcal{F}$ and C_X and C_Y are regular. If $\rho(d) > 0$, then* *there exists $\hat{x} \in X_d$ and positive scalars r and R satisfying*

$$\begin{array}{ll} (i) & B(\hat{x}, r) \subset X_d \\ (ii) & B(\hat{x}, r) \subset B(0, R) \\ (iii) & \frac{R}{r} \leq 7 \frac{1}{\min\left\{\tau_{C_X}, \tau_{C_Y}\right\}} C(d) \\ (iv) & \frac{1}{r} \leq 6 \frac{1}{\min\left\{\tau_{C_X}, \tau_{C_Y}\right\}} C(d) \\ (v) & R \leq 4 \frac{1}{\tau_{C_Y}} C(d) \ . \end{array}$$

Theorem 2 corresponds to part (vi) of Theorem 15 in [14]. The result states that, for a well-posed problem, we can find a feasible ball that is not far from the origin and whose radius is bounded from below.

2.3.2 Complexity of algorithms

Herein we review theoretical results concerning the complexity of algorithms for solving (CP_d). All of the results presented here use the condition number of the problem to bound the number of iterations the algorithm takes. The complexity results presented are for an interior point algorithm and for the ellipsoid algorithm. This section also reviews complexity results for algorithms that solve the feasibility problem for (CP_d) which depend on the condition number of the problem.

All the complexity results below assume a consistent and well-posed data instance. This means $d \in \mathcal{F}$ and $\rho(d) > 0$.

In [29], Renegar presents a complexity analysis for an interior point method in terms of the condition number. The algorithm used in solving (CP_d) is a primal path-following algorithm. This interior point method requires a ϑ-self-concordant barrier function for the cones C_X and C_Y. The algorithm starts from a given pair of interior points, which we denote by $\check{x} \in \text{relint}C_X$ and $\check{b} \in \text{relint}C_Y$. These starting points affect the complexity through their distance to the boundary of C_X and C_Y respectively. This is expressed by $\tau(\check{x}, C_X)$ and $\tau(\check{b}, C_Y)$. The algorithm also uses a positive scalar $\bar{s}$ that is specified as part of the input.

For a given tolerance level $\varepsilon > 0$, the algorithm analyzed returns an interior ε-optimal solution, that is a point in the set

$$X_d^\varepsilon = \left\{ x \in X_d \mid c^t x \leq z_* + \varepsilon \right\} ,$$

in at most

$$O\left(\sqrt{\vartheta} \ln\left(\sqrt{\vartheta} + C(d) + \frac{\|d\|}{\varepsilon} + \frac{1}{\tau(\check{x}, C_X)} + \frac{1}{\tau(\check{b}, C_Y)} + \frac{\max\{\bar{s}, \|d\|\}}{\min\{\bar{s}, \|d\|\}}\right)\right)$$

Newton iterations, see Theorem 3.1 and Corollary 7.3 of [29].

A version of the ellipsoid algorithm for solving (CP_d) is analyzed in [13]. For $\varepsilon > 0$, this algorithm finds an ε-optimal solution by finding a point in X_d^ε. The algorithm solves a homogenized version of (CP_d) and is initiated with the Euclidean unit ball centered at the origin. Under the additional assumptions that the cones C_X and C_Y are regular, and $0 < \varepsilon < \|c\|_*$, the ellipsoid algorithm computes an ε-optimal solution of (CP_d) in

$$O\left(n^2 \ln\left(C(d) + \frac{1}{\tau_{C_X}} + \frac{1}{\tau_{C_Y}} + \frac{\|c\|_*}{\varepsilon}\right)\right)$$

iterations, see Theorem 5.1 in [13].

The feasibility problem for (CP_d) is to find a point $x \in X_d$. This problem is the first step in the interior point method described above. For this part of the algorithm, we need the ϑ-self-concordant barrier, and the pair of interior points $\check{x}$ and $\check{b}$. Theorem 3.1 of [29] proves that the first step of the algorithm returns an interior point $\tilde{x} \in \operatorname{relint} X_d$ in

$$O\left(\sqrt{\vartheta} \ln\left(\sqrt{\vartheta} + C(d) + \frac{1}{\tau(\check{x}, C_X)} + \frac{1}{\tau(\check{b}, C_Y)}\right)\right)$$

Newton steps.

The ellipsoid algorithm in [13] can also be applied to solve the feasibility problem. Theorem 5.1 in [13] implies that if cones C_X and C_Y are regular, the ellipsoid algorithm will return a feasible solution to (CP_d) in

$$O\left(n^2 \ln\left(C(d) + \frac{1}{\tau_{C_X}} + \frac{1}{\tau_{C_Y}}\right)\right)$$

iterations.

An elementary algorithm has also been shown to solve the feasibility problem of (CP_d) in a number of iterations that is bounded by the condition number. An iteration of this elementary algorithm consists only of matrix-vector, vector-vector multiplications and comparisons. Therefore the work per iteration is significantly less than that in the interior point and ellipsoid algorithms, where linear systems need to be solved. In [5], the authors describe the elementary algorithm, for the case when C_X is regular and $C_Y = \{0\}$, which takes, at most,

$$O\left(\frac{C(d)^2}{\tau^2_{C_X^*}} \ln\left(\frac{C(d)}{\tau_{C_X}\tau_{C_X^*}}\right)\right)$$

iterations to find a feasible point, see Lemma 5 in [5].

The result for any regular cone C_Y appears in [7]. There, Lemma 4.5 states that for regular cones C_X, C_Y and a positive scalar value ν, the elementary algorithm requires, at most,

$$O\left(C(d)^2\left(\frac{\max\{\|d\|/\nu, \nu/\|d\|\}}{\min\{\tau_{C_X}\tau_{C_Y}\}}\right)^2\right)$$

iterations to find a feasible point.

2.3.3 Characterization and computation of $\rho(d)$

To compute condition numbers $C(d)$, we need to compute the distance to ill-posedness $\rho(d)$, and the norm of the data $\|d\|$. In this section, we present a characterization of $\rho(d)$ for consistent convex optimization problems in conic form, i.e., problems for data instances such that $d \in \mathcal{F}$.

This characterization of the distance to ill-posedness, which is central to our work in computing condition numbers, is presented in [14] based on Theorem 3.5 of [29]. The characterization uses the fact that if $d \in \mathcal{F}$, then $\rho(d) = \min\{\rho_P(d), \rho_D(d)\}$, where $\rho_P(d)$ and $\rho_D(d)$ are the primal and dual distances to infeasibility, respectively. These distances to infeasibility can be characterized by the solution to an optimization problem if we select $\|\cdot\|_1$ as the norm in the constraint space $I\!R^m$ and $\|\cdot\|_\infty$ as the norm in the variable space $I\!R^n$. Theorem 1 and Remark 6 in [14] imply that the primal distance to infeasibility is characterized by

$$\rho_P(d) = \min_{i \in \{1\ldots m\}} \; \min_{j \in \{-1,1\}} \; \min_{\substack{y \in C_Y^* \\ q \in C_X^* \\ g \geq 0 \\ y_i = j}} \; \max\{\|A^t y - q\|_1, |b^t y + g|\} \; .$$

Likewise the dual distance to infeasibility can be characterized by

$$\rho_D(d) = \min_{i \in \{1\ldots n\}} \; \min_{j \in \{-1,1\}} \; \min_{\substack{x \in C_X \\ p \in C_Y \\ g \geq 0 \\ x_i = j}} \; \max\{\|Ax + p\|_1, |c^t x + g|\} \; .$$

Therefore, the characterization of $\rho_P(d)$ is equivalent to solving $2m$ convex optimization problems, and the characterization of $\rho_D(d)$ is equivalent to solving $2n$ convex optimization problems.

2.4 Alternate characterization and previous computational work

Previous computational work on the condition number of linear programs was reported by Peña in [24]. This work considers the conic problem (CP_d) with $C_Y = \{0\}$. In that work the author presented several schemes to approximate the distance to ill-posedness, and reported on the tightness of these approximation schemes over randomly generated problems.

These schemes are based on two ideas: solving a related analytic center problem and exhibiting infeasible perturbations for the system. Both approaches work with consistent instances, i.e., $d \in \mathcal{F}$, and consider Euclidean norms in $\mathbb{R}^n$ and $\mathbb{R}^m$. The interior point approach assumes the existence of a ϑ-self-concordant barrier function over the C_X cone. In theory, the interior point approach yields an approximation of $\rho(d)$ within a factor of ϑ, while the second scheme produces an approximation of $\rho(d)$ within a factor of $\sqrt{m}$. In practice the author reports that over the randomly generated instances the approximations performed much better than predicted by theory.

Although this work computes the condition number for conic problems with equality constraints, it does not address the validity or significance of any of the theoretical condition number bounds.

2.5 Computing $C(d)$

Given the different results that, in theory, relate $C(d)$ to important behavioral and computational characteristics of (CP_d) outlined in this chapter, it is natural to attempt to obtain computational evidence that could describe the typical $C(d)$ of problems that arise in practice and that could indicate if the theoretical bounds above are significant for practical problems.

In order to address this lack of computational experience, one can start by computing the condition numbers for a suitably representative set of linear optimization instances that arise in practice, such as the NETLIB suite of industrial and academic linear optimization problems, see [21]. Practical methods for computing (or approximately computing) condition numbers for convex optimization problems in conic format (CP_d) have been developed in [11] and [24]. The first method is briefly presented in the previous section, the second alternative is due to Peña; both of these methods are relatively easy to implement. It would then seem to be a simple task to compute condition numbers for the NETLIB suite. However, it turns out that there is a subtle catch that gets in the way of this simple strategy, and in fact necessitates using an extension of the condition number theory just a bit, as we now explain.

Linear optimization problem instances arising in practice are typically conveyed in the following format:

$$\begin{array}{ll} \min_x & c^t x \\ \text{s.t.} & A_i x \le b_i \ , i \in L \\ & A_i x = b_i \ , i \in E \\ & A_i x \ge b_i \ , i \in G \\ & x_j \ge l_j \ , \ j \in L_B \\ & x_j \le u_j \ , \ j \in U_B \ , \end{array} \tag{2.4}$$

where the first three sets of inequalities/equalities are the "constraints" and the last two sets of inequalities are the lower and upper bound conditions, and where $L_B, U_B \subset \{1, \ldots, n\}$. (LP problems in practice might also contain range constraints of the form "$b_{i,l} \le A_i x \le b_{i,u}$" as well. We ignore this for now.) By defining C_Y to be an appropriate cartesian product of nonnegative half-lines $\mathbb{R}_+$, nonpositive half-lines $-\mathbb{R}_+$, and the origin $\{0\}$, we can naturally consider the constraints to be in the conic format "$Ax - b \in C_Y$" where $C_Y \subset \mathbb{R}^m$ and $m = |L| + |E| + |G|$. However, for the lower and upper bounds on the variables, there are different ways to convert the problem into the required conic format for computation and analysis of condition numbers. One way is to convert the lower and upper bound constraints into ordinary constraints. Assuming for expository convenience that all original constraints are equality constraints and that all lower and upper bounds are finite, this conversion of (2.4) to conic format is:

$$
\begin{array}{rl}
P_1 : \min_x & c^t x \\
\text{s.t.} & Ax - b = 0 \\
 & Ix - l \ge 0 \\
 & Ix - u \le 0 \ .
\end{array}
$$

whose data for this now-conic format is:

$$
\bar{A} := \begin{pmatrix} A \\ I \\ I \end{pmatrix}, \bar{b} := \begin{pmatrix} b \\ l \\ u \end{pmatrix}, \bar{c} := c
$$

with cones:

$$
\bar{C}_Y := \{0\}^m \times \mathbb{R}^n_+ \times -\mathbb{R}^n_+ \quad \text{and} \quad \bar{C}_X := \mathbb{R}^n \ .
$$

Another way to convert the problem to conic format is to replace the variables x with nonnegative variables $s := x - l$ and $t := u - x$, yielding:

$$
\begin{array}{lll}
P_2 : \min_{s,t} & c^t s + c^t l \\
\text{s.t.} & As - (b - Al) = 0 \\
& Is + It - (u - l) = 0 \\
& s, t \geq 0 \ ,
\end{array}
$$

whose data for this now-conic format is:

$$
\tilde{A} := \begin{pmatrix} A & 0 \\ I & I \end{pmatrix}, \tilde{b} := \begin{pmatrix} b - Al \\ u - l \end{pmatrix}, \tilde{c} := c
$$

with cones:

$$
\tilde{C}_Y := \{0\}^m \times \{0\}^n \quad \text{and} \quad \tilde{C}_X := \mathbb{R}^n_+ \times \mathbb{R}^n_+ \ .
$$

These two different conic versions of the same original problem have different data and different cones, and so will generically have different condition numbers. This is illustrated on the following elementary example:

$$
\begin{array}{lll}
P : \min_{x_1, x_2} & x_1 \\
\text{s.t.} & x_1 + x_2 \geq 1 \\
& 400x_1 + x_2 \leq 420 \\
& 1 \leq x_1 \leq 5 \\
& -1 \leq x_2 \ .
\end{array}
$$

	P_1	P_2
$\|d\|$	428	405
$\rho_P(d)$	0.24450	0.90909
$\rho_D(d)$	0.00250	1.00000
$C(d)$	171,200	445

Table 2.1: Condition Numbers for two different conic conversions of the same problem.

Table 2.1 shows condition numbers for problem P under the two different conversion strategies of P_1 and P_2, using the L_∞-norm in the space of the variables and the L_1-norm in the space of the right-hand-side vector. (To compute the condition numbers of these problems in conic form we can use for example the characterization presented in Section 2.3.3, which is described in more detail for LP in Section 7.1.) As Table 2.1 shows, the choice of the conversion strategy can have a very large impact on the resulting condition number, thereby calling into question the practical significance of performing such conversions to conic format.

It is to get around this problem that we introduce in the following chapter a generalization of the condition number theory to consider problems that are naturally not in conic form. This generalization will allow us to compute the condition number for linear programs that arise in practice without requiring a conversion of the problem that could alter the structure that is present in the data.

Chapter 3

Ground Set Model

3.1 Introduction and working assumptions

Recall from the previous discussion, that condition number theory for convex optimization has been developed for convex optimization problems in conic form:

$$
(CP_d) \qquad \begin{aligned} z_* = \quad \min_x \quad & c^t x \\ \text{s.t.} \quad & Ax - b \in C_Y \\ & x \in C_X, \end{aligned} \tag{3.1}
$$

and that problem (CP_d) covers a very general class of convex problems; in fact any convex optimization problem can be transformed to an equivalent instance of (CP_d). Recall also that such transformations are not necessarily unique and are sometimes rather unnatural given the "natural" data for the problem. These reasons make the current characterizations of the condition number ambiguous, and therefore, impractical for computing the condition number of problems in practice.

It is for this reason that we consider the following more general format for convex optimization:

$$
(GP_d) \qquad \begin{array}{rll} z_*(d) = & \min_x & c^t x \\ & \text{s.t.} & Ax - b \in C_Y \\ & & x \in P, \end{array} \tag{3.2}
$$

where now P, which we call the "ground-set," is allowed to be any closed convex set, possibly unbounded, and possibly lacking an interior. For example, P could be the solution set of box constraints of the form $l \leq x \leq u$ where some components of l and/or u might be unbounded, or P might be the solution of network flow constraints of the form $Nx = g, x \geq 0$. And of course, P might also be a closed convex cone. We refer to problem (GP_d) as the ground-set model (GSM) format.

The definition of the condition number for the convex optimization problem (GP_d) depends on the Lagrangian dual of (GP_d) and on the notion of distance to ill-posedness. In this chapter, we present the definitions and notation needed to define the Lagrangian dual problem, the distance to ill-posedness, and the condition number for problems in the GSM format. We also present characterizations for the distance to ill-posedness and conditions for strong duality of the problems in the GSM format.

This work considers the finite dimensional GSM format. For concreteness, denote the variable space $\mathcal{X}$ by $\mathbb{R}^n$ and the constraint space $\mathcal{Y}$ by $\mathbb{R}^m$. Therefore, $P \subseteq \mathbb{R}^n$, $C_Y \subseteq \mathbb{R}^m$, A is an m by n real matrix, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. The Reisz-Frénchet representation theorem, see Theorem 5.5 of [2], proves that the spaces of linear functionals $\mathcal{X}^*$ and $\mathcal{Y}^*$, for the Hilbert spaces $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{Y} = \mathbb{R}^m$, can be identified with $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

The assumptions we make on the ground set P and the cone C_Y are just that $P \neq \emptyset$ and $C_Y \neq \emptyset$. For the moment, note that when $C_Y = \mathbb{R}^m$, problem (GP_d) will be feasible for all data d.

3.2 GSM duality

The following definitions, which depend on the ground-set P, are needed to present the Lagrange dual problem of (GP_d).

Recall that a vector $r \in \mathbb{R}^n$ is a ray of P if there is a vector $x \in P$ such that for all $\theta \geq 0$ the vector $x + \theta r \in P$. Let R denote the set of rays of P. Since P is a closed convex set, the set of rays R is a closed convex cone.

Define the set

$$C_P := \{(x,t) \mid x \in tP, t > 0\}$$

and let C denote the closed convex cone

$$C := \text{cl } C_P \tag{3.3}$$

where cl S denotes the closure of set S. Then it is straightforward to prove that

$$C = C_P \cup \{(r,0) \mid r \in R\}$$

and that

$$\begin{aligned} C^* &= \left\{(s,u) \mid s^t x + u \geq 0, \text{ for all } x \in P\right\} \\ &= \left\{(s,u) \mid \inf_{x \in P} s^t x + u \geq 0\right\} . \end{aligned}$$

The Lagrange dual of (GP_d) is:

$$(GD_d) \qquad \begin{aligned} z^*(d) = \quad \max\nolimits_{y,u} \quad & b^t y - u \\ \text{s.t.} \quad & (c - A^t y, u) \in C^* \\ & y \in C_Y^* \ . \end{aligned} \tag{3.4}$$

Note that (GD_d) is technically in conic form; however evaluating the inclusion $(s, u) \in C^*$ is typically not an easy task. An equivalent form of the dual problem (GD_d) uses the convex function defined by

$$u(s) := -\inf_{x \in P} s^t x \ . \tag{3.5}$$

The convexity of function $u(\cdot)$ is due to the fact that the set P is convex and that

$$u(s) = \sup_{x \in P} (-s)^t x = \delta^*(-s \mid P) \ ,$$

where $\delta^*(\ \cdot \mid P)$ is the support function of the set P as defined in [31]; Theorem 5.5 of [31] shows that $u(\cdot)$ is convex. The epigraph of function $u(\cdot)$ is the set defined by

$$\text{epi } u(\cdot) := \{(s, v) \in I\!R^n \times I\!R \mid v \geq u(s)\} \ .$$

The projection of this set onto the space of the domain of $u(\cdot)$ is the effective domain of $u(\cdot)$:

$$\text{dom } u(\cdot) := \{s \in I\!R^n \mid u(s) < \infty\} \ .$$

Note that

$$\begin{aligned} C^* &= \left\{(s, u) \mid \inf_{x \in P} s^t x + u \geq 0\right\} \\ &= \{(s, u) \mid u \geq u(s)\} \\ &= \text{epi } u(\cdot) \ . \end{aligned}$$

With the definitions above, it is straightforward to show that the dual problem (GD_d)

is equivalent to the following optimization problem:

$$\begin{array}{rll} z^*(d) = & \max_y & b^t y - u(c - A^t y) \\ & \text{s.t.} & c - A^t y \in \text{dom } u(\cdot) \\ & & y \in C_Y^* \ . \end{array} \tag{3.6}$$

The following proposition, which is exactly Corollary 14.2.1 of [31], relates the domain of the function $u(\cdot)$ to the recession cone of the underlying ground set P.

Proposition 2 *[31] For a convex set P,* cl dom $u(\cdot) = R^*$. ■

This proposition implies in the case that dom $u(\cdot)$ is closed that

$$\begin{array}{rll} z^*(d) = & \max_y & b^t y - u(c - A^t y) \\ & \text{s.t.} & c - A^t y \in R^* \\ & & y \in C_Y^* \ . \end{array}$$

Consider now the case when the ground set P is a bounded set. In this case, for every vector $s \in I\!R^n$, the value $u(s) = -\inf_{x \in P} s^t x$ is a finite real number. As a consequence we have $(s, u(s)) \in C^*$ for any $s \in I\!R^n$. The above implies that problem (GD_d) is feasible for all data instances d.

Let X_d and Y_d denote the feasible regions of (GP_d) and (GD_d), respectively:

$$X_d := \{x \in I\!R^n \mid Ax - b \in C_Y, x \in P\} \tag{3.7}$$

and

$$Y_d := \{(y, u) \in I\!R^m \times I\!R \mid (c - A^t y, u) \in C^*, y \in C_Y^*\}. \tag{3.8}$$

Remark 1 *Weak duality holds between (GP_d) and (GD_d), that is, $z^*(d) \leq z_*(d)$.*

Proof: Let us consider x and (y, u) feasible for (GP_d) and (GD_d) respectively. Then

$$0 \leq \left(c - A^t y\right)^t x + u = c^t x - y^t A x + u \leq c^t x - b^t y + u,$$

where the last inequality follows from the fact that $y^t(Ax - b) \geq 0$. Therefore $z_*(d) \geq z^*(d)$. ■

3.3 Slater points and strong duality

A primal-dual pair of problems satisfies strong duality if the optimal objective function values of both problems coincide. In particular for the GSM format, problems (GP_d) and (GD_d) satisfy strong duality if $z_*(d) = z^*(d)$. The conditions under which a pair of primal-dual problems satisfies strong duality vary, for instance, in the case of linear programming if both the primal and the dual problems are feasible, then they satisfy strong duality. In convex optimization however, the existence of strong duality between a pair of primal-dual problems usually depends on additional hypotheses, these are known as constraint qualifications. A classic constraint qualification is the existence of a Slater point in the feasible region, see for example Theorem 30.4 of [31] or Chapter 5 of [1].

Below we present results that show that the existence of a Slater point is sufficient for strong duality in the GSM case. These proofs use a separating hyperplane argument to directly imply that the optimal objective function values coincide. This argument is based on the convexity of problems (GP_d) and (GD_d).

First let us make the notion of a Slater point precise for the GSM format. The relative interior of a set S is noted by relint$S$.

Definition 1 *A point x is a Slater point for problem (GP_d) if*

$$x \in \text{relint} P \quad \text{and } Ax - b \in \text{relint} C_Y \ .$$

Likewise, a point (y, u) is a Slater point for problem (GD_d) if

$$y \in \text{relint} C_Y^* \quad \text{and } (c - A^t y, u) \in \text{relint} C^* \ .$$

Theorem 3 *If x' is a Slater point for problem (GP_d), then $z_*(d) = z^*(d)$ and problem (GD_d) attains its optimum.*

Proof: For simplicity, let z_* and z^* denote the primal and dual optimal objective values respectively. Consider the set

$$S := \left\{ (p, q, \alpha) \mid \exists x \text{ s.t. } x + p \in P, Ax - b + q \in C_Y, c^t x - \alpha < z_* \right\},$$

which is a nonempty convex set. We can properly separate $(0, 0, 0)$ from S, since $(0, 0, 0) \notin S$. Therefore, there exists $(\gamma, y, \pi) \neq 0$, which satisfies $\gamma^t p + y^t q + \pi\alpha \geq 0$ for all $(p, q, \alpha) \in S$.

For any $x \in I\!R^n$, $\tilde{p} \in P$, $\tilde{q} \in C_Y$, and $\varepsilon > 0$ define $p = -x + \tilde{p}$, $q = -Ax + b + \tilde{q}$, and $\alpha = c^t x - z_* + \varepsilon$. Then $(p, q, \alpha) \in S$.

From the proper separation,

$$\gamma^t(-x + \tilde{p}) + y^t(-Ax + b + \tilde{q}) + \pi(c^t x - z_* + \varepsilon) \geq 0 \qquad \text{for all } x, \tilde{p} \in P, \tilde{q} \in C_Y, \varepsilon > 0 \ .$$

Because $\varepsilon > 0$ is not bounded, the multiplier $\pi \geq 0$. If $\pi > 0$, re-scale (γ, y, π) such that $\pi = 1$, and then

$$\gamma^t(-x + \tilde{p}) + y^t(-Ax + b + \tilde{q}) + c^t x - z_* + \varepsilon \geq 0 \qquad \text{for all } x, \tilde{p} \in P, \tilde{q} \in C_Y, \varepsilon > 0 \ .$$

Rearranging,

$$(-A^t y + c - \gamma)^t x + \gamma^t \tilde{p} + y^t \tilde{q} + y^t b - z_* + \varepsilon \geq 0 \qquad \text{for all } x, \tilde{p} \in P, \tilde{q} \in C_Y, \varepsilon > 0 \ .$$

This last expression implies that $c - A^t y = \gamma$ and $y \in C_Y^*$. Set $\tilde{q} = 0$, $u = y^t b - z_*$, and take the limit as $\varepsilon \to 0$, then the last expression implies $(c - A^t y, u) \in C^*$. Therefore (y, u) is feasible for (GD_d) and $z^* \geq b^t y - u = b^t y - y^t b + z_* = z_* \geq z^*$, which implies that $z^* = z_*$ and the dual feasible point (y, u) attains the dual optimum.

If $\pi = 0$, proper separation gives the following:

$$\gamma^t(-x + \tilde{p}) + y^t(-Ax + b + \tilde{q}) \geq 0 \qquad \text{for all } x, \tilde{p} \in P, \tilde{q} \in C_Y \ .$$

Rearranging,

$$(-A^t y - \gamma)^t x + \gamma^t \tilde{p} + y^t \tilde{q} + y^t b \geq 0 \qquad \text{for all } x, \tilde{p} \in P, \tilde{q} \in C_Y \ .$$

This implies that $-A^t y = \gamma$ and $y \in C_Y^*$. The last expression becomes $-y^t A \tilde{p} + y^t \tilde{q} + y^t b \geq 0$, for any $\tilde{p} \in P, \tilde{q} \in C_Y$. Proper separation also guarantees that there exists $(\hat{p}, \hat{q}, \hat{\alpha})$ and $\hat{x}$ such that $\hat{x} + \hat{p} \in P$, $A\hat{x} - b + \hat{q} \in C_Y$, $c^t \hat{x} - \hat{\alpha} < z_*$, and $\gamma^t \hat{p} + y^t \hat{q} + \pi \hat{\alpha} > 0$. Now $\pi = 0$ by supposition, and we noted that $\gamma = -A^t y$, and $y \in C_Y^*$. Therefore $-y^t A \hat{p} + y^t \hat{q} > 0$.

Let x' be the Slater point. For all $|\xi|$ sufficiently small, $x' + \xi(\hat{x} + \hat{p} - x') \in P$ and

$Ax' - b + \xi(A\hat{x} - b + \hat{q} - (Ax' - b)) \in C_Y$. Therefore

$$\begin{aligned} 0 &\leq -y^t A \left(x' + \xi(\hat{x} + \hat{p} - x')\right) + y^t \left(Ax' - b + \xi(A\hat{x} - b + \hat{q} - (Ax' - b))\right) + y^t b \\ &= \xi \left(-y^t A\hat{x} - y^t A\hat{p} + y^t Ax' + y^t A\hat{x} - y^t b + y^t \hat{q} - y^t Ax' + y^t b\right) \\ &= \xi \left(-y^t A\hat{p} + y^t \hat{q}\right) \ , \end{aligned}$$

which is a contradiction, since ξ can be negative and $-y^t A\hat{p} + y^t\hat{q} > 0$. Therefore $\pi \neq 0$, and the proof is complete. ■

Theorem 4 *If (y', u') is a Slater point for problem (GD_d), then $z_*(d) = z^*(d)$ and problem (GP_d) attains its optimum.*

Proof: For simplicity, let z_* and z^* denote the primal and dual optimal objective values respectively. Consider the nonempty convex set

$$S := \left\{(s, v, q, \alpha) \mid \exists y, u \text{ s.t. } (c - A^t y, u) + (s, v) \in C^*, y + q \in C_Y^*, b^t y - u + \alpha > z^*\right\} \ .$$

We can properly separate $(0,0,0,0)$ from S, since $(0,0,0,0) \notin S$. Therefore, there exists $(x, \beta, \gamma, \delta) \neq 0$, which satisfies $x^t s + \beta v + \gamma^t q + \delta\alpha \geq 0$ for all $(s, v, q, \alpha) \in S$.

For any $y \in I\!R^m$, $u \in I\!R$, $(\tilde{s}, \tilde{v}) \in C^*$, $\tilde{q} \in C_Y^*$, and $\varepsilon > 0$, define $s = -c + A^t y + \tilde{s}$, $v = -u + \tilde{v}$, $q = -y + \tilde{q}$, and $\alpha = z^* - b^t y + u + \varepsilon$. Then $(s, v, q, \alpha) \in S$.

From the proper separation,

$$x^t(-c + A^t y + \tilde{s}) + \beta(-u + \tilde{v}) + \gamma^t(-y + \tilde{q}) + \delta(z^* - b^t y + u + \varepsilon) \geq 0 \ ,$$

for all $y, u, (\tilde{s}, \tilde{v}) \in C^*, \tilde{q} \in C_Y^*, \varepsilon > 0$. Because $\varepsilon > 0$ and not bounded, then $\delta \geq 0$. If $\delta > 0$, re-scale $(x, \beta, \gamma, \delta)$ such that $\delta = 1$, and

$$x^t(-c + A^t y + \tilde{s}) + \beta(-u + \tilde{v}) + \gamma^t(-y + \tilde{q}) + z^* - b^t y + u + \varepsilon \geq 0 \ .$$

Rearranging,

$$(Ax - b - \gamma)^t y + (x, \beta)^t(\tilde{s}, \tilde{v}) + (1 - \beta)u + \gamma^t \tilde{q} - c^t x + z^* + \varepsilon \geq 0 \ ,$$

for any y, u and any $(\tilde{s}, \tilde{v}) \in C^*, \tilde{q} \in C_Y^*, \varepsilon > 0$. This last expression implies that $Ax - b = \gamma \in C_Y$, $\beta = 1$, and $(x, 1) \in C$, which means that $x \in P$. Therefore x is feasible for (GP_d). Set $(\tilde{s}, \tilde{v}) = (0, 0)$, $\tilde{q} = 0$, and take the limit $\varepsilon \to 0$, then the above expression implies $z^* \geq c^t x \geq z_* \geq z^*$, which implies that $z^* = z_*$ and the primal feasible point x attains the optimum.

If $\delta = 0$, proper separation gives the following:

$$x^t(-c + A^t y + \tilde{s}) + \beta(-u + \tilde{v}) + \gamma^t(-y + \tilde{q}) \geq 0 \qquad \text{for all } y, u, (\tilde{s}, \tilde{v}) \in C^*, \tilde{q} \in C_Y^* \ .$$

Rearranging,

$$(Ax - \gamma)^t y + (x, \beta)^t(\tilde{s}, \tilde{v}) - \beta u + \gamma^t \tilde{q} - c^t x \geq 0 \qquad \text{for all } y, u, (\tilde{s}, \tilde{v}) \in C^*, \tilde{q} \in C_Y^* \ .$$

This implies that $\beta = 0$ and $Ax = \gamma \in C_Y$. The above inequality also implies that $(x, 0) \in C$, which means that $x \in R$, and $x \neq 0$ (for otherwise $(x, \beta, \gamma, \delta) = 0$, a contradiction). The proper separation states that there exists $(\hat{s}, \hat{v}, \hat{q}, \hat{\alpha})$ and $(\hat{y}, \hat{u})$ satisfying $(c - A^t\hat{y} + \hat{s}, \hat{u} + \hat{v}) \in C^*$, $\hat{y} + \hat{q} \in C_Y^*$, $b^t\hat{y} - \hat{u} + \hat{\alpha} > z^*$, and $x^t\hat{s} + x^t A^t \hat{q} > 0$.

Consider the Slater point (y', u'). Then $(c - A^t y', u') \in \text{relint} C^*$ and $y' \in \text{relint} C_Y^*$. Then for all $|\xi|$ sufficiently small, we have $y' + \xi\left(\hat{y} + \hat{q} - y'\right) \in C_Y^*$ and

$$\left(c - A^t y' + \xi\left(c - A^t\hat{y} + \hat{s} - c + A^t y'\right), u' + \xi\left(\hat{u} + \hat{v} - u'\right)\right) \in C^* \ .$$

Therefore

$$x^t\left(c - A^t y' + \xi\left(c - A^t\hat{y} + \hat{s} - c + A^t y'\right)\right) + x^t A^t\left(y' + \xi\left(\hat{y} + \hat{q} - y'\right)\right) - c^t x \geq 0 \ .$$

Simplifying and canceling, we obtain

$$\begin{aligned}0 &\leq \xi\left(-x^tA^t\hat{y} + x^t\hat{s} + x^tA^ty' + x^tA^t\hat{y} + x^tA^t\hat{q} - x^tA^ty'\right) \\ &= \xi\left(x^t\hat{s} + x^tA^t\hat{q}\right) .\end{aligned}$$

However, by choosing $\xi < 0$ we have a contradiction, since $x^t\hat{s} + x^tA^t\hat{q} > 0$. Therefore $\delta \neq 0$, and the proof is complete. ■

3.4 Distance to ill-posedness and Condition Number

In this section we introduce the definitions of the distance to ill-posedness and condition number for a convex optimization problem in the GSM format. These definitions are the natural extension of the definitions introduced by Renegar for conic problems in [28] and [29].

The vector space of all data instances $d = (A, b, c)$ for the GSM format is

$$\mathcal{D} = \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^n.$$

A norm on $\mathcal{D}$ can be constructed as follows. Consider given norms $\|x\|$ and $\|y\|$ on $\mathbb{R}^n$ and $\mathbb{R}^m$, let $\|A\|$ denote the usual operator norm, and let $\|v\|_*$ denote the dual norm associated with the norm $\|w\|$ on $\mathbb{R}^n$ or $\mathbb{R}^m$. We define the norm of any data instance $d = (A, b, c) \in \mathcal{D}$ by $\|d\| := \max\{\|A\|, \|b\|, \|c\|_*\}$. Let $B(d, r) \subset \mathcal{D}$ denote the ball with center d and radius r, for the norm defined on $\mathcal{D}$.

We now pause to remark that there are four different normed vector spaces present in the GSM framework. There are the two normed vector spaces present in the definition of the primal problem: the variable space $(\mathbb{R}^n, \|\cdot\|)$ and the constraint space $(\mathbb{R}^m, \|\cdot\|)$. There are also the two normed vector spaces present in the definition of the dual problem:

the linear functionals over the constraint space $(\mathbb{R}^m, \|\cdot\|_*)$ and over the variable space $(\mathbb{R}^n, \|\cdot\|_*)$. The fact that elements of each dual space can be associated to elements of the corresponding original space is due to Theorem 5.5 of [2]. In the remainder of the thesis, we let $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ denote the primal points and $s \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ denote the dual points. This convention will remind us to make a distinction between primal and dual points in $\mathbb{R}^n$ or $\mathbb{R}^m$ in order to use the appropriate norm.

Define the sets $\mathcal{F}_P$ of primal feasible data instances:

$$\mathcal{F}_P := \{d \in \mathcal{D} \mid X_d \neq \emptyset\},$$

and $\mathcal{F}_D$ of dual feasible data instances:

$$\mathcal{F}_D := \{d \in \mathcal{D} \mid Y_d \neq \emptyset\}.$$

A data instance that is both primal and dual feasible is a consistent data instance. The set $\mathcal{F}$ of consistent data instances is therefore

$$\mathcal{F} := \mathcal{F}_P \cap \mathcal{F}_D = \{d \in \mathcal{D} \mid X_d \neq \emptyset, Y_d \neq \emptyset\}.$$

In general terms, a problem with data instance d is well-posed if its consistency status is unchanged by small perturbations of the data. In other words, $d \in \mathcal{F}$ is well-posed if, for some $\varepsilon > 0$, $d + B(0, \varepsilon) \subseteq \mathcal{F}$. Likewise, $d \in \mathcal{F}^C$ is well-posed if, for some $\varepsilon > 0$, $d + B(0, \varepsilon) \in \mathcal{F}^C$. The well-posed instances are the data instances which lay in the interior of either $\mathcal{F}$ or $\mathcal{F}^C$.

Therefore, the set $\mathcal{I}$ of ill-posed instances is made up of exactly those instances for which an arbitrarily small perturbation of the data will change its consistency status.

Ill-posed instances lie on the boundary of $\mathcal{F}$ and $\mathcal{F}^C$:

$$\mathcal{I} := \partial\mathcal{F} = \partial\mathcal{F}^C = \mathrm{cl}\mathcal{F} \cap \mathrm{cl}\mathcal{F}^C.$$

The distance of any instance d to the set $\mathcal{I}$, noted $\rho(d)$, is the distance to ill-posedness of the instance. For $d \in \mathcal{F}$, the distance to ill-posedness is

$$\rho(d) := \inf\left\{\|\Delta d\| \mid X_{d+\Delta d} = \emptyset \text{ or } Y_{d+\Delta d} = \emptyset\right\},$$

the size of the smallest data perturbation Δd that would make problem $(GP_{d+\Delta d})$ and/or problem $(GD_{d+\Delta d})$ infeasible. For $d \in \mathcal{F}^C$, the distance to ill-posedness is

$$\rho(d) := \inf\left\{\|\Delta d\| \mid X_{d+\Delta d} \neq \emptyset \text{ and } Y_{d+\Delta d} \neq \emptyset\right\},$$

the size of the smallest data perturbation Δd that would make both problems $(GP_{d+\Delta d})$, and $(GD_{d+\Delta d})$ feasible.

The existence of the set of ill-posed instances $\mathcal{I}$ depends on the ground set P and the cone C_Y. The pathological case is when both $C_Y = \mathbb{R}^m$ and the set P is bounded. The fact that $C_Y = \mathbb{R}^m$ implies that $X_d \neq \emptyset$ for all d, and the boundedness of P implies that $Y_d \neq \emptyset$ for all d. Therefore, in this pathological case, $\mathcal{F} = \mathcal{D}$ and the set of ill-posed instances does not exist. We will not consider this case in the remainder of this work, and so we make the following assumption:

Assumption 1 *Either $C_Y \neq \mathbb{R}^m$ or the ground set P is not bounded (or both).*

The condition number of instance d is defined as

$$C(d) := \begin{cases} \dfrac{\|d\|}{\rho(d)} & \rho(d) > 0 \\ \infty & \rho(d) = 0, \end{cases}$$

which is a scale invariant reciprocal of the distance to ill-posedness.

Note that $C(d)$ increases as the problem instance d becomes closer to the set of ill-posed instances $\mathcal{I}$.

Remark 2 *For any data instance $d \in \mathcal{D}$, the condition number satisfies $C(d) \geq 1$.*

Proof: Consider the data instance $d_0 = (0,0,0)$. Note that $X_{d_0} = P \neq \emptyset$ and $Y_{d_0} = C_Y^* \times I\!R_+ \neq \emptyset$, therefore $d_0 \in \mathcal{F}$. If $C_Y \neq I\!R^m$, consider $b \in I\!R^m \setminus C_Y$, $b \neq 0$, and for any $\varepsilon > 0$, define the instance $d_\varepsilon = (0, -\varepsilon b, 0)$. This instance is such that for any $\varepsilon > 0$, $X_{d_\varepsilon} = \emptyset$, which means that $d_\varepsilon \in \mathcal{F}^C$ and therefore $d_0 \in \mathcal{I}$.

If $C_Y = I\!R^m$, then P is unbounded. This means that there exists $r \in R$, $r \neq 0$, a ray of P. For any $\varepsilon > 0$ the instance $d_\varepsilon = (0,0,-\varepsilon r)$ is such that $Y_{d_\varepsilon} = \emptyset$, which means that $d_\varepsilon \in \mathcal{F}^C$ and therefore $d_0 \in \mathcal{I}$.

Since d_0 is always ill-posed, we know that $\rho(d) \leq \|d - d_0\| = \|d\|$. ■

3.5 Characterizations of $\rho(d)$ as the solution of associated optimization problems

For the remainder of this work we will consider consistent data instances, that is $d \in \mathcal{F}$. The characterization of the distance to ill-posedness, $\rho(d)$, in the case when $d \in \mathcal{F}^C$ is not a direct extension.

3.5.1 Definitions and preliminary results

To characterize the distance to ill-posedness we need a few more definitions. Define, for $d \in \mathcal{D}$, the primal distance to infeasibility $\rho_P(d)$:

$$\rho_P(d) := \inf \left\{ \|\Delta d\| \;\mid X_{d+\Delta d} = \emptyset \right\},$$

the size of the smallest data perturbation Δd that makes $(GP_{d+\Delta d})$ infeasible. Define also, the dual distance to infeasibility $\rho_D(d)$:

$$\rho_D(d) := \inf \left\{ \|\Delta d\| \;\mid Y_{d+\Delta d} = \emptyset \right\},$$

the size of the smallest data perturbation Δd that makes $(GD_{d+\Delta d})$ infeasible.

Now, for $d \in \mathcal{F}$, the distance to ill-posedness $\rho(d)$ becomes:

$$\begin{aligned} \rho(d) &:= \inf \left\{ \|\Delta d\| \;\mid X_{d+\Delta d} = \emptyset \text{ or } Y_{d+\Delta d} = \emptyset \right\} \\ &= \min \left\{ \rho_P(d), \rho_D(d) \right\}. \end{aligned}$$

This shows that to characterize of $\rho(d)$ it is sufficient to express $\rho_P(d)$ and $\rho_D(d)$ in a convenient form. Below we show that these distances to infeasibility can be obtained as the solutions of certain associated optimization problems. The characterization presented below follows ideas used to characterize the condition number for problems in conic form introduced in [29] and [14].

Before doing so, we first present a number of technical results that are used in the characterizations of the distances to infeasibility. Proposition 3 is a special case of the Hahn-Banach Theorem; the proof below appears originally in Proposition 2 of [14].

Proposition 3 *Consider X an n-dimensional linear vector space. For every $x \in X$,* *there exists $\bar{x} \in X^*$ with the property that $\|\bar{x}\|_* = 1$ and $\|x\| = \bar{x}^t x$.*

Proof: If $x = 0$, then any $\bar{x} \in X^*$ such that $\|\bar{x}\|_* = 1$ satisfies the proposition. Suppose now that $x \neq 0$. Consider the real valued convex function $f(x) = \|x\|$. Its sub-differential operator $\partial f(x)$ is non-empty for all $x \in X$. Let $z \in \partial f(x)$. Then

$$f(w) \geq f(x) + z^t(w - x) \quad \text{for any } w \in X \ .$$

Setting $w = 0$ we have $z^t x \geq f(x) = \|x\|$, and setting $w = 2x$ implies $z^t x \leq f(2x) - f(x) = f(x) = \|x\|$, whereby $z^t x = \|x\|$. Using the Hölder inequality in this last equality means that $\|z\|_* \geq 1$. Now for any $u \in X$, set $w = x + u$, then $z^t u + f(x) \leq f(x + u) \leq f(x) + f(u)$, therefore $z^t u \leq \|u\|$ for any $u \in X$. This implies from the definition of the dual norm that $\|z\|_* \leq 1$. In conclusion $\|z\|_* = 1$. ■

The following weak alternative lemmas will be used in the proofs for the characterizations of $\rho_P(d)$ and $\rho_D(d)$.

Lemma 1 *Consider the systems*

$$(X_d) \quad \begin{array}{l} Ax - b \in C_Y \\ x \in P \end{array} \ , \qquad (A1) \quad \begin{array}{l} (-A^t y, u) \in C^* \\ b^t y \geq u \\ y \neq 0 \\ y \in C_Y^* \end{array} \ , \qquad (A2) \quad \begin{array}{l} (-A^t y, u) \in C^* \\ b^t y > u \\ y \in C_Y^* \end{array}$$

If system (X_d) is infeasible, then system $(A1)$ is feasible. Conversely, if system $(A2)$ is feasible, then system (X_d) is infeasible.

Proof: Let us assume that system (X_d) is infeasible. This implies that

$$b \notin S := \{Ax - v \mid x \in P, v \in C_Y\} \ ,$$

and S is a nonempty convex set. Therefore the vector b can be separated from S, and therefore there exists $y \neq 0$ and u such that

$$\begin{aligned} & y^t b \geq u \\ & y^t(Ax - v) \leq u \quad \forall x \in P, v \in C_Y . \end{aligned}$$

Since $v \in C_Y$ we see that $y \in C_Y^*$, and setting $v = 0 \in C_Y$ we obtain

$$(-A^t y)^t x + u \geq 0, \forall x \in P \Rightarrow (-A^t y, u) \in C^*,$$

therefore (y, u) satisfies system (A1).

To prove the other implication, assume both $(A2)$ and (X_d) are feasible, then we obtain the following contradiction:

$$0 \leq y^t(Ax - b) = (A^t y)^t x - b^t y < -\Big((-A^t y)^t x + u\Big) \leq 0 \ . \qquad \blacksquare$$

Lemma 2 *Consider the systems*

$$(Y_d) \quad \begin{array}{l} (c - A^t y, u) \in C^* \\ y \in C_Y^* \end{array} , \qquad (B1) \quad \begin{array}{l} Ax \in C_Y \\ c^t x \leq 0 \\ x \neq 0 \\ x \in R \end{array} , \qquad (B2) \quad \begin{array}{l} Ax \in C_Y \\ c^t x < 0 \\ x \in R \end{array}$$

If system (Y_d) is infeasible, then system $(B1)$ is feasible. Conversely, if system $(B2)$ is feasible, then system (Y_d) is infeasible.

Proof: Assume that system (Y_d) is infeasible, this implies that

$$(0, 0, 0) \notin S := \Big\{(s, v, q) \mid \exists\ y, u \text{ s.t. } (c - A^t y, u) + (s, v) \in C^*, y + q \in C_Y^*\Big\} \ ,$$

and S is a nonempty convex set. The point $(0,0,0)$ can be separated from S, that is, there exists $(x, \delta, z) \neq 0$ such that $x^t s + \delta v + z^t q \geq 0$ for all $(s, v, q) \in S$.

Consider any (y, u), $(\tilde{s}, \tilde{v}) \in C^*$, and $\tilde{q} \in C_Y^*$, and define $s = -(c - A^t y) + \tilde{s}$, $v = -u + \tilde{v}$, and $q = -y + \tilde{q}$. Then $(s, v, q) \in S$. This implies that

$$-x^t c + (Ax - z)^t y + x^t \tilde{s} - \delta u + \delta \tilde{v} + z^t \tilde{q} \geq 0 \quad \text{for all } y, u, (\tilde{s}, \tilde{v}) \in C^*, \tilde{q} \in C_Y^* \ .$$

The above expression implies that $\delta = 0$, $Ax = z \in C_Y$, and $(x, \delta) = (x, 0) \in C$. This last inclusion implies that $x \in R$. Setting $(\tilde{s}, \tilde{v}) = (0,0)$ and $\tilde{q} = 0$ we obtain $c^t x \leq 0$. It is necessary that $x \neq 0$ because otherwise, $(x, \delta, z) = (x, \delta, Ax) = 0$. This means that $(B1)$ is feasible.

To prove the other implication, assume both $(B2)$ and (Y_d) are feasible. Then we obtain the following contradiction:

$$0 \leq x^t(c - A^t y) = c^t x - y^t Ax < -y^t Ax \leq 0 \ . \qquad \blacksquare$$

The next strong duality result will also be used in the characterizations of the distances to infeasibility.

Lemma 3 *Consider two closed convex cones $C \subseteq I\!R^n$ and $C_Y \subseteq I\!R^m$, and data $(M, v) \in I\!R^{m \times n} \times I\!R^m$. Strong duality holds between*

$$\begin{array}{llll} (P): & z_* = & \min & \|M^t y + q\|_* \\ & & s.t. & y^t v \geq 1 \\ & & & y \in C_Y^* \\ & & & q \in C^* \end{array} \qquad \textit{and} \qquad \begin{array}{llll} (D): & z^* = & \max & \theta \\ & & s.t. & Mx - \theta v \in C_Y \\ & & & \|x\| \leq 1 \\ & & & \theta \geq 0 \\ & & & x \in C \ . \end{array}$$

Proof: For any (y,q) feasible for (P) and (x,θ) feasible for (D) we have:

$$\theta \le \theta y^t v + x^t q \le x^t\left(M^t y + q\right) \le \|x\|\|M^t y + q\|_* \le \|M^t y + q\|_* \ .$$

Therefore $z^* \le z_*$. Suppose $z^* < z_*$, then we have that $0 \le z^* < z_* - \varepsilon$, for $\varepsilon > 0$ and small enough. Consider the nonempty convex set S defined as follows:

$$S := \left\{(u,\delta,\alpha) \mid \exists y, q \text{ s.t. } y + u \in C_Y^*, q + \delta \in C^*, y^t v \ge 1 - \alpha, \left\|M^t y + q\right\|_* \le z_* - \varepsilon\right\} \ .$$

Since $(0,0,0) \notin S$, we can separate $(0,0,0)$ from S. Therefore there exists $(z,x,\theta) \neq 0$ such that, $z^t u + x^t \delta + \theta\alpha \ge 0$ for any $(u,\delta,\alpha) \in S$.

Consider any $y \in \mathbb{R}^m$, $\tilde{u} \in C_Y^*$, $\tilde{\delta} \in C^*$, and $\tilde{q}$ that satisfies $\|\tilde{q}\|_* \le z_* - \varepsilon$, and define $q = -M^t y + \tilde{q}$, $u = -y + \tilde{u}$, $\delta = -q + \tilde{\delta}$, and $\alpha = 1 - y^t v$. Then $(u,\delta,\alpha) \in S$, and so

$$z^t(-y + \tilde{u}) + x^t(M^t y - \tilde{q} + \tilde{\delta}) + \theta(1 - y^t v) \ge 0 \ .$$

Rearranging yields

$$y^t\left(Mx - \theta v - z\right) + z^t\tilde{u} + x^t\tilde{\delta} - x^t\tilde{q} + \theta \ge 0$$

for any y, $\tilde{u} \in C_Y^*$, $\tilde{\delta} \in C^*$, and any $\|\tilde{q}\|_* \le z_* - \varepsilon$. This last expression implies that $Mx - \theta v = z \in C_Y$, $x \in C$, and $\theta \ge 0$. If $x \neq 0$, re-scale (z,x,θ) such that $\|x\| = 1$ and then (x,θ) is feasible for (D). From Proposition 3, there exists $\hat{q} \in \mathbb{R}^n$ such that $\|\hat{q}\|_* = 1$ and $\hat{q}^t x = \|x\| = 1$; use $\hat{q}$ to define $\tilde{q} = (z_* - \varepsilon)\hat{q}$. Then (x,θ) satisfies $z^* \ge \theta \ge x^t\tilde{q} = z_* - \varepsilon > z^*$, which is a contradiction.

If $x = 0$, the above expression implies $-\theta v = z \in C_Y$, and $\theta \ge 0$. If $\theta > 0$ then $-v \in C_Y$, which means that the point $(0,\theta)$ is feasible for (D) for any $\theta \ge 0$, implying that $z^* = \infty$, a contradiction. The last case is $\theta = 0$, but this means that $z = 0$, which is a contradiction since $(z,x,\theta) \neq 0$. ■

3.5.2 Characterizations of $\rho_P(d)$ and $\rho_D(d)$

Before delving into the characterizations of the primal and dual distances to infeasibility, we show the following:

Proposition 4 *For any data instance $d = (A, b, c) \in \mathcal{D}$,*

1. $\rho_P(d) = \infty$ *if and only if* $C_Y = \mathbb{R}^m$.

2. $\rho_D(d) = \infty$ *if and only if P is bounded.*

Proof: Clearly $C_Y = \mathbb{R}^m$ implies that $\rho_P(d) = \infty$. Also, if P is bounded, then $R = \{0\}$ and $R^* = \mathbb{R}^n$, whereby (GD_d) is feasible for any d, and so $\rho_D(d) = \infty$. Therefore in both points we only need to prove the converse implication.

Assume that $\rho_P(d) = \infty$, and suppose that $C_Y^* \neq \{0\}$. Consider a point $\tilde{y} \in C_Y^*$, $\tilde{y} \neq 0$, and define the perturbation $\Delta d = (\Delta A, \Delta b, \Delta c)$, where $\Delta A = -A$, $\Delta b = -b + \tilde{y}$, and $\Delta c = -c$. System $(A2)$ of Lemma 1 is feasible for the data $\bar{d} = d + \Delta d = (0, \tilde{y}, 0)$, in fact the point $(y, u) = \left(\tilde{y}, \frac{\tilde{y}^t \tilde{y}}{2}\right)$ satisfies $(A2)$. This means, from Lemma 1, that $X_{\bar{d}}$ is infeasible and $\|\bar{d} - d\| \geq \rho_P(d) = \infty$, a contradiction. Therefore $C_Y^* = \{0\}$, and so $C_Y = \mathbb{R}^m$.

Assume that $\rho_D(d) = \infty$, and suppose that $R \neq \{0\}$. Consider $\tilde{x} \in R$, $\tilde{x} \neq 0$, and define the perturbation $\Delta A = -A$, $\Delta b = -b$, and $\Delta c = -c - \tilde{x}$. System $(B2)$ of Lemma 2 is feasible for the data $\bar{d} = d + \Delta d = (0, 0, -\tilde{x})$, in fact the point $\tilde{x}$ satisfies all constraints. Then, from Lemma 2, we have that $Y_{\bar{d}}$ is infeasible and so $\|\bar{d} - d\| \geq \rho_D(d) = \infty$, a contradiction. Therefore $R = \{0\}$, which implies that the ground set P is bounded. ∎

The two theorems below present characterizations of the distance to infeasibility as optimization problems. Each of the following results is an extension to problems not in conic form of Theorem 3.5 of [29], and Theorems 1 and 2 of [14].

Theorem 5 *Suppose $d \in \mathcal{F}_P$. Then $\rho_P(d) = j_P(d) = r_P(d)$, where*

$$
\begin{aligned}
j_P(d) = \quad \min \quad & \max\left\{\|A^t y + s\|_*, |b^t y - u|\right\} \\
\|y\|_* = 1 & \\
y \in C_Y^* & \\
(s,u) \in C^* &
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
r_P(d) = \quad \min \quad & \max \quad \theta \\
\|v\| \le 1 \quad & Ax - bt - v\theta \in C_Y \\
v \in \mathbb{R}^m \quad & \|x\| + |t| \le 1 \\
& (x,t) \in C \ .
\end{aligned}
\tag{3.10}
$$

Proof: Assume that $j_P(d) > \rho_P(d)$, then there is an infeasible data instance $\bar{d} = (\bar{A}, \bar{b})$ such that $\left\|A - \bar{A}\right\| < j_P(d)$ an $\left\|b - \bar{b}\right\| < j_P(d)$. From Lemma 1, for the data instance $\bar{d}$, there is a point $(\bar{y}, \bar{u})$ that satisfies the following:

$$
\begin{aligned}
& (-\bar{A}^t \bar{y}, \bar{u}) \in C^* \\
& \bar{b}^t \bar{y} \ge \bar{u} \\
& \bar{y} \ne 0 \\
& \bar{y} \in C_Y^* .
\end{aligned}
$$

Scale $\bar{y}$ such that $\|\bar{y}\|_* = 1$, then $(y, s, u) = (\bar{y}, -\bar{A}^t\bar{y}, \bar{b}^t\bar{y})$ is feasible for (3.9) and

$$
\begin{aligned}
\left\|A^t y + s\right\|_* &= \left\|A^t\bar{y} - \bar{A}^t\bar{y}\right\|_* \le \left\|A - \bar{A}\right\| \|\bar{y}\|_* < j_P(d) \\
|b^t y - u| &= |b^t\bar{y} - \bar{b}^t\bar{y}| \le \left\|b - \bar{b}\right\| \|\bar{y}\|_* < j_P(d).
\end{aligned}
$$

In the first inequality above we used the fact that $\|A^t\|_* = \|A\|$. Therefore $j_P(d) \le \max\left\{\|A^t y + s\|_*, |b^t y - u|\right\} < j_P(d)$, a contradiction.

Let us now assume that $j_P(d) < \gamma < \rho_P(d)$ for some γ. This means that there exists $(\bar{y}, \bar{s}, \bar{u})$ such that $\bar{y} \in C_Y^*$, $\|\bar{y}\|_* = 1$, $(\bar{s}, \bar{u}) \in C^*$, and that

$$\left\|A^t\bar{y} + \bar{s}\right\|_* < \gamma, \quad |b^t\bar{y} - \bar{u}| < \gamma.$$

From Proposition 3, consider $\hat{y}$ such that $\|\hat{y}\| = 1$ and $\hat{y}^t\bar{y} = \|\bar{y}\|_* = 1$, and define, for $\varepsilon > 0$,

$$\begin{aligned} \bar{A} &= A - \hat{y}\left((A^t\bar{y})^t + \bar{s}^t\right) \\ \bar{b}_\varepsilon &= b - \hat{y}\left(b^t\bar{y} - \bar{u} - \varepsilon\right). \end{aligned}$$

We have that $\bar{y} \in C_Y^*$, $-\bar{A}^t\bar{y} = \bar{s}$, $\bar{b}_\varepsilon^t\bar{y} = \bar{u} + \varepsilon > \bar{u}$, and $(-\bar{A}^t\bar{y}, \bar{u}) \in C^*$. This implies that for any $\varepsilon > 0$, the problem $(A2)$ in Lemma 1 is feasible with data $\bar{d}_\varepsilon = (\bar{A}, \bar{b}_\varepsilon, 0)$. Lemma 1 then implies that $X_{\bar{d}_\varepsilon} = \emptyset$ and therefore $\rho_P(d) \leq \left\|d - \bar{d}_\varepsilon\right\|$. To finish the proof we compute the size of the perturbation:

$$\begin{aligned} \left\|A - \bar{A}\right\| &= \left\|\hat{y}\left((A^t\bar{y})^t + \bar{s}^t\right)\right\| \leq \left\|A^t\bar{y} + \bar{s}\right\|_* \|\hat{y}\| < \gamma \\ \left\|b - \bar{b}_\varepsilon\right\| &= |b^t\bar{y} - \bar{u} - \varepsilon| \, \|\hat{y}\| \leq |b^t\bar{y} - \bar{u}| + \varepsilon < \gamma + \varepsilon, \end{aligned}$$

which implies, $\rho_P(d) \leq \max\left\{\left\|A - \bar{A}\right\|, \left\|b - \bar{b}_\varepsilon\right\|\right\} < \gamma + \varepsilon < \rho_P(d)$, for ε small enough. This is a contradiction, whereby $j_P(d) = \rho_P(d)$.

To prove the other characterization, we invoke Lemma 3 to rewrite problem (3.10) as

$$\begin{array}{rccl} r_P(d) = & \min & \min & \max\left\{\|A^ty + s\|_*, |-b^ty + u|\right\} \\ & \|v\| \leq 1 & y^tv \geq 1 & \\ & v \in I\!R^m & y \in C_Y^* & \\ & & (s, u) \in C^*. & \end{array}$$

The above problem can be written as the following equivalent optimization problem:

$$
\begin{aligned}
r_P(d) = \quad & \min \quad \max\left\{\|A^t y + s\|_*, |-b^t y + u|\right\} \\
& \|y\|_* \geq 1 \\
& y \in C_Y^* \\
& (s,u) \in C^* .
\end{aligned}
$$

The equivalence of these problems is verified by combining the minimization operations in the first problem and using the Cauchy-Schwartz inequality. The converse makes use of Proposition 3. To finish the proof, we note that this last problem will be optimized at a point which also satisfies $\|y\|_* = 1$, whereby making it equivalent to (3.9). Therefore

$$
\begin{aligned}
r_P(d) = \quad & \min \quad \max\left\{\|A^t y + s\|_*, |-b^t y + u|\right\} \quad = j_P(d) \; . \\
& \|y\|_* = 1 \\
& y \in C_Y^* \\
& (s,u) \in C^* \qquad \blacksquare
\end{aligned}
$$

Theorem 6 *Suppose $d \in \mathcal{F}_D$. Then $\rho_D(d) = j_D(d) = r_D(d)$, where*

$$
\begin{aligned}
j_D(d) = \quad & \min \quad \max\left\{\|Ax - p\|, |c^t x + g|\right\} \\
& \|x\| = 1 \\
& x \in R \\
& p \in C_Y \\
& g \geq 0
\end{aligned}
\tag{3.11}
$$

and

$$r_D(d) = \min_{\substack{\|v\|_* \le 1 \\ v \in \mathbb{R}^n}} \quad \begin{array}{ll} \max & \theta \\ & -A^t y + c\delta - \theta v \in R^* \\ & \|y\|_* + |\delta| \le 1 \\ & y \in C_Y^* \\ & \delta \ge 0 \ . \end{array} \tag{3.12}$$

Proof: Assume that $\rho_D(d) < j_D(d)$. From Lemma 2, for data $\bar{d} = (\bar{A}, \bar{c})$, there exists $x \in R$ such that $x \neq 0$, $\bar{A}x \in C_Y$, $\bar{c}^t x \le 0$, $\|A - \bar{A}\| < j_D(d)$, and $\|c - \bar{c}\|_* < j_D(d)$. We can scale x such that $\|x\| = 1$. Set $p = \bar{A}x$ and $g = -\bar{c}^t x$. Then (x, p, g) is feasible for (3.11), and

$$\begin{aligned} \|Ax - p\| &= \left\|Ax - \bar{A}x\right\| \le \left\|A - \bar{A}\right\| \|x\| < j_D(d) \\ |c^t x + g| &= |c^t x - \bar{c}^t x| \le \|c - \bar{c}\|_* \|x\| < j_D(d) \ . \end{aligned}$$

Therefore, $j_D(d) \le \max\{\|Ax - p\|, |c^t x + g|\} < j_D(d)$, which is a contradiction.

Assume now that $\rho_D(d) > \delta > j_D(d)$ for some δ. Then there exists $(\bar{x}, \bar{p}, \bar{g})$ such that $\bar{x} \in R$, $\|\bar{x}\| = 1$, $\bar{p} \in C_Y$, and $\bar{g} \ge 0$, and that $\|A\bar{x} - \bar{p}\| \le \delta$ and $|c^t\bar{x} + \bar{g}| \le \delta$. From Proposition 3, consider $\hat{x}$ such that $\|\hat{x}\|_* = 1$ and $\hat{x}^t\bar{x} = \|\bar{x}\| = 1$, and define: $\bar{A} = A - (A\bar{x} - \bar{p})\,\hat{x}^t$ and $\bar{c}_\varepsilon = c - \hat{x}(c^t\bar{x} + \bar{g} + \varepsilon)$, for $\varepsilon > 0$. We have $\bar{A}\bar{x} = \bar{p} \in C_Y$ and $\bar{c}_\varepsilon^t\bar{x} = -\bar{g} - \varepsilon < 0$, for any $\varepsilon > 0$. With data $\bar{d}_\varepsilon = (\bar{A}, 0, \bar{c}_\varepsilon)$, Lemma 2 implies that $Y_{\bar{d}_\varepsilon} = \emptyset$. We can then bound $\rho_D(d)$ as follows:

$$\begin{aligned} \rho_D(d) &\le \left\|d - \bar{d}_\varepsilon\right\| \le \max\left\{\left\|(A\bar{x} - \bar{p})\,\hat{x}^t\right\|, \left\|\hat{x}(c^t\bar{x} + \bar{g} + \varepsilon)\right\|_*\right\} \\ &\le \max\{\delta, \delta + \varepsilon\} = \delta + \varepsilon < \rho_D(d) \end{aligned}$$

for ε small enough, which is a contradiction. Therefore $\rho_D(d) = j_D(d)$.

To prove the other characterization, we invoke Lemma 3 to rewrite problem (3.12) as

$$
r_D(d) = \min_{\substack{\|v\|_* \leq 1 \\ v \in I\!R^n}} \; \min_{\substack{x^t v \geq 1 \\ x \in R \\ p \in C_Y \\ g \geq 0}} \; \max\{\|-Ax + p\|, |c^t x + g|\} .
$$

The above problem can be written as the following equivalent optimization problem:

$$
r_D(d) = \min_{\substack{\|x\| \geq 1 \\ x \in R \\ p \in C_Y \\ g \geq 0}} \; \max\{\|-Ax + p\|, |c^t x + g|\} .
$$

The equivalence of these problems is verified by combining the minimization operations in the first problem and using the Cauchy-Schwartz inequality. The converse makes use of Proposition 3. To finish the proof, we note that this last problem will be optimized at a point which also satisfies $\|x\| = 1$, whereby making it equivalent to (3.11). Therefore

$$
r_D(d) = \min_{\substack{\|x\| = 1 \\ x \in R \\ p \in C_Y \\ g \geq 0}} \; \max\{\|-Ax + p\|, |c^t x + g|\} = j_D(d) . \qquad \blacksquare
$$

3.6 Well-posed instances and strong duality

We now show that a positive distance to ill-posedness gives us a sufficient condition for the strong duality between (GP_d) and (GD_d). We proceed by showing that a positive primal or dual distance to infeasibility implies the existence of a primal or dual Slater point, respectively.

Theorem 7 *Suppose that $d \in \mathcal{F}_P$ and that $\rho_P(d) > 0$. Then X_d contains a Slater point.*

Proof: If $\rho_P(d) > 0$, assume that X_d contains no Slater point. This implies that $\text{relint} C_Y \cap \{Ax - b \mid x \in \text{relint} P\} = \emptyset$. Then there exists $y \neq 0$ that separates these two nonempty convex sets, and so

$$y^t s \geq y^t (Ax - b) \quad \text{for any } s \in C_Y, x \in P \ .$$

This equation implies $y \in C_Y^*$ and $y^t Ax - y^t b \leq 0$ for any $x \in P$. Therefore, $(-A^t y, b^t y) \in C^*$. Then the point $(\tilde{y}, \tilde{s}, \tilde{u}) = \frac{1}{\|y\|_*}(y, -A^t y, b^t y)$ is feasible for problem (3.9) with an objective function value of zero. Therefore Theorem 5 implies $\rho_P(d) \leq 0$, which is a contradiction. ■

Theorem 8 *Suppose that $d \in \mathcal{F}_D$ and that $\rho_D(d) > 0$. Then Y_d contains a Slater point.*

Proof: If $\rho_D(d) > 0$, assume that Y_d has no Slater point. Consider the nonempty convex set S defined by:

$$S := \left\{ \left(c - A^t y, u\right) \mid y \in \text{relint} C_Y^*, u \in I\!R \right\} \ .$$

No Slater point in the dual implies that $\text{relint} C^* \cap S = \emptyset$. Therefore we can properly separate these two nonempty convex sets. This means that there exists $(r, t) \neq 0$ such that

$$r^t s + tv \geq r^t \left(c - A^t y\right) + tu \quad \text{for all } (s,v) \in C^*, y \in C_Y^*, u \in I\!R\ .$$

Since $u \in I\!R$ is unconstrained, this means that $t = 0$ and $r \neq 0$. Re-scale r so that $\|r\| = 1$. The left side of the inequality states that for any $(s,v) \in C^*$, the quantity $(r,0)^t(s,v) \geq r^t c$, which is bounded. Therefore $(r,0) \in C$, which means that $r \in R$. Setting $(s,v) = 0 \in C^*$, the right hand side of the inequality above can be written as $(Ar)^t y \geq r^t c$ for all $y \in C_Y^*$, which implies $Ar \in C_Y$. This same equation evaluated at $y = 0 \in C_Y^*$ states that $r^t c \leq 0$. Then the point $(x,p,g) = (r, Ar, -c^t r)$ is feasible for problem (3.11) with an objective function value of zero. Therefore Theorem 6 implies $\rho_D(d) \leq 0$, which is a contradiction. ■

In the next chapter, we show that a positive distance to infeasibility not only implies the existence of a Slater point for the problem, but it also establishes the existence of a Slater point that has bounds on the norm, its distance to the boundary, and the ratio between them. In other words, the next chapter proves the existence of a Slater point with good geometric properties.

Corollary 1 *(Strong Duality) Suppose that $d \in \mathcal{F}$, and that $\rho_P(d) > 0$ or $\rho_D(d) > 0$. Then $z_*(d) = z^*(d)$. If $\rho_P(d) > 0$ and $\rho_D(d) > 0$, then both the primal and the dual attain their respective optimal values.*

Proof: The proof of this result is a straightforward consequence of Theorems 3, 4, 7, and 8. ■

Note that the contrapositive of this result says that if $d \in \mathcal{F}$ and $z_*(d) > z^*(d)$, then $\rho(d) = 0$, which means that the instance is ill-posed. In other words the contrapositive states that a data instance for which the primal and dual problems are feasible and have a nonzero duality gap, must be ill-posed.

Chapter 4

Geometric Properties of the Primal and Dual Feasible Regions

In the previous chapter we showed among other things that a positive primal and/or dual distance to infeasibility implies the existence of a primal and/or dual Slater point, respectively. We now show that a positive distance to infeasibility also implies that the corresponding feasible region has a *reliable solution*, where by reliable solution we mean a relatively interior solution which has good geometric properties: it is not too far from the origin, it is not too close to the boundary of the feasible region, and the ratio of its norm to the distance to the boundary is not too large.

A primal solution x in the relative interior of the feasible region $\mathrm{relint} X_d$ is a reliable solution if its norm $\|x\|$, its distance to the boundary $\mathrm{dist}(x, \mathrm{rel}\partial X_d)$, and the ratio $\frac{\|x\|}{\mathrm{dist}(x,\mathrm{rel}\partial X_d)}$ are all not too large. The results in this chapter show that if a problem has a positive distance to ill-posedness, then a reliable solution exists and the above quantities are bounded by a function that depends naturally on the reciprocal of the distance to ill-posedness of the problem.

The next section presents the notation that is used in this chapter; the subsequent two sections present the results that imply the existence of reliable solutions for the primal problem and the dual problem, respectively.

4.1 Affine hull, norms, and width of a cone

We first present the definition of the affine hull of a set; this concept is useful to work in the vector subspaces where the (possibly lower dimensional) feasible regions lie.

An affine set T is the translation of a vector subspace L, i.e. $T = a + L$ for some a. The minimal affine set that contains a given set S is known as the affine hull of S. We denote this affine set by L_S, and it is characterized by

$$L_S = \left\{ \sum_{i \in I} \alpha_i x_i \mid \alpha_i \in I\!R,\ x_i \in S,\ \sum_{i \in I} \alpha_i = 1,\ I \text{ a finite set} \right\} ,$$

see for example [31]. Also note that if the set S contains 0 then the affine hull L_S is a subspace. We denote by $\hat{L}_S$ the vector subspace obtained when the affine hull L_S is translated to contain the origin. That is, for any $x \in S$, $\hat{L}_S := L_S - x$.

Let us now mention which norms are defined in the primal and dual feasible regions. Recall that there are given norms $\|x\|$ and $\|y\|$ on $I\!R^n$ and $I\!R^m$, and that $\|v\|_*$ denotes the dual norm associated with the norm $\|w\|$ on $I\!R^n$ or $I\!R^m$. Consistent with the characterization of $\rho_P(d)$ given by Problem (3.10) in Theorem 5, we define the following norm in the augmented space $I\!R^n \times I\!R$.

Definition 2 *For any $(x, t) \in I\!R^n \times I\!R$ define*

$$\|(x, t)\| := \|x\| + |t| \ .$$

This primal norm $\|x\| + |t|$ is for the points in $C \subset I\!R^n \times I\!R$ and implies that

$\max\{\|s\|_*, |u|\}$ is the dual norm used for points in $C^* \subset \mathbb{R}^n \times \mathbb{R}$.

The given norm on $\mathbb{R}^n$ is used to measure distances in the primal feasible region X_d. For the dual feasible region Y_d we have to define a norm on $\mathbb{R}^m \times \mathbb{R}$. Consistent with the characterization of $\rho_D(d)$ given by Problem (3.12) in Theorem 6, we define the following dual norm in the augmented space $\mathbb{R}^m \times \mathbb{R}$.

Definition 3 *For any* $(y, \delta) \in \mathbb{R}^m \times \mathbb{R}$ *define*

$$\|(y, \delta)\|_* := \|y\|_* + |\delta| \ .$$

It is clear that the above defines a norm on the vector space that contains Y_d, and that it implies that $\max\{\|w\|, |g|\}$ is the primal norm in $\mathbb{R}^m \times \mathbb{R}$.

The purpose of Lemma 5 below is to show how this dual norm on the dual feasible region is related to the norms on $\mathbb{R}^m$ and to the norms on $\mathbb{R}^n \times \mathbb{R}$. Before this lemma, we define the distance to the relative boundary of a set from a point in the set.

Definition 4 *Given a non-empty set* S *and a point* $x \in S$*, the distance from* x *to the relative boundary of* S *is denoted by* $\mathrm{dist}(x, \mathrm{rel}\partial S)$ *and given by*

$$\begin{array}{rll} \mathrm{dist}(x, \mathrm{rel}\partial S) := & \inf_{\bar{x}} & \|x - \bar{x}\| \\ & \text{s.t.} & \bar{x} \in L_S \setminus S \ . \end{array} \tag{4.1}$$

Define also the projection of Y_d onto $\mathbb{R}^m$,

$$\Pi Y_d := \{y \in \mathbb{R}^m \mid (y, u) \in Y_d \text{ for some } u\} \ , \tag{4.2}$$

and the set formed by one of the two constraints of Y_d:

$$\tilde{Y}_d := \left\{(y,u) \in I\!R^m \times I\!R \mid (c - A^t y, u) \in C^*\right\} \ . \tag{4.3}$$

Note that the dual feasible region is recovered by $Y_d = \tilde{Y}_d \cap (C^*_Y \times I\!R)$. This fact motivates the following Lemma:

Lemma 4 *Given sets A and B, and a point $x \in A \cap B$, then*

$$\mathrm{dist}(x, \mathrm{rel}\partial(A \cap B)) \geq \min\{\mathrm{dist}(x, \mathrm{rel}\partial A),\ \mathrm{dist}(x, \mathrm{rel}\partial B)\} \ .$$

Proof: The proof of this lemma is based in showing that $L_{A\cap B} \setminus (A \cap B) \subset (L_A \setminus A) \cup (L_B \setminus B)$. If this inclusion is true then

$$\begin{aligned}
\mathrm{dist}(x, \mathrm{rel}\partial(A \cap B)) &= \inf_{\bar{x} \in L_{A\cap B} \setminus (A \cap B)} \|x - \bar{x}\| \\
&\geq \inf_{\bar{x} \in (L_A \setminus A) \cup (L_B \setminus B)} \|x - \bar{x}\| \\
&= \min\left\{\inf_{\bar{x} \in L_A \setminus A} \|x - \bar{x}\|,\ \inf_{\bar{x} \in L_B \setminus B} \|x - \bar{x}\|\right\} \\
&= \min\{\mathrm{dist}(x, \mathrm{rel}\partial A),\ \mathrm{dist}(x, \mathrm{rel}\partial B)\} \ ,
\end{aligned}$$

which proves the lemma. Therefore we now prove the inclusion. Consider some $\bar{x} \in L_{A\cap B}$, this means that there exists $\alpha_i \in I\!R$, $x_i \in A \cap B$, $i \in I$ a finite set, and $\sum_{i \in I} \alpha_i = 1$, such that $\bar{x} = \sum_{i \in I} \alpha_i x_i$. Since $x_i \in A$, $\bar{x} \in L_A$ and since $x_i \in B$, $\bar{x} \in L_B$. Therefore $L_{A\cap B} \subset L_A \cap L_B$. The desired inclusion is then obtained with a little algebra:

$$\begin{aligned}
L_{A\cap B} \setminus (A \cap B) &\subset L_A \cap L_B \cap (A \cap B)^C \\
&= L_A \cap L_B \cap \left(A^C \cup B^C\right) \\
&= \left(L_A \cap L_B \cap A^C\right) \cup \left(L_A \cap L_B \cap B^C\right) \\
&\subset \left(L_A \cap A^C\right) \cup \left(L_B \cap B^C\right)
\end{aligned}$$

$$= \quad (L_A \setminus A) \cup (L_B \setminus B) \ . \qquad \blacksquare$$

Lemma 5 *Given a dual feasible point* $(y,u) \in Y_d$*, let* $s = c - A^t y \in \mathrm{dom}\, u(\cdot)$*. Then the following hold*

1. $\mathrm{dist}\,((y,u), \mathrm{rel}\partial(C_Y^* \times I\!R)) = \mathrm{dist}\,(y, \mathrm{rel}\partial C_Y^*)$.
2. $\mathrm{dist}\left((y,u), \mathrm{rel}\partial \widetilde{Y}_d\right) \geq \frac{1}{\max\{1,\|A\|\}} \mathrm{dist}((s,u), \mathrm{rel}\partial C^*)$.
3. $\mathrm{dist}\,((y,u), \mathrm{rel}\partial Y_d) \geq \frac{1}{\max\{1,\|A\|\}} \min\left\{\mathrm{dist}\,((s,u), \mathrm{rel}\partial C^*), \mathrm{dist}\,(y, \mathrm{rel}\partial C_Y^*)\right\}$.
4. $\mathrm{dist}(y, \mathrm{rel}\partial \Pi Y_d) \geq \mathrm{dist}((y,u), \mathrm{rel}\partial Y_d)$.

Proof: Equality (1.) is a consequence of the fact that $(y,u) \in L_{C_Y^* \times I\!R} \setminus (C_Y^* \times I\!R)$ if and only if $y \in L_{C_Y^*} \setminus C_Y^*$, and that $\|(y,u) - (\bar{y},u)\|_* = \|y - \bar{y}\|_* + |u - u| = \|y - \bar{y}\|_*$.

To prove inequality (2.), we first show that if $(\bar{y},\bar{u}) \in L_{\widetilde{Y}_d} \setminus \widetilde{Y}_d$, then $(c - A^t\bar{y}, \bar{u}) \in L_{C^*} \setminus C^*$. First note that if $(\bar{y},\bar{u}) \notin \widetilde{Y}_d$ then, by the definition of $\widetilde{Y}_d$, $(c - A^t\bar{y}, \bar{u}) \notin C^*$; therefore we only need to show that if $(\bar{y},\bar{u}) \in L_{\widetilde{Y}_d}$ then $(c - A^t\bar{y}, \bar{u}) \in L_{C^*}$. Let $(\bar{y},\bar{u}) \in L_{\widetilde{Y}_d}$. Then there exists $\alpha_i \in I\!R$, $(y_i,u_i) \in \widetilde{Y}_d$, $i \in I$ a finite set, and $\sum_{i\in I} \alpha_i = 1$, such that $(\bar{y},\bar{u}) = \sum_{i\in I} \alpha_i (y_i,u_i)$. Consider

$$(c - A^t\bar{y}, \bar{u}) = (c - A^t \sum_{i\in I} \alpha_i y_i, \sum_{i\in I} \alpha_i u_i) = \sum_{i\in I} \alpha_i (c - A^t y_i, u_i) \ .$$

Since $(y_i,u_i) \in \widetilde{Y}_d$ then $(c - A^t y_i, u_i) \in C^*$ and therefore $(c - A^t\bar{y}, \bar{u}) \in L_{C^*}$.

The inclusion above means that

$$\begin{aligned} \mathrm{dist}((s,u), \mathrm{rel}\partial C^*) &= \inf_{(\bar{s},\bar{u}) \in L_{C^*} \setminus C^*} \|(s,u) - (\bar{s},\bar{u})\| \\ &\leq \inf_{(\bar{y},\bar{u}) \in L_{\widetilde{Y}_d} \setminus \widetilde{Y}_d} \|(s,u) - (c - A^t\bar{y}, \bar{u})\| \\ &= \inf_{(\bar{y},\bar{u}) \in L_{\widetilde{Y}_d} \setminus \widetilde{Y}_d} \max\{\|s - (c - A^t\bar{y})\|_*, |u - \bar{u}|\} \end{aligned}$$

$$
\begin{aligned}
&= \inf_{(\bar{y},\bar{u})\in L_{\widetilde{Y}_d}\setminus \widetilde{Y}_d} \max\{\|A^t\bar{y} - A^t y\|_*, |u-\bar{u}|\} \\
&\leq \inf_{(\bar{y},\bar{u})\in L_{\widetilde{Y}_d}\setminus \widetilde{Y}_d} \max\{\|A\|\|\bar{y} - y\|_*, |u-\bar{u}|\} \\
&\leq \inf_{(\bar{y},\bar{u})\in L_{\widetilde{Y}_d}\setminus \widetilde{Y}_d} \max\{\|A\|, 1\} \max\{\|y - \bar{y}\|_*, |u-\bar{u}|\} \\
&\leq \max\{\|A\|, 1\} \inf_{(\bar{y},\bar{u})\in L_{\widetilde{Y}_d}\setminus \widetilde{Y}_d} \|y - \bar{y}\|_* + |u-\bar{u}| \\
&= \max\{\|A\|, 1\} \inf_{(\bar{y},\bar{u})\in L_{\widetilde{Y}_d}\setminus \widetilde{Y}_d} \|(y,u) - (\bar{y},\bar{u})\|_* \\
&= \max\{\|A\|, 1\} \mathrm{dist}((y,u), \mathrm{rel}\partial \widetilde{Y}_d) \ .
\end{aligned}
$$

Inequality (3.) follows from the observation that $Y_d = \widetilde{Y}_d \cap (C_Y^* \times I\!R)$, Lemma 4, and the bounds obtained in (1.) and (2.).

The proof of item (4.) uses the fact (soon to be proved) that if $\bar{y} \in L_{\Pi Y_d} \setminus \Pi Y_d$ then for any $\bar{u}$, $(\bar{y},\bar{u}) \in L_{Y_d} \setminus Y_d$. Then from the definition of the relative distance to the boundary we have

$$
\begin{aligned}
\mathrm{dist}\,((y,u), \mathrm{rel}\partial Y_d) &= \inf_{(\bar{y},\bar{u})\in L_{Y_d}\setminus Y_d} \|(y,u) - (\bar{y},\bar{u})\|_* \\
&\leq \inf_{\bar{y}\in L_{\Pi Y_d}\setminus \Pi Y_d, \bar{u}} \|(y,u) - (\bar{y},\bar{u})\|_* \\
&= \inf_{\bar{y}\in L_{\Pi Y_d}\setminus \Pi Y_d, \bar{u}} \|y - \bar{y}\|_* + |u-\bar{u}| \\
&= \inf_{\bar{y}\in L_{\Pi Y_d}\setminus \Pi Y_d} \|y - \bar{y}\|_* \\
&= \mathrm{dist}\,(y, \mathrm{rel}\partial \Pi Y_d) \ ,
\end{aligned}
$$

which proves inequality (4.). To finish the proof we now show that if $\bar{y} \in L_{\Pi Y_d} \setminus \Pi Y_d$ then for any $\bar{u}$, $(\bar{y},\bar{u}) \in L_{Y_d} \setminus Y_d$. The fact that $\bar{y} \notin \Pi Y_d$ implies that for any $\bar{u}$, $(\bar{y},\bar{u}) \notin Y_d$. Now since $\bar{y} \in L_{\Pi Y_d}$, there exists a finite number of dual feasible points $\{(y_i,u_i)\}_{i\in I} \subset Y_d$ such that $\bar{y} = \sum_{i\in I} \alpha_i y_i$ and $\sum_{i \in I} \alpha_i = 1$. Since for any $(y,u) \in Y_d$ and $\beta \geq 0$ the point

$(y, u + \beta) \in Y_d$, we can express the point $(\bar{y}, \bar{u})$ for any $\bar{u}$ by the following sum

$$(\bar{y}, \bar{u}) = \left(\sum_{i \in I} \alpha_i y_i + y - y, \sum_{i \in I} \alpha_i u_i + u + \beta_1 - u - \beta_2 \right) ,$$

where the positive values β_1 and β_2 are set so that the second component of the right side equals $\bar{u}$. This shows that for any $\bar{u}$, $(\bar{y}, \bar{u}) \in L_{Y_d}$, finishing the proof. ■

We finish this introductory section by reviewing the measure of the minimum width of a convex cone, defined in (2.3) for cones other than $\{0\}$. This measure plays a role in the bounds of the reliable solutions that we present in the following sections.

Definition 5 *For a convex cone K, the minimum width of K is defined by*

$$\tau_K := \sup \left\{ \frac{\text{dist}(y, \text{rel}\partial K)}{\|y\|} \mid y \in K, y \neq 0 \right\} ,$$

for $K \neq \{0\}$, and $\tau_K := \infty$ if $K = \{0\}$.

For a subspace $K \neq \{0\}$, the value of the minimum width is $\tau_K = \infty$. This can be deduced since in this case, for any $y \in K$, the distance $\text{dist}(y, \text{rel}\partial K) = \infty$. For a cone K which is not a subspace, the quantity τ_K measures the width of the largest spherical cone contained in K, and τ_K satisfies $0 < \tau_K \leq 1$, taking on larger values to the extent that K has larger minimum width.

4.2 Solutions in the relative interior of X_d

In this section we present a number of results that relate the primal distance to infeasibility $\rho_P(d)$, and a known point in the ground set $x^0 \in P$, to the existence of a feasible solution $\bar{x}$ in the relative interior of X_d that has nice geometric properties, in other words, the existence of a reliable solution of X_d.

We prove that under the hypothesis that $\rho_P(d) > 0$, there exists a Slater point in X_d which has norm, distance to the relative boundary of the feasible region, and the ratio of its norm to the distance to the boundary that are bounded by the condition number of the problem. Recall that a point $\bar{x}$ is a Slater point for (GP_d) if $A\bar{x} - b \in \mathrm{relint} C_Y$ and $\bar{x} \in \mathrm{relint} P$, whereby $\bar{x}$ also satisfies $\bar{x} \in \mathrm{relint} X_d$.

Many results in this section involve the distance of a point $x \in S$ to the relative boundary of the set S. In the case when S is an affine set, the distance to the relative boundary is infinite. This is due to the fact that in this case the affine hull equals the affine set, i.e. $L_S = S$, and therefore the distance to the relative boundary, from its definition in (4.1), is

$$\mathrm{dist}(x, \mathrm{rel}\partial S) := \begin{array}{ll} \inf_{\bar{x}} & \|x - \bar{x}\| \\ \text{s.t.} & \bar{x} \in L_S \setminus S \end{array} = \begin{array}{ll} \inf_{\bar{x}} & \|x - \bar{x}\| \\ \text{s.t.} & \bar{x} \in \emptyset \end{array} = \infty \ .$$

Note that a singleton $\{x\}$ is a special case of an affine set, which is obtained by translating the vector subspace $\{0\}$ by x. Recall that if the affine set S contains zero then it is a linear subspace.

The results in this section, besides showing the existence of a reliable solution $\bar{x} \in X_d$, also bound $\mathrm{dist}(\bar{x}, \mathrm{rel}\partial P)$ and/or $\mathrm{dist}(\bar{x}, \mathrm{rel}\partial X_d)$, using $\mathrm{dist}(x^0, \mathrm{rel}\partial P)$ for a given point $x^0 \in P$. Therefore, when the set P is an affine set, both the quantity to be bounded and the bound used might be infinity. For clarity of exposition, we avoid this special case, which can be dealt with in a straightforward but careful analysis, and present here the results for the case when P is not an affine set. For each result we will also mention what can be proved without this additional assumption in a subsequent remark.

The inspiration for the results in this section are Lemma 2 and Lemma 3 in [14] where for a consistent instance of a conic linear system the authors show that there exists a reliable solution with good geometric properties. The results here extend these results to the GSM format and are stated regardless of any regularity condition on the cone or the ground set.

First we present a technical lemma that is used to relate feasible solutions in X_d with the distance to primal infeasibility of the problem. This lemma makes use of the program (PP), which we now define.

For given points $x^0 \in P$ and $w^0 \in C_Y$, define problem (PP) by

$$\begin{array}{lll} (PP) & \max_{x,t,w,\theta} & \theta \\ & \text{s.t.} & Ax - bt - w = \theta\,(b - Ax^0 + w^0) \\ & & \|x\| + |t| \le 1 \\ & & (x,t) \in C \\ & & w \in C_Y . \end{array} \tag{4.4}$$

Lemma 6 *Suppose that $d \in \mathcal{F}_P$ and $\rho_P(d) > 0$. Given $x^0 \in P$ and $w^0 \in C_Y$ such that $Ax^0 - w^0 \neq b$, then there is a point (x, t, w, θ) feasible for problem (PP) that satisfies*

$$\theta \ge \frac{\rho_P(d)}{\|b - Ax^0 + w^0\|} > 0 \ . \tag{4.5}$$

Proof: Note that problem (PP) is feasible for any x^0 and w^0 since $(x, t, w, \theta) = (0, 0, 0, 0)$ is always feasible, therefore it can either be unbounded or have a finite optimal objective value. If (PP) is unbounded, we can find feasible points with an objective function large enough such that (4.5) holds. If (PP) has a finite optimal value, say z^*, then it attains this value since it is a linear objective over a bounded domain (add bounds on θ, for example "$0 \le \theta \le z^* + 1$", which makes the domain bounded). From Theorem 5 the optimal solution $(x^*, t^*, w^*, \theta^*)$ for (PP) satisfies (4.5). ∎

The following result establishes the existence of a feasible point $\bar{x}$ whose norm and distance to the boundary of P is bounded by similar quantities involving a known point in the ground set P.

Theorem 9 *Suppose that $d \in \mathcal{F}_P$, $\rho_P(d) > 0$, and P is not an affine set. Let $x^0 \in P$ be given. Then there exists $\bar{x} \in X_d$ satisfying:*

1. (a) $\|\bar{x} - x^0\| \le \dfrac{\mathrm{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} \max\{1, \|x^0\|\}$

   (b) $\|\bar{x}\| \le \|x^0\| + \dfrac{\mathrm{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}$

2. $\dfrac{1}{\mathrm{dist}(\bar{x}, \mathrm{rel}\partial P)} \le \dfrac{1}{\mathrm{dist}(x^0, \mathrm{rel}\partial P)} \left(1 + \dfrac{\mathrm{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}\right)$

3. (a) $\dfrac{\|\bar{x} - x^0\|}{\mathrm{dist}(\bar{x}, \mathrm{rel}\partial P)} \le \dfrac{1}{\mathrm{dist}(x^0, \mathrm{rel}\partial P)} \left(\dfrac{\mathrm{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} \max\{1, \|x^0\|\}\right)$

   (b) $\dfrac{\|\bar{x}\|}{\mathrm{dist}(\bar{x}, \mathrm{rel}\partial P)} \le \dfrac{1}{\mathrm{dist}(x^0, \mathrm{rel}\partial P)} \left(\|x^0\| + \dfrac{\mathrm{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}\right)$

Proof: We assume $x^0 \notin X_d$, otherwise select $\bar{x} = x^0$. The case $\rho_P(d) = +\infty$, which implies that $C_Y = \mathbb{R}^m$, is verified also by selecting $\bar{x} = x^0$, since then $X_d = P$. With this, all inequalities hold.

The non-trivial case is when $\rho_P(d)$ is finite. This implies that $C_Y \neq \mathbb{R}^m$. Set $w^0 \in C_Y$ such that $\|Ax^0 - b - w^0\| = \mathrm{dist}(Ax^0 - b, C_Y) > 0$ and let $r_{x^0} = \mathrm{dist}(x^0, \mathrm{rel}\partial P)$. With the above x^0 and w^0 use Lemma 6 to obtain the point (x, t, w, θ), feasible for (PP) and that from inequality (4.5) satisfies

$$0 < \frac{1}{\theta} \le \frac{\mathrm{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} \ . \tag{4.6}$$

Define the following:

$$\bar{x} = \frac{x + \theta x^0}{t + \theta} \ , \quad \bar{w} = \frac{w + \theta w^0}{t + \theta} \ , \quad r_{\bar{x}} = \frac{\theta r_{x^0}}{t + \theta} \ .$$

By construction $\mathrm{dist}(\bar{x}, \mathrm{rel}\partial P) \ge r_{\bar{x}}$ and $\bar{w} \in C_Y$. Note also that

$$A\bar{x} - b \ = \ \frac{1}{t + \theta}\left(Ax + \theta Ax^0\right) - b$$

$$
\begin{aligned}
&= \frac{1}{t+\theta}\left(Ax - tb - \theta\left(b - Ax^0 + w^0\right)\right) + \frac{\theta w^0}{t+\theta} \\
&= \frac{1}{t+\theta}\left(w + \theta w^0\right) \\
&= \bar{w} \in C_Y \ .
\end{aligned}
$$

Therefore the point $\bar{x} \in X_d$ and, if $r_{x^0} > 0$, then $\bar{x}$ is in the relative interior of P, such that $\text{dist}(\bar{x}, \text{rel}\partial P) \geq r_{\bar{x}}$. Note also that, by definition, $r_{\bar{x}} \leq r_{x^0}$.

To finish the proof, we just have to bound the different expressions from the statement of the theorem; here we make use of inequality (4.6):

1. (a) $\|\bar{x} - x^0\| = \dfrac{\|x - tx^0\|}{t+\theta} \leq \dfrac{1}{\theta}\max\{1, \|x^0\|\} \leq \dfrac{\text{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}\max\{1, \|x^0\|\}$.

   (b) $\|\bar{x}\| \leq \dfrac{1}{\theta}\|x\| + \|x^0\| \leq \dfrac{1}{\theta} + \|x^0\| \leq \|x^0\| + \dfrac{\text{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}$.

2. $\dfrac{1}{\text{dist}(\bar{x}, \text{rel}\partial P)} \leq \dfrac{1}{r_{\bar{x}}} = \dfrac{t+\theta}{\theta r_{x^0}} \leq \dfrac{1}{r_{x^0}}\left(1 + \dfrac{1}{\theta}\right) \leq \dfrac{1}{r_{x^0}}\left(1 + \dfrac{\text{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}\right)$.

3. (a)
$$
\begin{aligned}
\frac{\|\bar{x} - x^0\|}{\text{dist}(\bar{x}, \text{rel}\partial P)} &\leq \frac{\|x - tx^0\|}{\theta r_{x^0}} \leq \frac{1}{r_{x^0}}\frac{1}{\theta}\max\{1, \|x^0\|\} \\
&\leq \frac{1}{r_{x^0}}\frac{\text{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}\max\{1, \|x^0\|\} \ .
\end{aligned}
$$

   (b)
$$
\begin{aligned}
\frac{\|\bar{x}\|}{\text{dist}(\bar{x}, \text{rel}\partial P)} &\leq \frac{\|x + \theta x^0\|}{\theta r_{x^0}} \leq \frac{1}{r_{x^0}}\left(\|x^0\| + \frac{1}{\theta}\right) \\
&\leq \frac{1}{r_{x^0}}\left(\|x^0\| + \frac{\text{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}\right) \ .
\end{aligned}
$$
■

Remark 3 *Theorem 9 is valid when P is an affine set. The same proof shows that items 1.(a) and 1.(b) are still true, and the fact that* $\text{dist}(\bar{x}, \text{rel}\partial P) = \text{dist}(x^0, \text{rel}\partial P) = \infty$ *with the convention $\frac{1}{\infty} = 0$, implies that the bounds in 2, 3.(a), and 3.(b) have both sides equal to zero and are therefore trivially satisfied but uninformative.* ■

Theorem 9 shows that there exists a feasible point $\bar{x}$ which has some good geometric properties with respect to the ground set P. However, $\bar{x}$ could lie on the relative boundary of the feasible region X_d. The corollary below establishes the existence of a reliable solution for the primal problem, under the additional assumption that the cone C_Y is a subspace. This result shows that a Slater point exists which has norm, distance to relative boundary and ratio of these two quantities bounded by the distance to primal infeasibility and the properties of a relatively interior point $x^0 \in P$. Since Slater points belong to the relative interior of X_d this point is a reliable solution.

Corollary 2 *Suppose that C_Y is a subspace, $d \in \mathcal{F}_P$, $\rho_P(d) > 0$, and P is not an affine set. Let $x^0 \in \text{relint}P$ be given. Then there exists a Slater point $\bar{x}$ for (GP_d) which satisfies*

1. (a) $\|\bar{x} - x^0\| \le \dfrac{\text{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} \max\{1, \|x^0\|\}$

   (b) $\|\bar{x}\| \le \|x^0\| + \dfrac{\text{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}$

2. $$\frac{1}{\text{dist}(\bar{x}, \text{rel}\partial X_d)} \le \frac{1}{\text{dist}(x^0, \text{rel}\partial P)} \left(1 + \frac{\text{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}\right)$$

3. (a) $$\frac{\|\bar{x} - x^0\|}{\text{dist}(\bar{x}, \text{rel}\partial X_d)} \le \frac{1}{\text{dist}(x^0, \text{rel}\partial P)} \left(\frac{\text{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} \max\{1, \|x^0\|\}\right)$$

   (b) $$\frac{\|\bar{x}\|}{\text{dist}(\bar{x}, \text{rel}\partial X_d)} \le \frac{1}{\text{dist}(x^0, \text{rel}\partial P)} \left(\|x^0\| + \frac{\text{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}\right)$$

Proof: This is a consequence of Theorem 9 and the fact that if C_Y is a linear subspace, every point $w \in C_Y$ belongs to the relative interior of C_Y and $\text{dist}(w, \text{rel}\partial C_Y) = \infty$, which implies that $\text{dist}(\bar{x}, \text{rel}\partial X_d) = \text{dist}(\bar{x}, \text{rel}\partial P)$. ■

Remark 4 *Like Theorem 9, this corollary is also valid if P is an affine set, in which case bounds 2, 3.(a), and 3.(b) are uninformative.* ■

Remark 5 *The bound 1.(b) of Theorem 9 cannot be improved. Consider the example with data*

$$A = \begin{bmatrix} \frac{1}{2}t & t \\ 1 & 1 \end{bmatrix} \quad b = \begin{pmatrix} 1 \\ 2(1-t) \end{pmatrix} \quad c = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} ,$$

where $t \in [0,1)$, $C_Y = \mathbb{R}^2_+$ *and* $P = \left\{ x \in \mathbb{R}^2_+ \mid x_i \geq (1-\varepsilon) \ , \ i \in \{1,2\} \right\}$ *for any* $\varepsilon > 0$. *Consider also the* $\|\cdot\|_1$ *in* $\mathbb{R}^n$ *and* $\|\cdot\|_\infty$ *in* $\mathbb{R}^m$. *We show that for* $x^0 = (1,1) \in \operatorname{relint} P$, *every* $x \in X_d$ *must satisfy*

$$\|x\|_1 \geq \|x^0\|_1 + \frac{\operatorname{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} - \frac{1}{2}\varepsilon \ .$$

Proof: We show below that this example satisfies $\rho_P(d) = t$. Note that $\operatorname{dist}(Ax^0 - b, C_Y) = \operatorname{dist}\left((-1 + \frac{3}{2}t, 2t)^t, C_Y\right) = 1 - \frac{3}{2}t$, and thus

$$\|x^0\|_1 + \frac{\operatorname{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} = 2 + \frac{1 - \frac{3}{2}t}{t} = \frac{1}{2} + \frac{1}{t} \ .$$

If $x \in X_d$ then the first inequality of the constraints of the example implies that $\|x\|_1 = x_1 + x_2 \geq x_1 + \frac{1}{t} - \frac{1}{2}x_1 = \frac{1}{2}x_1 + \frac{1}{t}$. Now since $x \in P$ we can further bound this norm by

$$\|x\|_1 \geq \frac{1}{2}(1-\varepsilon) + \frac{1}{t} = \|x^0\|_1 + \frac{\operatorname{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} - \frac{1}{2}\varepsilon \ .$$

To complete the proof, we prove the claim that $\rho_P(d) = t$. The perturbation

$$\Delta A = \begin{bmatrix} -\frac{1}{2}t & -t \\ 0 & 0 \end{bmatrix}$$

makes the problem infeasible, and due to the norms chosen the size of this perturbation is $\|\Delta A\| = \sup_{\|x\|_1 \leq 1} \|\Delta A x\|_\infty = \max_{i,j \in \{1,2\}} |\Delta A_{i,j}| = t$. To show that any perturbation such that $\|\Delta d\| < t$ does not make the primal infeasible, we note that from Theorem 5 that is the same as showing that the system

$$\begin{aligned} & Ax - br - v \in C_Y \\ & \|x\|_1 + r \leq 1 \\ & x_i \geq (1-\varepsilon)r, \quad i \in \{1,2\} \end{aligned}$$

is feasible for any v such that $\|v\|_\infty < t$, which can be verified easily. ■

In what follows we extend the Slater point result of Corollary 2 to any cone C_Y. In order to do this the notion of minimum width of a cone, introduced in Definition 5, is central. Also, the following theorem will be used in the proof of the result.

Theorem 10 *(Theorem 6.3 of [31]) For any convex set* $Q \subseteq I\!\!R^n$*,* $\mathrm{cl}\ \mathrm{relint} Q = \mathrm{cl}\ Q$*, and* $\mathrm{relint}\ \mathrm{cl}\ Q = \mathrm{relint}\ Q$*.*

Theorem 11 *Suppose that* $d \in \mathcal{F}_P$*,* $\rho_P(d) > 0$*, and* P *is not an affine set. Let* $x^0 \in P$ *be given. Then there exists* $\bar{x} \in X_d$ *satisfying:*

1. (a) $\|\bar{x} - x^0\| \leq \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)} \max\{1, \|x^0\|\}$

   (b) $\|\bar{x}\| \leq \|x^0\| + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}$

2. (a) $\dfrac{1}{\mathrm{dist}(\bar{x}, \mathrm{rel}\partial P)} \leq \dfrac{1}{\mathrm{dist}(x^0, \mathrm{rel}\partial P)} \left(1 + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right)$

   (b) $\dfrac{1}{\mathrm{dist}(\bar{x}, \mathrm{rel}\partial X_d)} \leq \dfrac{1}{\min\left\{\mathrm{dist}(x^0, \mathrm{rel}\partial P), \tau_{C_Y}\right\}} \left(1 + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right)$

3. (a) $\dfrac{\|\bar{x} - x^0\|}{\mathrm{dist}(\bar{x}, \mathrm{rel}\partial P)} \leq \dfrac{1}{\mathrm{dist}(x^0, \mathrm{rel}\partial P)} \left(\dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)} \max\{1, \|x^0\|\}\right)$

   (b) $\dfrac{\|\bar{x} - x^0\|}{\mathrm{dist}(\bar{x}, \mathrm{rel}\partial X_d)} \leq \dfrac{1}{\min\left\{\mathrm{dist}(x^0, \mathrm{rel}\partial P), \tau_{C_Y}\right\}} \left(\dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)} \max\{1, \|x^0\|\}\right)$

   (c) $\dfrac{\|\bar{x}\|}{\mathrm{dist}(\bar{x}, \mathrm{rel}\partial P)} \leq \dfrac{1}{\mathrm{dist}(x^0, \mathrm{rel}\partial P)} \left(\|x^0\| + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right)$

$$(d)\ \frac{\|\bar{x}\|}{\text{dist}(\bar{x}, \text{rel}\partial X_d)} \leq \frac{1}{\min\{\text{dist}(x^0, \text{rel}\partial P), \tau_{C_Y}\}}\left(\|x^0\| + \frac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right)$$

Proof: In the case C_Y is a subspace, this is just a special case of Theorem 9. First note that, in this case, statement 2.a is equivalent to 2.b, 3.a is equivalent to 3b, and 3.c is equivalent to 3.d. Also note that $\text{dist}(Ax^0 - b, C_Y) \leq \|Ax^0 - b\| \leq \|Ax^0 - b\| + \|A\|$.

Therefore we only need to consider the case when C_Y is not a subspace. The proof is similar to the proof of Theorem 9. Note that now $\rho_P(d)$ is finite, for otherwise Proposition 4 shows that $C_Y = I\!R^m$ which is a subspace. We also assume $x^0 \notin X_d$, otherwise select $\bar{x} = x^0$ to satisfy all conditions. We now set $w^0 \in C_Y$ such that $\|w^0\| = \|A\|$ and $\tau_{C_Y} = \frac{\text{dist}(w^0, \text{rel}\partial C_Y)}{\|w^0\|}$. Let $r_{w^0} = \text{dist}(w^0, \text{rel}\partial C_Y) = \|A\|\tau_{C_Y}$ and let also $r_{x^0} = \text{dist}(x^0, \text{rel}\partial P)$. With the above x^0 and w^0 use Lemma 6 to obtain the point (x, t, w, θ), feasible for (PP) and that from inequality (4.5) satisfies

$$0 < \frac{1}{\theta} \leq \frac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)} . \tag{4.7}$$

Define the following:

$$\bar{x} = \frac{x + \theta x^0}{t + \theta}, \quad \bar{w} = \frac{w + \theta w^0}{t + \theta}, \quad r_{\bar{x}} = \frac{\theta r_{x^0}}{t + \theta}, \quad r_{\bar{w}} = \frac{\theta \tau_{C_Y}}{t + \theta}.$$

By construction $\text{dist}(\bar{x}, \text{rel}\partial P) \geq r_{\bar{x}}$, $\text{dist}(\bar{w}, \text{rel}\partial C_Y) \geq r_{\bar{w}}\|A\|$, and $A\bar{x} - b = \bar{w} \in C_Y$. Therefore the point $\bar{x} \in X_d$ and, if $r_{x^0} > 0$, $\bar{x}$ is in the relative interior of P, such that $\text{dist}(\bar{x}, \text{rel}\partial P) \geq r_{\bar{x}}$. Let us now show that it is in the relative interior of the feasible region and bound its distance to the boundary.

Consider any $v \in \hat{L}_P \cap \{y | Ay \in L_{C_Y}\}$ such that $\|v\| \leq 1$, then

$$\bar{x} + \alpha v \in P, \text{ for any } |\alpha| \leq r_{\bar{x}} ,$$

and

$$A(\bar{x}+\alpha v)-b=\bar{w}+\alpha(Av)\in C_Y,\ \text{ for any } |\alpha|\leq r_{\bar{w}}\ .$$

Therefore $(\bar{x}+\alpha v)\in X_d$ for any $|\alpha|\leq\min\{r_{\bar{x}},r_{\bar{w}}\}$, and the distance to the relative boundary of X_d is then $\operatorname{dist}(\bar{x},\operatorname{rel}\partial X_d)\geq|\alpha|\|v\|\geq|\alpha|$, for any $|\alpha|\leq\min\{r_{\bar{x}},r_{\bar{w}}\}$. Therefore $\operatorname{dist}(\bar{x},\operatorname{rel}\partial X_d)\geq\min\{r_{\bar{x}},r_{\bar{w}}\}\geq\frac{\theta\min\{r_{x^0},\tau_{C_Y}\}}{t+\theta}$.

To finish the proof, we just have to bound the different expressions from the statement of the theorem; here we make use of inequality (4.7):

1. (a) $\|\bar{x}-x^0\|=\dfrac{\|x-tx^0\|}{t+\theta}\leq\dfrac{1}{\theta}\max\{1,\|x^0\|\}\leq\dfrac{\|b-Ax^0\|+\|A\|}{\rho_P(d)}\max\{1,\|x^0\|\}\ .$

   (b) $\|\bar{x}\|\leq\dfrac{1}{\theta}\|x\|+\|x^0\|\leq\dfrac{1}{\theta}+\|x^0\|\leq\|x^0\|+\dfrac{\|b-Ax^0\|+\|A\|}{\rho_P(d)}\ .$

2. (a) $\dfrac{1}{\operatorname{dist}(\bar{x},\operatorname{rel}\partial P)}\leq\dfrac{1}{r_{\bar{x}}}=\dfrac{t+\theta}{\theta r_{x^0}}\leq\dfrac{1}{r_{x^0}}\left(1+\dfrac{1}{\theta}\right)\leq\dfrac{1}{r_{x^0}}\left(1+\dfrac{\|b-Ax^0\|+\|A\|}{\rho_P(d)}\right)\ .$

   (b) $$\begin{aligned}\frac{1}{\operatorname{dist}(\bar{x},\operatorname{rel}\partial X_d)}&\leq\frac{1}{\min\{r_{x^0},\tau_{C_Y}\}}\frac{t+\theta}{\theta}\leq\frac{1}{\min\{r_{x^0},\tau_{C_Y}\}}\left(1+\frac{1}{\theta}\right)\\&\leq\frac{1}{\min\{r_{x^0},\tau_{C_Y}\}}\left(1+\frac{\|b-Ax^0\|+\|A\|}{\rho_P(d)}\right)\ .\end{aligned}$$

3. (a) $$\begin{aligned}\frac{\|\bar{x}-x^0\|}{\operatorname{dist}(\bar{x},\operatorname{rel}\partial P)}&\leq\frac{\|x-tx^0\|}{\theta r_{x^0}}\leq\frac{1}{r_{x^0}}\frac{1}{\theta}\max\{1,\|x^0\|\}\\&\leq\frac{1}{r_{x^0}}\frac{\|b-Ax^0\|+\|A\|}{\rho_P(d)}\max\{1,\|x^0\|\}\ .\end{aligned}$$

   (b) $$\begin{aligned}\frac{\|\bar{x}-x^0\|}{\operatorname{dist}(\bar{x},\operatorname{rel}\partial X_d)}&\leq\frac{\|x-tx^0\|}{\theta\min\{r_{x^0},\tau_{C_Y}\}}\leq\frac{1}{\min\{r_{x^0},\tau_{C_Y}\}}\frac{1}{\theta}\max\{1,\|x^0\|\}\\&\leq\frac{1}{\min\{r_{x^0},\tau_{C_Y}\}}\frac{\|b-Ax^0\|+\|A\|}{\rho_P(d)}\max\{1,\|x^0\|\}\ .\end{aligned}$$

   (c) $$\begin{aligned}\frac{\|\bar{x}\|}{\operatorname{dist}(\bar{x},\operatorname{rel}\partial P)}&\leq\frac{\|x+\theta x^0\|}{\theta r_{x^0}}\leq\frac{1}{r_{x^0}}\left(\|x^0\|+\frac{1}{\theta}\right)\\&\leq\frac{1}{r_{x^0}}\left(\|x^0\|+\frac{\|b-Ax^0\|+\|A\|}{\rho_P(d)}\right)\ .\end{aligned}$$

$$\text{(d)}\quad \frac{\|\bar{x}\|}{\operatorname{dist}(\bar{x}, \operatorname{rel}\partial X_d)} \le \frac{\|x+\theta x^0\|}{\theta \min\{r_{x^0}, \tau_{C_Y}\}} \le \frac{1}{\min\{r_{x^0}, \tau_{C_Y}\}}\left(\|x^0\| + \frac{1}{\theta}\right)$$
$$\le \frac{1}{\min\{r_{x^0}, \tau_{C_Y}\}}\left(\|x^0\| + \frac{\|b - Ax^0\| + \|A\|}{\rho_P(d)}\right). \qquad \blacksquare$$

Remark 6 *Theorem 11 is valid when P is an affine set using essentially the same proof. In this case both sides of bounds 2.(a) and 3.(a) are zero; if in addition C_Y is a subspace, then bounds 2.(b) and 3.(b) are also uninformative.* $\blacksquare$

The next result, which is a direct consequence of Theorem 11, guarantees the existence of a reliable solution for the primal problem, $\bar{x} \in X_d$, under the hypothesis that the primal problem is feasible and $\rho_P(d) > 0$.

Corollary 3 *Suppose that $d \in \mathcal{F}_P$, $\rho_P(d) > 0$, and P is not an affine set. Then, given $x^0 \in \operatorname{relint} P$, there exists a Slater point $\bar{x}$ for (GP_d) which satisfies:*

1. $\|\bar{x}\| \le \|x^0\| + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}$

2. $\dfrac{1}{\operatorname{dist}(\bar{x}, \operatorname{rel}\partial X_d)} \le \dfrac{1}{\min\{\operatorname{dist}(x^0, \operatorname{rel}\partial P), \tau_{C_Y}\}}\left(1 + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right)$

3. $\dfrac{\|\bar{x}\|}{\operatorname{dist}(\bar{x}, \operatorname{rel}\partial X_d)} \le \dfrac{1}{\min\{\operatorname{dist}(x^0, \operatorname{rel}\partial P), \tau_{C_Y}\}}\left(\|x^0\| + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right).$ $\blacksquare$

Remark 7 *Corollary 3 can also be proved when P is an affine set. If in addition C_Y is a subspace, then bounds 2 and 3 are uninformative.* $\blacksquare$

We end this section by presenting a pair of theorems which restate Theorem 11, emphasizing how the geometric properties of the point $x^0 \in \operatorname{relint} P$ impact the geometric properties of the reliable solution $\bar{x} \in X_d$ and the geometry of the feasible region, through condition number quantities.

For $x^0 \in \text{relint}P$, let us define the following measure

$$
\begin{aligned}
g_{P,C_Y}(x^0) := & \frac{\max\{\|x^0\|, 1\}}{\min\{1, \text{dist}(x^0, \text{rel}\partial P), \tau_{C_Y}\}} \\
= & \max\left\{\|x^0\|, \frac{\|x^0\|}{\text{dist}(x^0, \text{rel}\partial P)}, \frac{1}{\text{dist}(x^0, \text{rel}\partial P)}, \frac{\|x^0\|}{\tau_{C_Y}}, \frac{1}{\tau_{C_Y}}, 1\right\} .
\end{aligned}
$$

Theorem 12 *Suppose that $d \in \mathcal{F}_P$, $\rho_P(d) > 0$, and P is not an affine set. Let $x^0 \in \text{relint}P$ be given. Then there exists $\bar{x} \in X_d$ satisfying:*

1. (a) $\|\bar{x} - x^0\| \leq g_{P,C_Y}(x^0) \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}$

   (b) $\|\bar{x}\| \leq g_{P,C_Y}(x^0) \left(1 + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right)$

2. (a) $\dfrac{1}{\text{dist}(\bar{x}, \text{rel}\partial P)} \leq g_{P,C_Y}(x^0) \left(1 + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right)$

   (b) $\dfrac{1}{\text{dist}(\bar{x}, \text{rel}\partial X_d)} \leq g_{P,C_Y}(x^0) \left(1 + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right)$

3. (a) $\dfrac{\|\bar{x} - x^0\|}{\text{dist}(\bar{x}, \text{rel}\partial P)} \leq g_{P,C_Y}(x^0) \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}$

   (b) $\dfrac{\|\bar{x} - x^0\|}{\text{dist}(\bar{x}, \text{rel}\partial X_d)} \leq g_{P,C_Y}(x^0) \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}$

   (c) $\dfrac{\|\bar{x}\|}{\text{dist}(\bar{x}, \text{rel}\partial P)} \leq g_{P,C_Y}(x^0) \left(1 + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right)$

   (d) $\dfrac{\|\bar{x}\|}{\text{dist}(\bar{x}, \text{rel}\partial X_d)} \leq g_{P,C_Y}(x^0) \left(1 + \dfrac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right)$. ■

Let us now define a geometric measure of the feasible region X_d:

$$
g_{X_d} := \min_{x \in X_d} \max\left\{\|x\|, \frac{\|x\|}{\text{dist}(x, \text{rel}\partial X_d)}, \frac{1}{\text{dist}(x, \text{rel}\partial X_d)}\right\} .
$$

The following theorem is a byproduct of Theorem 12.

Theorem 13 *Suppose $d \in \mathcal{F}_P$, $\rho_P(d) > 0$, and P is not an affine set. Let $x^0 \in \text{relint}P$ be given. Then:*

$$g_{X_d} \leq g_{P,C_Y}(x^0)\left(1 + \frac{\|Ax^0 - b\| + \|A\|}{\rho_P(d)}\right) . \qquad \blacksquare$$

Remark 8 *Under the additional assumption that C_Y is a subspace, the measure $g_{P,C_Y}(x^0)$ becomes*

$$g_{P,C_Y}(x^0) = \max\left\{\|x^0\|, \frac{\|x^0\|}{\text{dist}(x^0, \text{rel}\partial P)}, \frac{1}{\text{dist}(x^0, \text{rel}\partial P)}, 1\right\} ,$$

and we can replace $\|Ax^0 - b\| + \|A\|$ by $\text{dist}(Ax^0 - b, C_Y)$ in Theorems 12 and 13.

Finally, Theorems 12 and 13 are valid when P is an affine set because Theorems 9 and 11 are valid in this case. $\blacksquare$

4.3 Solutions in the relative interior of Y_d

This section presents a number of results that relate the dual distance to infeasibility $\rho_D(d)$, and a known point $y^0 \in C_Y^*$, to the existence of a feasible dual solution $(\bar{y}, \bar{u})$ in the relative interior of Y_d that has nice geometric properties.

Here we prove that under the hypothesis that $\rho_D(d) > 0$ there exists a Slater point in Y_d which has good geometric properties. Recall that a point $(\bar{y}, \bar{u})$ is a Slater point for (GD_d) if $(c - A^t\bar{y}, \bar{u}) \in \text{relint}C^*$ and $\bar{y} \in \text{relint}C_Y^*$. It then follows that if $(\bar{y}, \bar{u})$ is a Slater point it also satisfies $(\bar{y}, \bar{u}) \in \text{relint}Y_d$.

Recall also that for points in the dual feasible region $(y, u) \in Y_d$, we use the dual norm in $\mathbb{R}^m \times \mathbb{R}$

$$\|(y, u)\|_* = \|y\|_* + |u| .$$

The results below, like the results in Section 4.2, involve the distance from a point $x \in S$ to the relative boundary of the set S. In this section, for a reliable dual point $(\bar{y}, \bar{u}) \in Y_d$, we bound either $\mathrm{dist}(\bar{y}, \mathrm{rel}\partial C_Y^*)$ or $\mathrm{dist}((\bar{y}, \bar{u}), \mathrm{rel}\partial Y_d)$ with $\mathrm{dist}(y^0, \mathrm{rel}\partial C_Y^*)$, for a given point $y^0 \in C_Y^*$. Therefore, when the cone C_Y is a subspace, which is equivalent to C_Y^* being a subspace, the quantity to be bounded and the bound used can become infinity. For clarity of exposition, we avoid this special case, which can be dealt with in a straightforward but careful analysis, and present here the results for the case when C_Y is not a subspace. For each result we will also mention what can be proved without this additional assumption.

The results in this section are, like the ones in Section 4.2, also an extension of Lemma 2 and Lemma 3 of [14] to problems of a more general format. First we present a technical lemma that is used to relate feasible solutions in Y_d with the distance to dual infeasibility of the problem. This lemma makes use of the program (DP), which we now define.

For given points $y^0 \in C_Y^*$ and $s^0 \in R^*$, define problem (DP) by

$$\begin{array}{lll} (DP) & \max_{y,\delta,s,\theta} & \theta \\ & \text{s.t.} & -A^t y + \delta c - s = \theta\,(A^t y^0 - c + s^0) \\ & & \|y\|_* + |\delta| \le 1 \\ & & y \in C_Y^* \\ & & \delta \ge 0 \\ & & s \in R^*. \end{array} \tag{4.8}$$

Lemma 7 *Suppose that $d \in \mathcal{F}_D$ and $\rho_D(d) > 0$. Given $y^0 \in C_Y^*$ and $s^0 \in R^*$ such that $A^t y^0 + s^0 \neq c$, then there is a point (y, δ, s, θ) feasible for problem (DP) that satisfies*

$$\theta \ge \frac{\rho_D(d)}{\|A^t y^0 - c + s^0\|_*} > 0\ . \tag{4.9}$$

Proof: Note that problem (DP) is feasible for any y^0 and s^0 since $(y, \delta, s, \theta) = (0, 0, 0, 0)$ is always feasible. Therefore it can either be unbounded or have a finite optimal objective value. If (DP) is unbounded, we can find feasible points with an objective function large enough such that (4.9) holds. If (DP) has a finite optimal value, say z^*, then it attains this value since it is a linear objective over a bounded domain (add bounds on θ, for example "$0 \leq \theta \leq z^* + 1$", which makes the domain bounded). From Theorem 6, the optimal solution $(y^*, \delta^*, s^*, \theta^*)$ for (DP) satisfies (4.9). ■

The following result establishes the existence of a feasible point $(\bar{y}, \bar{u}) \in Y_d$ that has norm and distance to the boundary of C_Y^* bounded by properties of a known point $y^0 \in C_Y^*$.

Theorem 14 *Suppose that $d \in \mathcal{F}_D$, $\rho_D(d) > 0$, and C_Y is not a subspace. Let $y^0 \in C_Y^*$ be given. Then for any $\varepsilon > 0$, there exists $(\bar{y}, \bar{u}) \in Y_d$ that satisfies the following:*

1. (a) $$\|\bar{y} - y^0\|_* \leq \frac{\text{dist}(c - A^t y^0, R^*) + \varepsilon}{\rho_D(d)} \max\{1, \|y^0\|_*\}$$

   (b) $$\|\bar{y}\|_* \leq \|y^0\|_* + \frac{\text{dist}(c - A^t y^0, R^*) + \varepsilon}{\rho_D(d)}$$

2. $$\frac{1}{\text{dist}(\bar{y}, \text{rel}\partial C_Y^*)} \leq \frac{1}{\text{dist}(y^0, \text{rel}\partial C_Y^*)} \left(1 + \frac{\text{dist}(c - A^t y^0, R^*) + \varepsilon}{\rho_D(d)}\right)$$

3. (a) $$\frac{\|\bar{y} - y^0\|_*}{\text{dist}(\bar{y}, \text{rel}\partial C_Y^*)} \leq \frac{1}{\text{dist}(y^0, \text{rel}\partial C_Y^*)} \left(\frac{\text{dist}(c - A^t y^0, R^*) + \varepsilon}{\rho_D(d)} \max\{1, \|y^0\|_*\}\right)$$

   (b) $$\frac{\|\bar{y}\|_*}{\text{dist}(\bar{y}, \text{rel}\partial C_Y^*)} \leq \frac{1}{\text{dist}(y^0, \text{rel}\partial C_Y^*)} \left(\|y^0\|_* + \frac{\text{dist}(c - A^t y^0, R^*) + \varepsilon}{\rho_D(d)}\right)$$

Proof: We assume that $c - A^t y^0 \notin \text{dom} u(\cdot) \subset R^*$, from Proposition 2, otherwise select $\bar{y} = y^0$ and $\bar{u} = u(c - A^t y^0)$ to satisfy the assertions of the theorem. The case $\rho_D(d) = \infty$, which implies $R^* = \mathbb{R}^n$, is also verified by selecting $\bar{y} := y^0$ and $\bar{u} = u(c - A^t y^0)$, since then $c - A^t y^0 \in \mathbb{R}^n = R^* = \text{dom} u(\cdot)$.

Consider now the case when $\rho_D(d)$ is finite. This case implies that $R^* \neq \mathbb{R}^n$. Set $s^0 \in \text{relint} R^*$ such that $0 < \|c - A^t y^0 - s^0\|_* \leq \text{dist}(c - A^t y^0, R^*) + \varepsilon$, let $r_{y^0} = \text{dist}(y^0, \text{rel}\partial C_Y^*)$, and let $r_{s^0} = \text{dist}(s^0, \text{rel}\partial R^*)$. With the points y^0 and s^0 above, invoke Lemma 7 to obtain a point (y, δ, s, θ), feasible for (DP) and that from inequality (4.9) satisfies

$$0 < \frac{1}{\theta} \leq \frac{\text{dist}(c - A^t y^0, R^*) + \varepsilon}{\rho_D(d)} \ . \tag{4.10}$$

Define the following:

$$\bar{y} = \frac{y + \theta y^0}{\delta + \theta} \ , \quad \bar{s} = \frac{s + \theta s^0}{\delta + \theta} \ , \quad r_{\bar{y}} = \frac{\theta r_{y^0}}{\delta + \theta} \ , \quad r_{\bar{s}} = \frac{\theta r_{s^0}}{\delta + \theta} \ .$$

By construction $\text{dist}(\bar{y}, \text{rel}\partial C_Y^*) \geq r_{\bar{y}}$ and $\text{dist}(\bar{s}, \text{rel}\partial R^*) \geq r_{\bar{s}}$. We now show that $u(\bar{s})$ is finite, by considering two cases. If $R^* = \{0\}$, then $c - A^t \bar{y} = \bar{s} = 0$ and therefore $u(\bar{s}) = 0$. If $R^* \neq \{0\}$, then any point in $\text{relint} R^*$ has a positive relative distance to the boundary, therefore $r_{s^0} > 0$. This and (4.10) show that $r_{\bar{s}} > 0$, which in turn implies $c - A^t \bar{y} = \bar{s} \in \text{relint} R^* \subseteq \text{dom } u(\cdot)$, because $\text{relint} R^* = \text{relint cl dom } u(\cdot) = \text{relint dom } u(\cdot) \subseteq \text{dom } u(\cdot)$. The first equality here follows from Proposition 2 and the second equality is from Theorem 10. Therefore, in both cases $u(\bar{s})$ is finite, and so $(\bar{y}, \bar{u}) = (\bar{y}, u(\bar{s})) \in Y_d$.

To finish the proof, we just have to bound the different expressions from the statement of the theorem. Here we make use of inequality (4.10):

1. (a) $$\|\bar{y} - y^0\|_* = \frac{\|y - \delta y^0\|_*}{\delta + \theta} \leq \frac{1}{\theta} \max\{1, \|y^0\|_*\} \leq \frac{\text{dist}(c - A^t y^0, R^*) + \varepsilon}{\rho_D(d)} \max\{1, \|y^0\|_*\} \ .$$

   (b) $$\|\bar{y}\|_* \leq \frac{1}{\theta}\|y\|_* + \|y^0\|_* \leq \frac{1}{\theta} + \|y^0\|_* \leq \|y^0\|_* + \frac{\text{dist}(c - A^t y^0, R^*) + \varepsilon}{\rho_D(d)} \ .$$

2. $\dfrac{1}{\text{dist}(\bar{y}, \text{rel}\partial C_Y^*)} \le \dfrac{1}{r_{\bar{y}}} = \dfrac{\delta+\theta}{\theta r_{y^0}} \le \dfrac{1}{r_{y^0}}\left(1+\dfrac{1}{\theta}\right) \le \dfrac{1}{r_{y^0}}\left(1+\dfrac{\text{dist}(c-A^ty^0, R^*)+\varepsilon}{\rho_D(d)}\right)$.

3. (a)
$$\frac{\|\bar{y}-y^0\|_*}{\text{dist}(\bar{y}, \text{rel}\partial C_Y^*)} \le \frac{\|y-\delta y^0\|_*}{\theta r_{y^0}} \le \frac{1}{r_{y^0}\theta}\max\{1, \|y^0\|_*\}$$
$$\le \frac{1}{r_{y^0}}\frac{\text{dist}(c-A^ty^0, R^*)+\varepsilon}{\rho_D(d)}\max\{1, \|y^0\|_*\} \ .$$

(b)
$$\frac{\|\bar{y}\|_*}{\text{dist}(\bar{y}, \text{rel}\partial C_Y^*)} \le \frac{\|y+\theta y^0\|_*}{\theta r_{y^0}} \le \frac{1}{r_{y^0}}\left(\|y^0\|_* + \frac{1}{\theta}\right)$$
$$\le \frac{1}{r_{y^0}}\left(\|y^0\|_* + \frac{\text{dist}(c-A^ty^0, R^*)+\varepsilon}{\rho_D(d)}\right) \ .$$
■

Remark 9 *Theorem 14 is valid in the case that the cone C_Y is a subspace. The same proof can be used to show that items* 1.(a) *and* 1.(b) *hold, while the fact that* $\text{dist}(\bar{y}, \text{rel}\partial C_Y^*) = \text{dist}(y^0, \text{rel}\partial C_Y^*) = \infty$ *implies that both sides in bounds* 2, 3.(a), *and* 3.(b) *are zero, and therefore are trivially satisfied but uninformative.* ■

Theorem 14 shows that there exists a feasible point $(\bar{y}, \bar{u})$ which has good geometric properties with respect to C_Y^*. However, $(\bar{y}, \bar{u})$ could lie on the boundary of the feasible region Y_d. The corollary below establishes the existence of a reliable solution for the dual problem, under the additional assumption that the cone R is a subspace. This result shows that a Slater point exists which has norm, distance to relative boundary and ratio of these two quantities bounded by the distance to dual infeasibility and the properties of a relatively interior point $y^0 \in C_Y^*$. Since a Slater point belongs to the relative interior of Y_d this point is a reliable solution.

Corollary 4 *Suppose that R is a subspace, $d \in \mathcal{F}_D$, $\rho_D(d) > 0$, and C_Y is not a subspace. Let $y^0 \in \text{relint}C_Y^*$ be given. Then for any $\varepsilon > 0$, there exists a Slater point $(\bar{y}, \bar{u})$ for (GD_d), which satisfies*

1. *(a)* $\|\bar{y}-y^0\|_* \le \dfrac{\text{dist}(c-A^ty^0, R^*)+\varepsilon}{\rho_D(d)}\max\{1, \|y^0\|_*\}$

(b) $\|\bar{y}\|_* \le \|y^0\|_* + \dfrac{\text{dist}(c-A^ty^0, R^*)+\varepsilon}{\rho_D(d)}$

2. $\dfrac{1}{\text{dist}((\bar{y},\bar{u}),\text{rel}\partial Y_d)} \leq \dfrac{\max\{1,\|A\|\}}{\text{dist}(y^0,\text{rel}\partial C_Y^*)}\left(1+\dfrac{\text{dist}(c-A^ty^0,R^*)+\varepsilon}{\rho_D(d)}\right)$

3. (a) $\dfrac{\|\bar{y}-y^0\|_*}{\text{dist}((\bar{y},\bar{u}),\text{rel}\partial Y_d)} \leq \dfrac{\max\{1,\|A\|\}}{\text{dist}(y^0,\text{rel}\partial C_Y^*)}\left(\dfrac{\text{dist}(c-A^ty^0,R^*)+\varepsilon}{\rho_D(d)}\max\{1,\|y^0\|_*\}\right)$

   (b) $\dfrac{\|\bar{y}\|_*}{\text{dist}((\bar{y},\bar{u}),\text{rel}\partial Y_d)} \leq \dfrac{\max\{1,\|A\|\}}{\text{dist}(y^0,\text{rel}\partial C_Y^*)}\left(\|y^0\|_*+\dfrac{\text{dist}(c-A^ty^0,R^*)+\varepsilon}{\rho_D(d)}\right)$

Proof: Recall that for any set S, if S is a subspace, then so is S^*, and any $s \in S^*$ belongs to the relative interior of S^*, and $\text{dist}(s,\text{rel}\partial S^*) = \infty$. Consider the points $\bar{y} \in \text{relint}C_Y^*$ and $\bar{s} = c - A^t\bar{y} \in \text{dom } u(\cdot)$ constructed in Theorem 14. The fact that R is a subspace implies that R^* is a subspace and $R^* = \text{dom } u(\cdot)$. This equality follows because $R^* = \text{relint } R^* = \text{relint cl dom } u(\cdot) = \text{relint dom } u(\cdot) \subseteq \text{dom } u(\cdot) \subseteq \text{cl dom } u(\cdot) = R^*$, where the second and third equalities here follow from Proposition 2 and Theorem 10, respectively.

For a scalar $\kappa \geq 0$, define the function $\mu(\kappa)$ by

$$\begin{array}{rcll} \mu(\kappa) := & \kappa & + & \sup_s \quad u(s) \\ & & & \|s-c+A^t\bar{y}\|_* \leq \kappa \\ & & & s \in R^* \ . \end{array}$$

For every $\kappa \geq 0$, this function $\mu(\kappa)$ is finite, since $u(\cdot)$ is continuous on the relative interior of its effective domain, see Theorem 10.1 of [31], which here is R^* as noted above. By construction $\text{dist}((\bar{s},\mu(\kappa)),\text{rel}\partial C^*) \geq \kappa$, and the point $(\bar{s},\mu(\kappa)) \in \text{relint}C^*$. Therefore for every $\kappa > 0$ the point $(\bar{y},\mu(\kappa))$ is a Slater point of Y_d. From item 3 of Lemma 5, we can bound the distance of this Slater point to the relative boundary of Y_d:

$$\begin{aligned} \text{dist}((\bar{y},\mu(\kappa)),\text{rel}\partial Y_d) &\geq \frac{1}{\max\{1,\|A\|\}}\min\left\{\text{dist}\left((\bar{s},\mu(\kappa)),\text{rel}\partial C^*\right),\text{dist}\left(\bar{y},\text{rel}\partial C_Y^*\right)\right\} \\ &\geq \frac{1}{\max\{1,\|A\|\}}\min\{\kappa,\ \text{dist}\left(\bar{y},\text{rel}\partial C_Y^*\right)\} \ . \end{aligned}$$

Setting $\kappa = \text{dist}(\bar{y},\text{rel}\partial C_Y^*)$, $\bar{u} = \mu(\kappa)$, and using Theorem 14 yields the corollary. ■

Remark 10 *If the cone C_Y is a subspace then the same proof can be used to show that items* 1.(a) *and* 1.(b) *hold. In this case however, bounds* 2*,* 3.(a)*, and* 3.(b) *are not valid. But following the proof of Corollary 4, using the same $\bar{y}$ and setting $\bar{u} = \mu(\kappa)$ for a large enough κ, we can ensure that $(\bar{y}, \bar{u}) \in Y_d$ satisfies instead*

2'. $\dfrac{1}{\text{dist}((\bar{y}, \bar{u}), \text{rel}\partial Y_d)} \leq \varepsilon.$

3'. (a) $\dfrac{\|\bar{y} - y^0\|_*}{\text{dist}((\bar{y}, \bar{u}), \text{rel}\partial Y_d)} \leq \varepsilon.$

(b) $\dfrac{\|\bar{y}\|_*}{\text{dist}((\bar{y}, \bar{u}), \text{rel}\partial Y_d)} \leq \varepsilon.$ ■

The following remark shows that Theorem 14 finds a dual feasible solution whose norm can be bounded by $C(d)$ plus an arbitrarily small term. This bound is similar to what is obtained in the conic case, where there exists a feasible solution with norm bounded by $C(d)$, and illustrates a difference between the dual and the primal ground set problem, where the bounds on the norms of feasible solutions need the norm of a point in P, as illustrated by Remark 5.

Remark 11 *Suppose that $d \in \mathcal{F}_D$ and $\rho_D(d) > 0$. Then for any $\varepsilon > 0$ there exists $(\bar{y}, \bar{u}) \in Y_d$ such that*

$$\|\bar{y}\|_* \leq \frac{\|c\|_* + \varepsilon}{\rho_D(d)} .$$

Proof: Observe that if $y^0 = 0$ then the proof of Theorem 14 holds, and the only informative bounds are items 1.a and 1.b, which are equivalent and imply this remark. This argument is also valid in the case when C_Y is a subspace. ■

In what follows we extend the Slater point result of Corollary 4 to any cone R. Now the quantity that will play a central role is the minimum width of the cone R^*.

Theorem 15 *Suppose $d \in \mathcal{F}_D$, $\rho_D(d) > 0$, and C_Y is not a subspace. Let $y^0 \in C_Y^*$ be given and $r_{y^0} = \mathrm{dist}(y^0, \mathrm{rel}\partial C_Y^*)$. For any $\varepsilon > 0$, there exists $(\bar{y}, \bar{u}) \in Y_d$ with the following properties:*

1. (a) $\|\bar{y} - y^0\|_* \le \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)} \max\{1, \|y^0\|_*\}$

   (b) $\|\bar{y}\|_* \le \|y^0\|_* + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}$

2. (a) $\dfrac{1}{\mathrm{dist}(\bar{y}, \mathrm{rel}\partial C_Y^*)} \le \dfrac{1}{\mathrm{dist}(y^0, \mathrm{rel}\partial C_Y^*)} \left(1 + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right)$

   (b) $\dfrac{1}{\mathrm{dist}((\bar{y}, \bar{u}), \mathrm{rel}\partial Y_d)} \le \dfrac{(1+\varepsilon)\max\{1, \|A\|\}}{\min\left\{\mathrm{dist}(y^0, \mathrm{rel}\partial C_Y^*), \tau_{R^*}\right\}} \left(1 + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right)$

3. (a) $\dfrac{\|\bar{y} - y^0\|_*}{\mathrm{dist}(\bar{y}, \mathrm{rel}\partial C_Y^*)} \le \dfrac{1}{\mathrm{dist}(y^0, \mathrm{rel}\partial C_Y^*)} \left(\dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)} \max\{1, \|y^0\|_*\}\right)$

   (b) $$\frac{\|\bar{y} - y^0\|_*}{\mathrm{dist}((\bar{y}, \bar{u}), \mathrm{rel}\partial Y_d)} \le$$
   $$\le \frac{(1+\varepsilon)\max\{1, \|A\|\}}{\min\left\{\mathrm{dist}(y^0, \mathrm{rel}\partial C_Y^*), \tau_{R^*}\right\}} \left(\frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)} \max\{1, \|y^0\|_*\}\right)$$

   (c) $\dfrac{\|\bar{y}\|_*}{\mathrm{dist}(\bar{y}, \mathrm{rel}\partial C_Y^*)} \le \dfrac{1}{\mathrm{dist}(y^0, \mathrm{rel}\partial C_Y^*)} \left(\|y^0\|_* + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right)$

   (d) $$\frac{\|\bar{y}\|_*}{\mathrm{dist}((\bar{y}, \bar{u}), \mathrm{rel}\partial Y_d)} \le$$
   $$\le \frac{(1+\varepsilon)\max\{1, \|A\|\}}{\min\left\{\mathrm{dist}(y^0, \mathrm{rel}\partial C_Y^*), \tau_{R^*}\right\}} \left(\|y^0\|_* + \frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right)$$

Proof: If R is a subspace we have the hypothesis needed to invoke Corollary 4. Define $\varepsilon' \le \min\{\|A\|, \varepsilon\}$, and use this ε' in Theorem 14 and Corollary 4 to obtain a point $(\bar{y}, \bar{u}) \in Y_d$ which, by the proof of the corollary, also satisfies Theorem 14. The bounds in Corollary 4 and Theorem 14, and the fact that $\mathrm{dist}(c - A^t y^0, R^*) + \varepsilon' \le \|c - A^t y^0\|_* + \|A\|$ imply this theorem.

Therefore we consider the case when R is not a subspace. The following is similar to the proof of Theorem 14. Note that now $\rho_D(d)$ is finite since otherwise $R = \{0\}$ which is a subspace. Set $s^0 \in R^*$ such that $\|s^0\|_* = \|A\|$ and $\tau_{R^*} = \frac{\text{dist}(s^0, \text{rel}\partial R^*)}{\|s^0\|_*}$. We also assume for now that $c - A^t y^0 \neq s^0$. We show later in the proof how to handle the case when $c - A^t y^0 = s^0$. Denote $r_{y^0} = \text{dist}(y^0, \text{rel}\partial C_Y^*)$, and $r_{s^0} = \text{dist}(s^0, \text{rel}\partial R^*) = \tau_{R^*}\|A\| > 0$.

With the points y^0 and s^0, use Lemma 7 to obtain a point (y, δ, s, θ) feasible for (DP) such that from inequality (4.9) satisfies

$$0 < \frac{1}{\theta} \leq \frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)} \ . \tag{4.11}$$

Define the following:

$$\bar{y} = \frac{y + \theta y^0}{\delta + \theta}, \quad \bar{s} = \frac{s + \theta s^0}{\delta + \theta}, \quad r_{\bar{y}} = \frac{\theta r_{y^0}}{\delta + \theta}, \quad r_{\bar{s}} = \frac{\theta r_{s^0}}{\delta + \theta}.$$

By construction $\text{dist}(\bar{y}, \text{rel}\partial C_Y^*) \geq r_{\bar{y}}$, $\text{dist}(\bar{s}, \text{rel}\partial R^*) \geq r_{\bar{s}}$, and $c - A^t\bar{y} = \bar{s}$. Therefore the point $(\bar{y}, u(\bar{s})) \in Y_d$. We now will choose $\bar{u}$ so that $(\bar{y}, \bar{u}) \in \text{relint} Y_d$ and bound its distance to the relative boundary. Since relint $R^* \subseteq$ dom $u(\cdot)$, from Proposition 2 and Theorem 10, we have that for any $\varepsilon > 0$, the ball $B\left(\bar{s}, \frac{r_{\bar{s}}}{1+\varepsilon}\right) \cap L_{R^*} \subset$ relint dom $u(\cdot)$. Similar to the function $\mu(\cdot)$ defined in Corollary 4, define the function

$$\begin{array}{rlll} \tilde{\mu}(\bar{s}, \kappa) := & \frac{1}{\|A\|}\kappa & + & \sup_s u(s) \\ & & & \|s - \bar{s}\|_* \leq \kappa \\ & & & s \in R^* \end{array} .$$

This function $\tilde{\mu}(\cdot, \cdot)$ is finite for every $\bar{s} \in$ relint dom $u(\cdot)$ and $\kappa \in [0, \text{dist}(\bar{s}, \text{rel}\partial R^*))$. It is finite for these arguments because the supremum is over a set contained in the relative interior of the effective domain of $u(\cdot)$, where $u(\cdot)$ is continuous, see Theorem 10.1 of [31]. With this function $\tilde{\mu}$, we can define $\bar{u} = \tilde{\mu}\left(\bar{s}, \frac{r_{\bar{s}}}{1+\varepsilon}\right)$, since by construction $r_{\bar{s}} > 0$. Therefore the point $(c - A^t\bar{y}, \bar{u}) = (\bar{s}, \bar{u}) \in C^*$ belongs to the relative interior of C^*, since the norm on C^* is $\|(s, u)\| = \max\{\|s\|_*, |u|\}$.

Let us now show that $(\bar{y}, \bar{u}) \in \operatorname{relint} Y_d$, and bound its distance to the relative boundary. Consider any vector $v \in L_{C_Y^*} \cap \{y | -A^t y \in L_{R^*}\}$ such that $\|v\|_* \leq 1$, then

$$\bar{y} + \alpha v \in C_Y^* \text{ for any } |\alpha| \leq r_{\bar{y}} ,$$

and

$$c - A^t(\bar{y} + \alpha v) = \bar{s} + \alpha(-A^t v) \in B\left(\bar{s}, \frac{r_{\bar{s}}}{1+\varepsilon}\right) \cap L_{R^*} \text{ for any } |\alpha| \leq \frac{r_{\bar{s}}}{\|A\|(1+\varepsilon)} .$$

This last inclusion implies that $(c - A^t(\bar{y} + \alpha v), \bar{u} + \beta) = (\bar{s} + \alpha(-A^t v), \bar{u} + \beta) \in C^*$ for any $|\alpha|, |\beta| \leq \frac{r_{\bar{s}}}{\|A\|(1+\varepsilon)}$. We have shown that $\operatorname{dist}(\bar{y}, \operatorname{rel}\partial C_Y^*) \geq r_{\bar{y}}$ and $\operatorname{dist}((c - A^t\bar{y}, \bar{u}), \operatorname{rel}\partial C^*) \geq \frac{r_{\bar{s}}}{\|A\|(1+\varepsilon)}$. Therefore item 3 of Lemma 5 implies

$$\begin{aligned}
\operatorname{dist}((\bar{y}, \bar{u}), \operatorname{rel}\partial Y_d) &\geq \frac{1}{\max\{1, \|A\|\}} \min\left\{r_{\bar{y}}, \frac{r_{\bar{s}}}{\|A\|(1+\varepsilon)}\right\} \\
&\geq \frac{1}{(1+\varepsilon)\max\{1, \|A\|\}} \frac{\theta}{\delta+\theta} \min\left\{r_{y^0}, \frac{r_{s^0}}{\|A\|}\right\} \\
&= \frac{\theta \min\{r_{y^0}, \tau_{R^*}\}}{(1+\varepsilon)\max\{1, \|A\|\}(\delta+\theta)} .
\end{aligned}$$

For the case $c - A^t y^0 = s^0$, define $\bar{y} = y^0$ and $\bar{u} = \tilde{\mu}\left(s^0, \frac{\tau_{R^*}\|A\|}{1+\varepsilon}\right)$. The same analysis done for the general case shows that now $\operatorname{dist}((c - A^t\bar{y}, \bar{u}), \operatorname{rel}\partial C^*) \geq \frac{\tau_{R^*}}{1+\varepsilon}$, this implies that $\operatorname{dist}((c - A^t\bar{y}, \bar{u}), \operatorname{rel}\partial Y_d) \geq \frac{1}{\max\{1,\|A\|\}(1+\varepsilon)} \min\{\tau_{R^*}, \operatorname{dist}(y^0, \operatorname{rel}\partial C_Y^*)\}$ from item 3 of Lemma 5, this is the last ingredient to show that the theorem also holds in this case.

To finish the proof, we bound the different expressions in the statement of the theorem; let $\xi = \max\{1, \|A\|\}$ for simplicity. Here we use inequality (4.11):

1. (a) $\|\bar{y} - y^0\|_* = \dfrac{\|y - \delta y^0\|_*}{\delta+\theta} \leq \dfrac{\max\{1, \|y^0\|_*\}}{\theta} \leq \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)} \max\{1, \|y^0\|_*\}$.

   (b) $\|\bar{y}\|_* \leq \dfrac{1}{\theta}\|y\|_* + \|y^0\|_* \leq \dfrac{1}{\theta} + \|y^0\|_* \leq \|y^0\|_* + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}$.

2. (a) $$\frac{1}{\mathrm{dist}(\bar{y}, \mathrm{rel}\partial C_Y^*)} \le \frac{1}{r_{\bar{y}}} = \frac{\delta+\theta}{\theta r_{y^0}} \le \frac{1}{r_{y^0}}\left(1+\frac{1}{\theta}\right) \le \frac{1}{r_{y^0}}\left(1+\frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right).$$

(b) $$\begin{aligned}\frac{1}{\mathrm{dist}((\bar{y},\bar{u}), \mathrm{rel}\partial Y_d)} &\le \frac{(1+\varepsilon)\xi}{\min\{r_{y^0}, \tau_{R^*}\}}\frac{\delta+\theta}{\theta} \le \frac{(1+\varepsilon)\xi}{\min\{r_{y^0}, \tau_{R^*}\}}\left(1+\frac{1}{\theta}\right)\\ &\le \frac{(1+\varepsilon)\xi}{\min\{r_{y^0}, \tau_{R^*}\}}\left(1+\frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right).\end{aligned}$$

3. (a) $$\begin{aligned}\frac{\|\bar{y}-y^0\|_*}{\mathrm{dist}(\bar{y}, \mathrm{rel}\partial C_Y^*)} &\le \frac{\|y-\delta y^0\|_*}{\theta r_{y^0}} \le \frac{1}{r_{y^0}\theta}\max\{1, \|y^0\|_*\}\\ &\le \frac{1}{r_{y^0}}\frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\max\{1, \|y^0\|_*\}.\end{aligned}$$

(b) $$\begin{aligned}\frac{\|\bar{y}-y^0\|_*}{\mathrm{dist}((\bar{y},\bar{u}), \mathrm{rel}\partial Y_d)} &\le \frac{(1+\varepsilon)\xi\|y-\delta y^0\|_*}{\theta\min\{r_{y^0}, \tau_{R^*}\}} \le \frac{(1+\varepsilon)\xi}{\min\{r_{y^0}, \tau_{R^*}\}}\frac{1}{\theta}\max\{1, \|y^0\|_*\}\\ &\le \frac{(1+\varepsilon)\xi}{\min\{r_{y^0}, \tau_{R^*}\}}\frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\max\{1, \|y^0\|_*\}.\end{aligned}$$

(c) $$\begin{aligned}\frac{\|\bar{y}\|_*}{\mathrm{dist}(\bar{y}, \mathrm{rel}\partial C_Y^*)} &\le \frac{\|y+\theta y^0\|_*}{\theta r_{y^0}} \le \frac{1}{r_{y^0}}\left(\|y^0\|_* + \frac{1}{\theta}\right)\\ &\le \frac{1}{r_{y^0}}\left(\|y^0\|_* + \frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right).\end{aligned}$$

(d) $$\begin{aligned}\frac{\|\bar{y}\|_*}{\mathrm{dist}((\bar{y},\bar{u}), \mathrm{rel}\partial Y_d)} &\le \frac{\|y+\theta y^0\|_*(1+\varepsilon)\xi}{\theta\min\{r_{y^0}, \tau_{R^*}\}} \le \frac{(1+\varepsilon)\xi}{\min\{r_{y^0}, \tau_{R^*}\}}\left(\|y^0\|_* + \frac{1}{\theta}\right)\\ &\le \frac{(1+\varepsilon)\xi}{\min\{r_{y^0}, \tau_{R^*}\}}\left(\|y^0\|_* + \frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right).\end{aligned}$$ ■

Remark 12 *In the case that C_Y is a subspace, by inspection we note that items* 2.(a), 3.(a), *and* 3.(c) *have both sides of the inequality equal to zero, and are therefore uninformative. In this case the same proof can be used to show that items* 1.(a) *and* 1.(b) *are true. If C_Y is a subspace and the cone R is not a subspace, then the proof above also implies that* 2.(b), 3.(b), *and* 3.(d) *are true. These last three bounds are not true if C_Y and the cone R are subspaces; however in this case, as in Remark 10, we can define $\bar{u}$ such that $(\bar{y}, \bar{u}) \in Y_d$ satisfies* 2', 3'.(a), *and* 3'.(b) *of Remark 10.* ■

The next result, which is a direct consequence of Theorem 15, demonstrates the existence of a reliable solution $(\bar{y}, \bar{u})$ for the dual problem, under the hypothesis that the dual problem is feasible and $\rho_D(d) > 0$.

Corollary 5 *Suppose that $d \in \mathcal{F}_D$, $\rho_D(d) > 0$, and C_Y is not a subspace. Let $y^0 \in \text{relint}C_Y^*$ be given. For any $\varepsilon > 0$, there exists a Slater point $(\bar{y}, \bar{u})$ for (GD_d), such that*

1. $\|\bar{y}\|_* \leq \|y^0\|_* + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}$
2. $\dfrac{1}{\text{dist}((\bar{y}, \bar{u}), \text{rel}\partial Y_d)} \leq \dfrac{(1+\varepsilon)\max\{1, \|A\|\}}{\min\{\text{dist}(y^0, \text{rel}\partial C_Y^*), \tau_{R^*}\}} \left(1 + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right)$
3. $\dfrac{\|\bar{y}\|_*}{\text{dist}((\bar{y}, \bar{u}), \text{rel}\partial Y_d)} \leq \dfrac{(1+\varepsilon)\max\{1, \|A\|\}}{\min\{\text{dist}(y^0, \text{rel}\partial C_Y^*), \tau_{R^*}\}} \left(\|y^0\|_* + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right)$ ■

Remark 13 *In the case that C_Y is a subspace and R is not a subspace, this corollary is valid, but if both C_Y and R are subspaces, then we can construct a Slater point $(\bar{y}, \bar{u})$ that satisfies item* 1 *above and also satisfies* 2' *and* 3'.(b) *of Remark 10.* ■

We end this section by presenting a pair of theorems which restate Theorem 15, emphasizing how the geometric properties of the point $y^0 \in \text{relint}C_Y^*$ impact the geometric properties of the reliable solution $\bar{y} \in \text{dom } u(\cdot)$ and the geometry of the feasible region Y_d, through condition number theory.

For $y^0 \in \text{relint}C_Y^*$, let us define the following measure

$$\begin{aligned} g_{C_Y^*, R^*}(y^0) := & \frac{\max\{\|y^0\|_*, 1\}}{\min\{1, \text{dist}(y^0, \text{rel}\partial C_Y^*), \tau_{R^*}\}} \\ & = \max\left\{\|y^0\|_*, \frac{\|y^0\|_*}{\text{dist}(y^0, \text{rel}\partial C_Y^*)}, \frac{1}{\text{dist}(y^0, \text{rel}\partial C_Y^*)}, \frac{\|y^0\|_*}{\tau_{R^*}}, \frac{1}{\tau_{R^*}}, 1\right\}. \end{aligned}$$

Theorem 16 *Suppose that $d \in \mathcal{F}_D$, $\rho_D(d) > 0$ and C_Y is not a subspace. Let $y^0 \in \text{relint}C_Y^*$ be given. Then for all $\varepsilon > 0$ there exists $(\bar{y}, \bar{u}) \in Y_d$ satisfying:*

1. (a) $\|\bar{y} - y^0\|_* \leq g_{C_Y^*, R^*}(y^0) \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}$
   (b) $\|\bar{y}\|_* \leq g_{C_Y^*, R^*}(y^0) \left(1 + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right)$

2. (a) $\dfrac{1}{\text{dist}(\bar{y}, \text{rel}\partial C_Y^*)} \leq g_{C_Y^*,R^*}(y^0)\left(1 + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right)$

   (b) $\dfrac{1}{\text{dist}((\bar{y}, \bar{u}), \text{rel}\partial Y_d)} \leq (1+\varepsilon)\max\{1, \|A\|\} g_{C_Y^*,R^*}(y^0)\left(1 + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right)$

3. (a) $\dfrac{\|\bar{y} - y^0\|_*}{\text{dist}(\bar{y}, \text{rel}\partial C_Y^*)} \leq g_{C_Y^*,R^*}(y^0)\dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}$

   (b) $\dfrac{\|\bar{y} - y^0\|_*}{\text{dist}((\bar{y}, \bar{u}), \text{rel}\partial Y_d)} \leq (1+\varepsilon)\max\{1, \|A\|\} g_{C_Y^*,R^*}(y^0)\dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}$

   (c) $\dfrac{\|\bar{y}\|_*}{\text{dist}(\bar{y}, \text{rel}\partial C_Y^*)} \leq g_{C_Y^*,R^*}(y^0)\left(1 + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right)$

   (d) $\dfrac{\|\bar{y}\|_*}{\text{dist}((\bar{y}, \bar{u}), \text{rel}\partial Y_d)} \leq (1+\varepsilon)\max\{1, \|A\|\} g_{C_Y^*,R^*}(y^0)\left(1 + \dfrac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right)$

   . ■

We now define a geometric measure for the dual feasible region. We will not consider the whole set Y_d; instead we consider only the projection onto the first m coordinates. We define a geometric measure to describe the relative interior of the set ΠY_d. Note that the set ΠY_d corresponds exactly to the feasible region in the alternate formulation of the dual problem (3.6).

We define the following measure,

$$g_{Y_d} := \inf_{(y,u)\in Y_d} \max\left\{\|y\|_*, \frac{\|y\|_*}{\text{dist}(y, \text{rel}\partial\Pi Y_d)}, \frac{1}{\text{dist}(y, \text{rel}\partial\Pi Y_d)}\right\} .$$

The following theorem is a byproduct of Theorem 16.

Theorem 17 *Suppose $d \in \mathcal{F}_D$, $\rho_D(d) > 0$ and C_Y is not a subspace. Let $y^0 \in \text{relint}C_Y^*$ be given. Then*

$$g_{Y_d} \leq \max\{1, \|A\|\} g_{C_Y^*,R^*}(y^0)\left(1 + \frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right) .$$

Proof: From Lemma 5, item 4, $\operatorname{dist}(\bar{y}, \operatorname{rel}\partial \Pi Y_d) \geq \operatorname{dist}((\bar{y}, \bar{u}), \operatorname{rel}\partial Y_d)$. From Theorem 16 use items *1.(b)*, *2.(b)*, and *3.(d)* and apply the definition of g_{Y_d} to obtain

$$g_{Y_d} \leq (1+\varepsilon)\max\{1, \|A\|\} g_{C_Y^*, R^*}(y^0)\left(1 + \frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right) .$$

Since now the left side is independent of ε, take the limit as $\varepsilon \to 0$. ■

Remark 14 *Under the additional assumption that R^* is a subspace, the measure $g_{C_Y^*, R^*}(y^0)$ becomes*

$$g_{C_Y^*, R^*}(y^0) = \max\left\{\|y^0\|_*, \frac{\|y^0\|_*}{\operatorname{dist}(y^0, \operatorname{rel}\partial C_Y^*)}, \frac{1}{\operatorname{dist}(y^0, \operatorname{rel}\partial C_Y^*)}, 1\right\} ,$$

and we can replace $\|A^t y^0 - c\|_* + \|A\|$ by $\operatorname{dist}(c - A^t y^0, R^*)$ in Theorems 16 and 17.

Finally, from Theorem 15 and the discussion following it, we see that if C_Y is a subspace and R is not a subspace then Theorem 16 is valid, but if C_Y and R are subspaces, then we can sharpen items 2.(b), 3.(b), *and* 3.(d) *and replace them by* 2', 3'.(a), *and* 3'.(b) *of Remark 10, respectively.*

This implies that if C_Y is a subspace and R is not a subspace then Theorem 17 holds, and if C_Y and R are subspaces, then we can sharpen Theorem 17 to obtain

$$g_{Y_d} \leq g_{C_Y^*, R^*}(y^0)\left(1 + \frac{\|A^t y^0 - c\|_* + \|A\|}{\rho_D(d)}\right) .$$

■

Chapter 5

Size of Optimal Solutions and Sensitivity Analysis

In this chapter we continue to explore the geometric properties of the GSM format in terms of the condition number $C(d)$ of the problem. Here we provide bounds on the norms of solutions, on the optimal objective function value, and on the rate of deformation of solutions due to data perturbation in terms of $C(d)$.

In order to construct these theoretical bounds for the GSM format, we first need to define a geometric measure of the ground set P. This measure, which depends on a scalar parameter r and is noted by $g^*(r)$, reduces to 0 in the conic case.

5.1 Geometry measure of support function

In this section we define the measure $g^*(r)$, prove that it is well defined, and build some intuition on it by presenting some examples. Recall from Section 4.1 that we denote the affine hull of a set S by L_S, and that for a convex cone K, τ_K denotes the minimum width of K.

Definition 6 *For any $r \in [0, \tau_{R^*}]$ we define the measure $g^*(r)$ by*

$$\begin{array}{rlllll} g^*(r) := & \sup_{x,s} & -s^t x & = & \sup_s & u(s) \\ & \text{s.t.} & x \in P & & \text{s.t.} & \|s\|_* = 1 \\ & & \|s\|_* = 1 & & & \text{dist}(s, \text{rel}\partial R^*) \geq r \\ & & \text{dist}(s, \text{rel}\partial R^*) \geq r & & & s \in R^* \ , \\ & & s \in R^* & & & \end{array} \tag{5.1}$$

if $R^* \neq \{0\}$, and by $g^*(r) = 0$ for all $r \geq 0$ if $R^* = \{0\}$.

Note that if $R^* = \{0\}$, then $P = R = \mathbb{R}^n$ which implies that (GP_d) is in conic form. Proposition 6 below shows that for any problem in conic form, i.e. P is a cone R, then $g^*(r) = 0$ for all $r \geq 0$.

The parameter $r > 0$ in the definition above causes the optimization problem to consider only points s in the relative interior of R^*. The idea is that points in relintR^* are in the relative interior of the domain of the function $u(\cdot)$, while this function might be unbounded at points on the relative boundary of R^*. A small positive r will bound the points away from the relative boundary of R^* and allows us to bound the function $u(\cdot)$. Note that from Definition 5, there is no point $s \in R^*$ such that $\|s\|_* = 1$ and $\text{dist}(s, \text{rel}\partial R^*) > \tau_{R^*}$; therefore τ_{R^*} is the largest value of r for which Problem (5.1) is feasible, which explains why $g^*(r)$ is only defined for $r \in [0, \tau_{R^*}]$.

Theorem 18 *For any $r \in (0, \tau_{R^*}]$, the measure $g^*(r)$ is finite.*

Proof: Consider $R^* \neq \{0\}$. The upper bound on r, ensures that the problem that defines $g^*(r)$ is feasible, therefore providing a finite lower bound on $g^*(r)$.

For the upper bound, assume $g^*(r)$ is unbounded, that means there exist sequences $\{s_i\}$ and $\{x_i\}$ such that: $x_i \in P$, $s_i \in R^*$, $\|s_i\|_* = 1$, $\text{dist}(s_i, \text{rel}\partial R^*) \geq r$, and $-s_i^t x_i \rightarrow$

∞. From the sequence $\{x_i\}$, we can construct a sequence $\{a_i\}$ which satisfies $a_i \in P \cap L_{R^*}$ and $-s_i^t a_i \to \infty$, as follows: for every i, let $x_i = a_i + b_i$, where $a_i \in L_{R^*}$ and $b_i \in (L_{R^*})^\perp$. Therefore $-b_i \in (L_{R^*})^\perp \subset R$, which means that the point $x_i - b_i = a_i \in P$. Since $-s_i^t a_i = -s_i^t x_i + s_i^t b_i = -s_i^t x_i \to \infty$.

The above implies that $\|a_i\| \to \infty$ and $a_i/\|a_i\| \to a \in R \cap L_{R^*}$, $a \neq 0$. Considering a subsequence, if necessary, $s_i \to \check{s}$, such that $\|\check{s}\|_* = 1$, $\check{s} \in R^*$, and $\text{dist}(\check{s}, \text{rel}\partial R^*) \geq r$. This implies that $\check{s}^t a > 0$. But since $-s_i^t a_i \to \infty$, we know that $\frac{s_i^t a_i}{\|a_i\|} \leq 0$, which taking limit, implies $\check{s}^t a \leq 0$. This is a contradiction, and therefore $g^*(r)$ is bounded. ■

Proposition 5 *Let $s \in R^*$, be such that $s \neq 0$ and $\dfrac{\|s\|_*}{\text{dist}(s, \text{rel}\partial R^*)} \leq \dfrac{1}{r}$. Then for any $\bar{x} \in P$, we have*

$$-s^t \bar{x} \leq u(s) \leq \|s\|_* g^*(r) \ .$$

Proof: The definition of function $u(\cdot)$ implies the left-hand inequality (which in fact holds for any $s \in \mathbb{R}^n$), and

$$u(s) = -\|s\|_* \inf_{x \in P} \frac{s^t x}{\|s\|_*} = \|s\|_* \sup_{x \in P} -\frac{s^t x}{\|s\|_*} \leq \|s\|_* g^*(r) \ .$$

The last inequality follows from the definition of $g^*(r)$ since the hypothesis is equivalent to $\text{dist}\left(\dfrac{s}{\|s\|_*}, \text{rel}\partial R^*\right) \geq r$. ■

Note that this result is also valid for $s = 0$, since $u(0) = 0$. This then includes the case $R^* = \{0\}$.

5.1.1 Examples

Proposition 6 *If $P = R$ is a cone, then $g^*(r) = 0$ for all $r \geq 0$.*

Proof: For any x, s feasible for Problem (5.1), the fact that $x \in P = R$ and $s \in R^*$ implies that $-s^t x \leq 0$. Since $0 \in P = R$, there is a feasible solution such that $-s^t x = 0$. ∎

Proposition 7 *If $P = E + R$, and $|E| := \max_x \{\|x\| \mid x \in E\}$ is bounded, then $g^*(r) \leq |E|$ for any $r \in [0, \tau_{R^*}]$.*

Proof: Any $x \in P = E + R$, can be written as $x = \bar{x} + d$, with $\bar{x} \in E$ and $d \in R$. The definition of $g^*(r)$ becomes

$$
\begin{array}{rlllll}
g^*(r) := & \sup_{s,\bar{x},d} & -s^t\bar{x} - s^t d & \leq \; \sup_{s,\bar{x}} & -s^t\bar{x} & \leq \; |E| \\
 & \text{s.t.} & \bar{x} \in E & \quad\;\; \text{s.t.} & \bar{x} \in E & \\
 & & d \in R & & \|s\|_* = 1 & \\
 & & \|s\|_* = 1 & & & \\
 & & \text{dist}(s, \text{rel}\partial R^*) \geq r & & & \\
 & & s \in R^* \; . & & &
\end{array}
$$
∎

Proposition 8 *If R is a subspace, then $g^*(r)$ is independent of r, that is $g^*(r) = g^* \geq 0$ for all $r \geq 0$, where*

$$
\begin{array}{rll}
g^* = & \sup_{x,s} & s^t x \\
 & s.t. & x \in P \\
 & & \|s\|_* = 1 \\
 & & s \in R^\perp \; ,
\end{array}
\tag{5.2}
$$

and this program satisfies

$$
\begin{array}{rll}
g^* \leq & \max_x & \|x\| \\
 & s.t. & x \in P \cap R^\perp \; .
\end{array}
$$

Proof: The case $R = I\!R^n$ implies $R^* = \{0\}$ and is therefore true by definition. If $R \neq I\!R^n$, then $R^* \neq \{0\}$. The fact that R is a subspace implies that R^* is a subspace, $R^\perp = R^*$, and for every $s \in R^*$, $\text{dist}(s, \text{rel}\partial R^*) = \infty$ and $-s \in R^*$. These conditions prove the equivalence between the optimization problems that define $g^*(r)$ and g^* for all $r \geq 0$. The fact that $-s \in R^*$ for any $s \in R^*$ implies that $g^* \geq 0$.

The proof of the bound on g^* uses the fact that $P = P \cap R^\perp + R$, which we now prove. The inclusion $P \cap R^\perp + R \subseteq P$ is true trivially. For the other inclusion consider any $x \in P$, and its decomposition $x = x_1 + x_2$, where $x_1 \in R^\perp$ and $x_2 \in R$. As the cone R is a subspace, then $-x_2 \in R$, which implies that $x_1 = x - x_2 \in P$ and therefore $x \in P \cap R^\perp + R$. The bound is then due to Proposition 7 and the fact that the set $P \cap R^\perp$ is bounded. To prove that $P \cap R^\perp$ is a bounded set, consider an unbounded sequence $\{x_i\} \subset P \cap R^\perp$, and note that there exists a subsequence for which $\frac{x_i}{\|x_i\|} \to r \in R$, $\|r\| = 1$, which is a contradiction since, by construction, $r \in R^\perp$. ∎

Remark 15 *If R is a subspace and the dual norm on $I\!R^n$ is the dot product norm, i.e. $\|s\|_* = \sqrt{s^t s}$, then $g^* = \max_x\{\|x\| \mid x \in P \cap R^\perp\}$.*

Proof: Let $x^* \in P \cap R^\perp$ be of maximum norm, i.e. $\|x^*\| = \max_x\{\|x\| \mid x \in P \cap R^\perp\}$. From Proposition 3, there exists $\bar{s} \in I\!R^n$ such that $\|\bar{s}\|_* = 1$ and $\bar{s}^t x^* = \|x^*\|$. Let $\bar{s} = s_1 + s_2$, with $s_1 \in R$ and $s_2 \in R^\perp$. By construction we have $s_2^t x^* = \bar{s}^t x^* = \|x^*\| \geq 0$. Then $\|\bar{s}\|_*^2 = s_1^t s_1 + 2 s_1^t s_2 + s_2^t s_2 = \|s_1\|_*^2 + \|s_2\|_*^2 \geq \|s_2\|_*^2$. If $s_2 = 0$, then $\|x^*\| = \bar{s}^t x^* = s_2^t x^* = 0$, therefore $g^* = 0$ and the remark is true. If $s_2 \neq 0$, then $\left(x^*, \frac{s_2}{\|s_2\|_*}\right)$ is feasible for (5.2), and $g^* \geq \frac{s_2^t x^*}{\|s_2\|_*} \geq s_2^t x^* = \bar{s}^t x^* = \|x^*\|$. ∎

5.1.2 Asymptotic behavior of $g^*(r)$ for small r

It is straightforward to check from the definition of $g^*(r)$ that it is a decreasing function of r. Denote by F_r the feasible region of (5.1). The monotonicity of $g^*(\cdot)$ is due to the fact that a larger value r means a smaller feasible region F_r.

As r approaches the boundaries of its domain $[0, \tau_{R^*}]$, the behavior of function $g^*(r)$, is determined by the ground set P. The upper limit of the domain is exactly the min-width of R^*. Theorem 18 shows that this will be a finite quantity. The behavior at the other extreme of the domain of r is not so clear. We now illustrate the behavior of $g^*(r)$ as r approaches zero using three examples. These examples consider the same linear problem in two variables, where we only change the definition of the ground set P. In these examples we will use $\|\cdot\|_1$ as the dual norm on $I\!R^n$. Since the set P is what determines the function $g^*(\cdot)$, it is the only thing we make explicit in the examples.

1. Let $P = \{(x_1, x_2) \mid x_2 x_1 \geq 1,\ x_1 \geq 0\}$. This ground set has a recession cone given by $R = \{(x_1, x_2) \mid x_1, x_2 \geq 0\}$, and $R^* = R$ with $\tau_{R^*} = 1/2$. By considering $r \in (0, 1/2)$ and careful algebraic manipulation, we obtain the following:

$$
\begin{array}{rll}
g^*(r) = & \max_{x,s} & -s_1 x_1 - s_2 x_2 \quad = \quad -2\sqrt{r(1-r)}\ . \\
& \text{s.t.} & x_1 x_2 \geq 1 \\
& & x_1, x_2 \geq 0 \\
& & s_1 + s_2 = 1 \\
& & s_1, s_2 \geq r
\end{array}
$$

   The above implies that $\lim_{r \to 0} g^*(r) = 0$.

2. Let $P = \{(x_1, x_2) \mid x_2 \geq x_1^2\}$. This ground set has a recession cone given by $R = \{(x_1, x_2) \mid x_1 = 0, x_2 \geq 0\}$, and $R^* = \{(x_1, x_2) \mid x_2 \geq 0\}$ with $\tau_{R^*} = 1$. By considering $r \in (0, 1)$ and careful algebraic manipulation, we obtain the following:

$$
\begin{array}{rll}
g^*(r) = & \max_{x,s} & -s_1 x_1 - s_2 x_2 \quad = \quad \frac{(1-r)^2}{4r}\ . \\
& \text{s.t.} & x_2 \geq x_1^2 \\
& & |s_1| + s_2 = 1 \\
& & s_2 \geq r
\end{array}
$$

The above implies that $\lim_{r\to 0} g^*(r) = \infty$. In fact,

$$\lim_{r\to 0} r^\alpha g^*(r) = \begin{cases} \infty & \alpha < 1 \\ 1/4 & \alpha = 1 \\ 0 & \alpha > 1\ . \end{cases}$$

3. Let $P = \{(x_1, x_2) \mid x_2 \geq -\ln(x_1),\ x_1 \geq 0\}$. This ground set has a recession cone given by $R = \{(x_1, x_2) \mid x_1, x_2 \geq 0\}$, and $R^* = R$ with $\tau_{R^*} = 1/2$. By considering $r \in (0, 1/2)$ and careful algebraic manipulation, we obtain the following:

$$\begin{array}{rll} g^*(r) = & \max_{x,s} & -s_1x_1 - s_2x_2 \quad = (1-r)(\ln(1-r) - \ln(r) - 1)\ . \\ & \text{s.t.} & x_2 \geq -\ln(x_1) \\ & & x_1 \geq 0 \\ & & s_1 + s_2 = 1 \\ & & s_1, s_2 \geq r \end{array}$$

The above implies that $\lim_{r\to 0} g^*(r) = \infty$ and also

$$\lim_{r\to 0} r^\alpha g^*(r) = \begin{cases} \infty & \alpha \leq 0 \\ 0 & \alpha > 0\ . \end{cases}$$

The examples above show that as r tends to zero, the function $g^*(\cdot)$ can either diverge or converge, all at different rates, depending on the shape of the ground set P.

5.2 Bounds on norms of optimal solutions and the optimal objective value

The following theorem bounds the optimal objective function value using condition number quantities and the $g^*(\cdot)$ measure. The rest of this section deals with condition number bounds on the norms of optimal solutions.

The next theorem, as well as the other results in this section, are an extension to the GSM format of the results developed for the conic case by Renegar in [28]. For example, Theorem 19 below extends to the GSM format a result which appears in Assertion 3 of Theorem 1.1 in [28] for the conic case. In the remainder of the section we also mention which result of [28] the current theorem or proposition is related to.

Theorem 19 *Suppose $d \in \mathcal{F}$ and $\rho(d) > 0$. Then problem (GP_d) has an optimal solution. Consider $x^0 \in P$, then $z_*(d)$ will satisfy:*

$$|z_*(d)| \leq \|d\| C(d) \left(3 \max\{g^*(\tilde{r})^+, \|x^0\|\} + 2\right) \ ,$$

where $\tilde{r} = \dfrac{\tau_{R^*}\|A\|}{3\|d\|C(d)}$ if R is not a subspace, and $\tilde{r} \geq 0$ if R is a subspace.

Proof: The existence of an optimal solution is a consequence of $\rho(d) > 0$ and Corollary 1. In fact, there is no duality gap and both the primal and the dual problems attain their common optimal value.

We assume that R is not a subspace. To bound the optimal objective function value, we use Theorem 9, which asserts the existence of a feasible $\bar{x} \in X_d$ which satisfies

$$\|\bar{x}\| \leq \|x^0\| + \frac{\mathrm{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} \leq \|x^0\| + \frac{\|Ax^0 - b\|}{\rho_P(d)} \ .$$

We also use Theorem 15 with $y^0 = 0$ to obtain a feasible $(\bar{y}, \bar{u}) \in Y_d$ that satisfies

$$\|\bar{y}\|_* \leq \frac{\|c\|_* + \|A\|}{\rho_D(d)} \ .$$

We use the claim, yet to be proved, that when R^* is not a subspace, the dual feasible solution $(\bar{y}, \bar{u})$ also satisfies

$$\frac{\|c - A^t\bar{y}\|_*}{\mathrm{dist}(c - A^t\bar{y}, \mathrm{rel}\partial R^*)} \leq \frac{1}{\tilde{r}} \ , \tag{5.3}$$

for the $\tilde{r}$ defined in the statement of the theorem.

First note that

$$\begin{aligned} z_*(d) \leq c^t\bar{x} \leq \|c\|_*\|\bar{x}\| &\leq \|c\|_* \left(\|x^0\| + \frac{\|Ax^0 - b\|}{\rho_P(d)} \right) \\ &\leq \|d\|C(d)\left(2\|x^0\| + 1\right) \ . \end{aligned}$$

Now, let us look at the lower bound:

$$z_*(d) = z^*(d) \geq b^t\bar{y} - u(c - A^t\bar{y}) \geq -\|b\|\|\bar{y}\|_* - \|c - A^t\bar{y}\|_* g^*(\tilde{r}) \ ,$$

here the bound on $u(c - A^t\bar{y})$ uses Proposition 5, where equation (5.3) implies the necessary hypothesis. Therefore

$$\begin{aligned} z_*(d) &\geq -\|b\|\|\bar{y}\|_* - \|c - A^t\bar{y}\|_* g^*(\tilde{r})^+ \\ &\geq -\|c\|_* g^*(\tilde{r})^+ - \left(\|b\| + \|A\|g^*(\tilde{r})^+\right)\|\bar{y}\|_* \\ &\geq -\left[\|c\|_* g^*(\tilde{r})^+ + \left(\|b\| + \|A\|g^*(\tilde{r})^+\right)\left(\frac{\|c\|_* + \|A\|}{\rho_D(d)}\right)\right] \\ &\geq -\|d\|\left[g^*(\tilde{r})^+ + \left(1 + g^*(\tilde{r})^+\right)2C(d)\right] \\ &\geq -\|d\|C(d)\left(3g^*(\tilde{r})^+ + 2\right) \end{aligned}$$

The theorem is obtained by combining the upper and lower bounds of the optimal objective value.

We now prove that if R^* is not a subspace then (5.3) holds. Using the notation in

Theorem 15, we know that the feasible solution $(\bar{y}, \bar{u})$ also satisfies

$$\operatorname{dist}(c - A^t\bar{y}, \operatorname{rel}\partial R^*) \geq r_{\bar{s}} = \frac{\theta\tau_{R^*}\|A\|}{\delta + \theta} \quad \text{and} \quad \bar{y} = \frac{y}{\theta + \delta} \ ,$$

where $\delta \geq 0$, $\|y\|_* + \delta \leq 1$, and θ satisfies (4.11). Therefore

$$\begin{aligned}
\frac{\|c - A^t\bar{y}\|_*}{\operatorname{dist}(c - A^t\bar{y}, \operatorname{rel}\partial R^*)} &\leq \frac{\|d\|\,(1 + \|\bar{y}\|_*)}{r_{\bar{s}}} \\
&\leq \|d\| \left(\frac{\theta + \delta}{\theta\tau_{R^*}\|A\|} + \frac{\|y\|_*}{\theta\tau_{R^*}\|A\|} \right) \\
&\leq \frac{\|d\|}{\tau_{R^*}\|A\|} \left(1 + \frac{1}{\theta} \right) \\
&\leq \frac{\|d\|}{\tau_{R^*}\|A\|} \left(1 + \frac{\|c\|_* + \|A\|}{\rho_D(d)} \right) \\
&\leq \frac{\|d\|}{\tau_{R^*}\|A\|} \,(2C(d) + 1) \\
&\leq \frac{3\|d\|C(d)}{\tau_{R^*}\|A\|} \\
&= \frac{1}{\tilde{r}} \ .
\end{aligned}$$

Assume now that R is a subspace. In this case the proof presented above is valid, and is in fact simplified since, under this assumption, Proposition 5 is valid regardless of the value of $\tilde{r}$. ■

We now turn our attention to bounding the sizes of optimal primal and dual solutions. This will require a couple of different intermediate results and will require an additional assumption on the ground set P. The next two propositions, which do not require this additional assumption, are exactly Proposition 3.4 and Proposition 3.5 in [28] for problems in conic form.

Proposition 9 *If $\rho_P(d) > 0$ and (y, u) is feasible for (GD_d), then*

$$\|y\|_* \leq \frac{\max\{\|c\|_*, -(b^t y - u)\}}{\rho_P(d)}.$$

Proof: If $y = 0$ the result is true. If $y \neq 0$, then there exists $\hat{y}$ such that $\|\hat{y}\| = 1$ and $\hat{y}^t y = \|y\|_*$, see Proposition 3. For any $\varepsilon > 0$, define the following perturbation:

$$\Delta A = -\frac{1}{\|y\|_*}\hat{y}c^t, \qquad \Delta b = \frac{((-b^t y + u)^+ + \varepsilon)}{\|y\|_*}\hat{y}, \qquad \Delta c = 0\ .$$

We note that (y, u) satisfies the system

$$\begin{aligned}
&(-\bar{A}^t y, u) \in C^* \\
&\bar{b}^t y > u \\
&y \in C_Y^*
\end{aligned}$$

for data $\bar{d} = (\bar{A}, \bar{b}, \bar{c}) = d + \Delta d$. Therefore $(GP_{d+\Delta d})$ is infeasible, from Lemma 1. We conclude that $\rho_P(d) \leq \|\Delta d\|$, which implies

$$\rho_P(d) \quad \leq \quad \frac{\max\{\|c\|_*, (-b^t y + u)^+ + \varepsilon\}}{\|y\|_*}$$

and so

$$\rho_P(d) \quad \leq \quad \frac{\max\{\|c\|_*, -(b^t y - u)\}}{\|y\|_*}\ . \qquad \blacksquare$$

Proposition 10 *If $\rho_D(d) > 0$ and $x = \hat{x} + r$ is feasible for (GP_d), where $\hat{x} \in P$ and $r \in R$, then*

$$\|r\| \leq \frac{1}{\rho_D(d)} \max\left\{\|A\hat{x} - b\|, c^t r\right\}\ .$$

Proof: If $r = 0$ the result is true. If $r \neq 0$, then there exists $\hat{r}$ such that $\|\hat{r}\|_* = 1$ and $\hat{r}^t r = \|r\|$, from Proposition 3. Consider the perturbation

$$\Delta A = \frac{1}{\|r\|}\left(A\hat{x} - b\right)\hat{r}^t, \qquad \Delta b = 0, \qquad \Delta c = \frac{-(c^t r)^+ - \varepsilon}{\|r\|}\hat{r} \ .$$

Note that r satisfies the system

$$\begin{aligned} &\bar{A}r \in C_Y \\ &\bar{c}^t r < 0 \\ &r \in R \ , \end{aligned}$$

for data $\bar{d} = (\bar{A}, \bar{b}, \bar{c}) = d + \Delta d$. Therefore $(GD_{d+\Delta d})$ is infeasible from Lemma 2. We conclude that $\rho_D(d) \leq \|\Delta d\|$, which implies

$$\rho_D(d) \quad \leq \quad \frac{\max\left\{\|A\hat{x} - b\|, (c^t r)^+ + \varepsilon\right\}}{\|r\|}$$

and so

$$\rho_D(d) \quad \leq \quad \frac{\max\left\{\|A\hat{x} - b\|, c^t r\right\}}{\|r\|} \ . \qquad \blacksquare$$

In order to be able to bound the norm of the optimal solutions, we now make an additional assumption on the set P.

Assumption 2 *The ground set P can be separated as a sum of a bounded set E and a cone of rays R, in other words $P = E + R$ where*

$$\begin{aligned} |E| := \quad & \sup_x \quad \|x\| \\ & \text{s.t.} \quad x \in E \ , \end{aligned}$$

and $|E|$ is finite.

The following theorem, which provides condition number bounds for the optimal solution size and optimal objective function value for problems in GSM format is an extension of the results given by Assertions 3 and 4 of Theorem 1.1 in [28]. The bound on the dual optimal solution size for conic problems, although is not explicitly stated in [28], is obtained combining Propositions 3.4 and 3.7 in that paper.

Theorem 20 *Suppose $d \in \mathcal{F}$ and $\rho(d) > 0$. If P satisfies Assumption 2, then problems (GP_d) and (GD_d) have optimal solutions. All optimal solutions x^* of (GP_d) satisfy:*

$$\|x^*\| \le C(d)^2 \left(4|E| + 1\right) \ ,$$

and all optimal solutions (y^*, u^*) of (GD_d) satisfy

$$\|y^*\|_* \le C(d)^2 \left(2|E| + 1\right) \ .$$

Furthermore

$$|z_*(d)| = |z^*(d)| \le \|d\| C(d)(2|E| + 1) \ .$$

Proof: From the previous two propositions, we know that for any feasible $(y, u) \in Y_d$

$$\|y\|_* \le \frac{\max\{\|c\|_*, -(b^t y - u)\}}{\rho_P(d)} \ ,$$

and that for any feasible $x \in X_d$, which is separated into

$$x = \hat{x} + r \quad \text{with } \hat{x} \in E \text{ and } r \in R \ , \tag{5.4}$$

$$\begin{aligned}
\|x\| &\le \|\hat{x}\| + \frac{\max\{\|A\hat{x} - b\|, c^t x - c^t \hat{x}\}}{\rho_D(d)} \\
&\le \|\hat{x}\| + \frac{\max\{\|A\|\|\hat{x}\| + \|b\|, c^t x + \|c\|_* \|\hat{x}\|\}}{\rho_D(d)} \\
&\le \|\hat{x}\| \left(1 + \frac{\max\{\|A\|, \|c\|_*\}}{\rho_D(d)}\right) + \frac{\max\{\|b\|, c^t x\}}{\rho_D(d)}
\end{aligned}$$

$$\leq \quad 2|E|C(d) + \frac{\max\{\|b\|, c^t x\}}{\rho_D(d)} \ .$$

For optimal solutions the above inequalities imply

$$\|y^*\|_* \ \leq \ \frac{\max\{\|c\|_*, -z^*\}}{\rho_P(d)} \tag{5.5}$$

$$\|x^*\| \ \leq \ 2|E|C(d) + \frac{\max\{\|b\|, z^*\}}{\rho_D(d)} \ , \tag{5.6}$$

where $z^* = z^*(d) = z_*(d)$ from Corollary 1. Suppose that $z^* \leq 0$. Then (5.6) implies that $\|x^*\| \leq 2|E|C(d) + \frac{\|b\|}{\rho_D(d)} \leq C(d)(2|E|+1)$. This in turn can be used to lower bound the objective value: $z^* \geq -\|c\|_*\|x^*\| \geq -\|d\|C(d)(2|E|+1)$, and it then follows that $\|y^*\|_* \leq C(d)^2(2|E|+1)$.

Now suppose that $z^* \geq 0$. Then (5.5) implies that $\|y^*\|_* \leq \frac{\|c\|_*}{\rho_P(d)} \leq C(d)$. This can be used to bound the objective value: $z^* \leq \|b\|\|y^*\|_* - u(c - A^t y^*) \leq \|d\|C(d) - u(c - A^t y^*)$. The right most term can be bounded by $-u(c - A^t y^*) \leq -(c - A^t y^*)^t x \leq \|c - A^t y^*\|_* \|x\| \leq \|d\|(C(d)+1)\|x\|$ for any $x \in P$. By selecting some $x \in E$ we can further bound the objective value by $z^* \leq \|d\|C(d) + \|d\|(C(d)+1)|E| \leq \|d\|C(d)\,(2|E|+1)$.

This then implies that

$$\begin{aligned} \|x^*\| \ &\leq \ 2C(d)|E| + C(d)^2\,(2|E|+1) \\ &\leq \ C(d)^2\,(4|E|+1) \ . \qquad \blacksquare \end{aligned}$$

Without Assumption 2 we can use (5.5) and Theorem 19 to bound the norm of the dual optimal solution as follows:

$$\|y^*\|_* \ \leq \ \frac{\max\{\|c\|_*, -z_*(d)\}}{\rho_P(d)}$$

$$
\begin{aligned}
&\leq \frac{\max\{\|c\|_*, 3\|d\|C(d)\kappa\}}{\rho_P(d)} \\
&\leq 3C(d)^2\kappa \ ,
\end{aligned}
$$

where $\kappa = \max\{g^*(\tilde{r})^+, \|x^0\|\} + 1$, and $x^0 \in P$ is given. Although this bound does not require Assumption 2, it is a looser bound than the one provided by Theorem 20 since $\kappa \leq |E| + 1$ under Assumption 2.

For a primal optimal solution x^* the bound we can obtain without Assumption 2 is weaker. If we separate the optimal solution into $x^* = \hat{x} + r$ as in (5.4), note that $|E|$ in inequality (5.6) is simply a bound on $\|\hat{x}\|$, and using Theorem 19 we obtain

$$
\begin{aligned}
\|x^*\| &\leq 2\|\hat{x}\|C(d) + \frac{\max\{\|b\|, z_*(d)\}}{\rho_D(d)} \\
&\leq 2\|\hat{x}\|C(d) + \frac{\max\{\|b\|, 3\|d\|C(d)\kappa\}}{\rho_D(d)} \\
&\leq 2\|\hat{x}\|C(d) + 3C(d)^2\kappa \ ,
\end{aligned}
$$

where $\hat{x}$ depends on the optimal solution x^*. Here is where some additional assumption on the ground set, or a different proof, is necessary.

5.3 Relative error bounds and sensitivity under data perturbation

The results in this section are also inspired in the results obtained by Renegar for problems in conic form in [28]. The next theorem is an extension to problems in GSM format of Assertion 1 of Theorem 1.1 in [28].

Theorem 21 *Suppose that $d \in \mathcal{F}_P$ and $\rho_P(d) > 0$. Let $x' \in P$ be given. Then there* *exists $\bar{x} \in X_d$ satisfying*

$$\|\bar{x} - x^{'}\| \leq \frac{\operatorname{dist}(Ax^{'} - b, C_Y)}{\rho_P(d)} \max\left\{1, \|x^{'}\|\right\} \ .$$

Proof: This is just part *1.a* of Theorem 9. ■

In the next two theorems we present bounds on the change in primal feasible solutions when the data is perturbed. These bounds depend on the condition number of the problem prior to the data perturbation. Theorem 22 below extends Assertion 2 of Theorem 1.1 in [28] to problems in GSM format and is the basis for Theorem 23, which presents the sensitivity of primal solutions to a general data perturbation.

Theorem 22 *Suppose that $d \in \mathcal{F}_P$ and $\rho_P(d) > 0$. Let $\Delta d = (0, \Delta b, 0)$. Then, for every $x' \in X_{d+\Delta d}$, there exists $\bar{x} \in X_d$ satisfying*

$$\|\bar{x} - x'\| \leq \frac{\|\Delta b\| \max\{1, \|x'\|\}}{\rho_P(d)}$$

Proof: We consider problem (PP), defined by (4.4), with $x^0 = x'$ and w^0 such that $Ax' - b - \Delta b = w^0 \in C_Y$. From Lemma 6 we have that there exists a point (x, t, θ, w) feasible for (PP) that satisfies

$$\theta \geq \frac{\rho_P(d)}{\|b - Ax^0 + w^0\|} = \frac{\rho_P(d)}{\|\Delta b\|} \ . \tag{5.7}$$

We define

$$\bar{x} = \frac{x + \theta x'}{t + \theta} \ , \quad \bar{w} = \frac{w + \theta w^0}{t + \theta} \ .$$

By construction we have that $\bar{x} \in P$, $A\bar{x} - b = \bar{w} \in C_Y$, therefore $\bar{x} \in X_d$, and

$$\|\bar{x} - x'\| = \frac{\|x - tx'\|}{t + \theta} \leq \frac{(\|x\| + t)\max\{1, \|x'\|\}}{\theta} \leq \frac{\|\Delta b\|}{\rho_P(d)} \max\{1, \|x'\|\} \ . \qquad \blacksquare$$

Theorem 23 *Suppose that $d \in \mathcal{F}_P$ and $\rho_P(d) > 0$. Let $\Delta d = (\Delta A, \Delta b, \Delta c)$. Then for every $x' \in X_{d+\Delta d}$, there exists $\bar{x} \in X_d$ satisfying*

$$\|\bar{x} - x^{'}\| \leq (\|\Delta b\| + \|\Delta A\| \|x'\|) \frac{\max\{1, \|x'\|\}}{\rho_P(d)} \ .$$

Proof: If $x' \in X_{d+\Delta d}$ then it also belongs to $X_{d+\Delta \tilde{d}}$, where $\Delta \tilde{d} = (0, \Delta b - \Delta A x', 0)$. Theorem 22 with data $\Delta \tilde{d}$ gives

$$\|\bar{x} - x^{'}\| \leq \frac{\max\{1, \|x'\|\}}{\rho_P(d)} \|\Delta b - \Delta A x'\| \leq \frac{\max\{1, \|x'\|\}}{\rho_P(d)} (\|\Delta b\| + \|\Delta A\| \|x'\|) \ . \qquad \blacksquare$$

The next two results present bounds on the change of the optimal objective function value when the data of the problem is perturbed. These bounds depend on the condition number of the problem prior to the data perturbation. Proposition 11 and Theorem 24 below respectively extend to the GSM format the results for problems in conic form of Lemma 3.9 and Assertion 5 of Theorem 1.1 in [28].

Proposition 11 *Suppose that $d \in \mathcal{F}$ and $\rho(d) > 0$. Let $\Delta d = (0, \Delta b, 0)$ satisfy $X_{d+\Delta d} \neq \emptyset$. Then,*

$$z_*(d + \Delta d) - z_*(d) \geq -\|\Delta b\| \frac{\max\{\|c\|_*, -z^*(d)\}}{\rho_P(d)} \ .$$

Proof: The hypothesis that $\rho(d) > 0$ implies that the GSM format problem with data d has zero duality gap and (GP_d) and (GD_d) attain their optimal values, see Corollary 1. Also, since $Y_{d+\Delta d} = Y_d \neq \emptyset$ has a Slater point (since $\rho_D(d) > 0$), and

$X_{d+\Delta d} \neq \emptyset$, then $(GP_{d+\Delta d})$ and $(GD_{d+\Delta d})$ have no duality gap and attain their optimal values, see Corollary 1. Let $(y,u) \in Y_d$ be an optimal solution of (GD_d), due to the form of the perturbation, point $(y,u) \in Y_{d+\Delta d}$, and therefore

$$z^*(d+\Delta d) \geq (b+\Delta b)^t y - u = z^*(d) + \Delta b^t y \geq z^*(d) - \|\Delta b\| \|y\|_* \ .$$

The result now follows using the bound on the norm of dual feasible solutions from Proposition 9. ■

Theorem 24 *Suppose that $d \in \mathcal{F}$ and $\rho(d) > 0$. Let $\Delta d = (\Delta A, \Delta b, \Delta c)$ satisfy $\|\Delta d\| < \rho(d)$. Then, if x^* and $\hat{x}$ are optimal solutions for (GP_d) and $(GP_{d+\Delta d})$ respectively,*

$$|z_*(d+\Delta d) - z_*(d)| \leq \|\Delta b\| \frac{\max\{\|c\|_* + \|\Delta c\|_*, -z^*(d)\}}{\rho_P(d) - \|\Delta d\|} +$$
$$+ \left(\|\Delta c\|_* + \|\Delta A\| \frac{\max\{\|c\|_* + \|\Delta c\|_*, -z^*(d)\}}{\rho_P(d) - \|\Delta d\|} \right) \max\{\|x^*\|, \|\hat{x}\|\} \ .$$

Proof: The hypothesis that $\rho(d) > 0$ and $\rho(d+\Delta d) > 0$ imply that the GSM format problems with data d and $d + \Delta d$ both have zero duality gap and all problems attain their optimal values, see Corollary 1.

Let $\hat{x} \in X_{d+\Delta d}$ be an optimal solution for $(GP_{d+\Delta d})$. Define the perturbation $\Delta\tilde{d} = (0, \Delta b - \Delta A\hat{x}, 0)$. Then by construction the point $\hat{x} \in X_{d+\Delta\tilde{d}}$. Therefore

$$z_*(d+\Delta d) = (c+\Delta c)^t \hat{x} \geq -\|\Delta c\|_* \|\hat{x}\| + c^t \hat{x} \geq -\|\Delta c\|_* \|\hat{x}\| + z_*(d+\Delta\tilde{d}) \ .$$

Invoking Proposition 11, we bound the optimal objective function value for the problem instance $d + \Delta\tilde{d}$:

$$z_*(d+\Delta d) + \|\Delta c\|_* \|\hat{x}\| \geq z_*(d+\Delta\tilde{d}) \geq z_*(d) - \|\Delta b - \Delta A\hat{x}\| \frac{\max\{\|c\|_*, -z^*(d)\}}{\rho_P(d)} \ .$$

Therefore

$$z_*(d+\Delta d) - z_*(d) \geq -\|\Delta c\|_* \|\hat{x}\| - (\|\Delta b\| + \|A\|\|\hat{x}\|)\,\frac{\max\{\|c\|_*, -z^*(d)\}}{\rho_P(d)}\ .$$

Changing the roles of d and $d+\Delta d$ we can construct the following upper bound:

$$z_*(d+\Delta d) - z_*(d) \leq \|\Delta c\|_* \|x^*\| + (\|\Delta b\| + \|A\|\|x^*\|)\,\frac{\max\{\|c+\Delta c\|_*, -z^*(d+\Delta d)\}}{\rho_P(d+\Delta d)}\ .$$

The value $-z^*(d+\Delta d)$ can be replaced by $-z^*(d)$ on the right side of the previous bound. To see this consider two cases. If $-z^*(d+\Delta d) \leq -z^*(d)$, then we can do the replacement since it yields a larger bound. If $-z^*(d+\Delta d) > -z^*(d)$, the inequality above has a negative left side and a positive right side after the replacement. To finish the proof note that because of the hypothesis $\|\Delta d\| < \rho(d)$, the distance to infeasibility satisfies $\rho_P(d+\Delta d) \geq \rho_P(d) - \|\Delta d\| > 0$. Incorporating this and combining these previous two bounds we finish the proof. ■

Chapter 6

Condition Number Complexity of solving (GP_d) using an Interior-Point Algorithm

In this chapter we present an interior-point algorithm to approximately solve problem (GP_d). The algorithm is comprised of two phases: Phase 1 will obtain a feasible point, $\bar{x}$, which will have good geometry with respect to the feasible region. Then Phase 2 will obtain an ε-optimal solution starting from the point $\bar{x}$ obtained in Phase 1. The overall number of Newton iterations of the combined Phase 1 and Phase 2 algorithm is bounded by a function that depends on the condition number of the problem.

6.1 Notation, assumptions, and interior point complexity

The interior point algorithm presented here is inspired in the algorithm and complexity analysis present in [12]. This algorithm uses as input an interior point $x^0 \in \text{relint}P$, with $r_{x^0} = \text{dist}(x^0, \text{rel}\partial P)$, and an interior point w^0 in the cone C_Y, $w^0 \in \text{relint}C_Y$ with $r_{w^0} = \text{dist}(w^0, \text{rel}\partial C_Y)$. From [12], we borrow the definition of the following geometric measure, which is also used as input to the algorithm to bound the iteration count:

$$
\begin{aligned}
G_\varepsilon := \quad \max_x \quad & \|x - x^0\| \\
\text{s.t.} \quad & Ax - b \in C_Y \\
& x \in P \\
& c^t x \leq z_*(d) + \varepsilon \ .
\end{aligned}
\tag{6.1}
$$

The geometric measure G_ε is equal to the maximum distance of points in the ε-optimal level set to the interior point x^0.

Recall that for a given set S, L_S denotes its affine hull and $\widehat{L}_S$ denotes the linear subspace obtained by the translation of L_S to the origin. Define the quantity

$$
\tilde{s} := \max_w \{c^t w \mid Aw \in L_{C_Y}, w \in \widehat{L}_P, \|w\| \leq 1\} \ . \tag{6.2}
$$

This constant appears in the complexity bound of the algorithm but can be replaced by $\|c\|_*$ since $\tilde{s} \leq \|c\|_*$ from its definition.

This algorithm uses self-concordant barrier functions for the sets involved. We assume there exist a ϑ_P-self-concordant barrier function for the set P, a ϑ_{C_Y}-self-concordant barrier function for the cone C_Y, and a $\vartheta_{\|\ \|}$-self-concordant barrier function over the unit ball for the primal norm $\|\cdot\|$ defined in $I\!R^n$.

The algorithm we present in this chapter uses the interior point machinery as developed by Renegar in [29] and [30], based on the theory of self-concordant functions of Nesterov and Nemirovskii [20]. We refer to this machinery as the barrier method. It is designed to approximately solve a problem of the type

$$(OP) \qquad z^* = \min\{f^t x \mid x \in S\} \ ,$$

where S is a compact convex set and $f \in \mathbb{R}^n$ is a linear objective function vector.

The barrier method requires the knowledge of a self-concordant barrier function $F(x)$ on the interior of set S, such that $\mathrm{dom}F(\cdot) = \mathrm{relint}S$. The method solves a sequence of problems of the form

$$(OP_\mu) \qquad z_\mu^* = \min\{f^t x + \mu F(x) \mid x \in \mathrm{relint}S\} \ ,$$

for a decreasing sequence of the parameter μ.

In very general terms, the barrier method creates a sequence of iterates, which are interior to the set S and approach the optimal solution set. The sequence is started at an interior point $x^0 \in \mathrm{relint}S$. The barrier method is separated into two stages. In stage I, the method computes iterates according to Newton's method ending when it has computed a point $\hat{x}$ which is an approximate solution to problem $(OP_{\bar{\mu}})$ for some $\bar{\mu}$ computed internally in stage I. Stage II, sequentially constructs approximate solutions x^k of (OP_{μ_k}) using Newton's method, for a sequence of parameters μ_k converging to zero. One of the key properties of the iterates in stage II is that

$$f^t x^k - 2\mu_k \vartheta \leq z^* \leq f^t x^k \ , \tag{6.3}$$

where ϑ is the complexity parameter of the self-concordant barrier function $F(\cdot)$. In general, the barrier method is stopped when the current iterate is an ε-optimal solution, for a given tolerance ε. An ε-optimal solution is a point $x \in S$ such that $f^t x \le z^* + \varepsilon$.

The complexity of the barrier method for approximately solving problem (OP) can be stated in the following result.

- *Assume that S is a bounded set, and that $x^0 \in \mathrm{relint} S$ is given. The barrier method requires*

$$O\left(\sqrt{\vartheta} \ln \left(\vartheta + \frac{1}{\mathrm{sym}(x^0, S)} + \frac{\bar{R}}{\varepsilon}\right)\right) \tag{6.4}$$

iterations of Newton's method to compute an ε-optimal solution of (OP).

The above result is based on the convergence results for the barrier method presented in [29] and has been stated in this form in [11] and [12]. Here ϑ is the complexity parameter of the self-concordant barrier function defined on S, $\bar{R}$ is the range of the objective function over the set S, that is $\bar{R} = z^u - z^*$ where $z^u = \max\{f^t x \mid x \in S\}$, and the quantity $\mathrm{sym}(x, S)$ measures the symmetry of the point x with respect to the set S, defined by

$$\mathrm{sym}(x, S) := \max\left\{t \mid y \in S \Rightarrow x - t(y - x) \in S\right\} .$$

The next two sections describe and analyze the complexity of the interior point algorithm we use to solve (GP_d). The first section covers Phase I, which computes a feasible interior point with good geometry. Phase II, which is covered in the second section, is initialized at this good interior feasible solution and computes an ε-optimal solution. Both phases of the algorithm use the barrier method described above.

6.2 Phase I

The inputs for Phase I are an interior point $x^0 \in \mathrm{relint}P$, with a distance $r_{x^0} = \mathrm{dist}(x^0, \mathrm{rel}\partial P) > 0$, and $w^0 \in \mathrm{relint}C_Y$, with a distance $r_{w^0} = \mathrm{dist}(w^0, \mathrm{rel}\partial C_Y) > 0$. The output for Phase I will be a feasible solution $\bar{x} \in X_d$ with good geometric properties.

To simplify notation, we define the following measure for a point x in the relative interior of a set S, with $r_x = \mathrm{dist}(x, \mathrm{rel}\partial S)$:

$$g_S(x) := \max\left\{\|x\|\ , \frac{\|x\|}{r_x}\ , \frac{1}{r_x}\right\}\ .$$

The main result in this section will show that Phase I will require

$$O\left(\sqrt{\vartheta}\ln\left(\vartheta + g_P(x^0) + g_{C_Y}(w^0) + \|d\| + C(d) + \frac{\|w^0\|}{\rho_P(d)}\right)\right)$$

iterations of Newton's method. In this complexity result, the value ϑ is the expression

$$\vartheta := 800\vartheta_P + \vartheta_{C_Y} + 800\vartheta_{\|\ \|} + 2\ ,$$

and therefore depends on the complexity parameters of the barrier functions for the set P, the cone C_Y, and the unit ball in $\mathbb{R}^n$ with norm $\|\cdot\|$. In other words, ϑ is $O\left(\vartheta_P + \vartheta_{C_Y} + \vartheta_{\|\ \|}\right)$.

Phase I will solve, using the barrier method, a modified version of problem (PP), and obtain a point $\bar{x} \in \mathrm{relint}X_d$ such that $r_{\bar{x}} = \mathrm{dist}(\bar{x}, \mathrm{rel}\partial X_d) > 0$. This solution will be the input to the Phase II of the algorithm. Let us recall problem (PP) from (4.4):

$$
\begin{array}{lll}
(PP) & \max_{x,t,w,\theta} & \theta \\
 & \text{s.t.} & Ax - bt - w = \theta\,(b - Ax^0 + w^0) \\
 & & \|x\| + |t| \leq 1 \\
 & & (x,t) \in C \\
 & & w \in C_Y .
\end{array}
$$

Let us first assume that the given interior points x^0 and w^0 satisfy $Ax^0 - b - w^0 \in C_Y$. This means that $Ax^0 - b \in w^0 + C_Y \subset \mathrm{relint} C_Y$, and therefore $x^0 \in X_d$. The following shows that in this case $x^0 \in \mathrm{relint} X_d$, which implies that the original point x^0 is already an interior feasible point and no computation of phase I is necessary.

Proposition 12 *Consider a point* $x \in X_d$ *such that* $\mathrm{dist}(x, \mathrm{rel}\partial P) = r_x > 0$ *and* $\mathrm{dist}(Ax - b, \mathrm{rel}\partial C_Y) = r_w > 0$. *Then* $x \in \mathrm{relint} X_d$ *and* $\mathrm{dist}(x, \mathrm{rel}\partial X_d) \geq \min\{r_x, \frac{1}{\|A\|} r_w\}$.

Proof: Consider $\xi \in I\!R^n$ such that $\|\xi\| \leq 1$ and $\xi \in \hat{L}_P \cap \{x \mid Ax \in L_{C_Y}\}$. Since this vector lies in the appropriate linear spaces we have that

$$
\begin{array}{rl}
x + \alpha\xi \in P & \quad \text{for } |\alpha| \leq r_x \\
A(x + \alpha\xi) - b = Ax - b + \alpha A\xi \in C_Y & \quad \text{for } |\alpha| \leq \dfrac{1}{\|A\|} r_w \ .
\end{array}
$$

Therefore $x + \alpha\xi \in X_d$ for any $|\alpha| \leq \min\{r_x, \frac{1}{\|A\|} r_w\}$, which completes the proof. ■

We consider now the case in which $Ax^0 - b - w^0 \notin C_Y$. We first present a result that shows that in this case problem (PP) has a bounded optimal objective, and therefore a bounded feasible region. Then we present a variation of problem (PP) which has an explicitly bounded feasible region, this modification will provide a simple expression for the complexity bound. We finalize this section by analyzing the complexity of solving this new problem with the barrier method and proving that the output of Phase I is an interior primal feasible solution with good geometry.

Lemma 8 *Suppose that $d \in \mathcal{F}_P$, $x^0 \in \operatorname{relint} P$, and $w^0 \in \operatorname{relint} C_Y$. If $Ax^0 - b - w^0 \notin C_Y$ then problem (PP), defined in (4.4) is bounded and the optimal solution θ^* satisfies*

$$-1 \leq \theta^* \leq \frac{\|d\|}{\operatorname{dist}(Ax^0 - b - w^0, C_Y)} \ .$$

Proof: For the lower bound, consider the point $(x, t, w, \theta) = \frac{1}{2\|x^0\|+2}(x^0, 1, w^0, -1)$, and note that it is feasible for (PP) and that $-1 \leq \frac{-1}{2\|x^0\|+2}$. For the upper bound first assume there is a sequence $(x_i, t_i, w_i, \theta_i)$ of feasible points for (PP), such that $\theta_i \to \infty$. Rearranging the feasibility conditions of these points we can write

$$\frac{w_i}{\theta_i} = \frac{Ax_i - t_i b}{\theta_i} + Ax^0 - b - w^0 \ .$$

Since θ_i tends to infinity, we can consider that $\theta_i \geq 1$, which implies that $\|\frac{w_i}{\theta_i}\|$ is bounded by $\|d\| + \|Ax^0 - b - w^0\|$ and therefore (taking a subsequence if need be) we have $\frac{w_i}{\theta_i} \to \bar{w} \in C_Y$. The equation above then implies that $Ax^0 - b - w^0 = \bar{w} \in C_Y$, which contradicts the hypothesis.

Therefore there is no feasible sequence with $\theta_i \to \infty$ and the optimal solution of (PP) is bounded. To bound the optimal solution, now consider a feasible sequence of points $(x_i, t_i, w_i, \theta_i)$ such that $\theta_i \to \theta^*$. Due to the bounded domain in x and t, and restricting to a subsequence if necessary, we have that $(x_i, t_i) \to (\bar{x}, \bar{t}) \in C$. Then $\bar{w} := A\bar{x} - \bar{t}b - \theta^*(b - Ax^0 + w^0)$ belongs to C_Y, since by construction we have that $w_i \to \bar{w}$. Rearranging and taking norms in the definition of $\bar{w}$ we can bound θ^* by

$$\theta^* = \frac{\|A\bar{x} - \bar{t}b\|}{\|b - Ax^0 + w^0 + \frac{1}{\theta^*}\bar{w}\|} \leq \frac{\|A\|\|\bar{x}\| + \|b\||\bar{t}|}{\operatorname{dist}(Ax^0 - b - w^0, C_Y)} \leq \frac{\|d\|}{\operatorname{dist}(Ax^0 - b - w^0, C_Y)}$$

where the first inequality is due to the fact that $\frac{1}{\theta^*}\bar{w} \in C_Y$. ■

Therefore problem (PP) has a bounded domain under the hypothesis of Lemma 8. We modify problem (PP) by adding explicit bounds on the variable θ. The main purpose of this modification is to provide a nicer expression for the symmetry of the starting point for the barrier method that will solve the problem. We consider the following variation of (PP):

$$
\begin{array}{lll}
(P1) & \max_{x,t,w,\theta} & \theta \\
 & \text{s.t.} & Ax - bt - w = \theta\,(b - Ax^0 + w^0) \\
 & & \|x\| + |t| \leq 1 \\
 & & (x,t) \in C \\
 & & w \in C_Y \\
 & & -1 \leq \theta \leq 1 \ .
\end{array}
\tag{6.5}
$$

The proposition below shows that the optimal solution to problem $(P1)$ has a lower bound similar to Equation 4.5 for problem (PP). This lower bound is used to show that the solution provided by Phase I has good geometric properties.

Proposition 13 *Suppose that $d \in \mathcal{F}_P$ and $\rho_P(d) > 0$. Let $(x^*, t^*, w^*, \theta^*)$ be an optimal solution of $(P1)$. Then*

$$\theta^* \geq \min\left\{1\ ,\ \frac{\rho_P(d)}{\|b - Ax^0 + w^0\|}\right\}\ .$$

Proof: If $\theta^* = 1$ the result is true, therefore we assume that $\theta^* < 1$. In this case $(x^*, t^*, w^*, \theta^*)$ is also optimal for (PP), and this in turn implies the result, since

$$\theta^* \geq \frac{\rho_P(d)}{\|b - Ax^0 + w^0\|}$$

from Equation (4.5) in Lemma 6. ■

To study the complexity of solving problem $(P1)$ using the barrier method we have to bound the symmetry of the starting point, the complexity parameter of the barrier function over the feasible region, and the range of the objective. The next proposition concerns the symmetry of the starting point with respect to the feasible region of $(P1)$. Let us call S_1 the feasible region of problem $(P1)$, namely

$$\begin{aligned} S_1 := \ \{(x,t,w,\theta) \mid \ & Ax - bt - w - \theta(b - Ax^0 + w^0) = 0 \ , \\ & \|x\| + |t| \leq 1 \ , \\ & (x,t) \in C \ , \\ & w \in C_Y \ , \\ & -1 \leq \theta \leq 1 \ \} \ . \end{aligned}$$

We will denote points in the set S_1 by $z = (x,t,w,\theta)$.

Proposition 14 *Assume $d \in \mathcal{F}_P$ and $\rho_P(d) > 0$. Then* $\mathrm{sym}\,(z^0, S_1) \geq \beta$*, where* $z^0 := \frac{1}{2\|x^0\|+2}(x^0, 1, w^0, -1)$ *and*

$$\beta = \frac{1}{3\|x^0\| + 3} \min \left\{ \frac{r_{x^0}}{r_{x^0} + \|x^0\| + 1} \ , \ \frac{r_{w^0}}{\|d\|(\|x^0\| + 2) + \|w^0\|} \ , 1 \right\} \ .$$

Proof: First we assume that P is not an affine set and C_Y is not a subspace.

Note that $z^0 \in S_1$ and for simplicity let $\eta = 2\|x^0\| + 2$. For a point $z = (x,t,w,\theta)$ which satisfies that $z^0 + z \in S_1$, we have to show that $z^0 - \beta z \in S_1$. Note that since $z^0 + z \in S_1$ then $Ax - bt - w - \theta(b - Ax^0 + w^0) = 0$, therefore $z^0 - \beta z$ satisfies the equality constraint: $A\left(\frac{x^0}{\eta} - \beta x\right) - b\left(\frac{1}{\eta} - \beta t\right) - \left(\frac{w^0}{\eta} - \beta w\right) - \left(\frac{-1}{\eta} - \beta\theta\right)(b - Ax^0 + w^0) = 0.$

The hypothesis $z^0 + z \in S_1$ also implies the following four expressions:

$$-1 \le \frac{-1}{\eta} + \theta \le 1 \tag{6.6}$$

$$\left\| \frac{1}{\eta} x^0 + x \right\| + \left| \frac{1}{\eta} + t \right| \le 1 \tag{6.7}$$

$$\left(\frac{1}{\eta} x^0 + x, \frac{1}{\eta} + t \right) \in C \tag{6.8}$$

$$\frac{1}{\eta} w^0 + w \in C_Y \ . \tag{6.9}$$

Equation (6.6) is equivalent to $-\frac{2\|x^0\|+1}{2\|x^0\|+2} \le \theta \le \frac{2\|x^0\|+3}{2\|x^0\|+2}$. Also, from its definition we see that β satisfies $0 < \beta \le \frac{1}{3} \le \frac{2\|x^0\|+1}{2\|x^0\|+3} \le 1$. Then

$$\frac{-1}{2\|x^0\|+2} - \beta\theta \le \frac{-1}{2\|x^0\|+2} + 1\frac{2\|x^0\|+1}{2\|x^0\|+2} = \frac{2\|x^0\|}{2\|x^0\|+2} \le 1$$

and

$$\frac{-1}{2\|x^0\|+2} - \beta\theta \ge \frac{-1}{2\|x^0\|+2} - \left(\frac{2\|x^0\|+1}{2\|x^0\|+3}\right)\left(\frac{2\|x^0\|+3}{2\|x^0\|+2}\right) = -1 \ ,$$

which means that $-1 \le \frac{1}{\eta} - \beta\theta \le 1$.

From equation (6.7) we note that $\|x\| + |t| \le 1 + \frac{1}{\eta}(\|x^0\| + 1) = 3/2$. Then

$$\left\| \frac{1}{\eta}(x^0, 1) - \beta(x, t) \right\| \le \frac{1}{2} + \beta(\|x\| + |t|) \le \frac{1}{2} + \left(\frac{1}{3}\right)\left(\frac{3}{2}\right) = 1 \ .$$

The hypotheses $z^0 \in S_1$ and $z^0 + z \in S_1$ imply that $(x, t) \in L_C$. Now consider the expression

$$\frac{\frac{1}{\eta}x^0 - \beta x}{\frac{1}{\eta} - \beta t} = \frac{x^0 - \eta\beta x}{1 - \eta\beta t} = x^0 + \frac{\eta\beta}{1 - \eta\beta t}\left(x^0 t - x\right) \ ,$$

which belongs to P if $\frac{\eta\beta}{|1-\eta\beta t|}\|x^0 t - x\| \le r_{x^0}$. To prove that this condition is true, first note that from its definition $\beta \le \frac{2r_{x^0}}{3\eta(r_{x^0}+\|x^0\|+1)}$. We combine this bound on β with the inequality $-3/2 \le t \le 3/2$ to show that $1 - \eta\beta t \ge 1 - \frac{3}{2}\eta\beta \ge \frac{\|x^0\|+1}{r_{x^0}+\|x^0\|+1} > 0$. By applying these inequalities to the expression we want to bound, we obtain

$$\frac{\eta\beta\|x^0 t - x\|}{|1-\eta\beta t|} \le \frac{2r_{x^0}}{3(r_{x^0}+\|x^0\|+1)} \frac{r_{x^0}+\|x^0\|+1}{\|x^0\|+1} (\|x\|+|t|) \left(\|x^0\|+1\right) \le r_{x^0} \ .$$

We have therefore proved the condition that $\left(\frac{1}{\eta}x^0 - \beta x, \frac{1}{\eta} - \beta t\right) \in C$.

To finish the proof have to show that $\frac{1}{\eta}w^0 - \beta w \in C_Y$. We note from $w^0 \in C_Y$ and $w^0 + w \in C_Y$, that $w \in L_{C_Y}$. To show that $\frac{1}{\eta}w^0 - \beta w = \frac{1}{\eta}\left(w^0 - \eta\beta w\right)$ belongs to C_Y we only need to prove that $\|\eta\beta w\| \le r_{w^0}$.

From the definition of β, we bound $\beta \le \frac{2r_{w^0}}{3\eta(\|d\|(\|x^0+2\|)+\|w^0\|)}$. From $z^0 + z \in C$ we have $A\left(\frac{x^0}{\eta} + x\right) - b\left(\frac{1}{\eta} + t\right) - \left(\frac{w^0}{\eta} + w\right) - \left(\frac{-1}{\eta} + \theta\right)\left(b - Ax^0 + w^0\right) = 0$. Rearranging and taking norms in this expression gives

$$\begin{aligned}
\|w\| &\le \|d\| + \frac{1}{\eta}\|w^0\| + \|b - Ax^0 + w^0\| \\
&\le \|d\| + \frac{1}{\eta}\|w^0\| + \|d\| + \|d\|\|x^0\| + \|w^0\| \\
&= 2\|d\| + \|d\|\|x^0\| + \frac{2\|x^0\|+3}{2\|x^0\|+2}\|w^0\| \\
&\le \|d\|(\|x^0\|+2) + \frac{3}{2}\|w^0\| \\
&\le \frac{3}{2}\left(\|d\|(\|x^0\|+2) + \|w^0\|\right) \ .
\end{aligned}$$

We can now prove that $\|\eta\beta w\| \le r_{w^0}$, since

$$\begin{aligned}
\|\eta\beta w\| &= \eta\beta\|w\| \\
&\le \frac{2r_{w^0}}{3(\|d\|(\|x^0\|+2)+\|w^0\|)} \frac{3}{2}\left(\|d\|(\|x^0\|+2)+\|w^0\|\right) \\
&\le r_{w^0} \ .
\end{aligned}$$

Assume now that P is an affine set and/or C_Y is a subspace. In this case the proof presented above is still valid, with simpler arguments to show that $\left(\frac{1}{\eta}x^0 - \beta x, \frac{1}{\eta} - \beta t\right) \in C$ and/or $\left(\frac{1}{\eta}w^0 - \beta w\right) \in C_Y$. ■

We now turn our attention to the desirable geometric properties of the point computed by the Phase I of the algorithm. These properties as well as the assumptions needed to construct it are seen in the next proposition.

Proposition 15 *Suppose that* $d \in \mathcal{F}_P$ *and* $\rho_P(d) > 0$. *Let* $x^0 \in \text{relint}X_d$ *and* $w^0 \in \text{relint}C_Y$ *be given. If* (x, t, w, θ) *is feasible for* $(P1)$ *and* $\theta \leq \theta^* \leq 2\theta$ *then the point* $\bar{x} := \frac{x+\theta x^0}{t+\theta} \in X_d$ *satisfies:*

1. $\|\bar{x} - x^0\| \leq 2\max\{1, \|x^0\|\}\left(C(d)(\|x^0\| + 1) + \dfrac{\|w^0\|}{\rho_P(d)}\right)$

2. $\dfrac{1}{\text{dist}(\bar{x}, \text{rel}\partial X_d)} \leq \dfrac{2}{\min\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\}}\left(\dfrac{1}{2} + C(d)(\|x^0\| + 1) + \dfrac{\|w^0\|}{\rho_P(d)}\right)$

3. $\dfrac{\|\bar{x} - x^0\|}{\text{dist}(\bar{x}, \text{rel}\partial X_d)} \leq \dfrac{2\max\{1, \|x^0\|\}}{\min\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\}}\left(C(d)(\|x^0\| + 1) + \dfrac{\|w^0\|}{\rho_P(d)}\right)$.

Proof: Proposition 13 provides the following bound:

$$\frac{1}{\theta} \leq \frac{2}{\theta^*} \leq 2\max\left\{1\ ,\ \frac{\|b - Ax^0 + w^0\|}{\rho_P(d)}\right\} \leq 2C(d)(\|x^0\| + 1) + \frac{2\|w^0\|}{\rho_P(d)}\ . \tag{6.10}$$

Note that the above bound implies that $\theta > 0$. Define the following:

$$\bar{x} = \frac{x + \theta x^0}{t + \theta}, \quad \bar{w} = \frac{w + \theta w^0}{t + \theta}, \quad r_{\bar{x}} = \frac{\theta r_{x^0}}{t + \theta}, \quad r_{\bar{w}} = \frac{\theta r_{w^0}}{t + \theta}.$$

By construction $\bar{x} \in P$ and $\bar{w} \in C_Y$, also $\text{dist}(\bar{x}, \text{rel}\partial P) \geq r_{\bar{x}} > 0$ and $\text{dist}(\bar{w}, \text{rel}\partial C_Y) \geq r_{\bar{w}} > 0$. We also have that $A\bar{x} - b = \bar{w} \in C_Y$. Therefore $\bar{x} \in X_d$, and from Proposition

12 we have that

$$\operatorname{dist}(\bar{x}, \operatorname{rel}\partial X_d) \geq \min\left\{r_{\bar{x}}, \frac{1}{\|A\|} r_{\bar{w}}\right\} = \frac{\theta}{t+\theta} \min\left\{r_{x^0}, \frac{1}{\|A\|} r_{w^0}\right\} > 0 \ .$$

We finish the proof showing the that the point $\bar{x}$ satisfies the inequalities in the statement of the proposition:

1. $\|\bar{x} - x^0\| = \dfrac{\|x - tx^0\|}{t+\theta} \leq \dfrac{\|x\| + |t|\|x^0\|}{\theta} \leq \max\{1, \|x^0\|\} \left(2C(d)(\|x^0\| + 1) + \dfrac{2\|w^0\|}{\rho_P(d)}\right)$.

2. $\dfrac{1}{\operatorname{dist}(\bar{x}, \operatorname{rel}\partial X_d)} \leq \dfrac{1}{\min\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\}} \left(1 + \dfrac{1}{\theta}\right) \leq \dfrac{1 + 2C(d)(\|x^0\| + 1) + \frac{2\|w^0\|}{\rho_P(d)}}{\min\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\}}$.

3. $$\frac{\|\bar{x} - x^0\|}{\operatorname{dist}(\bar{x}, \operatorname{rel}\partial X_d)} \leq \frac{\|x - tx^0\|}{t+\theta} \frac{t+\theta}{\theta \min\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\}} \leq \frac{\|x\| + |t|\|x^0\|}{\theta} \frac{1}{\min\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\}}$$
$$\leq \frac{\max\{1, \|x^0\|\} \left(2C(d)(\|x^0\| + 1) + \frac{2\|w^0\|}{\rho_P(d)}\right)}{\min\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\}}. \quad \blacksquare$$

In order to solve problem $(P1)$ using the barrier method, we must specify the barrier function on the feasible set S_1. This barrier can be constructed using barrier calculus from the barrier functions defined on P, C_Y and the unit ball in $I\!R^n$ of norm $\|\cdot\|$. To construct the barriers functions over the sets C and $\{(x,t) \mid \|x\| \leq 1 - |t|\}$, use Proposition 5.1.4 of [20], which shows that for a given ϑ-self-concordant barrier function $F(\cdot)$ over a set S, a self-concordant barrier function over the set $\{(x,t) \mid x \in tS, t > 0\}$ is $400\left[F\left(\frac{x}{t}\right) - 2\vartheta \ln t\right]$, with complexity value of 800ϑ. The barrier function over set S_1 is therefore induced by

$$\begin{aligned} F(x,t,w,\theta) := \ & 400\left[F_P\left(\tfrac{x}{t}\right) - 2\vartheta_P \ln t\right] + F_{C_Y}(w) - \ln(1-\theta) - \ln(1+\theta) + \\ & +400\left[F_{\|\cdot\|}\left(\tfrac{x}{1-t}\right) - 2\vartheta_{\|\cdot\|} \ln(1-t)\right] \ , \end{aligned}$$

and therefore the induced barrier has complexity value at most

$$\vartheta := 800\vartheta_P + \vartheta_{C_Y} + 2 + 800\vartheta_{\|\cdot\|} \ .$$

Before presenting the Phase I algorithm, we will discuss the termination criteria for the barrier method. In this case we are not interested in solving problem $(P1)$ approximately because, from Proposition 15, if the current iterate satisfies $\theta \le \theta^* \le 2\theta$ we can construct an interior point with good geometry. The iterates of the barrier method satisfy inequalities (6.3), which for problem $(P1)$ are equivalent to

$$\theta^k \le \theta^* \le \theta^k + 2\mu_k \vartheta \ .$$

Therefore, taking $2\mu_k\vartheta \le \frac{1}{2}\theta^k$ implies the condition of Proposition 15 strictly.

Definition 7 *Algorithm Phase I:*

1. *If* $Ax^0 - b - w^0 \in C_Y$*, stop, return* $\bar{x} = x^0$.
2. *Starting from* $\frac{1}{2\|x^0\|+2}(x^0, 1, w^0, -1)$*, use the interior-point method to solve* $(P1)$.
3. *Stop when the current iterate of the interior-point method satisfies* $4\mu_k\vartheta \le \theta^k$. *Return* $\bar{x} = \frac{x^k + \theta^k x^0}{t^k + \theta^k}$.

Theorem 25 *(Complexity of Algorithm Phase I.) Consider* $d \in \mathcal{F}_P$*, such that* $\rho_P(d) > 0$*. Algorithm Phase I finds returns a feasible point* $\bar{x} \in \mathrm{relint} X_d$ *that satisfies*

1. $\|\bar{x} - x^0\| \le 2\max\{1, \|x^0\|\}\left(C(d)(\|x^0\|+1) + \dfrac{\|w^0\|}{\rho_P(d)}\right)$
2. $\dfrac{1}{\mathrm{dist}(\bar{x}, \mathrm{rel}\partial X_d)} \le \dfrac{2}{\min\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\}}\left(\dfrac{1}{2} + C(d)(\|x^0\|+1) + \dfrac{\|w^0\|}{\rho_P(d)}\right)$
3. $\dfrac{\|\bar{x} - x^0\|}{\mathrm{dist}(\bar{x}, \mathrm{rel}\partial X_d)} \le \dfrac{2\max\{1, \|x^0\|\}}{\min\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\}}\left(C(d)(\|x^0\|+1) + \dfrac{\|w^0\|}{\rho_P(d)}\right)$.

Algorithm Phase I will terminate after

$$O\left(\sqrt{\vartheta}\ln\left(\vartheta + g_P(x^0) + g_{C_Y}(w^0) + \|d\| + C(d) + \frac{\|w^0\|}{\rho_P(d)}\right)\right) \ ,$$

Newton iterations.

Proof: The inequalities *1.*, *2.*, and *3.* are a consequence of Proposition 15 and the fact that the termination criteria implies that $\theta^k \le \theta^* \le 2\theta^k$. Also note that the point x^0 satisfies the inequalities of the theorem trivially.

For the complexity bound on this algorithm first recall that the barrier function on the feasible region of $(P1)$ has complexity parameter ϑ, then use Proposition 14 to bound the symmetry of the starting point, this gives

$$\begin{aligned}\frac{1}{\mathrm{sym}(z^0, S_1)} &\le 3(\|x^0\| + 1) \max\left\{1 + \frac{\|x^0\|}{r_{x^0}} + \frac{1}{r_{x^0}} \ , \ \frac{\|d\|(\|x^0\| + 2)}{r_{w^0}} + \frac{\|w^0\|}{r_{w^0}}\right\} \\ &\le 3(g_P(x^0) + 1) \max\left\{1 + 2g_P(x^0) \ , \ g_{C_Y}(w^0) g_P(x^0)\|d\| + g_{C_Y}(w^0)(2\|d\| + 1)\right\}.\end{aligned}$$

The definition of S_1 shows that the range of the objective function of $(P1)$ over S_1 is $\bar{R} = 2$. Also, the stopping criteria for the algorithm is equivalent to $\varepsilon = \frac{\theta^*}{2}$, since if the iterate satisfies $4\mu_k\vartheta \le \theta^k$, then $\theta^k \ge \theta^* - 2\mu_k\vartheta \ge \theta^* - \frac{\theta^k}{2} \ge \frac{\theta^*}{2} = \theta^* - \varepsilon$. Therefore we can bound

$$\frac{\bar{R}}{\varepsilon} \le \frac{4}{\theta^*} \le \frac{4}{\theta^k} \le 4\left(C(d)(\|x^0\| + 1) + \frac{\|w^0\|}{\rho_P(d)}\right) ,$$

using equation (6.10). These bounds and properties of the logarithm function complete the proof. ■

6.3 Phase II

The second phase of the algorithm uses the barrier method with $\bar{x} \in X_d$ obtained from Phase I as the starting point. Phase II applies the barrier method to problem (GP_d) with an additional constraint in order to bound the feasible region. The problem solved is

$$
\begin{aligned}
(P2) \quad \min_x \quad & c^t x \\
\text{s.t.} \quad & Ax - b \in C_Y \\
& x \in P \\
& c^t x \leq c^t \bar{x} + \bar{s} \ ,
\end{aligned}
\tag{6.11}
$$

where the quantity $\bar{s}$ is a positive offset, which keeps $\bar{x}$ in the interior of the feasible region of $(P2)$. The complexity analysis done in [29] uses this type of additional constraint, the explicit construction of the offset $\bar{s}$, which we use here, is due to [12]. The offset is defined by

$$
\bar{s} := \left(\frac{6\vartheta_{\|\cdot\|} + 1}{\sqrt{2}} \right) \max \left\{ c^t x \mid Ax \in L_{C_Y} \ , x \in \widehat{L}_P \ , x^t H(0) x \leq 1 \right\} \ ,
$$

where $H(0)$ is the Hessian matrix of the $\vartheta_{\|\cdot\|}$-self-concordant barrier function for the unit ball in $\mathbb{R}^n$ of norm $\|\cdot\|$ evaluated at zero. Also define

$$
\tilde{s} := \max \left\{ c^t x \mid Ax \in L_{C_Y} \ , x \in \widehat{L}_P \ , \|x\| \leq 1 \right\} \ .
$$

The main result in this section shows that we can compute an ε-optimal solution in at most

$$
O\left(\sqrt{\vartheta_P + \vartheta_{C_Y}} \ln \left(\vartheta_P + \vartheta_{C_Y} + \vartheta_{\|\cdot\|} + G_\varepsilon + \max\left\{1, \frac{1}{\bar{r}}\right\} + \|\bar{x} - x^0\| + \max\left\{1, \frac{\tilde{s}}{\varepsilon}\right\} \right) \right) \ ,
$$

iterations of Newton's method, where G_ε is the maximum distance of points in the ε-optimal level set to the interior point x^0, see (6.1). We denote by S_2 the feasible region of problem $(P2)$, namely

$$
S_2 := \left\{ x \mid Ax - b \in C_Y, \ x \in P, \ c^t x \leq c^t \bar{x} + \bar{s} \right\} \ .
$$

The next lemma establishes the relationship between the practical quantity we use in the algorithm $\bar{s}$ and the theoretical quantity $\tilde{s}$.

Lemma 9 $\frac{1}{6\vartheta_{\|\cdot\|}+1}\bar{s} \le \tilde{s} \le \bar{s}$.

Proof: Since $F_{\|\cdot\|}(x)$ is a $\vartheta_{\|\cdot\|}$-self-concordant barrier for $B(0,1)$, then $\bar{F}(x) := F_{\|\cdot\|}(x) + F_{\|\cdot\|}(-x)$ is a $2\vartheta_{\|\cdot\|}$-self-concordant barrier for $B(0,1)$, whose analytic center is $x^c = 0$, and note that the Hessian of $\bar{F}(x)$ at $x^c = 0$ is $2H(0)$ where $H(x)$ is the Hessian of $F_{\|\cdot\|}(\cdot)$ at x. Then from Proposition 2.3.2 of [20] it follows that

$$\left\{x \mid \sqrt{x^t 2H(0)x} \le 1\right\} \subset B(0,1) \subset \left\{x \mid \sqrt{x^t 2H(0)x} \le 3(2\vartheta_{\|\cdot\|})+1\right\} .$$

These set inclusions imply the result. ■

The next lemma presents a property which is central in the complexity bound obtained here. This lemma implies that if the ε-optimal level set is bounded then the entire set S_2 is bounded, this helps bound the symmetry of the starting point $\bar{x}$. This lemma is the extension of Lemma 4.1 of [12], which assumes $C_Y = \{0\}$, to the case of a general cone C_Y. Although the proof is analogous, we present it here for completeness.

Lemma 10 *Consider* $\tilde{x} \in X_d$ *such that* $\operatorname{dist}(\tilde{x}, \operatorname{rel}\partial X_d) = \tilde{r} > 0$ *and* $c^t\tilde{x} \le \alpha$. *If* Q *satisfies*

$$Q \ge \max\left\{\|x-\tilde{x}\| \mid Ax-b \in C_Y \ , x \in P \ , c^t x \le \alpha\right\} ,$$

then, for every $t \ge 0$, *if* x *is such that* $x \in X_d$ *and* $c^t x \le \alpha + t$ *then*

$$\|x-\tilde{x}\| \le Q\left(1+\frac{2t}{\tilde{s}\tilde{r}}\right) .$$

Proof: Let $\breve{x} := \operatorname{argmax}_x\{c^t x \mid Ax \in L_{C_Y}, x \in \widehat{L}_P, \|x\| \le 1\}$. Then by definition $\tilde{s} = c^t\breve{x} \ge 0$. Then $\tilde{x} - \tilde{r}\breve{x} \in X_d$ and $c^t(\tilde{x}-\tilde{r}\breve{x}) = c^t\tilde{x} - \tilde{r}c^t\breve{x} \le \alpha$. Note that if $\tilde{s} = 0$

this lemma is trivial since the right hand side is infinity; therefore we consider the case $\tilde{s} > 0$ in which it is easy to show that $\|\check{x}\| = 1$.

Let $x \in X_d$ be such that $c^t x \leq \alpha + t$. Consider the convex combination of x and $\tilde{x} - \tilde{r}\check{x}$:

$$x_\lambda = \lambda(\tilde{x} - \tilde{r}\check{x}) + (1-\lambda)x \in X_d \quad \text{for any } \lambda \in [0,1] \ .$$

Define $s' := c^t x - \alpha \leq t$. If $s' \leq 0$ then $c^t x \leq \alpha$ and $\|x - \tilde{x}\| \leq Q \leq Q\left(1 + \frac{2t}{\tilde{s}\tilde{r}}\right)$ by the definition of Q. Consider then the case $s' > 0$, this implies that $s' + \alpha - c^t\tilde{x} + \tilde{r}\tilde{s} > 0$. Set $\lambda = \frac{s'}{s' + \alpha - c^t\tilde{x} + \tilde{r}\tilde{s}}$, which means that $1 - \lambda = \frac{\alpha - c^t\tilde{x} + \tilde{r}\tilde{s}}{s' + \alpha - c^t\tilde{x} + \tilde{r}\tilde{s}}$, and $c^t x_\lambda = \alpha$. Therefore

$$Q \geq \|x_\lambda - \tilde{x}\| = \|-\tilde{r}\lambda\check{x} + (1-\lambda)(x - \tilde{x})\| \geq (1-\lambda)\|x - \tilde{x}\| - \tilde{r}\lambda \ .$$

Rearranging this expression we obtain

$$\begin{aligned}
\|x - \tilde{x}\| &\leq \frac{s' + \alpha - c^t\tilde{x} + \tilde{r}\tilde{s}}{\alpha - c^t\tilde{x} + \tilde{r}\tilde{s}}\left(Q + \frac{\tilde{r}s'}{s' + \alpha - c^t\tilde{x} + \tilde{r}\tilde{s}}\right) \\
&\leq \left(1 + \frac{s'}{\tilde{r}\tilde{s}}\right)Q + \frac{\tilde{r}s'}{\tilde{r}\tilde{s}} \\
&\leq \left(1 + \frac{2s'}{\tilde{r}\tilde{s}}\right)Q \ ,
\end{aligned}$$

where the last inequality is because $Q \geq \|\tilde{x} - \tilde{r}\check{x} - \tilde{x}\| = \tilde{r}\|\check{x}\| = \tilde{r}$. ■

The following two propositions are used to provide a bound on the symmetry of the point $\bar{x}$ with respect to the set S_2. This bound is a polynomial function of $\bar{r}$, $\|\bar{x} - x^0\|$, $\vartheta_{\|\cdot\|}$, $\tilde{s}$, ε, and G_ε. When the ε-optimal level set of (GP_d) is unbounded (G_ε is infinite,) then this bound is also infinite and therefore so will be the bound on the symmetry and the number of iterations. The interesting case is when the ε-optimal level set is bounded, and G_ε is finite.

Proposition 16 *If $\bar{x} \in X_d$ is such that $\bar{r} := \operatorname{dist}(\bar{x}, \operatorname{rel}\partial X_d) > 0$, then any $x \in S_2$*

satisfies $\|x - \bar{x}\| \le h(\bar{x}, \bar{r})$, *where*

$$h(\bar{x}, \bar{r}) := 4G_\varepsilon + \frac{1}{\bar{r}} 8G_\varepsilon \left(\|\bar{x} - x^0\| + G_\varepsilon + 6\vartheta_{\|\cdot\|} + 1\right) \max\left\{1, \frac{\tilde{s}}{\varepsilon}\left(\|\bar{x} - x^0\| + G_\varepsilon\right)\right\} \ .$$

Proof: If $G_\varepsilon = \infty$ the result is trivial, therefore assume G_ε is finite. First we show that there exists $\tilde{x} \in X_d$ such that $c^t\tilde{x} \le z^* + \varepsilon$ and $\operatorname{dist}(\tilde{x}, \operatorname{rel}\partial X_d) \ge \tilde{r} := \min\left\{1, \frac{\varepsilon}{c^t\bar{x} - z^*}\right\} \bar{r} > 0$.

If $c^t\bar{x} \le z^* + \varepsilon$ then set $\tilde{x} = \bar{x}$ and $\operatorname{dist}(\tilde{x}, \operatorname{rel}\partial X_d) = \bar{r} \ge \tilde{r}$. Therefore we now consider the case $c^t\bar{x} > z^* + \varepsilon$. Let $x^* \in X_d$ be such that $z^* = c^t x^*$, this x^* exists since the optimal level set is bounded by G_ε. Define

$$\tilde{x} = \frac{c^t\bar{x} - z^* - \varepsilon}{c^t\bar{x} - z^*} x^* + \frac{\varepsilon}{c^t\bar{x} - z^*} \bar{x} \ ,$$

which satisfies $c^t\tilde{x} = z^* + \varepsilon$ and $\operatorname{dist}(\tilde{x}, \operatorname{rel}\partial X_d) \ge \frac{\varepsilon}{c^t\bar{x} - z^*}\bar{r} \ge \tilde{r}$.

Note that for any $y \in X_d$ such that $c^t y \le z^* + \varepsilon$, the definition of G_ε implies that

$$\|y - \tilde{x}\| \le \|y - x^0\| + \|\tilde{x} - x^0\| \le 2G_\varepsilon \ .$$

Consider the case when $c^t\bar{x} + \bar{s} < z^* + \varepsilon$. This case implies that $\bar{x} = \tilde{x}$ and that any $x \in S_2$ is such that $c^t x \le c^t\bar{x} + \bar{s} < z^* + \varepsilon$. Then we conclude that for any $x \in S_2$, $\|x - \bar{x}\| = \|x - \tilde{x}\| \le 2G_\varepsilon$.

The case when $c^t\bar{x} + \bar{s} \ge z^* + \varepsilon$ is handled using Lemma 10. Set $Q = 2G_\varepsilon$, then Lemma 10 shows that for any $x \in S_2$ the norm

$$\|x - \tilde{x}\| \le 2G_\varepsilon \left(1 + \frac{2(c^t\bar{x} + \bar{s} - z^* - \varepsilon)}{\tilde{s}\tilde{r}}\right) \ .$$

This then implies that for any $x \in S_2$,

$$\|x - \bar{x}\| \leq \|x - \tilde{x}\| + \|\bar{x} - \tilde{x}\| \leq 4G_\varepsilon \left(1 + \frac{2(c^t\bar{x} + \bar{s} - z^* - \varepsilon)}{\tilde{s}\tilde{r}}\right) .$$

Now since $c^t\bar{x} - z^* = c^t\bar{x} - c^t x^* \leq \tilde{s}\|\bar{x} - x^*\| \leq \tilde{s}(\|\bar{x} - x^0\| + G_\varepsilon)$, then $c^t\bar{x} - z^* + \bar{s} - \varepsilon \leq \tilde{s}\left(\|\bar{x} - x^0\| + G_\varepsilon + 6\vartheta_{\|\cdot\|} + 1\right)$, and $\tilde{r} = \bar{r}\min\left\{1, \frac{\varepsilon}{c^t\bar{x} - z^*}\right\} \geq \bar{r}\min\left\{1, \frac{\varepsilon}{\tilde{s}(\|\bar{x} - x^0\| + G_\varepsilon)}\right\}$, we obtain an upper bound, for every $x \in S_2$

$$\|x - \bar{x}\| \leq 4G_\varepsilon \left(1 + 2\frac{1}{\bar{r}}\max\left\{1, \frac{\tilde{s}}{\varepsilon}\left(\|\bar{x} - x^0\| + G_\varepsilon\right)\right\}\left(\|\bar{x} - x^0\| + G_\varepsilon + 6\vartheta_{\|\cdot\|} + 1\right)\right) . \qquad \blacksquare$$

Proposition 17 *If $\bar{x} \in X_d$ is such that $\bar{r} := \mathrm{dist}(\bar{x}, \mathrm{rel}\partial X_d) > 0$, then*

$$\frac{1}{\mathrm{sym}(\bar{x}, S_2)} \leq \frac{h(\bar{x}, \bar{r})}{\min\{1, \bar{r}\}} .$$

Proof: Let $\beta = \frac{\min\{1,\bar{r}\}}{h(\bar{x},\bar{r})}$, and consider v such that $\bar{x} + v \in S_2$, we have to show that $\bar{x} - \beta v \in S_2$. Since $\bar{x} + v \in S_2$, from Proposition 16 we know that $\|v\| = \|\bar{x} + v - \bar{x}\| \leq h(\bar{x}, \bar{r})$, and since $\bar{x} \in X_d$ and $\bar{x} + v \in X_d$ we have that $v \in \widehat{L}_P$ and $Av \in L_{C_Y}$. This implies that $\bar{x} - \beta v \in X_d$ since $\|\beta v\| \leq \beta h(\bar{x}, \bar{r}) = \min\{1, \bar{r}\} \leq \bar{r}$. To check that $\bar{x} - \beta v \in S_2$, it remains to verify that $c^t(\bar{x} - \beta v) \leq c^t\bar{x} + \bar{s}$, which we now verify:

$$\begin{aligned}
c^t\left(\bar{x} - \beta v\right) &= c^t\bar{x} - \beta c^t v \\
&\leq c^t\bar{x} + \beta\tilde{s}\|v\| \\
&\leq c^t\bar{x} + \beta h(\bar{x}, \bar{r})\tilde{s} \\
&\leq c^t\bar{x} + \min\{1, \bar{r}\}\tilde{s} \\
&\leq c^t\bar{x} + \tilde{s} \\
&\leq c^t\bar{x} + \bar{s} .
\end{aligned}$$

Therefore $x - \beta v \in S_2$ and the proof is complete. ∎

In order to solve problem $(P2)$ using the barrier method, we must specify the barrier function on the feasible set S_2. This barrier can be constructed using barrier calculus from the barrier functions defined on P and C_Y. The barrier function over set S_2 is induced by

$$F(x, w) := F_P(x) + F_{C_Y}(w) - \ln(c^t\bar{x} + \bar{s} - c^t x) ,$$

therefore the induced barrier has a complexity parameter at most

$$\vartheta = \vartheta_P + \vartheta_{C_Y} + 1 .$$

Definition 8 *Algorithm Phase II.*

Given a starting point $\bar{x}$ such that $\mathrm{dist}(\bar{x}, \mathrm{rel}\partial X_d) \geq \bar{r} > 0$.

1. *If $\bar{s} = 0$, stop. All feasible points are optimal. Return $\bar{x}$ as the optimal solution.*
2. *Use the interior-point method to solve $(P2)$ starting at $x = \bar{x}$ to find an ε-optimal solution to (GP_d).*

Theorem 26 *(Complexity of Algorithm Phase II.) Suppose that $d \in \mathcal{F}_P$, and that there exists a point $\bar{x} \in X_d$ which has* $\mathrm{dist}(\bar{x}, \mathrm{rel}\partial X_d) \geq \bar{r} > 0$. *Then Algorithm Phase II will compute an ε-optimal solution of (GP_d) in at most*

$$O\left(\sqrt{\vartheta_P + \vartheta_{C_Y}} \ln\left(\vartheta_P + \vartheta_{C_Y} + \vartheta_{\|\cdot\|} + G_\varepsilon + \max\left\{1, \frac{1}{\bar{r}}\right\} + \|\bar{x} - x^0\| + \max\left\{1, \frac{\tilde{s}}{\varepsilon}\right\}\right)\right)$$

iterations of Newton's method.

Proof: First recall that the complexity parameter of the self-concordant barrier function used in Phase II is $\vartheta = \vartheta_P + \vartheta_{C_Y} + 1$.

Also from Proposition 17 we note that

$$\frac{1}{\mathrm{sym}(\bar{x}, S_2)} \leq \max\left\{1, \frac{1}{\bar{r}}\right\} h(\bar{x}, \bar{r}) \ .$$

We can therefore bound the symmetry of $\bar{x}$ with respect to the set S_2 by a polynomial in $\frac{1}{\bar{r}}$, G_ε, $\|\bar{x} - x^0\|$, $\vartheta_{\|\cdot\|}$, and $\max\left\{1, \frac{\tilde{s}}{\varepsilon}\right\}$.

Finally we bound

$$\begin{aligned} \frac{R}{\varepsilon} &= \frac{1}{\varepsilon}\left(c^t\bar{x} + \bar{s} - z^*\right) \\ &\leq \frac{1}{\varepsilon}\tilde{s}\left(\|\bar{x} - x^*\| + 6\vartheta_{\|\cdot\|} + 1\right) \\ &\leq \frac{\tilde{s}}{\varepsilon}\left(\|\bar{x} - x^0\| + G_\varepsilon + 6\vartheta_{\|\cdot\|} + 1\right) \end{aligned}$$

Combining the bounds on $\frac{1}{\mathrm{sym}(\bar{x}, S_2)}$ and $\frac{R}{\varepsilon}$ and the definition of ϑ we obtain the result. ■

6.4 Overall complexity

We now present the complexity bound of running both Algorithm Phase I and Algorithm Phase II. Given interior points $x^0 \in \mathrm{relint}P$ and $w^0 \in \mathrm{relint}C_Y$ we execute Algorithm Phase I, which outputs $\bar{x} \in \mathrm{relint}X_d$. We use this point as input to Algorithm Phase II, which returns an ε-optimal point for (GP_d).

Theorem 27 *Algorithm Phase I and Algorithm Phase II will find an ε-optimal solution of (GP_d) in at most*

$$O\left(\sqrt{\vartheta}\ln\left(\begin{array}{l} \vartheta + \frac{1}{\min\left\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\right\}} + g_P(x^0) + g_{C_Y}(w^0) + \max\left\{\frac{\tilde{s}}{\varepsilon}, 1\right\} + \\ C(d) + G_\varepsilon + \|d\| + \frac{\|w^0\|}{\rho_P(d)} \end{array}\right)\right)$$

iterations of the Newton method, where $\vartheta = \vartheta_P + \vartheta_{C_Y} + \vartheta_{\|\cdot\|}$.

Proof: The proof is a consequence of Theorem 25 and Theorem 26, which give bounds of

$$O\left(\sqrt{\vartheta}\ln\left(\vartheta + g_P(x^0) + g_{C_Y}(w^0) + \|d\| + C(d) + \frac{\|w^0\|}{\rho_P(d)}\right)\right) ,$$

and

$$O\left(\sqrt{\vartheta_P + \vartheta_{C_Y}}\ln\left(\vartheta_P + \vartheta_{C_Y} + \vartheta_{\|\cdot\|} + G_\varepsilon + \max\left\{1, \frac{1}{\bar{r}}\right\} + \|\bar{x} - x^0\| + \max\left\{1, \frac{\tilde{s}}{\varepsilon}\right\}\right)\right) ,$$

respectively. To finish the proof we note that bounds from Theorem 25 imply

$$\begin{aligned}
\frac{1}{\bar{r}} &\leq \text{polynomial}\left(\frac{1}{\min\left\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\right\}}, C(d), g_P(x^0), \frac{\|w^0\|}{\rho_P(d)}\right) \\
\|\bar{x} - x^0\| &\leq \text{polynomial}\left(C(d), g_P(x^0), \frac{\|w^0\|}{\rho_P(d)}\right) . \qquad \blacksquare
\end{aligned}$$

6.5 Condition Number bound on G_ε

We now present a result which proves that for a consistent instance, we can bound the measure G_ε by condition number quantities if the ground set P satisfies Assumption 2 and the instance is well posed.

Recall that Assumption 2 states that $P = E + R$, that is the ground set is the direct sum of a bounded set E and a recession cone R. Recall also that we denote by $|E| = \max\{\|x\| \mid x \in E\}$.

Proposition 18 *Assume that* P *satisfies Assumption 2,* $d \in \mathcal{F}$*, and* $\rho(d) > 0$*. Then* *for any $\varepsilon > 0$, the measure G_ε defined in (6.1) satisfies*

$$G_\varepsilon \le \|x^0\| + C(d)^2\,(4|E|+1) + \frac{\varepsilon}{\rho(d)} \ .$$

Proof: Consider any x in the ε-optimal level set, that is $x \in X_d$ and $c^t x \le z_*(d) + \varepsilon$. Then Assumption 2 implies that there exist $\hat{x} \in E$ and $r \in R$ such that $x = \hat{x} + r$. Then

$$\begin{aligned}
\|x - x^0\| &\le \|\hat{x} - x^0\| + \|r\| \\
&\le \|\hat{x} - x^0\| + \frac{\max\{\|A\hat{x} - b\|, c^t r\}}{\rho_D(d)} \\
&= \|\hat{x} - x^0\| + \frac{\max\{\|A\hat{x} - b\|, c^t x - c^t \hat{x}\}}{\rho_D(d)} \\
&\le \|\hat{x} - x^0\| + \frac{\max\{\|A\hat{x} - b\|, z_*(d) + \varepsilon + \|c\|_* \|\hat{x}\|\}}{\rho_D(d)} \\
&\le \|\hat{x} - x^0\| + C(d)\max\left\{\|\hat{x}\| + 1, \|\hat{x}\| + \frac{z_*(d) + \varepsilon}{\|d\|}\right\} \ ,
\end{aligned}$$

where the second inequality is from Proposition 10. We can bound the optimal objective function value using Theorem 20 to obtain $|z_*(d)| \le \|d\| C(d)(2|E| + 1)$. Incorporating this bound in the previous expression we obtain

$$\begin{aligned}
\|x - x^0\| &\le \|\hat{x} - x^0\| + C(d)\max\left\{\|\hat{x}\| + 1, \|\hat{x}\| + C(d)(2|E|+1) + \frac{\varepsilon}{\|d\|}\right\} \\
&\le \|\hat{x} - x^0\| + C(d)\left(\|\hat{x}\| + C(d)(2|E|+1) + \frac{\varepsilon}{\|d\|}\right) \\
&\le \|x^0\| + 2C(d)\|\hat{x}\| + C(d)^2(2|E|+1) + \frac{\varepsilon}{\rho(d)} \\
&\le \|x^0\| + C(d)^2(4|E|+1) + \frac{\varepsilon}{\rho(d)} \ . \qquad \blacksquare
\end{aligned}$$

Note that if $x^0 = 0$, the previous result shows that $G_\varepsilon \leq C(d)^2(4|E|+1) + \frac{\varepsilon}{\|d\|}C(d)$. The first term in this bound is exactly the bound on the size of the optimal primal solution according to Theorem 20. The second term shows that increasing ε expands the ε-optimal level set at most proportional to $\frac{1}{\rho(d)}$.

A consequence of Proposition 18 is that under Assumption 2 we can state the following complexity bound to solve (GP_d).

Corollary 6 *Assume that P satisfies Assumption 2, $d \in \mathcal{F}$, and $\rho(d) > 0$. Algorithm Phase I and Algorithm Phase II will find an ε-optimal solution of (GP_d) in at most*

$$O\left(\sqrt{\vartheta}\ln\left(\begin{array}{l}\vartheta + \frac{1}{\min\left\{r_{x^0}, \frac{r_{w^0}}{\|A\|}\right\}} + g_P(x^0) + g_{C_Y}(w^0) + \max\left\{\frac{\tilde{s}}{\varepsilon}, 1\right\} + \\ C(d) + \|d\| + |E| + \frac{\|w^0\|}{\rho_P(d)} + \frac{\varepsilon}{\rho(d)}\end{array}\right)\right)$$

iterations of the Newton method, where $\vartheta = \vartheta_P + \vartheta_{C_Y} + \vartheta_{\|\cdot\|}$. ∎

Chapter 7

Computational Experience and the Explanatory Value of Condition Numbers for LP

The last three chapters contain a number of results which show that the condition number of a convex optimization problem is a quantity which is of interest in theory. In order to investigate the relevance of condition numbers for problems that arise in practice, we start by computing the condition numbers for a suitably representative set of linear optimization instances that arise in practice, such as the NETLIB suite of industrial and academic linear optimization problems, see [21]. Practical methods for computing (or approximately computing) condition numbers for convex optimization problems in conic format (CP_d) have been developed in [11] and [24], and such methods are relatively easy to implement. It would then seem to be a simple task to compute condition numbers for the NETLIB suite. But as noted in Chapter 1, it turns out that there is a subtle catch that gets in the way of this simple strategy, and in fact necessitates using the GSM format extension presented.

Linear optimization problem instances arising in practice are typically conveyed in the following format:

$$
\begin{array}{ll}
\min_x & c^t x \\
\text{s.t.} & A_i x \leq b_i \ , i \in L \\
& A_i x = b_i \ , i \in E \\
& A_i x \geq b_i \ , i \in G \\
& x_j \geq l_j \ , \ j \in L_B \\
& x_j \leq u_j \ , \ j \in U_B \ ,
\end{array}
\tag{7.1}
$$

where the first three sets of inequalities/equalities are the "constraints" and the last two sets of inequalities are the lower and upper bound conditions, and where $L_B, U_B \subset \{1, \ldots, n\}$. (LP problems in practice might also contain range constraints of the form "$b_{i,l} \leq A_i x \leq b_{i,u}$" as well. We ignore this for now.) By defining C_Y to be an appropriate cartesian product of nonnegative half-lines $I\!R_+$, nonpositive half-lines $-I\!R_+$, and the origin $\{0\}$, we can naturally consider the constraints to be in the conic format "$Ax - b \in C_Y$" where $C_Y \subset I\!R^m$ and $m = |L| + |E| + |G|$. Note that we are amending the notation, previously in the thesis, the set E denoted the bounded subset of P in Assumption 2. (To avoid confusion, for this chapter, we denote by E the set of indices for which the constraint is an equality; and by $\tilde{E}$ for the bounded subset of the ground set P, as in Assumption 2.)

We will treat linear optimization problems conveyed in the format (7.1) to be an instance of (GP_d) by setting the ground-set P to be defined by the lower and upper bounds:

$$
P := \{x \mid x_j \geq l_j \text{ for } j \in L_B, \ x_j \leq u_j \text{ for } j \in U_B\} \ , \tag{7.2}
$$

and by re-writing the other constraints in conic format as described earlier. But now notice the data d does not include the lower and upper bound data $l_j, j \in L_B$ and

$u_j, j \in U_B$. This is somewhat advantageous since in many settings of linear optimization the lower and/or upper bounds on most variables are 0 or 1 or other scalars that are "fixed" and are not generally thought of as subject to modification. (Of course, there are other settings where keeping the lower and upper bounds fixed independent of the other constraints is not as natural.)

7.1 Computation of $\rho_P(d)$, $\rho_D(d)$, and $C(d)$ via convex optimization

In this section we show how to compute $\rho_P(d)$ and $\rho_D(d)$ for linear optimization data instances $d = (A, b, c)$ of the ground-set model format, as well as how to estimate $\|d\|$ and $C(d)$. The methodology presented herein builds on Theorems 5 and 6 and is an extension of the methodology for computing $\rho_P(d)$ and $\rho_D(d)$ developed in [11]. We will make the following choice of norms throughout this chapter:

Assumption 3 *The norm on the space of the x variables in $\mathbb{R}^n$ is the L_∞-norm, and the norm on the space of the right-hand-side vector in $\mathbb{R}^m$ is the L_1-norm.*

Using this choice of norms, we will show in this chapter how to compute $\rho(d)$ for linear optimization problems by solving $2n + 2m$ LPs of size roughly that of the original problem. As is discussed in [11], the complexity of computing $\rho(d)$ very much depends on the chosen norms, with the norms given in Assumption 3 being particularly appropriate for efficient computation. We begin our analysis with a seemingly innocuous proposition which will prove to be very useful.

Proposition 19 *Consider the problem:*

$$\begin{aligned} z_1 = \quad \min_{v,w} \quad & f(v,w) \\ s.t. \quad & \|v\|_\infty = 1 \\ & (v,w) \in K \ , \end{aligned} \tag{7.3}$$

where $v \in \mathbb{R}^k$, $w \in \mathbb{R}^l$, K is a closed convex cone in $\mathbb{R}^{k+l}$, and $f(\cdot) : \mathbb{R}^{k+l} \mapsto \mathbb{R}_+$ is positively homogeneous of degree one ($f(\alpha(v,w)) = |\alpha| f(v,w)$ for any $\alpha \in \mathbb{R}$ and $(v,w) \in \mathbb{R}^{k+l}$). Then problem (7.3) and (7.4) have the same optimal values, i.e., $z_1 = z_2$, where

$$z_2 = \min_{\substack{i,j \\ i \in \{1,\ldots,n\}, j \in \{-1,1\}}} \quad \begin{aligned} \min_{v,w} \ & f(v,w) \\ & v_i = j \\ & (v,w) \in K \ . \end{aligned} \tag{7.4}$$

Proof: Let (v^*, w^*) be an optimal solution of (7.3). Since $\|v^*\|_\infty = 1$, there exist $i^* \in \{1,\ldots,n\}$ and $j^* \in \{-1,1\}$ such that $v^*_{i^*} = j^*$. Therefore (v^*, w^*) is feasible for the inner problem in (7.4) for $i = i^*$ and $j = j^*$, and so $z_2 \le z_1$.

If (v^*, w^*) is an optimal solution of (7.4) with $i = i^*$ and $j = j^*$, then $\|v^*\|_\infty \ge 1$. If $\|v^*\|_\infty = 1$, the point (v^*, w^*) is feasible for (7.3) which means that $z_1 \le z_2$, completing the proof. Therefore, assume that $\|v^*\|_\infty > 1$, and consider the new point $(\tilde{v}, \tilde{w}) := \frac{1}{\|v^*\|_\infty}(v^*, w^*) \in K$. Then $(\tilde{v}, \tilde{w})$ is feasible for an inner problem in (7.4) for some $i = \hat{i} \ne i^*$ and $j = \hat{j}$, and so $z_2 \le f(\tilde{v}, \tilde{w}) = f\left(\frac{1}{\|v^*\|_\infty}(v^*, w^*)\right) = \frac{1}{\|v^*\|_\infty} f(v^*, w^*) \le z_2$, which now implies that $(\tilde{v}, \tilde{w})$ is also an optimal solution of (7.4). Since $\|\tilde{v}\|_\infty = 1$, the previous argument implies that $z_1 \le z_2$, completing the proof. ■

7.1.1 Computing $\rho_P(d)$ and $\rho_D(d)$

Assume that $d \in \mathcal{F}$ and that the norms are chosen as in Assumption 3. Recall that in this case, Theorem 5 and Theorem 6 characterize the primal and dual distance to ill-infeasibility by $\rho_P(d) = j_P(d)$ and $\rho_D(d) = j_D(d)$, where

$$\begin{array}{rlll} j_P(d) = & \min_{y,s,v} & \max\left\{\|A^t y + s\|_1, |b^t y - v|\right\} & \\ & \text{s.t.} & \|y\|_\infty = 1 & \\ & & y \in C_Y^* & \\ & & (s,v) \in C^* \ , & \end{array} \tag{7.5}$$

and

$$\begin{array}{rlll} j_D(d) = & \min_{x,p,g} & \max\left\{\|Ax - p\|_1, |c^t x + g|\right\} & \\ & \text{s.t.} & \|x\|_\infty = 1 & \\ & & x \in R & \\ & & p \in C_Y & \\ & & g \geq 0 \ . & \end{array} \tag{7.6}$$

Neither (7.5) nor (7.6) are convex problems. However, both (7.5) and (7.6) are of the form (7.3), and so we can invoke Proposition 19, and solve (7.5) and (7.6) using problem (7.4). From Proposition 19, we have:

$$\rho_P(d) = \min_{i \in \{1,\ldots,m\}, j \in \{-1,1\}} \begin{array}{ll} \min_{y,s,v} & \max\left\{\|A^t y + s\|_1, |b^t y - v|\right\} \\ \text{s.t.} & y_i = j \\ & y \in C_Y^* \\ & (s,v) \in C^* \end{array} \tag{7.7}$$

and

$$\rho_D(d) = \min_{i \in \{1,\ldots,n\}, j \in \{-1,1\}} \begin{array}{ll} \min_{x,p,g} & \max\left\{\|Ax-p\|_1, |c^t x + g|\right\} \\ \text{s.t.} & x_i = j \\ & x \in R \\ & p \in C_Y \\ & g \geq 0. \end{array} \tag{7.8}$$

Taken together, (7.7) and (7.8) show that we can compute $\rho_P(d)$ by solving $2m$ convex optimization problems, and we can compute $\rho_D(d)$ by solving $2n$ convex optimization problems. In conclusion, we can compute $\rho(d)$ by solving $2n + 2m$ convex optimization problems, where all of the optimization problems involved are of the roughly the same size as the original problem GP_d.

Of course, each of the $2n + 2m$ convex problems in (7.7) and (7.8) will be computationally tractable only if we can conveniently work with the cones involved; we now show that for the special case of linear optimization models (7.1), there are convenient linear inequality characterizations of all of the cones involved in (7.7) and (7.8). The cone C_Y is easily seen to be:

$$C_Y = \{p \in \mathbb{R}^m \mid p_i \leq 0 \text{ for } i \in L, p_i = 0 \text{ for } i \in E, p_i \geq 0 \text{ for } i \in G\}, \tag{7.9}$$

and so

$$C_Y^* = \{y \in \mathbb{R}^m \mid y_i \leq 0 \text{ for } i \in L, y_i \in \mathbb{R} \text{ for } i \in E, y_i \geq 0 \text{ for } i \in G\}. \tag{7.10}$$

With the ground-set P defined in (7.2), we have:

$$R = \{x \in \mathbb{R}^n \mid x_j \geq 0 \text{ for } j \in L_B,\ x_j \leq 0 \text{ for } j \in U_B\}, \tag{7.11}$$

and also

$$C = \{(x,t) \in I\!R^n \times I\!R \mid t \geq 0, x_j \geq l_j t \text{ for } j \in L_B,\ x_j \leq u_j t \text{ for } j \in U_B\}\ . \qquad (7.12)$$

The only cone whose characterization is less than obvious is C^*, which we now characterize. Consider the following system of linear inequalities in the variables $(s, v, s^+, s^-) \in I\!R^n \times I\!R \times I\!R^n \times I\!R^n$:

$$\begin{array}{rcll} s - s^+ + s^- & = & 0 & \\ s^+ & \geq & 0 & \\ s^- & \geq & 0 & \\ s_j^- & = & 0 & \text{for } j \in N \setminus U_B \\ s_j^+ & = & 0 & \text{for } j \in N \setminus L_B \\ v + \sum\limits_{j \in L_B} l_j s_j^+ - \sum\limits_{j \in U_B} u_j s_j^- & \geq & 0\ , & \end{array} \qquad (7.13)$$

where we use the notation $N := \{1, \ldots, n\}$ and $S \setminus T$ is the set difference $\{k \mid k \in S, k \notin T\}$.

Proposition 20 *For the ground-set P defined in (7.2), the cone C^* is characterized by*

$$C^* = \{(s,v) \in I\!R^n \times I\!R \mid (s, v, s^+, s^-) \text{ satisfies (7.13) for some } s^+, s^- \in I\!R^n\}\ .$$

Proof: Suppose first that (s, v) together with some s^+, s^- satisfies (7.13). Then for all

$(x, t) \in C$ we have

$$
\begin{aligned}
(x,t)^t(s,v) &= \sum_{j \in L_B} s_j^+ x_j - \sum_{j \in U_B} s_j^- x_j + tv \\
&\geq \sum_{j \in L_B} s_j^+ l_j t - \sum_{j \in U_B} s_j^- u_j t + tv \qquad (7.14) \\
&\geq 0 \ ,
\end{aligned}
$$

and so $(s, v) \in C^*$. Conversely, suppose that $(s, v) \in C^*$. Then

$$
\begin{aligned}
-\infty < -v \leq \min_{x \in P} s^t x = \min \quad & \sum_{j=1}^{n} s_j x_j \\
\text{s.t.} \quad & x_j \geq l_j \text{ for } j \in L_B \qquad (7.15) \\
& x_j \leq u_j \text{ for } j \in U_B \ ,
\end{aligned}
$$

and define s^+ and s^- to be the positive and negative parts of s, respectively. Then $s = s^+ - s^-, s^+ \geq 0$, and $s^- \geq 0$, and (7.15) implies $s_j^+ = 0$ for $j \in N \setminus L_B$, $s_j^- = 0$ for $j \in N \setminus U_B$, as well as the last inequality of (7.13), whereby (s, v, s^+, s^-) satisfies all inequalities of (7.13). ∎

Taken together, we can use (7.9), (7.10), (7.11), (7.12), and Proposition 20 to re-write the right-most minimization problems of (7.7) and (7.8) and obtain:

$$
\rho_P(d) = \min_{\substack{i \in \{1,\ldots,m\} \\ j \in \{-1,1\}}} \quad \begin{array}{rl} \min_{y,s^+,s^-,v} & \max\left\{\left\|A^t y + s^+ - s^-\right\|_1, \left|b^t y - v\right|\right\} \\ \text{s.t.} & y_i = j \\ & y_l \leq 0 \quad \text{for } l \in L \\ & y_l \geq 0 \quad \text{for } l \in G \\ & s_k^- = 0 \quad \text{for } k \in N \setminus U_B \\ & s_k^+ = 0 \quad \text{for } k \in N \setminus L_B \\ & v + \sum_{k \in L_B} l_k s_k^+ - \sum_{k \in U_B} u_k s_k^- \geq 0 \\ & s^+, s^- \geq 0 \end{array} \tag{7.16}
$$

and

$$
\rho_D(d) = \min_{\substack{i \in \{1,\ldots,n\} \\ j \in \{-1,1\}}} \quad \begin{array}{rl} \min_{x,p,g} & \max\left\{\left\|Ax - p\right\|_1, \left|c^t x + g\right|\right\} \\ \text{s.t.} & x_i = j \\ & x_k \geq 0 \quad \text{if } k \in L_B \\ & x_k \leq 0 \quad \text{for } k \in U_B \\ & p_l \leq 0 \quad \text{for } l \in L \\ & p_l = 0 \quad \text{for } l \in E \\ & p_l \geq 0 \quad \text{for } l \in G \\ & g \geq 0\ , \end{array} \tag{7.17}
$$

whose right-most objective functions can then easily be converted to linear optimization problems by standard techniques. This then shows that we can indeed compute $\rho_P(d), \rho_D(d)$, and $\rho(d)$ by solving $2n + 2m$ LPs, under the choice of norms given in Assumption 3.

7.1.2 Computing $\|d\|$

In order to compute the condition number given by $C(d) := \|d\|/\rho(d)$, we must also compute $\|d\| = \max\{\|A\|, \|b\|, \|c\|_*\}$. Under Assumption 3 we have $\|b\| = \|b\|_1$ and $\|c\|_* = \|c\|_1$, which are both easy to compute. However, $\|A\|$ is the operator norm, and so $\|A\| := \|A\|_{\infty,1} := \max\{\|Ax\|_1 \mid \|x\|_\infty = 1\}$, whose computation is a hard combinatorial problem. We therefore will bound $\|A\|_{\infty,1}$ and hence $\|d\|$ from below and above, using the following elementary norm inequalities:

$$\max\left\{\|A\|_{1,1}, \|A\|_{2,2}, \|A\|_F, \|Ae\|_1, \|A\hat{x}\|_1\right\} \leq \|A\|_{\infty,1} \leq \max\left\{\|A\|_{L_1}, \sqrt{nm}\,\|A\|_{2,2}\right\},$$

where

$$\begin{aligned}
\|A\|_{1,1} &= \max_{j=1,\ldots,n} \|A_{\bullet j}\|_1, \\
\|A\|_{2,2} &= \sqrt{\lambda_{\max}(A^t A)}, \\
\|A\|_F &= \sqrt{\textstyle\sum_{i=1}^m \sum_{j=1}^n (A_{i,j})^2}, \\
\|A\|_{L_1} &= \textstyle\sum_{i=1}^m \sum_{j=1}^n |A_{i,j}|,
\end{aligned}$$

$e := (1, \ldots, 1)^t$, and $\hat{x}$ is defined using $\hat{x}_j = \mathrm{sign}(A_{i^*,j})$, where $i^* = \mathrm{argmax}_{i=1,\ldots,m} \|A_{i\bullet}\|_1$.

Most of the norm inequalities are easily verified due to the definition of the operator norm and the relations $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty$. The bounds that are not verified this way are $\|A\|_{\infty,1} \leq \|A\|_{L_1}$ and $\|A\|_F \leq \|A\|_{\infty,1}$.

Of these, the first is true because if $\|x\|_\infty \leq 1$, then $|x_j| \leq 1$ for all j and so $\|Ax\|_1 = \sum_{i=1}^m \left|\sum_{j=1}^n A_{i,j}x_j\right| \leq \sum_{i=1}^m \sum_{j=1}^n |A_{i,j}||x_j| \leq \|A\|_{L_1}$. The remaining bound to prove is a bit more involved, and requires the following property of matrices.

Lemma 11 *For any $M \in \mathbb{R}^{n\times n}$, $\displaystyle\max_{\|x\|_\infty \leq 1} x^t M x \geq \mathrm{tr}(M)$.*

Proof: The proof is by induction on the dimension n. If $n = 1$, then the result is verified since $\displaystyle\max_{-1\leq x\leq 1} m_{11}x^2 = m_{11} = \mathrm{tr}(m_{11})$.

Assume the result is true for matrices of order $n \times n$. Consider the $(n+1) \times (n+1)$ matrix

$$M = \begin{bmatrix} M_n & b_1 \\ b_2^t & M_{n+1,n+1} \end{bmatrix} ,$$

where $M_n \in I\!R^{n\times n}$, $b_1, b_2 \in I\!R^n$, and $M_{n+1,n+1}$ is a scalar. We also separate the $n+1$ dimensional vector $x = (x_n, x_{n+1})$, where $x_n \in I\!R^n$ and x_{n+1} is a scalar. Also, let $x^* \in I\!R^n$ be such that $x^* = \text{argmax}_{\|x_n\|_\infty \leq 1} x_n^t M_n x_n$. Then

$$\begin{aligned}
\max_{\|x\|_\infty \leq 1} x^t M x &= \max_{\|x_n\|_\infty \leq 1,\ |x_{n+1}| \leq 1} x_n^t M_n x_n + x_{n+1}(b_1^t x_n + b_2^t x_n) + M_{n+1,n+1} x_{n+1}^2 \\
&\geq \max_{|x_{n+1}| \leq 1} x^* M_n x^* + x_{n+1}(b_1^t x^* + b_2^t x^*) + M_{n+1,n+1} x_{n+1}^2 \\
&\geq \text{tr}(M_n) + \max_{|x_{n+1}| \leq 1} x_{n+1}(b_1^t x^* + b_2^t x^*) + M_{n+1,n+1} x_{n+1}^2 \\
&\geq \text{tr}(M_n) + \left| b_1^t x^* + b_2^t x^* \right| + M_{n+1,n+1} \\
&\geq \text{tr}(M_n) + M_{n+1,n+1} \\
&= \text{tr}(M_{n+1}) \ . \qquad \blacksquare
\end{aligned}$$

Proposition 21 *For $A \in I\!R^{m\times n}$, $\|A\|_F \leq \|A\|_{\infty,1}$.*

Proof: Let $M = A^t A$, and from Lemma 11 we know there exists $\bar{x}$ satisfying $\|\bar{x}\|_\infty \leq 1$ and $\bar{x}^t M \bar{x} \geq \text{tr}(M)$. Then

$$\|A\|_{\infty,1} \geq \|A\bar{x}\|_1 \geq \|A\bar{x}\|_2 = \sqrt{\bar{x}^t A^t A \bar{x}} \ ,$$

where the last equality is from the definition of the Euclidean norm. Rearranging the above expression we have

$$\begin{aligned}
\|A\|_{\infty,1} &\geq \sqrt{\bar{x}^t A^t A \bar{x}} \\
&= \sqrt{\bar{x}^t M \bar{x}} \\
&\geq \sqrt{\text{tr} M}
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\mathrm{tr} A^t A} \\
&= \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{m} A_{ij}^2} \\
&= \|A\|_F \ . \qquad \blacksquare
\end{aligned}
$$

7.2 Computational results on the NETLIB Suite of linear optimization problems

7.2.1 Condition Numbers for the NETLIB Suite prior to pre-processing

We computed the distances to ill-posedness and condition numbers for the NETLIB suite of linear optimization problems, using the methodology developed in Section 7.1. The NETLIB suite is comprised of 98 linear optimization problem instances from diverse application areas, collected over a period of many years. While this suite does not contain any truly large problems by today's standards, it is arguably the best publicly available collection of practical LP problems. The sizes of the problem instances in the NETLIB suite range from 32 variables and 28 constraints to problems with roughly 7,000 variables and 3,000 constraints. 44 of the 98 problems in the suite have non-zero lower bound constraints and/or upper bound constraints on the variables, and five problems have range constraints. We omitted the five problems with range constraints (boeing1, boeing2, forplan, nesm, seba) for the purposes of our analysis. We also omitted an additional five problems in the suite because they are not readily available in MPS format (qap8, qap12, qap15, stocfor3, truss). (These problems are each presented as a code that can be used to generate an MPS file.) Of the remaining 88 problems, there were five more problems (dfl001, fit2p, maros-r7, pilot, pilot87) for which our methodology was unable to compute the distance to ill-posedness in spite of over a full week of computation time. These problems were omitted as well, yielding a final sample set of 83 linear optimization problems. The burden of computing the distances to ill-posedness for the NETLIB suite via the solution of $2n + 2m$ LPs obviously grows with the dimensions of the problem instances. On afiro, which is a small problem instance ($n = 28$, $m = 32$), the total computation time amounted to only 0.28 seconds of machine time, whereas for d6cube ($n = 5,442$ and $m = 402$ after pre-processing), the total computation time was $77,152.43$ seconds of machine time (21.43 hours).

Table 7.1 shows the distances to ill-posedness and the condition number estimates for the 83 problems, using the methodology for computing $\rho_P(d)$ and $\rho_D(d)$ and for estimating $\|d\|$ presented in Section 7.1. All linear programming computation was performed using CPLEX 7.1 (function *primopt*).

Table 7.1: Condition Numbers for the NETLIB Suite prior to Pre-Processing

			$\|d\|$		$\log C(d)$	
			Lower	Upper	Lower	Upper
Problem	$\rho_P(d)$	$\rho_D(d)$	Bound	Bound	Bound	Bound
25fv47	0.000000	0.000000	30,778	55,056	∞	∞
80bau3b	0.000000	0.000000	142,228	142,228	∞	∞
adlittle	0.000000	0.051651	68,721	68,721	∞	∞
afiro	0.397390	1.000000	1,814	1,814	3.7	3.7
agg	0.000000	0.771400	5.51E+07	5.51E+07	∞	∞
agg2	0.000000	0.771400	1.73E+07	1.73E+07	∞	∞
agg3	0.000000	0.771400	1.72E+07	1.72E+07	∞	∞
bandm	0.000000	0.000418	10,200	17,367	∞	∞
beaconfd	0.000000	0.000000	15,322	19,330	∞	∞
blend	0.003541	0.040726	1,020	1,255	5.5	5.5
bnl1	0.000000	0.106400	8,386	9,887	∞	∞
bnl2	0.000000	0.000000	36,729	36,729	∞	∞
bore3d	0.000000	0.003539	11,912	12,284	∞	∞
brandy	0.000000	0.000000	7,254	10,936	∞	∞
capri	0.000252	0.095510	33,326	33,326	8.1	8.1
cycle	0.000000	0.000000	365,572	391,214	∞	∞
czprob	0.000000	0.008807	328,374	328,374	∞	∞

			$\|d\|$		$\log C(d)$	
Problem	$\rho_P(d)$	$\rho_D(d)$	Lower Bound	Upper Bound	Lower Bound	Upper Bound
d2q06c	0.000000	0.000000	171,033	381,438	∞	∞
d6cube	0.000000	2.000000	47,258	65,574	∞	∞
degen2	0.000000	1.000000	3,737	3,978	∞	∞
degen3	0.000000	1.000000	4,016	24,646	∞	∞
e226	0.000000	0.000000	22,743	37,344	∞	∞
etamacro	0.000000	0.200000	31,249	63,473	∞	∞
fffff800	0.000000	0.033046	1.55E+06	1.55E+06	∞	∞
finnis	0.000000	0.000000	31,978	31,978	∞	∞
fit1d	3.500000	∞	493,023	618,065	5.1	5.2
fit1p	1.271887	0.437500	218,080	384,121	5.7	5.9
fit2d	317.000000	∞	1.90E+06	2.25E+06	3.8	3.9
ganges	0.000000	1.000000	1.29E+06	1.29E+06	∞	∞
gfrd-pnc	0.000000	0.347032	1.63E+07	1.63E+07	∞	∞
greenbea	0.000000	0.000000	21,295	26,452	∞	∞
greenbeb	0.000000	0.000000	21,295	26,452	∞	∞
grow15	0.572842	0.968073	209	977	2.6	3.2
grow22	0.572842	0.968073	303	1,443	2.7	3.4
grow7	0.572842	0.968073	102	445	2.3	2.9
israel	0.027248	0.166850	2.22E+06	2.22E+06	7.9	7.9
kb2	0.000201	0.018802	10,999	11,544	7.7	7.8
lotfi	0.000306	0.000000	166,757	166,757	∞	∞
maros	0.000000	0.000000	2.51E+06	2.55E+06	∞	∞
modszk1	0.000000	0.108469	1.03E+06	1.03E+06	∞	∞
perold	0.000000	0.000943	703,824	2.64E+06	∞	∞
pilot.ja	0.000000	0.000750	2.67E+07	1.40E+08	∞	∞
pilot.we	0.000000	0.044874	5.71E+06	5.71E+06	∞	∞
pilot4	0.000000	0.000075	763,677	1.09E+06	∞	∞
pilotnov	0.000000	0.000750	2.36E+07	1.35E+08	∞	∞
recipe	0.000000	0.000000	14,881	19,445	∞	∞
sc105	0.000000	0.133484	3,000	3,000	∞	∞
sc205	0.000000	0.010023	5,700	5,700	∞	∞
sc50a	0.000000	0.562500	1,500	1,500	∞	∞
sc50b	0.000000	0.421875	1,500	1,500	∞	∞
scagr25	0.021077	0.034646	430,977	430,977	7.3	7.3
scagr7	0.022644	0.034646	120,177	120,177	6.7	6.7
scfxm1	0.000000	0.000000	21,425	22,816	∞	∞
scfxm2	0.000000	0.000000	44,153	45,638	∞	∞
scfxm3	0.000000	0.000000	66,882	68,459	∞	∞
scorpion	0.000000	0.949393	5,622	5,622	∞	∞
scrs8	0.000000	0.000000	68,630	69,449	∞	∞
scsd1	5.037757	1.000000	1,752	1,752	3.2	3.2

			$\|d\|$		$\log C(d)$	
			Lower	Upper	Lower	Upper
Problem	$\rho_P(d)$	$\rho_D(d)$	Bound	Bound	Bound	Bound
scsd6	1.603351	1.000000	2,973	2,973	3.5	3.5
scsd8	0.268363	1.000000	5,549	5,549	4.3	4.3
sctap1	0.032258	1.000000	8,240	17,042	5.4	5.7
sctap2	0.586563	1.000000	32,982	72,870	4.7	5.1
sctap3	0.381250	1.000000	38,637	87,615	5.0	5.4
share1b	0.000015	0.000751	60,851	87,988	9.6	9.8
share2b	0.001747	0.287893	19,413	23,885	7.0	7.1
shell	0.000000	1.777778	253,434	253,434	∞	∞
ship04l	0.000000	13.146000	811,956	811,956	∞	∞
ship04s	0.000000	13.146000	515,186	515,186	∞	∞
ship08l	0.000000	21.210000	1.91E+06	1.91E+06	∞	∞
ship08s	0.000000	21.210000	1.05E+06	1.05E+06	∞	∞
ship12l	0.000000	7.434000	794,932	794,932	∞	∞
ship12s	0.000000	7.434000	381,506	381,506	∞	∞
sierra	0.000000	∞	6.60E+06	6.61E+06	∞	∞
stair	0.000580	0.000000	976	1,679	∞	∞
standata	0.000000	1.000000	21,428	23,176	∞	∞
standgub	0.000000	0.000000	21,487	23,235	∞	∞
standmps	0.000000	1.000000	22,074	23,824	∞	∞
stocfor1	0.001203	0.011936	23,212	23,441	7.3	7.3
stocfor2	0.000437	0.000064	462,821	467,413	9.9	9.9
tuff	0.000000	0.017485	136,770	145,448	∞	∞
vtp.base	0.000000	0.500000	530,416	534,652	∞	∞
wood1p	0.000000	1.000000	3.66E+06	5.04E+06	∞	∞
woodw	0.000000	1.000000	9.86E+06	1.35E+07	∞	∞

Table 7.2 presents some summary statistics of the condition number computations from Table 7.1. As the table shows, 72% (60/83) of the problems in the NETLIB suite are ill-conditioned due to either $\rho_P(d) = 0$ or $\rho_D(d) = 0$ or both. Furthermore, notice that among these 60 ill-conditioned problems, that almost all of these (58 out of 60) have $\rho_P(d) = 0$. This means that for 70% (58/83) of the problems in the NETLIB suite, arbitrarily small changes in the data will render the primal problem infeasible.

Notice from Table 7.1 that there are three problems for which $\rho_D(d) = \infty$, namely fit1d, fit2d, and sierra. This can only happen when the ground-set P is bounded, which for linear optimization means that all variables have finite lower and upper bounds.

Table 7.2: Summary Statistics of Distances to Ill-Posedness for the NETLIB Suite prior to Pre-Processing.

		$\rho_D(d)$			
		0	Finite	∞	Totals
$\rho_P(d)$	0	18	39	1	58
	Finite	2	21	2	25
	∞	0	0	0	0
	Totals	20	60	3	83

7.2.2 Condition Numbers for the NETLIB Suite after pre-processing

Most commercial software packages for solving linear optimization problems perform pre-processing heuristics prior to solving a problem instance. These heuristics typically include row and/or column re-scaling, checks for linearly dependent equations, heuristics for identifying and eliminating redundant variable lower and upper bounds, etc. The original problem instance is converted to a post-processed instance by the processing heuristics, and it is this post-processed instance that is used as input to solution software. In CPLEX 7.1, the post-processed problem can be accessed using function *prslvwrite*.

In order to get a sense of the distribution of the condition numbers of the problems that are input to a modern IPM solver, we computed condition numbers for the post-processed versions of the 83 NETLIB suite problems, where the processing used was the default CPLEX preprocessing with the linear dependency check option activated. Table 7.3 shows the condition numbers in detail for the post-processed versions of the problems, and Table 7.4 presents some summary statistics of these condition numbers. Notice from Table 7.4 that only 19% (16/83) of the post-processed problems in the NETLIB suite are ill-posed. The pre-processing heuristics have increased the number of problems with finite condition numbers to 67 problems. In contrast to the original problems, the vast majority of post-processed problems have finite condition numbers.

Table 7.3: Condition Numbers for the NETLIB Suite after Pre-Processing by CPLEX 7.1

Problem	$\rho_P(d)$	$\rho_D(d)$	$\|d\|$ Lower Bound	$\|d\|$ Upper Bound	$\log C(d)$ Lower Bound	$\log C(d)$ Upper Bound
25fv47	0.000707	0.000111	35,101	54,700	8.5	8.7
80bau3b	0.000000	0.000058	126,355	126,355	∞	∞
adlittle	0.004202	1.000488	68,627	68,627	7.2	7.2
afiro	0.397390	1.000000	424	424	3.0	3.0
agg	0.000000	0.031728	3.04E+07	3.04E+07	∞	∞
agg2	0.000643	1.005710	1.57E+07	1.57E+07	10.4	10.4
agg3	0.000687	1.005734	1.56E+07	1.56E+07	10.4	10.4
bandm	0.001716	0.000418	7,283	12,364	7.2	7.5
beaconfd	0.004222	1.000000	6,632	6,632	6.2	6.2
blend	0.011327	0.041390	872	1,052	4.9	5.0
bnl1	0.000016	0.159015	8,140	9,544	8.7	8.8
bnl2	0.000021	0.000088	18,421	20,843	8.9	9.0
bore3d	0.000180	0.012354	8,306	8,306	7.7	7.7
brandy	0.000342	0.364322	4,342	7,553	7.1	7.3
capri	0.000375	0.314398	30,323	30,323	7.9	7.9
cycle	0.000021	0.009666	309,894	336,316	10.2	10.2
czprob	0.000000	0.001570	206,138	206,138	∞	∞
d2q06c	0.000000	0.003925	172,131	378,209	∞	∞
d6cube	0.945491	2.000000	43,629	60,623	4.7	4.8
degen2	0.000000	1.000000	2,613	3,839	∞	∞
degen3	0.000000	1.000000	4,526	24,090	∞	∞
e226	0.000737	0.021294	21,673	35,518	7.5	7.7
etamacro	0.001292	0.200000	55,527	87,767	7.6	7.8
fffff800	0.000000	0.033046	696,788	696,788	∞	∞
finnis	0.000000	0.000000	74,386	74,386	∞	∞
fit1d	3.500000	∞	493,023	617,867	5.1	5.2
fit1p	1.389864	1.000000	218,242	383,871	5.3	5.6
fit2d	317.000000	∞	1.90E+06	2.24E+06	3.8	3.8
ganges	0.000310	1.000000	143,913	143,913	8.7	8.7
gfrd-pnc	0.015645	0.347032	1.22E+07	1.22E+07	8.9	8.9
greenbea	0.000033	0.000004	65,526	65,526	10.2	10.2
greenbeb	0.000034	0.000007	43,820	43,820	9.8	9.8
grow15	0.572842	0.968073	209	977	2.6	3.2
grow22	0.572842	0.968073	303	1,443	2.7	3.4
grow7	0.572842	0.968073	102	445	2.3	2.9
israel	0.135433	0.166846	2.22E+06	2.22E+06	7.2	7.2
kb2	0.000201	0.026835	10,914	11,054	7.7	7.7
lotfi	0.000849	0.001590	170,422	170,422	8.3	8.3

Problem	$\rho_P(d)$	$\rho_D(d)$	$\|d\|$ Lower Bound	$\|d\|$ Upper Bound	$\log C(d)$ Lower Bound	$\log C(d)$ Upper Bound
maros	0.000000	0.006534	1.76E+06	1.80E+06	∞	∞
modszk1	0.016030	0.114866	1.03E+06	1.03E+06	7.8	7.8
perold	0.000000	0.002212	1.56E+06	2.35E+06	∞	∞
pilot.ja	0.000000	0.001100	2.36E+07	1.36E+08	∞	∞
pilot.we	0.000000	0.044874	5.71E+06	5.71E+06	∞	∞
pilot4	0.000399	0.002600	696,761	1.03E+06	9.2	9.4
pilotnov	0.000000	0.001146	2.36E+07	1.32E+08	∞	∞
recipe	0.063414	0.000000	13,356	15,815	∞	∞
sc105	0.778739	0.400452	3,000	3,000	3.9	3.9
sc205	0.778739	0.030068	5,700	5,700	5.3	5.3
sc50a	0.780744	1.000000	1,500	1,500	3.3	3.3
sc50b	0.695364	1.000000	1,500	1,500	3.3	3.3
scagr25	0.021191	0.049075	199,859	199,859	7.0	7.0
scagr7	0.022786	0.049075	61,259	61,259	6.4	6.4
scfxm1	0.000010	0.002439	20,426	21,811	9.3	9.3
scfxm2	0.000010	0.002439	38,863	43,630	9.6	9.6
scfxm3	0.000010	0.002439	57,300	65,449	9.8	9.8
scorpion	0.059731	0.995879	123,769	123,769	6.3	6.3
scrs8	0.009005	0.004389	66,362	68,659	7.2	7.2
scsd1	5.037757	1.000000	1,752	1,752	3.2	3.2
scsd6	1.603351	1.000000	2,973	2,973	3.5	3.5
scsd8	0.268363	1.000000	5,549	5,549	4.3	4.3
sctap1	0.032258	1.000000	7,204	15,186	5.3	5.7
sctap2	0.669540	1.000000	27,738	64,662	4.6	5.0
sctap3	0.500000	1.000000	32,697	78,415	4.8	5.2
share1b	0.000015	0.000751	1.67E+06	1.67E+06	11.0	11.0
share2b	0.001747	0.287893	19,410	23,882	7.0	7.1
shell	0.000263	0.253968	874,800	874,800	9.5	9.5
ship04l	0.000386	25.746000	881,005	881,005	9.4	9.4
ship04s	0.000557	25.746000	545,306	545,306	9.0	9.0
ship08l	0.000000	22.890000	1.57E+06	1.57E+06	∞	∞
ship08s	0.000000	22.890000	816,531	816,531	∞	∞
ship12l	0.000124	7.434000	748,238	748,238	9.8	9.8
ship12s	0.000149	7.434000	340,238	340,238	9.4	9.4
sierra	0.001039	47.190000	6.60E+06	6.61E+06	9.8	9.8
stair	0.003800	0.163162	7,071	7,071	6.3	6.3
standata	0.090909	1.000000	4,931	5,368	4.7	4.8
standgub	0.090909	1.000000	4,931	5,368	4.7	4.8
standmps	0.020000	1.000000	12,831	12,831	5.8	5.8
stocfor1	0.002130	0.109062	10,833	29,388	6.7	7.1
stocfor2	0.000811	0.000141	45,458	616,980	8.5	9.6

Problem	$\rho_P(d)$	$\rho_D(d)$	$\|d\|$ Lower Bound	$\|d\|$ Upper Bound	$\log C(d)$ Lower Bound	$\log C(d)$ Upper Bound
tuff	0.000025	0.047081	131,554	138,783	9.7	9.7
vtp.base	0.005287	3.698630	17,606	17,606	6.5	6.5
wood1p	0.059008	1.442564	2.11E+06	3.25E+06	7.6	7.7
woodw	0.009357	1.000000	5.68E+06	7.26E+06	8.8	8.9

Table 7.4: Summary Statistics of Distances to Ill-Posedness for the NETLIB Suite after Pre-Processing by CPLEX 7.1.

		$\rho_D(d)$ 0	$\rho_D(d)$ Finite	$\rho_D(d)$ ∞	Totals
$\rho_P(d)$	0	1	14	0	15
	Finite	1	65	2	68
	∞	0	0	0	0
	Totals	2	79	2	83

Figure 7-1 presents a histogram of the condition numbers of the post-processed problems taken from Table 7.3. The condition number of each problem is represented by the geometric mean of the upper and lower bound estimates in this histogram. The right-most column in the figure is used to tally the number of problems for which $C(d) = \infty$, and is shown to give a more complete picture of the data. This histogram shows that of the problems with finite condition number, $\log C(d)$ is fairly nicely distributed between 2.6 and 11.0. Of course, when $C(d) = 10^{11}$, it is increasingly difficult to distinguish between a finite and non-finite condition number.

7.2.3 Condition Numbers and the observed performance of interior-point methods on the NETLIB Suite

It is part of the folklore of linear optimization that the number of iterations of the simplex method tends to grow roughly linearly in the number of variables, see [32] for a survey of studies of simplex method computational performance. This observed

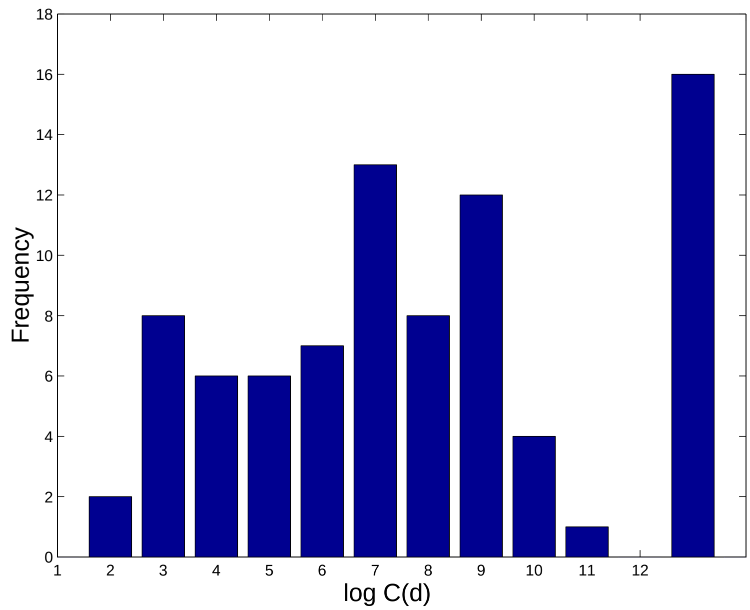

Figure 7-1: Histogram of Condition Numbers for the NETLIB Suite After Pre-Processing Heuristics were Applied (using the geometric mean of the lower and upper bound estimates of $C(d)$)

linear growth is (fortunately) in stark contrast to the worst-case iteration bound for the simplex method, which is exponential in the dimension of the problem, see [16]. Of course, even here one must bear in mind that implemented versions of the simplex algorithm (on which computational performance is assessed) do not correspond to the theoretical versions of the simplex algorithm (on which theory and complexity analysis is developed) in many respects, including the way degeneracy is handled, feasibility checks are performed, numerical tolerances are used, etc.

In the case of modern IPM algorithms for linear optimization, the number of IPM iterations needed to solve a linear optimization instance has been observed to be fairly constant over a huge range of problem sizes; for the NETLIB suite the number of iterations varies between 8 and 48 using CPLEX 7.1 *baropt*; for other codes the numbers are a bit different. Extensive computational experience over the past 15 years has shown that the IPM iterations needed to solve a linear optimization problem instance vary in the range between 10-100 iterations. There is some evidence that the number of IPM iterations grows roughly as $\log n$ on a particular class of structured problem instances, see for example [18].

The observed performance of modern IPM algorithms is fortunately superior to the worst-case bounds on IPM iterations that arise via theoretical complexity analysis. Depending on the complexity model used, one can bound the number of IPM iterations from above by $\sqrt{\vartheta}\tilde{L}$, where ϑ is the number of inequalities plus the number of variables with at least one bound in the problem instance:

$$\vartheta := |L| + |G| + |L_B| + |U_B| - |L_B \cap U_B| \ , \tag{7.18}$$

and $\tilde{L}$ is the bit-size of a binary encoding of the problem instance data, see [27] (subtraction of the final term of (7.18) is shown in [10]). The bit-size model was a motivating force for modern polynomial-time LP algorithms, but is viewed today as somewhat outdated in the context of linear and nonlinear optimization. Using instead the condition-number model for complexity analysis, one can bound the IPM iterations by $O(\sqrt{\vartheta}\log(C(d)+\cdots)$, where the other terms in the bound are of a more technical nature, see [29] for details. Similar to the case of the simplex algorithm, the IPM algorithms that are used in practice are different from the IPM algorithms that are used in the development of the complexity theory.

A natural question to ask is whether the observed variation in the number of IPM iterations (albeit already small) can be accounted for by the condition numbers of the problem instances? The finite condition numbers of the 67 post-processed problems from the NETLIB suite shown in Table 7.3 provide a rich set of data that can be used to explore this question. Here the goal is to assess whether or not condition numbers are relevant for understanding the practical performance of IPM algorithms (and is *not* aimed at validating the complexity theory).

In order to assess any relationship between condition numbers and IPM iterations for the NETLIB suite, we first solved and recorded the IPM iterations for the 83 problems from the NETLIB suite. The problems were pre-processed with the linear dependency check option and solved with CPLEX 7.1 function *baropt* with default parameters. The default settings use the standard barrier algorithm, include a starting heuristic that sets the initial dual solution to zero, and a convergence criteria of a relative complementarity smaller than 10^{-8}. The iteration counts are shown in Table 7.5. Notice that these iteration counts vary between 8 and 48.

Table 7.5: IPM Iterations for the NETLIB Suite using CPLEX 7.1 *baropt*

Problem	IPM Iterations	Problem	IPM Iterations	Problem	IPM Iterations
25fv47	22	ganges	13	scrs8	20
80bau3b	30	gfrd-pnc	18	scsd1	10
adlittle	12	greenbea	38	scsd6	11
afiro	9	greenbeb	33	scsd8	9
agg	22	grow15	12	sctap1	13
agg2	18	grow22	12	sctap2	15
agg3	21	grow7	10	sctap3	15
bandm	16	israel	23	share1b	22
beaconfd	8	kb2	17	share2b	14
blend	11	lotfi	14	shell	16
bnl1	25	maros	27	ship04l	13
bnl2	28	modszk1	23	ship04s	17
bore3d	16	perold	42	ship08l	14
brandy	19	pilot.ja	46	ship08s	14
capri	19	pilot.we	48	ship12l	19
cycle	25	pilot4	35	ship12s	17
czprob	32	pilotnov	19	sierra	16
d2q06c	28	recipe	9	stair	16
d6cube	22	sc105	10	standata	9
degen2	13	sc205	11	standgub	9
degen3	19	sc50a	10	standmps	13
e226	18	sc50b	9	stocfor1	10
etamacro	24	scagr25	14	stocfor2	16
fffff800	30	scagr7	13	tuff	21
finnis	19	scfxm1	18	vtp.base	10
fit1d	14	scfxm2	20	wood1p	13
fit1p	13	scfxm3	20	woodw	21
fit2d	18	scorpion	13		

Figure 7-2 shows a scatter plot of the number of IPM iterations taken by CPLEX 7.1 to solve the 83 problems in the NETLIB suite after pre-processing (from Table 7.5) and $\sqrt{\vartheta} \log C(d)$ of the post-processed problems (using the $\log C(d)$ estimates from columns 6 and 7 from Table 7.3). In the figure, the horizontal lines represent the range for $\sqrt{\vartheta} \log C(d)$ due to the lower and upper estimates of $C(d)$ from the last two columns of Table 7.3. Also, similar to Figure 7-1, problems with infinite condition number are shown in the figure on the far right as a visual aid.

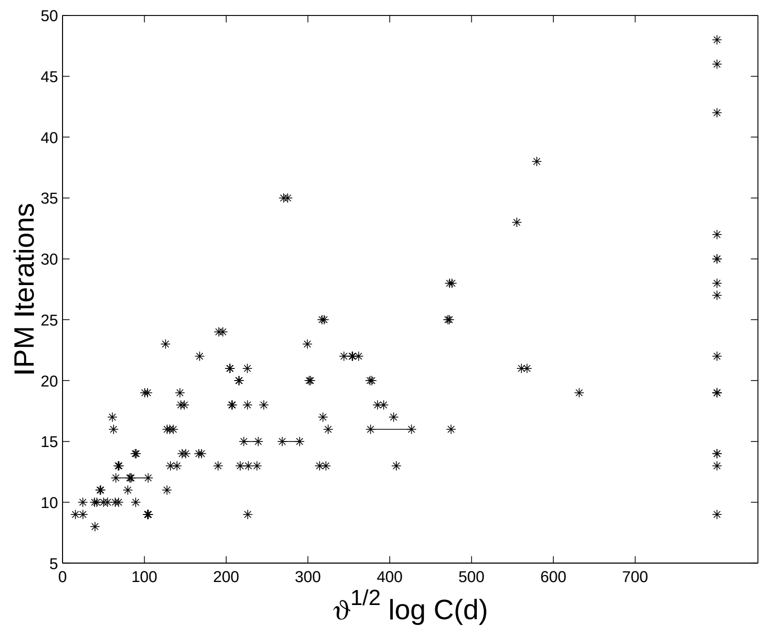

Figure 7-2: Scatter plot of IPM iterations and $\sqrt{\vartheta} \log C(d)$ for 83 NETLIB problems after pre-processing, using CPLEX 7.1

Figure 7-2 shows that as $\sqrt{\vartheta} \log C(d)$ increases, so does the number of IPM iterations needed to solve the problem (with exceptions, of course). Perhaps a more accurate summary of the figure is that if the number of IPM iterations is large, then the problem will tend to have a large value of $\sqrt{\vartheta} \log C(d)$. The converse of this statement is not supported by the scatter plot: if a problem has a large value of $\sqrt{\vartheta} \log C(d)$, one cannot state in general that the problem will take a large number of IPM iterations to solve.

In order to be a bit more definitive, we ran a simple linear regression with the IPM iterations of the post-processed problem as the dependent variable and $\sqrt{\vartheta}\log C(d)$ as the independent variable, for the 67 NETLIB problems which have a finite condition number after pre-processing. For the purposes of the regression computation we used the geometric mean of the lower and upper estimates of the condition number from the last two columns of Table 7.3. The resulting linear regression equation is:

$$\text{IPM Iterations } = 10.8195 + 0.0265\sqrt{\vartheta}\log C(d) \ ,$$

with $R^2 = 0.4267$. This indicates that over 42% of the variation in IPM iteration counts among the NETLIB suite problems is accounted for by $\sqrt{\vartheta}\log C(d)$. A plot of this regression line is shown in Figure 7-3, where once again the 16 problems that were ill-conditioned are shown in the figure on the far right as a visual aid. Both coefficients of this simple linear regression are significant at the 95% confidence level, see the regression statistics shown in Table 7.6.

Table 7.6: Statistics for the Linear Regression of IPM iterations and $\sqrt{\vartheta}\log C(d)$.

Coefficient	Value	t-statistic	95% Confidence Interval
β_0	10.8195	10.7044	[8.8009 , 12.8381]
β_1	0.0265	6.9556	[0.0189 , 0.0341]

The presence of $\sqrt{\vartheta}$ in complexity bounds for interior-point methods seems to be a fixture of the theory of self-concordant barrier functions, see [20], despite the belief that such dependence is not borne out in practice. The above regression analysis indicates that $\sqrt{\vartheta}\log C(d)$ does explain 42% of variation in IPM iteration counts among the NETLIB suite of linear optimization problems. Nevertheless, one can also ask whether the condition number alone (without the $\sqrt{\vartheta}$ factor) can account for the variation in IPM iteration counts among the NETLIB suite problems? Figure 7-4 shows a scatter plot of the number of IPM iterations taken by CPLEX 7.1 to solve the 83 problems in the NETLIB suite after pre-processing and $\log C(d)$ of the post-processed problems (the

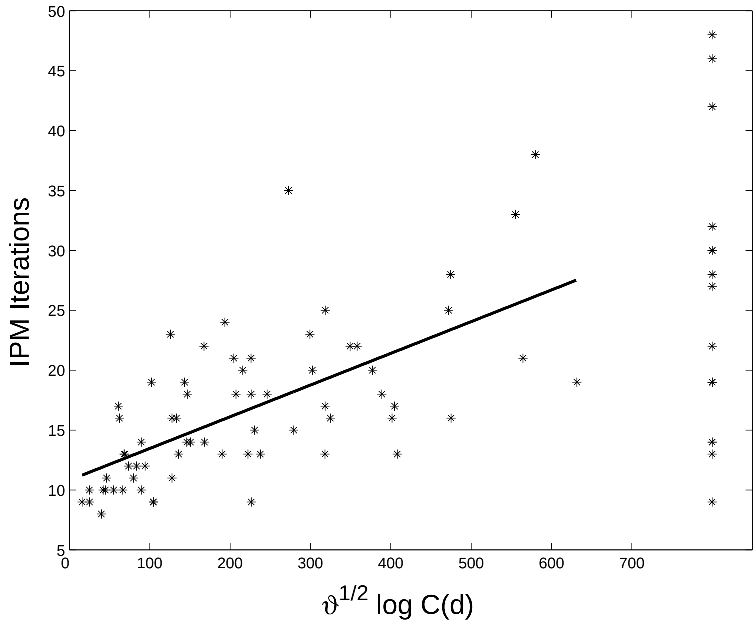

Figure 7-3: Linear regression of IPM iterations and $\sqrt{\vartheta}\log C(d)$ for 67 NETLIB problems with finite condition number after pre-processing, using CPLEX 7.1 (using the geometric mean of the lower and upper bound estimates of $C(d)$)

horizontal lines refer to the range of the lower and upper estimates of $C(d)$ from the last two columns of Table 7.3; also, problems with infinite condition number are shown in the figure on the far right as a visual aid). We also ran a simple linear regression of IPM iterations as the dependent variable and $\log C(d)$ as the independent variable. The resulting linear regression equation is:

$$\text{IPM Iterations } = 4.1389 + 1.7591\log C(d) \ ,$$

with $R^2 = 0.4258$. A plot of this regression is shown in Figure 7-5, and Table 7.7 shows the regression statistics. It turns out that this regression model is comparable to the linear regression with $\sqrt{\vartheta}\log C(d)$. Both regression models are significant at the 95% confidence level and account for just over 42% of the variance in the iterations of the NETLIB suite. These results indicate that $\log C(d)$ and $\sqrt{\vartheta}\log C(d)$ are essentially equally good at explaining the variation in IPM iteration counts among the NETLIB

Table 7.7: Statistics for the Linear Regression of IPM iterations and $\log C(d)$.

Coefficient	Value	t-statistic	95% Confidence Interval
β_0	4.1389	2.1999	[0.3814 , 7.8963]
β_1	1.7591	6.9427	[1.2531 , 2.2652]

suite of linear optimization instances.

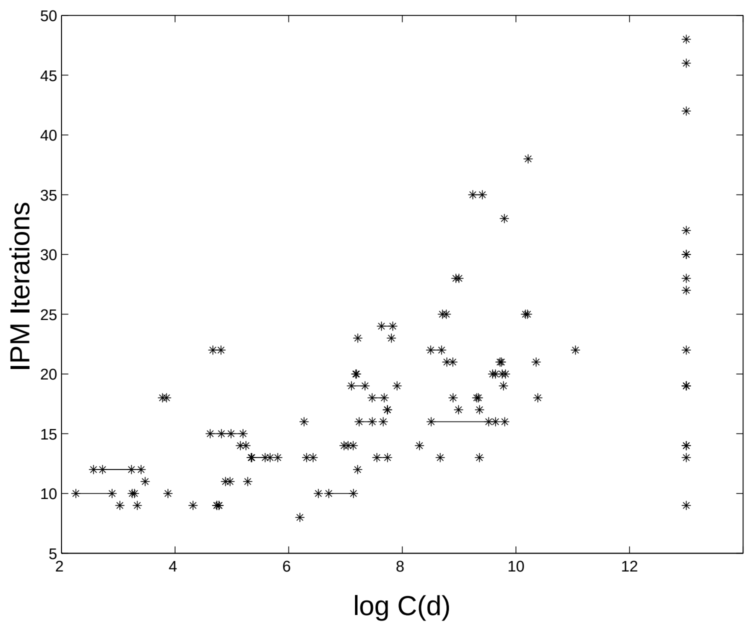

Figure 7-4: Scatter plot of IPM iterations and $\log C(d)$ for 83 NETLIB problems after pre-processing, using CPLEX 7.1

We also computed the sample correlation coefficients of the IPM iterations from Table 7.5 with the following dimensional measures for the 67 finitely-conditioned problems in the NETLIB suite: $\log m$, $\log n$, $\log \vartheta$, and $\sqrt{\vartheta}$. The resulting sample correlations are shown in Table 7.8. Observe from Table 7.8 that IPM iterations are better correlated with $\log C(d)$ than with any of the other measures. The closest other measure is $\log m$, for which $R = 0.520$, and so a linear regression of IPM iterations as a function of $\log m$ would yield $R^2 = (0.520)^2 = 0.270$, which is decidedly less than $R^2 = 0.4258$ for $\log C(d)$.

Note from Table 7.8 that $\log C(d)$ and $\log m$ themselves are somewhat correlated,

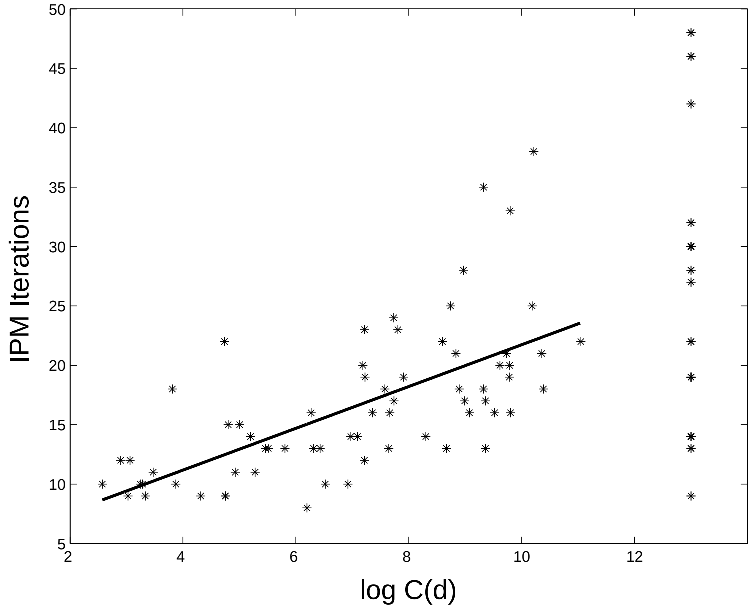

Figure 7-5: Linear regression of IPM iterations and $\log C(d)$ for 67 NETLIB problems with finite condition number after pre-processing, using CPLEX 7.1 (using the geometric mean of the lower and upper bound estimates of $C(d)$)

having a correlation coefficient of 0.431. We have no immediate explanation for this observed correlation, and this may be the subject of future study. Also, note from Table 7.8 that both $\log \vartheta$ and $\sqrt{\vartheta}$ by themselves are significantly less correlated with the IPM iterations than $\log C(d)$.

7.2.4 Controlled perturbations of problems in the NETLIB Suite

One potential drawback of the analysis in Subsection 7.2.3 is that in making comparisons of problem instances with different condition numbers one necessarily fails to keep the problem instance size or structure invariant. Herein, we attempt to circumvent this drawback by performing controlled perturbations of linear optimization problems which allows one to keep the problem size and structure intact.

Table 7.8: Sample correlations for 67 NETLIB problems (using the geometric mean of the lower and upper bound estimates of $C(d)$)

	IPM iterations	$\log C(d)$	$\log n$	$\log m$	$\log \vartheta$	$\sqrt{\vartheta}$
IPM iterations	1.000					
$\log C(d)$	0.653	1.000				
$\log n$	0.453	0.262	1.000			
$\log m$	0.520	0.431	0.732	1.000		
$\log \vartheta$	0.467	0.267	0.989	0.770	1.000	
$\sqrt{\vartheta}$	0.421	0.158	0.919	0.585	0.929	1.000

Consider a problem instance $d = (A, b, c)$ and the computation of the primal and dual distances to ill-posedness $\rho_P(d)$ and $\rho_D(d)$. It is fairly straightforward to show that if $(i^*, j^*, \lambda^*, (s^+)^*, (s^-)^*, v^*)$ is an optimal solution of (7.16), then the rank-1 data perturbation:

$$\Delta d = (\Delta A, \Delta b, \Delta c) := \left(-j^* e^{i^*}\left(A^t\lambda^* + (s^+)^* - (s^-)^*\right)^t, -j^* e^{i^*}\left(b^t\lambda^* - v^*\right), 0\right) \quad (7.19)$$

is a minimum-norm perturbation for which $\rho_P(d + \Delta d) = 0$ (where e^{i^*} denotes the $(i^*)^{\text{th}}$ unit vector in $\mathbb{R}^m$). That is, $\|\Delta d\| = \rho_P(d)$ and the data instance $\tilde{d} := d + \Delta d$ is primal ill-posed.

The simple construction shown in (7.19) allows one to construct a controlled perturbation of the data instance d. Consider the family of data instances $d_\alpha := d + \alpha \Delta d$ for $\alpha \in [0, 1]$. Then if $\rho_D(d) \geq \rho_P(d) > 0$ it follows that $\rho(d_\alpha) = (1 - \alpha)\rho(d)$ for $\alpha \in [0, 1]$, and we can bound the condition number of d_α as follows:

$$C(d_\alpha) = \frac{\|d + \alpha\Delta d\|}{(1-\alpha)\rho(d)} \geq \frac{\|d\| - \alpha\rho(d)}{(1-\alpha)\rho(d)} \ ,$$

where the numerator satisfies $\|d\| - \alpha\rho(d) \geq 0$ for $\alpha \in [0, 1]$. In the case when $\|d\| > \rho(d)$

(satisfied by all problem instances in the NETLIB suite) we can create a family of data instances for which $C(d_\alpha) \to \infty$ as $\alpha \to 1$ by varying α in the range $[0, 1]$, all the while keeping the problem dimensions, the structure of the cone C_Y, and the ground-set P invariant.

To illustrate, consider the problem scagr25 from the NETLIB suite, and let $\bar{d}$ denote the data for this problem instance after pre-processing. According to Table 7.3, $\rho_D(\bar{d}) = 0.049075 \geq 0.021191 = \rho_P(\bar{d}) > 0$. Now let $\Delta\bar{d}$ be the perturbation of this data instance according to (7.19). If we solve the resulting perturbed problem instances $\bar{d}_\alpha$ for select values of $\alpha \in [0, 1]$ and record the number of IPM iterations, we obtain the results portrayed in Figure 7-6. As the figure shows, the number of IPM iterations grows as the perturbed problem instance becomes more ill-conditioned.

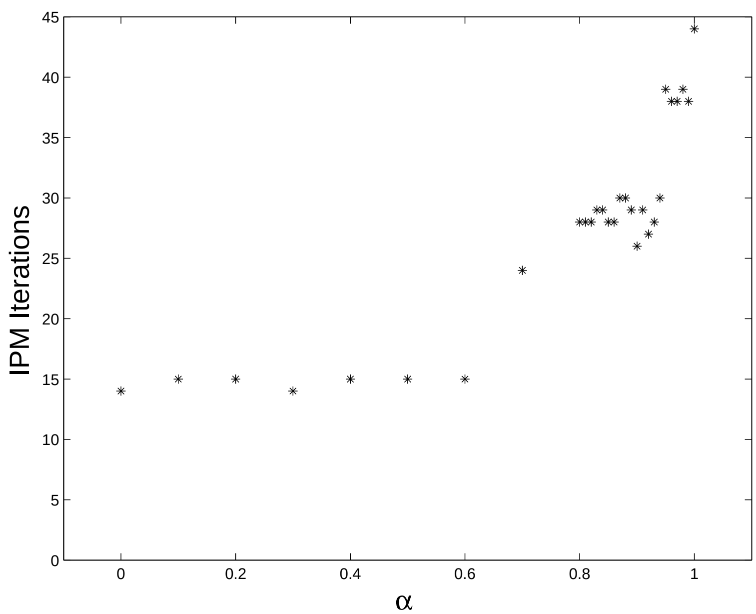

Figure 7-6: The number of IPM iterations needed to solve the perturbed post-processed problem instance scagr25, as a function of the perturbation scalar α.

The pattern of growth in IPM iterations as the perturbed problem becomes more ill-conditioned is not shared by all problem instances in the NETLIB suite. Figure 7-7 shows the plot of IPM iterations for problem e226, as the perturbed problem instance becomes more ill-conditioned. For this problem instance the growth in IPM iterations is not monotone.

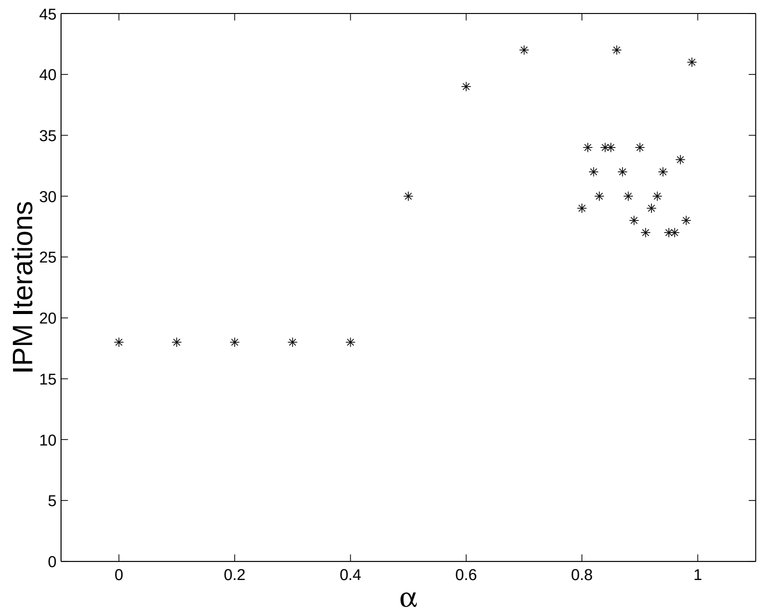

Figure 7-7: The number of IPM iterations needed to solve the perturbed post-processed problem instance e226, as a function of the perturbation scalar α.

Of the 67 post-processed problems in the NETLIB suite with finite condition number, 56 of these problems satisfy $\rho_D(d) \geq \rho_P(d) > 0$ and $\|d\| > \rho(d)$, and so are amenable to analysis via the construction described above. For a given problem instance in the NETLIB suite, let k_α denote the number of IPM iterations needed to solve the perturbed post-processed problem instance $\bar{d}_\alpha$. Then

$$\Delta k := k_1 - k_0$$

is the difference between the IPM iterations needed to solve the un-perturbed problem instance and the fully-perturbed problem instance. Table 7.9 shows some summary statistics of the distribution of Δk for the 56 problems in the NETLIB suite that are readily amenable to this analysis. As the table shows, the fully perturbed problem instance has a larger IPM iteration count in 68% (38 out of 56) of the problem instances. Curiously, the number of IPM iterations is actually less (but by at most three iterations) for the fully-perturbed problem instance in 18% (10 out of 56) problem instances amenable to this analysis. A rough summary of the results in Table 7.9 is that the number of IPM iterations for the fully perturbed problem increases dramatically (more than 10 iterations) on 30% of the problem instances, increases modestly (1-10 iterations) on 38% of the problem instances, and remains the same or decreases slightly on 32% of problem instances.

Table 7.9: The distribution of the change in IPM iterations needed to solve the unperturbed problem instance and the fully-perturbed problem instance, for 56 post-processed problems in the NETLIB suite.

Change in IPM Iterations (Δk)	Number of Problem Instances
-3 to -1	10
0	8
1 to 5	12
6 to 10	9
11 or more	17
Total	56

7.2.5 Condition Number bounds in practice

In this section we compare how the theoretical results that bound the minimum-norm solution size, the optimal solution size, and the optimal objective function value compare with the values obtained for linear programs in practice. We are concerned with the theoretical bounds presented in Theorem 9 and Theorem 14, which bound minimum-norm feasible solution sizes, and Theorem 20, which bounds the optimal solution sizes and optimal objective function value.

We use the problems in the NETLIB suite as the practical testbed of problems to study the significance of these theoretical bounds. These are linear programs of type (7.1), in which the ground set P is defined by (7.2), that is

$$P := \{x \in I\!R^n \mid x_j \geq l_j \text{ for } j \in L_B,\ x_j \leq u_j \text{ for } j \in U_B\}\ .$$

Recall that for this ground set P, the recession cone is given by (7.11)

$$R = \{x \in I\!R^n \mid x_j \geq 0 \text{ for } j \in L_B,\ x_j \leq 0 \text{ for } j \in U_B\}\ ,$$

and we can easily verify that P satisfies Assumption 2, that is $P = \widetilde{E} + R$ with the bounded set $\widetilde{E}$ defined by

$$\widetilde{E} := \left\{ x \in I\!R^n \;\middle|\; \begin{array}{ll} l_j \leq x_j \leq u_j & \text{for } j \in U_B \cap L_B \\ l_j \leq x_j \leq \max\{0, l_j\} & \text{for } j \in L_B \setminus U_B \\ \min\{0, u_j\} \leq x_j \leq u_j & \text{for } j \in U_B \setminus L_B \\ x_j = 0 & \text{for } j \notin U_B \cup L_B \end{array} \right\}\ . \tag{7.20}$$

Note that Assumption 2 is necessary for Theorem 20.

The theoretical bounds in Theorems 9, 14, 20 can be sharpened when considering an LP of form (7.1). In this case, we have a number of easily computable quantities that can be used to improve the theoretical bounds. For the set $\widetilde{E}$ given by (7.20) we define the quantity

$$\widehat{E} := \max\left\{ \max_{j \in L_B} -l_j,\ \ \max_{j \in U_B} u_j \right\}\ .$$

The following result shows how to compute other values also used to improve the theoretical bounds for practical LPs.

Lemma 12 *Consider a linear program of type (7.1), with $\|\cdot\|_\infty$ and $\|\cdot\|_1$ as the primal norms in $I\!R^n$ and $I\!R^m$ respectively. Then, for C_Y defined in (7.9), R defined in* (7.11)*, and $\widetilde{E}$ defined in (7.20) we have*

1. $\left|\widetilde{E}\right| := \max_{x \in \widetilde{E}} \|x\|_\infty = \max\left\{\max_{j \in L_B} |l_j|, \;\; \max_{j \in U_B} |u_j|\right\}$

2. $\mathrm{dist}(w, C_Y) := \min_{w' \in C_Y} \|w - w'\|_1 = \sum_{i \in L} w_i^+ + \sum_{i \in G} w_i^- + \sum_{i \in E} |w_i|$

3. $\mathrm{dist}(s, R^*) := \min_{s' \in R^*} \|s - s'\|_1 = \sum_{j \in U_B \setminus L_B} s_j^+ + \sum_{j \in L_B \setminus U_B} s_j^- + \sum_{j \notin U_B \cup L_B} |s_j|$

Proof: The definition of $\left|\widetilde{E}\right|$ is the same as the one in Assumption 2. From (7.20) we note that for any $x \in \widetilde{E}$ we have that $\max_j |x_j| \le \max\{\max_{j \in L_B} |l_j|, \;\; \max_{j \in U_B} |u_j|\}$. This shows that $\left|\widetilde{E}\right| \le \max\{\max_{j \in L_B} |l_j|, \;\; \max_{j \in U_B} |u_j|\}$. The equality is shown using the point $\tilde{x}$ defined by

$$\tilde{x}_j := \begin{cases} u_j & \text{if } j \in U_B \setminus L_B \text{ or } j \in U_B \cap L_B \text{ and } |u_j| \ge |l_j| \\ l_j & \text{if } j \in L_B \setminus U_B \text{ or } j \in U_B \cap L_B \text{ and } |l_j| > |u_j| \\ 0 & \text{otherwise .} \end{cases}$$

The point $\tilde{x}$ defined above belongs to $\widetilde{E}$ and has norm equal to the right hand side.

To show items *2* and *3*, note that C_Y from (7.9) is

$$C_Y = \{w \in I\!R^m \mid w_i \le 0 \text{ for } i \in L, w_i = 0 \text{ for } i \in E, w_i \ge 0 \text{ for } i \in G\} \;,$$

and from (7.11) we see that the polar of R is

$$R^* = \{s \in I\!R^n \mid s_j \le 0 \text{ for } j \in U_B \setminus L_B, s_j = 0 \text{ for } j \notin L_B \cup U_B, s_j \ge 0 \text{ for } j \in L_B \setminus U_B\} \;.$$

The formulas of items 2 and 3 follow easily from these characterizations of C_Y and R^*. ∎

We now present the result which states tighter theoretical bounds for the case of practical linear programs.

Theorem 28 *Consider an instance of problem (7.1) with* $d \in \mathcal{F}$ *and* P *defined by (7.2). Let* $x^0 \in P$ *be given and let* $\|\cdot\|_\infty$ *and* $\|\cdot\|_1$ *be the primal norms in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively. Then the following holds*

1. *If* $\rho_P(d) > 0$ *then there exists an* $x \in X_d$ *such that*

$$\|x\|_\infty \leq \|x^0\|_\infty + \frac{\operatorname{dist}(Ax^0 - b, C_Y)}{\rho_P(d)}$$

2. *If* $\rho_D(d) > 0$ *then for any* $\varepsilon > 0$ *there exists* $(y, u) \in Y_d$ *such that*

$$\|y\|_\infty \leq \frac{\operatorname{dist}(c, R^*) + \varepsilon}{\rho_D(d)}$$

If $\rho(d) > 0$ *then problems* (GP_d) *and* (GD_d) *have optimal solutions. If* $x^* \in X_d$ *and* $(y^*, u^*) \in Y_d$ *are optimal solutions of* (GP_d) *and* (GD_d) *respectively, then*

3. $-\|c\|_1 \left(\widehat{E} + \frac{\|A\| \widehat{E} + \|b\|_1}{\rho_D(d)} \right) \leq z^*(d) \leq \|c\|_1 \left(\|x^0\|_\infty + \frac{\operatorname{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} \right)$

4. $\|x^*\|_\infty \leq |E| \left(1 + \frac{\max\{\|A\|, \|c\|_1\}}{\rho_D(d)} \right) + \qquad + \frac{\max\{\|b\|_1, \|c\|_1\}}{\rho_D(d)} \max \left\{ 1, \|x^0\|_\infty + \frac{\operatorname{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} \right\}$

5. $\|y^*\|_\infty \leq \frac{\|c\|_1}{\rho_P(d)} \max \left\{ 1, \widehat{E} + \frac{\|A\| \widehat{E} + \|b\|_1}{\rho_D(d)} \right\}$

Proof: Items *1* and *2* are exactly items *1.(b)* of Theorem 9 and *1.(b)* of Theorem 14 using $y^0 = 0$, respectively.

The fact that $\rho(d) > 0$ implies the existence of optimal solutions with no duality gap is due to Corollary 1. To bound the optimal objective function value, we use a feasible solution $x \in X_d$ from item *1.* and bound using:

$$z^*(d) \leq c^t x \leq \|c\|_1 \|x\|_\infty \leq \|c\|_1 \left(\|x^0\|_\infty + \frac{\operatorname{dist}(Ax^0 - b, C_Y)}{\rho_P(d)} \right) .$$

For the lower bound we use item *2* and note that if $(y,u) \in Y_d$, then

$$z^*(d) \geq b^t y - u(c - A^t y) \geq -\|b\|_1 \|y\|_\infty - u(c - A^t y) \ ,$$

where since $c - A^t y \in \text{dom}\, u(\cdot) \subset R^*$ we know, from the definition of R^*, that if $(c - A^t y)_i > 0$ then $i \in L_B$, and if $(c - A^t y)_i < 0$ then $i \in U_B$. Therefore, denoting $s = c - A^t y$, we have that

$$\begin{aligned}
u(c - A^t y) &= \sup_{x \in P} \sum_{j=1}^{n} -s_j x_j \\
&= \sup_{x \in P} \left(\sum_{j:s_j>0} -s_j x_j + \sum_{j:s_j<0} -s_j x_j \right) \\
&= -\sum_{j:s_j>0} s_j l_j - \sum_{j:s_j<0} s_j u_j \\
&\leq \max\left\{ \max_{j \in L_B} -l_j, \ \max_{j \in U_B} u_j \right\} \left(\sum_{j:s_j>0} s_j + \sum_{j:s_j<0} -s_j \right) \\
&= \widehat{E} \|s\|_1 \ .
\end{aligned}$$

The above inequality then, using item *2*, implies that

$$\begin{aligned}
z^*(d) &\geq -\|b\|_1 \|y\|_\infty - \widehat{E} \|c - A^t y\|_1 \\
&\geq -\|b\|_1 \|y\|_\infty - \widehat{E} \|c\|_1 - \widehat{E} \|A\| \|y\|_\infty \\
&\geq -\widehat{E} \|c\|_1 - \left(\|b\|_1 + \widehat{E} \|A\| \right) \frac{\|c\|_1 + \varepsilon}{\rho_D(d)} \ .
\end{aligned}$$

To finish the proof of the lower bound in the optimal objective function value we take the limit as $\varepsilon \to 0$.

To prove the bound on the optimal solution size, we recall from the proof of Theorem 20 that x^* and y^* satisfy

$$\|x^*\|_\infty \leq \left| \widetilde{E} \right| \left(1 + \frac{\max\{\|A\|, \|c\|_1\}}{\rho_D(d)} \right) + \frac{\max\{\|b\|_1, z^*(d)\}}{\rho_D(d)}$$

and

$$\|y^*\|_\infty \leq \frac{\max\{\|c\|_1, -z^*(d)\}}{\rho_P(d)} \ .$$

The last two items in the theorem are obtained by using item *3* to bound the optimal objective function value in the above inequalities. ■

To assess the significance of the theoretical condition number bounds presented in Theorem 28 we solve and compute the theoretical bound for each of the 83 problems from the NETLIB suite. Below we compare the primal and dual optimal solution size to the bounds in Theorem 28 items *4* and *5*, for the NETLIB suite problems. We also analyze the difference between the optimal objective function value and the bounds from Theorem 28 item *3*. To analyze the bound on the size of a feasible solution, we compute and record the minimum norm primal feasible solution, that is $\min_{x \in X_d} \|x\|_\infty$, and compare it to the bound in Theorem 28 item *1* for each problem in the NETLIB suite.

To compute the bounds in Theorem 28, we define $x^0 \in E$ by

$$x_i^0 = \begin{cases} u_i & \text{if } i \in U_B \text{ and } u_i < 0 \\ l_i & \text{if } i \in L_B \text{ and } l_i > 0 \\ 0 & \text{otherwise} \ , \end{cases}$$

and we compute the quantities $|E|$, $\widehat{E}$, $\text{dist}(Ax^0 - b, C_Y)$, and $\|x^0\|_\infty$ for each problem in the NETLIB suite. The problems were pre-processed with the linear dependency check option and solved with CPLEX 7.1 function *baropt* with default parameters. The primal and dual optimal solution sizes, $\|x^*\|_\infty$ and $\|y^*\|_\infty$, and the corresponding theoretical bounds from Theorem 28, which we denote by $\text{Bound}(\|x^*\|_\infty)$ and $\text{Bound}(\|y^*\|_\infty)$ respectively, are shown in Table 7.10.

Table 7.10: Primal and dual optimal solution size and theoretical bounds for the NETLIB suite after pre-processing by CPLEX 7.1

Problem	$\|x^*\|_\infty$	Bound($\|x^*\|_\infty$)	$\|y^*\|_\infty$	Bound($\|y^*\|_\infty$)
25fv47	2,082	4.5E+15	46	1.3E+17
80bau3b	9,442	∞	116	∞
adlittle	320	8.7E+10	3,310	3.3E+11
afiro	500	20,919	1	178,979
agg	955,197	∞	186,640	∞
agg2	325,488	2.7E+16	3,816	1.0E+17
agg3	413,300	2.3E+16	3,816	1.0E+17
bandm	65	1.8E+13	18	1.2E+14
beaconfd	1,893	1.0E+10	4	2.6E+08
blend	22	670,420	6	3.1E+09
bnl1	1,545	2.4E+12	1,477	5.5E+15
bnl2	2,077	7.2E+16	39	5.9E+19
bore3d	4,696	8.0E+12	63	4.0E+14
brandy	1,445	5.1E+09	23	4.9E+09
capri	5,072	8.1E+12	432	5.2E+12
cycle	21,056	2.1E+11	1	4.8E+17
czprob	1,467	∞	19,060	∞
d2q06c	133,345	∞	116	∞
d6cube	48	2.6E+08	3	6.4E+07
degen2	4	∞	42	∞
degen3	10	∞	8	∞
e226	103	2.3E+09	29	1.3E+13
etamacro	77	6.6E+10	52	8.9E+13
fffff800	246,944	∞	132,633	∞
finnis	11,411	∞	1,050	∞
fit1d	3	3	38	94,869
fit1p	167	6.4E+08	3	4.8E+11
fit2d	4	32	9	56,972
ganges	15,299	3.8E+13	1,250	9.2E+12
gfrd-pnc	70,000	3.2E+14	98	1.0E+20
greenbea	3.3E+08	3.3E+19	116,003	2.9E+22
greenbeb	190,604	8.4E+18	17,582	1.6E+22
grow15	1.6E+06	1.1E+09	86	3.4E+11
grow22	1.5E+06	1.6E+09	86	7.5E+11
grow7	1.5E+06	5.1E+08	86	6.9E+10
israel	10,372	3.6E+10	365	2.7E+13
kb2	6,263	4.1E+07	17	1.7E+13
lotfi	13,905	1.8E+16	1	9.5E+13
maros	46,784	∞	101	∞

Problem	$\|x^*\|_\infty$	Bound($\|x^*\|_\infty$)	$\|y^*\|_\infty$	Bound($\|y^*\|_\infty$)
modszk1	687,209	5.7E+14	296	7.6E+13
perold	110,244	∞	354	∞
pilot.ja	152,417	∞	62	∞
pilot.we	143,857	∞	19,394	∞
pilot4	96,137	1.2E+17	48	3.1E+17
pilotnov	56,652	∞	1	∞
recipe	20	∞	2	∞
sc105	709	7,492	1.6E-01	7,950
sc205	2,380	189,570	1.6E-01	201,184
sc50a	300	1,500	3.1E-01	1,588
sc50b	195	1,500	3.2E-02	2,157
scagr25	22,937	4.0E+13	5,459	4.4E+14
scagr7	4,569	3.8E+12	3,374	2.0E+13
scfxm1	14,443	2.9E+15	183	3.6E+17
scfxm2	14,561	1.3E+16	183	1.4E+18
scfxm3	14,657	3.1E+16	183	3.2E+18
scorpion	1	2.1E+07	432	1.0E+08
scrs8	137	2.2E+10	1,499	6.5E+11
scsd1	1	1,752	9	348
scsd6	2	9,734	16	9,734
scsd8	23	930,439	42	930,439
sctap1	35	9.1E+07	293	1.1E+08
sctap2	16	2.4E+07	21	3.9E+07
sctap3	6	2.9E+07	21	6.2E+07
share1b	1.3E+06	2.1E+20	43	8.1E+18
share2b	58	3.2E+07	315	3.1E+09
shell	106,169	8.7E+15	4,435	2.6E+18
ship04l	187	3.4E+10	5,861	2.8E+11
ship04s	191	1.6E+10	5,602	3.0E+11
ship08l	71	∞	21,826	∞
ship08s	72	∞	26,494	∞
ship12l	136	5.6E+11	42,559	5.9E+11
ship12s	134	2.3E+11	42,214	5.3E+12
sierra	2,079	2.7E+12	40,454	5.5E+19
stair	283	2.2E+10	9	2.3E+07
standata	10	4.9E+06	21	3.0E+10
standgub	10	4.9E+06	21	3.0E+10
standmps	10	8.7E+07	22	7.7E+11
stocfor1	42	4.4E+09	474	8.5E+13
stocfor2	406	2.2E+13	257	3.4E+18
tuff	4,088	2.7E+11	1.2E-02	3.0E+11
vtp.base	1,856	1.6E+10	1,098	4.2E+10
wood1p	5.0E-01	178,084	5	178,084

Problem	$\|x^*\|_\infty$	Bound($\|x^*\|_\infty$)	$\|y^*\|_\infty$	Bound($\|y^*\|_\infty$)
woodw	2.7E-01	3.4E+08	2	7.5E+11

The results in Table 7.10 show that the theoretical bounds on the primal and dual optimal solution size can be quite large. Also, if a problem has a zero primal or dual distance to infeasibility, that is the problem is ill-posed, then it has an infinite bound. Define the ratio of the theoretical bound to the norm of the optimal solution:

$$\text{ratio}(x^*) = \frac{\text{Bound}(\|x^*\|_\infty)}{\|x^*\|_\infty}\ , \quad \text{and ratio}(y^*) = \frac{\text{Bound}(\|y^*\|_\infty)}{\|y^*\|_\infty}\ .$$

These ratios are numbers always greater than one, if a ratio is small it means that the corresponding bound is a good estimate of the size of that optimal solution. Note that these ratios are well defined for all problems in the NETLIB suite since $\|x^*\|_\infty > 0$ and $\|y^*\|_\infty > 0$ for these problems.

To illustrate the significance of these bounds, we plot the histogram of the ratios defined above. In Figure 7-8 we plot the histogram of $\log(\text{ratio}(x^*))$, and in Figure 7-9 we plot the histogram of $\log(\text{ratio}(y^*))$. We use a log scale in both graphs due to the large range in the values of the ratios. The rightmost column in the figures is used to tally the number of problems that have an infinite bound, which are exactly the instances with a zero primal or dual distance to infeasibility.

Table 7.11 presents the arithmetic means and standard deviations of distributions of the logarithm of ratios presented in Figures 7-8, and 7-9. These averages correspond to the geometric mean of the ratios, which due to the wide range of values is a more meaningful measure than a simple arithmetic mean of the ratios.

In Table 7.12 below we present the optimal objective function value, $z^*(d)$, and its upper and lower bounds from Theorem 28, which we denote by UpBound($z^*(d)$) and LoBound($z^*(d)$), respectively. The minimum norm primal feasible solution, $\|x\|_\infty$, and

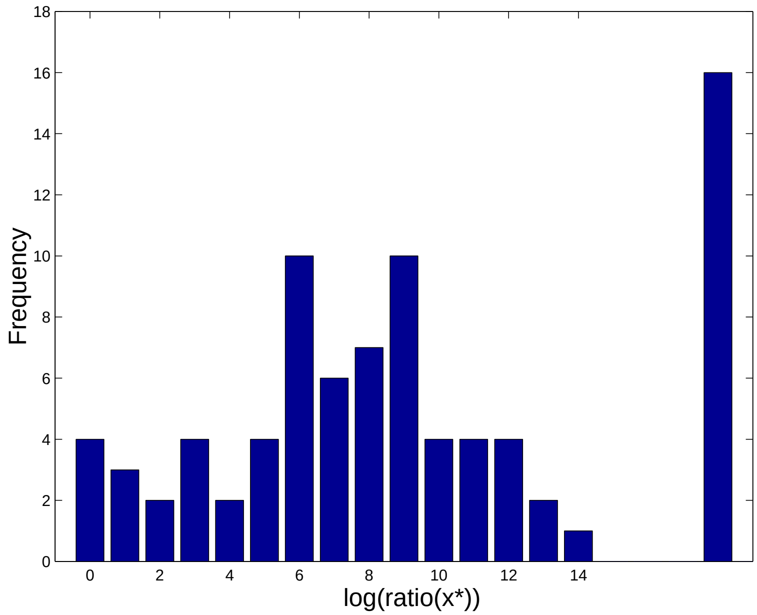

Figure 7-8: Ratio of theoretical bound to observed primal optimal solution size for the NETLIB suite after pre-processing by CPLEX 7.1

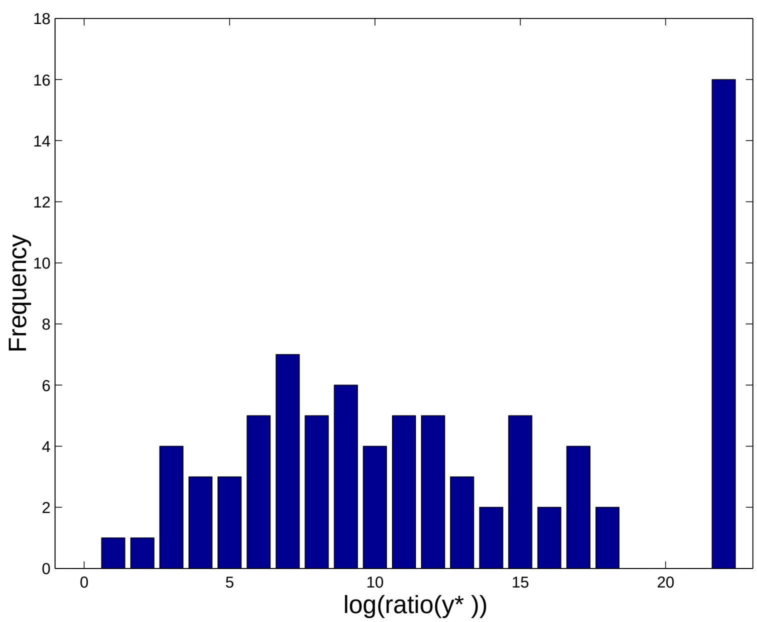

Figure 7-9: Ratio of theoretical bound to observed dual optimal solution size for the NETLIB suite after pre-processing by CPLEX 7.1

Table 7.11: Arithmetic mean and standard deviation of the finite values in log(ratio(·)) for primal and dual optimal solution sizes, for the NETLIB suite after pre-processing by CPLEX 7.1

	log(ratio(x^*))	log(ratio(y^*))
Mean	7.49	10.18
Std.Dev.	3.49	4.36

the bound on a primal feasible solution from Theorem 28, denoted by Bound($\|x\|_\infty$), also appear in Table 7.12.

Table 7.12: Optimal objective function value, minimum-norm primal feasible solution and theoretical bounds for the NETLIB suite after pre-processing by CPLEX 7.1

Problem	LoBound($z^*(d)$)	$z^*(d)$	UpBound($z^*(d)$)	$\|x\|_\infty$	Bound($\|x\|_\infty$)
25fv47	-9.3E+13	5,661	1.4E+10	1,414	1.4E+07
80bau3b	-4.7E+19	529,528	∞	2,400	∞
adlittle	-1.4E+09	225,495	8.7E+10	90	1.3E+06
afiro	-71,125	-465	0	0	0
agg	-4.7E+15	-3.6E+07	∞	308,235	∞
agg2	-6.6E+13	-2.3E+07	1.3E+13	181,638	1.7E+09
agg3	-6.8E+13	7.1E+06	1.1E+13	181,638	1.5E+09
bandm	-2.1E+11	-298	4.4E+08	39	1.8E+06
beaconfd	-1.1E+06	19,652	1.9E+08	947	1.6E+06
blend	-3.5E+07	-31	0	0	0
bnl1	-8.7E+10	1,978	1.8E+11	875	1.9E+08
bnl2	-1.2E+15	1,688	3.0E+11	950	3.4E+08
bore3d	-7.2E+10	5	1.7E+10	596	1.2E+07
brandy	-1.7E+06	1,519	3.4E+06	1,439	1.7E+06
capri	-2.0E+09	5,316	2.5E+10	2,626	8.4E+07
cycle	-1.0E+13	-5	0	0	0
czprob	-1.5E+14	966,733	∞	489	∞
d2q06c	-6.8E+15	-111,634	∞	62,668	∞
d6cube	-6.1E+07	314	1.3E+08	2	23,604
degen2	-1.1E+07	-794	∞	2	∞
degen3	-7.1E+07	-717	∞	3	∞
e226	-9.9E+09	-11	4.9E+07	8	135,375
etamacro	-1.2E+11	167	2.5E+09	17	973,770
fffff800	-4.6E+13	555,622	∞	246,943	∞
finnis	-∞	221,974	∞	3,860	∞
fit1d	-332,040	-9,146	0	0	0
fit1p	-6.7E+11	9,146	1.4E+07	80	10,195

Problem	LoBound($z^*(d)$)	$z^*(d)$	UpBound($z^*(d)$)	$\|x\|_\infty$	Bound($\|x\|_\infty$)
fit2d	-1.8E+07	-68,464	0	0	0
ganges	-2.9E+09	-108,347	1.3E+10	7,722	2.6E+08
gfrd-pnc	-1.6E+18	6.9E+06	1.1E+14	70,000	9.0E+06
greenbea	-9.5E+17	-7.3E+07	3.5E+13	15,496	2.0E+09
greenbeb	-5.3E+17	-4.2E+06	2.3E+13	9,948	1.3E+09
grow15	-1.9E+11	-1.1E+08	0	0	0
grow22	-4.3E+11	-1.6E+08	0	0	0
grow7	-4.0E+10	-4.8E+07	0	0	0
israel	-3.7E+12	-896,645	4.8E+07	1,100	2,577
kb2	-3.5E+09	-1,750	0	0	0
lotfi	-8.1E+10	-30	6.8E+09	2,851	1.7E+08
maros	-1.1E+16	-104,895	∞	13,724	∞
modszk1	-1.2E+12	321	8.7E+12	69,299	6.4E+07
perold	-4.5E+14	-8,747	∞	39,764	∞
pilot.ja	-2.5E+18	-5,597	∞	28,177	∞
pilot.we	-2.4E+19	-2.7E+06	∞	36,579	∞
pilot4	-1.2E+14	-2,581	6.9E+09	412	5.8E+08
pilotnov	-1.4E+17	-4,497	∞	6,556	∞
recipe	$-\infty$	-257	11,768	10	798
sc105	-6,191	-52	0	0	0
sc205	-156,670	-52	0	0	0
sc50a	-1,240	-65	0	0	0
sc50b	-1,500	-70	0	0	0
scagr25	-9.4E+12	-1.4E+07	7.8E+11	2,144	9.8E+06
scagr7	-4.5E+11	-2.1E+06	5.4E+10	1,996	3.1E+06
scfxm1	-3.6E+12	18,626	1.3E+11	14,120	6.0E+08
scfxm2	-1.4E+13	37,079	5.8E+11	14,120	1.3E+09
scfxm3	-3.2E+13	55,529	1.3E+12	14,120	2.0E+09
scorpion	-6.2E+06	1,696	2.1E+07	1	166
scrs8	-5.9E+09	904	9.5E+07	14	1,698
scsd1	-1,752	9	348	1.3E-01	2.0E-01
scsd6	-15,607	51	9,734	4.6E-01	3
scsd8	-249,695	905	930,439	8	168
sctap1	-3.6E+06	1,412	9.1E+07	10	16,523
sctap2	-2.6E+07	1,725	2.4E+07	3.7E-01	1,038
sctap3	-3.1E+07	1,424	2.9E+07	3.2E-01	1,072
share1b	-1.2E+14	-76,589	4.7E+13	122,892	9.3E+10
share2b	-5.4E+06	-416	2.1E+06	42	48,655
shell	-6.8E+14	1.2E+09	9.4E+14	60,801	2.5E+09
ship04l	-1.1E+08	1.8E+06	8.8E+11	74	994,244
ship04s	-1.7E+08	1.8E+06	4.2E+11	74	761,565
ship08l	-4.5E+07	1.9E+06	∞	32	∞
ship08s	-3.0E+08	1.9E+06	∞	32	∞

Problem	LoBound($z^*(d)$)	$z^*(d)$	UpBound($z^*(d)$)	$\|x\|_\infty$	Bound($\|x\|_\infty$)
ship12l	-7.3E+07	1.5E+06	4.2E+12	82	5.6E+06
ship12s	-7.9E+08	1.5E+06	1.7E+12	82	5.1E+06
sierra	-5.7E+16	1.5E+07	1.3E+14	957	3.1E+07
stair	-88,221	-251	508,805	209	508,805
standata	-2.7E+09	1,086	2.3E+06	10	802
standgub	-2.7E+09	1,086	2.3E+06	10	802
standmps	-1.5E+10	1,235	8.4E+07	10	6,568
stocfor1	-1.8E+11	-41,130	4.8E+08	40	44,472
stocfor2	-2.7E+15	-39,024	3.1E+09	49	303,137
tuff	-7.4E+06	2.9E-01	66,831	3,986	9.5E+06
vtp.base	-2.2E+08	48,931	7.1E+09	1,650	3.4E+06
wood1p	-10,508	1	256,898	1.1E-02	31
woodw	-7.0E+09	1	3.4E+08	6.9E-02	53,654

Table 7.12 shows that the range formed by the upper and lower bounds on the optimal objective function value can be quite large. The case of the minimal norm feasible solution is a bit more encouraging: although the bounds can be large, the values of the feasible norm bound are consistently smaller than the values of the optimal objective function value bound, in some instances significantly smaller. Which bounds in Table 7.12 are infinite is explained by Theorem 28, which shows that if the primal distance to infeasibility is zero then UpBound($z^*(d)$) and Bound($\|x\|_*$) are infinite, and if the dual distance to infeasibility is zero then LoBound($z^*(d)$) is infinite.

Similar to the exposition for the bounds on the optimal solution size, we define the ratios of theoretical bounds to the quantity bounded for the optimal objective function value and the feasible solution size. To quantify the bounds on the optimal objective function value we define the following two ratios:

$$\text{ratio1}(z^*) = \frac{\max\{|\text{LoBound}(z^*(d))|, |\text{UpBound}(z^*(d))|\}}{|z^*(d)|},$$

and

$$\text{ratio2}(z^*) = \begin{cases} \frac{\text{LoBound}(z^*(d))}{z^*(d)} & \text{if } z^*(d) < 0 \\ \frac{\text{UpBound}(z^*(d))}{z^*(d)} & \text{if } z^*(d) \geq 0 \; . \end{cases}$$

The main difference in these ratios is that one is an a priori bound and the second requires knowledge of the optimal solution. However, the second ratio is, by definition, consistently sharper than the first one.

We also define the following ratio to quantify the bound on the minimum-norm primal feasible solution:

$$\text{ratio}(x) = \frac{\text{Bound}(\|x\|_\infty)}{\|x\|_\infty} \,.$$

These ratios are also numbers always greater than one, and if a ratio has a small value, close to one, then it means that the corresponding bound is a good estimate of the quantity bounded. Here we use the convention that $\frac{0}{0} = 1$, to define the ratio when the minimum norm feasible solution and its bound are both zero. Since in this case the bound equals the quantity bounded a ratio of one is appropriate.

To illustrate the significance of the bounds in Table 7.12, we plot the histograms of the ratios defined above. In Figure 7-10 we plot the histogram of $\log(\text{ratio1}(z^*))$, in Figure 7-11 we plot the histogram of $\log(\text{ratio2}(z^*))$, and in Figure 7-12 we plot the histogram of $\log(\text{ratio}(x))$. We use a log scale in all graphs due to the large range in the values of the ratios. The rightmost column in the figures is used to tally the number of problems that have an infinite bound. Note that $\text{ratio2}(z^*)$ has 9 instances more that are finite compared to $\text{ratio1}(z^*)$. From Theorem 28, this is due to problem instances which have a finite dual distance to infeasibility, involved in $\text{LoBound}(z^*(d))$, and infinite primal distance to infeasibility.

Table 7.13 presents the arithmetic means and standard deviations of distributions of the logarithm of ratios presented in Figures 7-10, 7-11, and 7-12. These averages correspond to the geometric mean of the ratios, which due to the wide range of values is a more meaningful measure than a simple arithmetic mean of the ratios.

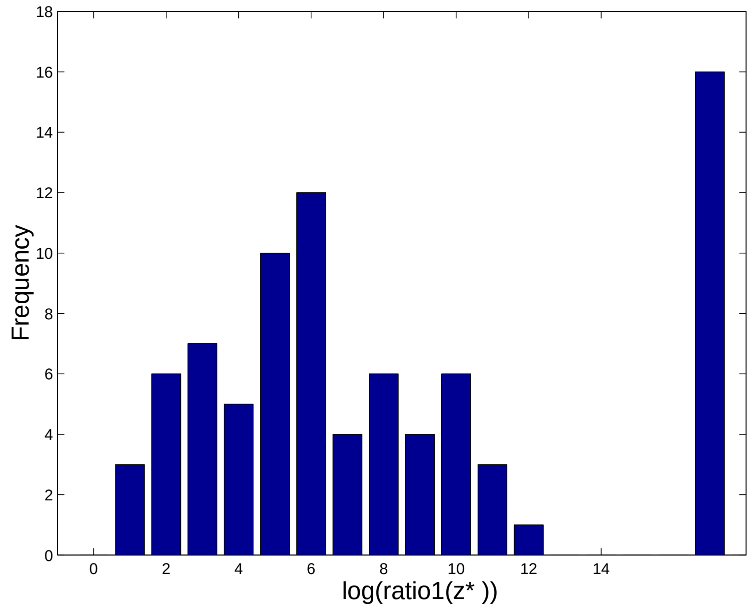

Figure 7-10: Ratio 1 of theoretical bound to observed optimal objective function value for the NETLIB suite after pre-processing by CPLEX 7.1

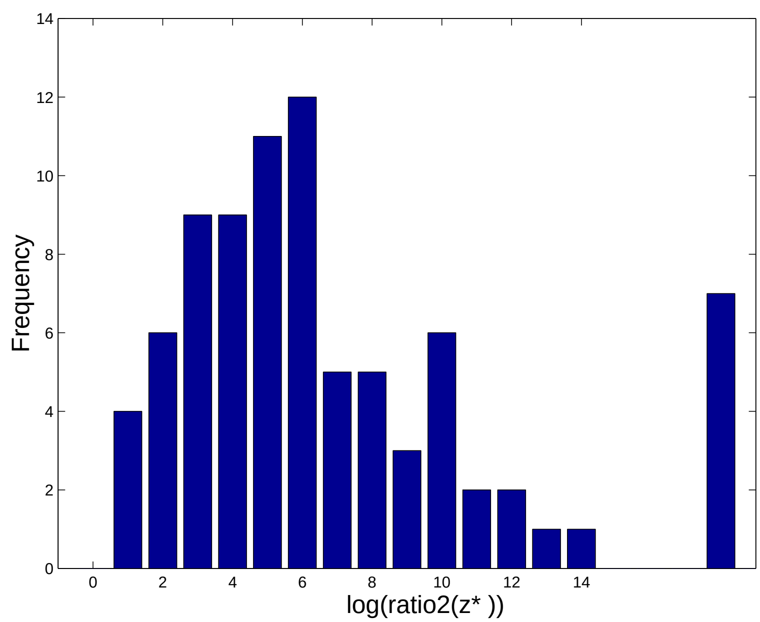

Figure 7-11: Ratio 2 of theoretical bound to observed optimal objective function value for the NETLIB suite after pre-processing by CPLEX 7.1

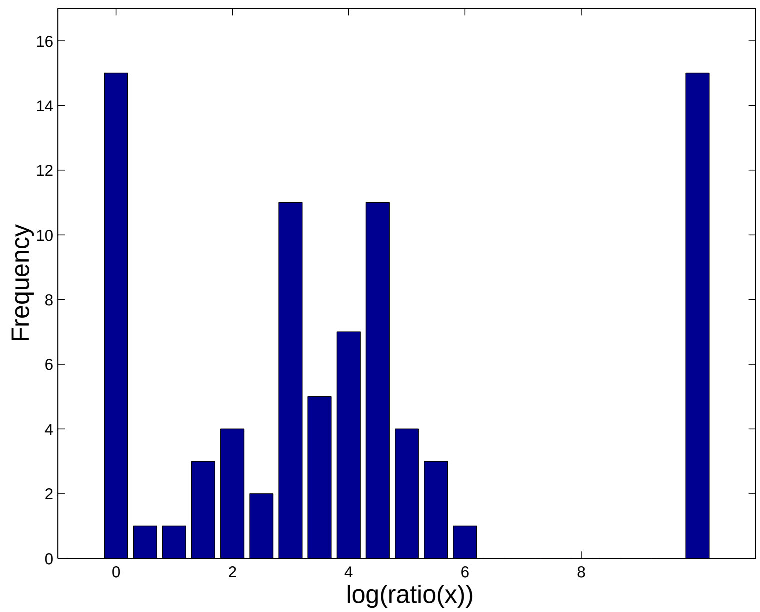

Figure 7-12: Ratio of theoretical bound to observed minimum-norm feasible solution size for the NETLIB suite after pre-processing by CPLEX 7.1

Table 7.13: Arithmetic mean and standard deviation of the finite values in log(ratio(·)) for optimal objective function value and minimum-norm primal feasible solution, for the NETLIB suite after pre-processing by CPLEX 7.1

	$\log(\text{ratio1}(z^*))$	$\log(\text{ratio2}(z^*))$	$\log(\text{ratio}(x))$
Mean	6.36	6.25	2.98
Std.Dev.	2.89	3.11	1.90

7.3 Summary conclusions

The purpose of this computational study has been to gain some computational experience and to test the practical relevance of condition numbers for linear optimization on problem instances that one might encounter in practice. We used the NETLIB suite of linear optimization problems as a test bed for condition number computation and analysis, and we computed condition numbers for 83 NETLIB suite problem instances both prior to and after the instance was pre-processed using CPLEX 7.1. This computation was done using the ground-set model format of convex optimization, where the ground-set was defined by the lower and upper bound constraints on the variables.

A summary of our computational findings is as follows:

1. 72% of the original problem instances in the NETLIB suite are ill-conditioned.
2. 70% of the original problem instances in the NETLIB suite are primal ill-posed, i.e., arbitrarily small data perturbations will render the primal problem infeasible.
3. After pre-processing of the problem instances by CPLEX 7.1, only 19% of problem instances are ill-posed.
4. $\log C(d)$ of the 67 post-processed problems with finite condition number is fairly nicely distributed in the range from $2.6 - 11.0$.
5. The number of IPM iterations needed to solve linear optimization problem instances is related to the condition numbers of the post-processed problem instances. If the number of IPM iterations is large for a given problem instance, then the problem will tend to have a large post-processed condition number. However, the converse of this statement is not supported by computational experience: if the post-processed problem instance has a large condition number, one cannot assert that the problem instance will need a large number of IPM iterations to solve.

6. A simple linear regression model of IPM iterations as the dependent variable and $\sqrt{\vartheta}\log C(d)$ as the independent variable yields a positive linear relationship between IPM iterations and $\sqrt{\vartheta}\log C(d)$, significant at the 95% confidence level, with $R^2 = 0.4267$. This means that 42% of the variation in IPM iterations among the NETLIB suite problems is accounted for by $\sqrt{\vartheta}\log C(d)$.

7. A simple linear regression model of IPM iterations as the dependent variable and $\log C(d)$ as the independent variable yields a very similar result, also significant at the 95% confidence level, and with $R^2 = 0.4258$. These results indicate that $\log C(d)$ and $\sqrt{\vartheta}\log C(d)$ are essentially equally good at explaining the variation in IPM iteration counts among the NETLIB suite of linear optimization instances.

8. The number of IPM iterations correlates better with $\log C(d)$ than with $\log n$, $\log m$, $\log \vartheta$, or $\sqrt{\vartheta}$.

9. Curiously, $\log C(d)$ is somewhat correlated with $\log m$, having a sample correlation of 0.431. This observation bears further scrutiny.

10. In controlled perturbations of problem instances to ill-conditioned perturbed instances, the number of IPM iterations of the ill-posed perturbed instances are larger than for the original instance in about 68% of the problems studied, significantly larger in about half of these. However in the other 32% of the problems studied there was no change or even a slight decrease in IPM iterations.

11. The theoretical bounds on the optimal solution size and optimal objective function value are in general large for the post-processed NETLIB suite of problems. The ratio of the theoretical bound to the quantity has geometric means of $10^{7.49}$ for the primal optimal solution size, $10^{10.18}$ for the dual optimal solution size, and $10^{6.25}$ for the optimal objective function value.

12. The theoretical bound on the minimum norm primal feasible solution is fairly sharp for the post-processed NETLIB suite of problems. The ratio of the theoretical

bound to the quantity has a mean of $10^{2.98}$ for the minimum norm primal feasible solution.

Chapter 8

Extensions

In this chapter first we present an adaptation of the characterization of the distance to ill-posedness to semi-definite programming problems. This extension should permit computational testing of condition number theory for semi-definite programming problems. We then list other questions and future research ideas that this thesis has brought to light.

8.1 Computation for semi-definite programming

This section outlines how to adapt the characterizations of the distance to ill-posedness presented in Theorems 5 and 6 to semi-definite programming (SDP) problems. The principal aim of this line of research is to perform computational experiments to assess the significance of condition number theory in a setting different from linear programming.

We consider a SDP problem in the form

$$\begin{aligned} \min\quad & C \bullet X \\ \text{s.t.}\quad & A_i \bullet X - b_i \le 0 \quad i = 1, \ldots, k \\ & A_i \bullet X - b_i = 0 \quad i = k+1, \ldots, m \\ & X \succeq 0 \ , \end{aligned} \tag{8.1}$$

where C, $A_i \in S^{n\times n}$ for $i \in \{1, \ldots, m\}$ and $S^{n\times n}$ is the space of $n \times n$ symmetric matrices. We define for matrices A, $B \in S^{n\times n}$ the usual dot product in $S^{n\times n}$ by $A \bullet B = \sum_{i,j=1}^{n} a_{ij} b_{ij} = \mathrm{trace}(A^t B)$. We also use the usual vector dot product in $I\!R^m$. Finally the constraint $X \succeq 0$ means that the matrix X is positive semi-definite.

8.1.1 Characterization of $\rho(d)$

Before presenting the characterization of the distance to infeasibility, we set the notation which will help us in determining which norms should be used in $S^{n\times n}$ and $I\!R^m$. The SDP problems of form (8.1) can be written as

$$\begin{aligned} \min\quad & C \bullet X \\ \text{s.t.}\quad & A(X) - b \in C_Y \\ & X \succeq 0 \ . \end{aligned} \tag{8.2}$$

With this notation, $X \in S^{n\times n}$ belongs to the semi-definite cone, $b \in I\!R^m$, C_Y is a closed convex cone in $I\!R^m$ that is made up of cross products of $I\!R_+$ and $\{0\}$, and we define the operator A by

$$\begin{aligned} A:\ S^{n\times n} &\mapsto I\!R^m \\ X &\mapsto A(X) = (A_1 \bullet X, \ldots, A_m \bullet X) \ . \end{aligned}$$

Given the dot products defined in $S^{n\times n}$ and $I\!R^m$ we note that the adjoint of operator A

is defined by

$$\begin{array}{rccl} A^*: & \mathbb{R}^m & \mapsto & S^{n\times n} \\ & v & \mapsto & A^*v = \sum_{i=1}^m v_i A_i \ . \end{array}$$

Therefore the data of the problem is the linear operator A, the right hand vector b and the cost matrix C, and we denote $d = (A, b, C)$.

To compute the distance to ill posedness $\rho(d)$ we compute the primal and the dual distance to infeasibility, which are $\rho_P(d)$ and $\rho_D(d)$ respectively. This approach is possible since for a feasible problem with a finite optimal value,

$$\rho(d) = \min\{\rho_P(d), \rho_D(d)\} \ .$$

The programs that characterize the distance to infeasibility are given by Theorems 5 and 6 for the general GSM format, and apply in particular for the conic case problem considered here. For the SDP problem given by (8.2), these theorems imply the following characterizations for the primal and dual distance to infeasibility:

$$\begin{array}{rll} \rho_P(d) = & \min & \max\{\|A^*y + Z\|_*, |b^t y - g|\} \\ & y \in C_Y^* & \\ & Z \succeq 0 & \\ & g \geq 0 & \\ & \|y\|_* = 1 & \end{array}$$

and

$$\begin{array}{rll} \rho_D(d) = & \min & \max\{\|A(X) - p\|, |C \bullet X + g|\} \\ & X \succeq 0 & \\ & p \in C_Y & \\ & g \geq 0 & \\ & \|X\| = 1. & \end{array}$$

What allows to obtain a tractable program, as in the case of LP, is the fact that by selecting appropriate norms, these non-convex programs can be partitioned into a small number of convex problems. We select the norm $\|\cdot\|_1$ as the primal norm on $I\!R^m$ and the norm $\|X\| := \max_{i,j}|X_{ij}|$ as the primal norm on $S^{n\times n}$. The dual norm on $S^{n\times n}$ is then given by

$$\|S\|_* := \sup_{\|X\|\leq 1} S \bullet X = \sum_{i,j} |S_{ij}|\ .$$

Proposition 22 *If* $X \succeq 0$ *then* $\|X\| = \max_i X_{ii}$.

Proof: The proof of this proposition is reduced to showing that for any pair i, j, we have $X_{ij}^2 \leq X_{ii}X_{jj}$, which means that $|X_{ij}| \leq \max\{X_{ii}, X_{jj}\}$, which in turn implies the result. The inequality is a consequence of the fact that the 2×2 submatrix formed by the i and j columns and rows is positive semi-definite. ■

The objective functions in the characterizations of the primal and dual distances can be simplified to become:

$$\begin{array}{rl}
\rho_P(d) = & \min \quad \gamma \\
& -S \leq A^*y - Z \leq S \\
& (ee^t)\bullet S \leq \gamma \\
& -b^t y \leq \gamma \\
& y \in C_Y^* \\
& Z \succeq 0 \\
& \|y\|_\infty = 1
\end{array}$$

and

$$\begin{array}{rl} \rho_D(d) = & \min \quad \gamma \\ & -s \leq A(X) + p \leq s \\ & e^t s \leq \gamma \\ & C \bullet X \leq \gamma \\ & X \succeq 0 \\ & p \in C_Y \\ & \|X\| = 1. \end{array}$$

From Proposition 19, the problem that characterizes the primal distance to infeasibility is equivalent to

$$\begin{array}{rcl} \rho_P(d) = & \min_{i,j} & \min_{y,Z,\gamma,S} \quad \gamma \\ & i \in \{1,\ldots,m\} & -S \leq A^* y - Z \leq S \\ & j \in \{-1,1\} & (ee^t) \bullet S \leq \gamma \\ & & -b^t y \leq \gamma \\ & & y \in C_Y^* \\ & & Z \succeq 0 \\ & & y_i = j \ . \end{array}$$

Proposition 19 also implies that the dual distance to infeasibility is equivalent to

$$\begin{array}{rcl} \rho_D(d) = & \min_{i} & \min_{X,p,\gamma,s} \quad \gamma \\ & i \in \{1,\ldots,n\} & -s \leq A(X) + p \leq s \\ & & e^t s \leq \gamma \\ & & C \bullet X \leq \gamma \\ & & X \succeq 0 \\ & & p \in C_Y \\ & & X_{ii} = 1 \ . \end{array}$$

In conclusion, to compute the distance to ill-posedness for a SDP, we need to solve the $2m+n$ SDP problems. This approach makes computable the distance to ill-posedness for SDP problems of type (8.1).

8.1.2 Computing $\|d\|$

The characterization of the condition number for SDP instances also requires computing the norm of the data. The characterization of the norm of the data is determined by the norms selected to compute the distance to ill-posedness. In the previous section we defined $\|\cdot\|_1$ as the primal norm in $\mathbb{R}^m$ and the norm $\|X\| = \max_{i,j} |X_{ij}|$ as the primal norm in $S^{n\times n}$. Therefore, the norm on the data $d = (A, b, C)$ is

$$\|d\| = \max\left\{\|A\|, \|b\|_1, \|C\|_*\right\} ,$$

where the dual norm on $S^{n\times n}$ is used for the objective cost matrix C, that is

$$\|C\|_* = \sum_{i,j=1}^{n} |C_{ij}| ,$$

and the operator norm on A is given by

$$\begin{array}{rl} \|A\| = \sup_X & \sum_{i=1}^{m} |A_i \bullet X| \\ \text{s.t.} & \max_{i,j} |X_{ij}| \leq 1 \\ & X \in S^{n\times n} . \end{array}$$

An efficient method of computing the operator norm is needed to compute the condition number for SDP problems. Currently we upper bound this norm by

$$\|A\| \leq \sum_{k=1}^{m} \|A_k\|_* = \sum_{k=1}^{m} \sum_{i,j=1}^{n} |(A_k)_{ij}| .$$

This upper bound is obtained by changing the supremum with the sum in the definition of the operator norm.

8.2 Future work

In this section we outline possible future research directions that arise from this thesis. We group these directions into three categories: (1) research to improve the data perturbation model used, (2) research aimed at expanding computational experience, and (3) research on using the GSM format to represent the structure of the data in specific problem classes and application areas.

8.2.1 On the data perturbation model

The GSM format developed in this thesis allows us to consider specific structures that can be present in the data of the problem. For instance in the case of linear programming, the upper and lower bound constraints are not considered part of the data, therefore the structure in these constraints is preserved under data perturbation. We now list some limitations of the GSM format presented here and mention ideas to overcome them.

First, the GSM format requires the additional Assumption 2 to be capable of extending the conic case condition number theory. This assumption, which is used to bound the size of the primal optimal solution, does not allow to consider ground sets P in full generality and is somewhat artificial. A different proof to obtain a condition number bound on the norm of the primal optimal solution, that does not require Assumption 2 would allow us to improve the generality of this model.

It is clear that not all structures that might be present in the data can be considered with the GSM format, for example a linear program with range constraints cannot be adequately represented in the current GSM format. To be capable of considering other types of structure present in the data we should develop different data perturbation models. An initial idea is a further extension of the GSM format which we call the Full GSM format. The Full GSM format considers a convex optimization problem, with data $d = (A, b, c)$, of the form

$$\begin{array}{ll} \min_x & c^t x \\ \text{s.t.} & Ax - b \in Q \\ & x \in P \ , \end{array}$$

where now the ground sets P and Q are closed convex sets no longer required to be cones. As noted for the linear problems studied, a non-conic constraint is not frequent in practice, however it does occurs. Out of the 98 NETLIB suite problems, only 5 problems have range constraints.

Note that the Full GSM format can be used to treat more forms of structure present in the data, however, the model still requires that the data is a linear constraint with a right hand side, and a linear objective. Alternate models of data perturbation should be studied to consider further structure that cannot be adequately represented in this fashion.

A different form of extension, which has not been studied in this thesis, is the characterization for inconsistent instances of a condition number which takes into account structure present in the data of the problem. The extension to characterizing the distance to ill-posedness for inconsistent instances is not a straightforward task. As mentioned in Chapter 3, the distance of an inconsistent data instance $d \in \mathcal{F}^C$ to the set of ill-posed instances is given by

$$\rho(d) := \inf \left\{ \|\Delta d\| \mid X_{d+\Delta d} \neq \emptyset \text{ and } Y_{d+\Delta d} \neq \emptyset \right\},$$

the size of the smallest data perturbation Δd that would make both problems $(GP_{d+\Delta d})$, and $(GD_{d+\Delta d})$ feasible.

An important reason for studying these instances arises from algorithms that can decide whether a problem is feasible or not. Conceivably the complexity of detecting that a problem is infeasible should depend on how infeasible that problem is, and the measure of the distance to ill-posedness for inconsistent instances measures this. The homogeneous self-dual algorithm, introduced in [37], is an example of algorithms that can decide the feasibility question.

8.2.2 On expanding computational experience

The purpose of improving the computational experience is twofold: first we need to develop a more efficient computational scheme to compute or even estimate the condition number of a problem. Second, we need to validate the usefulness of the condition number for problems that arise in practice.

The computational scheme presented in this thesis to compute the condition number of a problem requires more work than solving the original problem. Although this can be justifiable for some instances, for example if the problem will be solved repeatedly a detailed study of its complexity might prove useful, in general it does not make sense to spend more work than solving the problem to obtain information on its complexity, sizes of solutions and sensitivity analysis.

Alternate characterizations and computational schemes, for example those in [24] and [25], should be studied and compared to the computational scheme presented here. It can prove worthwhile to reduce the accuracy in computing the distance to ill-posedness in order to obtain a more efficient computation. Once a faster computational strategy is available, the analysis of examples that are larger and pose difficulties to solvers can be attempted.

8.2.3 On applications and problem classes

In order to validate the usefulness of condition number for problems that arise in practice, we need to perform more computational tests. By considering certain application areas with particular classes of problems we can explore if the condition number is more suited for certain structure present in the data or not.

As mentioned in Section 8.2.1, what the GSM format accomplishes, is to separate from consideration as data certain structure present in the description of the problem. In general terms the GSM creates a framework by which anything that is not data is included in the definition of the set P and could conceivably allow for richer data to be separated.

Some classes of problems have inherent structure which should not be considered subject to perturbation. Two elementary examples come to mind: network flow problems and inventory management problems.

For a given graph G, a network flow problem solves the problem of finding the flows x on the arcs of graph G that minimize

$$\begin{array}{ll} \min_x & c^t x \\ \text{s.t.} & Nx = b \\ & l \leq x \leq u \ , \end{array}$$

for the node-arc incidence matrix N of graph G, a vector of node demands b, a linear cost vector c, and upper and lower bounds on arc flow u and l, respectively. Besides the data structure present in the upper and lower bounds, here the data structure of graph G is encoded in the matrix N.

If we wish to consider as fixed the structure that defines graph G, we should define a ground set P that restricts the variables to the feasible flows for graph G. For example we might define the problem as

$$\begin{array}{ll} \min_{x,s} & c^t x \\ \text{s.t.} & s = b \\ & (x,s) \in P \ , \end{array}$$

where the set P is defined to be

$$P = \{(x,s) \mid Nx = s, \ l \leq x \leq u\} \ ,$$

which is the set of feasible flows through graph G; note that P is a convex set. For this version of the problem, we could study the condition number theory as a function of quantities like the cost vector c and the demand vector b.

We consider now inventory management problems. Without going into the specifics of which inventory management problem we are considering, we can identify variables $\bar{x}_i$, for $i \in \{1, \ldots, T\}$ which represent the amount of inventory (at a warehouse, a store, total inventory, etc.) for each stage i of the planning horizon. These inventory variables are related stage by stage through the following equation:

$$\bar{x}_i - \bar{x}_{i+1} + a_i - b_i = 0 \ , \tag{8.3}$$

where a_i accounts for the amount of inventory added at stage i and b_i accounts for the amount of inventory consumed at stage i.

Whatever are the dynamics that govern the specific inventory management problem we consider, and that determine a_i and b_i, we can always identify a constraint of the form (8.3). These constraints have integer data, and a perturbation of this inventory management problem should also have constraints with this structure. This structure is what defines a problem to be an inventory management problem and should be taken into account by a suitable format. The following ground set P incorporates the structure of these constraints and could be used to define a GSM format in which inventory management problems are studied:

$$P = \{x = (\bar{x}, a, b, \tilde{x}) \mid \bar{x}_i - \bar{x}_{i+1} + a_i - b_i = 0, \text{ for } i \in \{1, \ldots, T\}\} \ ,$$

where we assume that the decision variable x is at least of dimension $3T$ and that we can identify in x variables $\bar{x}$, a, and b; all other variables are grouped in $\tilde{x}$.

Bibliography

[1] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. *Nonlinear Programming, Theory and Algorithms*. John Wiley & Sons, Inc, New York, second edition, 1993.

[2] Haïm Brezis. *Analyse Fonctionnelle, Théorie et applications*. Masson, Paris, second edition, 1987.

[3] Felipe Cucker and Javier Peña. A primal-dual algorithm for solving polyhedral conic systems with a finite-precision machine. *SIAM Journal on Optimization*, 12(2):522–554, 2001.

[4] C. Eckart and G. Young. The approximation of one matrix by another of lower rank. *Psychometrika*, 1:211–218, 1936.

[5] Marina Epelman and Robert M. Freund. Condition number complexity of an elementary algorithm for computing a reliable solution of a conic linear system. *Mathematical Programming*, 88(3):451–485, 2000.

[6] Marina Epelman and Robert M. Freund. A new condition measure, preconditioners, and relations between different measures of conditioning for conic linear systems. *SIAM Journal on Optimization*, 12(3):627–655, 2002.

[7] Marina A. Epelman. *Complexity, Condition Numbers, and Conic Linear Systems*. PhD thesis, Massachusetts Institute of Technology, 1999.

[8] Sharon Filipowski. On the complexity of solving sparse symmetric linear programs specified with approximate data. *Mathematics of Operations Research*, 22(4):769–792, 1997.

[9] Sharon Filipowski. On the complexity of solving feasible linear programs specified with approximate data. *SIAM Journal on Optimization*, 9(4):1010–1040, 1999.

[10] R. Freund and M. Todd. Barrier functions and interior–point algorithms for linear programming with zero–, one–, or two–sided bounds on the variables. *Mathematics of Operations Research*, 20(2):415–440, 1995.

[11] R. M. Freund and J. R. Vera. On the complexity of computing estimates of condition measures of a conic linear system. Technical Report, Operations Research Center, MIT, August 1999.

[12] Robert M. Freund. Complexity of convex optimization using geometry-based measures and a reference point. Working Paper OR 358-01, Operations Research Center, MIT, September 2001.

[13] Robert M. Freund and Jorge R. Vera. Condition-based complexity of convex optimization in conic linear form via the ellipsoid algorithm. *SIAM Journal on Optimization*, 10(1):155–176, 1999.

[14] Robert M. Freund and Jorge R. Vera. Some characterizations and properties of the "distance to ill-posedness" and the condition measure of a conic linear system. *Mathematical Programming*, 86(2):225–260, 1999.

[15] W. Kahan. Numerical linear algebra. *Canadian Mathematical Bulletin*, 9:757–801, 1966.

[16] V. Klee and G.J. Minty. How good is the simplex algorithm? In O. Shisha, editor, *Inequalities*, pages 159–175. Academic Press, New York, 1972.

[17] David W. Lewis. *Matrix Theory.* World Scientific Publishing Co., 1991.

[18] I. Lustig, R. Marsten, and D. Shanno. The primal dual interior point method on the cray supercomputer. In Thomas F. Coleman and Yuying Li, editors, *Large-Scale Numerical Optimization, Papers from the Workshop held at Cornell University, Ithaca, NY, October 1989*, volume 46 of *SIAM Proceedings in Applied Mathematics*, pages 70–80. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1990.

[19] I. Lustig, R. Marsten, and D. Shanno. Interior point methods: computational state of the art. *ORSA Journal on Computing*, 6:1–14, 1994.

[20] Yurii Nesterov and Arkadii Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1993.

[21] NETLIB linear programming library. http://www.netlib.org/lp/.

[22] M. A. Nunez and R. M. Freund. Condition measures and properties of the central trajectory of a linear program. *Mathematical Programming*, 83(1):1–28, 1998.

[23] M. A. Nunez and R. M. Freund. Condition-measure bounds on the behavior of the central trajectory of a semi-definite program. *SIAM Journal on Optimization*, 11(3):818–836, 2001.

[24] Javier Peña. Computing the distance to infeasibility: theoretical and practical issues. Technical report, Center for Applied Mathematics, Cornell University, 1998.

[25] Javier Peña. Understanding the geometry of infeasible perturbations of a conic linear system. *SIAM Journal on Optimization*, 10(2):534–550, 2000.

[26] Javier Peña and James Renegar. Computing approximate solutions for convex conic systems of constraints. *Mathematical Programming*, 87(3):351–383, 2000.

[27] J. Renegar. A polynomial-time algorithm, based on Newnton's method, for linear programming. *Mathematical Programming*, 40(1):59–93, 1988.

[28] J. Renegar. Some perturbation theory for linear programming. *Mathematical Programming*, 65(1):73–91, 1994.

[29] J. Renegar. Linear programming, complexity theory, and elementary functional analysis. *Mathematical Programming*, 70(3):279–351, 1995.

[30] James Renegar. *A Mathematical View of Interior-Point Methods in Convex Optimization*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2001.

[31] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, New Jersey, 1997.

[32] Ron Shamir. The efficiency of the simplex method: a survey. *Management Science*, 33(3):301–334, 1987.

[33] Jorge R. Vera. Ill-posedness and the computation of solutions to linear programs with approximate data. Technical Report, Cornell University, May 1992.

[34] Jorge R. Vera. *Ill-Posedness in Mathematical Programming and Problem Solving with Approximate Data*. PhD thesis, Cornell University, 1992.

[35] Jorge R. Vera. Ill-posedness and the complexity of deciding existence of solutions to linear programs. *SIAM Journal on Optimization*, 6(3):549–569, 1996.

[36] Jorge R. Vera. On the complexity of linear programming under finite precision arithmetic. *Mathematical Programming*, 80(1):91–123, 1998.

[37] Yinyu Ye, Michael J. Todd, and Shinji Mizuno. An $O(\sqrt{n}L)$-iteration homogeneous and self-dual linear programming algorithm. *Mathematics of Operations Research*, 19(1):53–67, 1994.