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A robust optimization approach for the capacitated vehicle routing problem with demand uncertainty

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A robust optimization approach for the capacitated vehicle routing problem with demand uncertainty

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In this paper we introduce a robust optimization approach to solve the Vehicle Routing Problem (VRP) with demand uncertainty. This approach yields routes that minimize transportation costs while satisfying all demands in a given bounded uncertainty set. We show that for the Miller–Tucker–Zemlin formulation of the VRP and specific uncertainty sets, solving for the robust solution is no more difficult than solving a single deterministic VRP. Our computational results on benchmark instances and on families of clustered instances show that the robust solution can protect from unmet demand while incurring a small additional cost over deterministic optimal routes. This is most pronounced for clustered instances under moderate uncertainty, where remaining vehicle capacity is used to protect against variations within each cluster at a small additional cost. We compare the robust optimization model with classic stochastic VRP models for this problem to illustrate the differences and similarities between them. We also observe that the robust solution amounts to a clever management of the remaining vehicle capacity compared to uniformly and non-uniformly distributing this slack over the vehicles.

Keywords: Robust optimization, vehicle routing, demand uncertainty

1. Introduction

Many industrial applications deal with the problem of routing a fleet of vehicles from a depot to service a set of customers that are geographically dispersed. This type of problem is faced daily by courier services (e.g., Federal Express, United Parcel Service and the Overnight United States Postal Service), local trucking companies and demand responsive transportation services, just to name a few. These types of services have experienced tremendous growth in recent years. For example, both United Parcel Service and the Federal Express have annual revenue of well over $10 billion, and the dial-a-ride service for the disabled and handicapped is today a $1.2 billion industry (Palmer *et al.*, 2004). However, congestion and variability in demand and travel times affects these industries on three major service dimensions: (i) travel time; (ii) reliability; and (iii) cost (Meyer, 1996). Therefore, there is a need to develop routing and scheduling tools that directly account for the uncertainty. In this paper, we focus on the uncertainty in demand.

Generally speaking, current methods to represent the uncertainty in a Vehicle Routing Problem (VRP) either make strong assumptions regarding the distribution of the uncertain parameters, or use a discrete distribution repeating the problem for each scenario (Bertsimas and Simchi-Levi, 1996; Gendreau *et al.*, 1996). These methods then obtain a solution that minimizes the expected value or are used to analyze the performance of a routing policy in expected value or worst case situations. The resulting solution is potentially sensitive to the actual data that occur in the problem. Considering that the VRP solution in a given application will only face a single realization of the uncertainty, a reasonable goal is to obtain a *robust* solution, i.e., a solution that is *good* for all possible data uncertainty.

In this paper, we consider the Capacitated Vehicle Routing Problem (CVRP) with uncertain demand on a set of fixed demand points. We use the robust optimization methodology introduced by Ben-Tal and Nemirovski (1998) to formulate a new problem for the VRP with demand uncertainty, the Robust Vehicle Routing Problem (RVRP). The optimal solution for this problem is the route that optimizes the worst-case value over all data uncertainty. The expectation is that such a solution would be efficient in its worst case and thus efficient for every possible outcome. A key aspect of our work is to model demand uncertainty in the CVRP in such a way as to obtain a RVRP which can be solved efficiently. That is to say that it is not

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significantly more difficult than solving the deterministic CVRP.

The proposed RVRP identifies an optimal *a priori* route that is feasible for every demand realization and does not consider recourse actions. Such solutions are possible in cases where the demand uncertainty is small or the vehicle fleet has sufficient capacity. A different model is needed if there is no route that can meet all demand realizations and we say that the RVRP is infeasible. In problems with recourse there is the additional difficulty of determining optimal recourse actions. This leads either to augmenting the original deterministic model or requiring strong uncertainty assumptions to evaluate the recourse action. In both cases, the additional flexibility of the recourse solution comes with an increased complexity in the formulation and solution procedure. In this work we propose a model that uses a simple representation of the uncertainty and does not make the problem more difficult to solve. We concentrate on studying when this *a priori* RVRP can be beneficial, providing solutions that effectively address the uncertainty without being overly conservative, and leave to future research the extension of this model to infeasible instances. To illustrate the differences between the RVRP and existing stochastic VRP models we present a comparison with a chance constrained model and a stochastic model with recourse for the VRP with demand uncertainty. In addition to comparing the quality of the solutions we note that each method requires a different uncertainty model and set of assumptions. These differences are important in determining which is the best model to use for a specific application.

A natural method to address demand uncertainty is to reserve vehicle capacity to be able to adapt to cases when the realized demand is greater than the expected demand. In fact, if there is abundant vehicle capacity, such as in the uncapacitated VRP, the optimal routing solution can easily accommodate changes in the demand levels. However, in capacitated cases with little excess vehicle capacity, the difficult question is how to distribute this extra capacity among routes to better address the demand uncertainty. The RVRP distributes this slack by finding a minimum cost route that satisfies all possible demand realizations.

The structure of the paper is as follows. We discuss the relevant literature in the next section. In Section 3 we present the derivations of the RVRP formulations for problems with demand uncertainty and show that for the Miller–Tucker–Zemlin (MTZ) formulation and demand uncertainty sets constructed from combinations of scenarios the resulting RVRP is another instance of a CVRP. We present our computational results in Section 4. Here we show computation of robust solutions for a well-known suite of CVRP problems (Augerat *et al.*, 1995), and compare the robust solution against the deterministic solution on a family of clustered instances. We also contrast the robust optimization method to existing stochastic VRP methods and simple methods of distributing the unused vehicle capacity. We finish the paper with concluding remarks in Section 5.

2. Literature review

Problems where a given set of vehicles with finite capacity have to be routed to satisfy a geographically dispersed demand at minimum cost are known as VRPs. This class of problems was introduced by Dantzig and Ramser (1959) and a considerable amount of research on the VRP and its numerous extensions and applications has been performed. General surveys of vehicle routing research can be found in Fisher (1995), Laporte and Osman (1995), and Toth and Vigo (2002). The VRP is known to be NP-hard (Lenstra and Rinnooy Kan, 1981), but nevertheless, there has been considerable work on developing exact solution procedures, see for instance Ralphs *et al.* (2003), Baldacci *et al.* (2004), Lysgaard *et al.* (2004), and Fukasawa *et al.* (2006).

The most studied areas in the stochastic VRP literature are the VRP with Stochastic Demands (VRPSD) and the VRP with Stochastic Customers (VRPSC). A major contribution to VRPSD is that of Bertsimas (1992), where *a priori* solutions use different recourse policies to solve the VRPSD and bounds, asymptotic results and other theoretical properties are derived. Bertsimas and Simchi-Levi (1996) surveys work on VRPSD with an emphasis on the proposed algorithms and the insights gained from their use. Different solution algorithms are presented in Dror *et al.* (1989) and Dror (1993), including conventional stochastic programming and Markov decision processes for single and multi-stage stochastic models. Secomandi (2001) introduces a re-optimization-type routing policy for the VRPSD.

The VRPSC, where fixed demand customers have a probability p_i of being present, and the VRPs with Stochastic Customers and Demands (VRPSCD), which combines VRPSC and VRPSD, first appeared in Jézéquel (1985), Jaillet (1987) and Jaillet and Odoni (1988). Bertsimas (1988) gives a systematic analysis and presents several properties, bounds and heuristics. Gendreau *et al.* (1995) proposes the first exact solution, an L-shaped method and a tabu search meta heuristic for the VRPSCD.

This prior work on stochastic VRPs includes two important types of problem formulations: (i) chance constrained models; and (ii) stochastic models with recourse. A chance constrained model assumes that constraints are satisfied with a high probability given a known probability distribution of the uncertain parameters. These chance constrained models have been shown to be equivalent to deterministic VRPs for a number of routing problems and uncertainty assumptions (Stewart and Golden, 1983; Laporte *et al.*, 1989, 1992). Stochastic models with recourse allow for recourse actions that adjust an *a priori* solution after the uncertainty is revealed. See Gendreau *et al.* (1996) for a good survey. Different recourse actions have been proposed in the literature, such as skipping non-occurring customers, returning to the depot when the capacity is exceeded, or complete reschedule for occurring customers (Jaillet, 1988; Waters, 1989; Bertsimas *et al.*, 1990). These recourse actions allow

a broader feasible solution set but can increase the problem size and require specialized solution procedures.

Recent works by Morales (2006) and Erera *et al.* (2007) use robust optimization for the VRPSD with recourse. It is assumed that vehicles replenish at the depot, the worst-case value for the recourse action is computed by finding the longest path on an augmented network and the problem is solved using a tabu search heuristic. In contrast, the RVRP we propose avoids the additional complexity of recourse models and uses a simple model of uncertainty. The point is to investigate when this simple approach can provide an *a priori* solution that effectively adapts to the uncertainty without being overly conservative.

A priori solutions for the VRPSD vary in how they allocate the unused capacity. This allocation strategy is key in determining the success of the solution in coping with uncertainty, especially when there is little unused capacity. A few methods developed for the deterministic VRP have focused explicitly on how to distribute the vehicle capacity among routes. For instance, Daganzo (1988) proposes the use of a consolidation center as a strategy to better manage vehicle capacity. Charikar *et al.* (2001) introduces a constant ratio approximation algorithm which assigns each vehicle half of its capacity and uses the remaining capacity to improve routes with a matching algorithm. Branke *et al.* (2005) shows that managing the slack by waiting at strategic locations can increase the probability of meeting additional demand. Our work departs from these prior results since we consider a different problem domain: a standard CVRP with no transshipment nodes and with a small capacity-to-demand ratio. Zhong *et al.* (2007) developed a recourse model for the VRPSCD that uses the capacity that is not assigned in the first stage to adapt to the demand uncertainty in the second stage. They show that keeping unassigned customers near the depot is a good strategy for balancing the workload due to daily demand variations. Although our work differs from this because it focuses on *a priori* routing strategies with no recourse, we also notice that having customers near the depot facilitates the creation of efficient robust routes.

In this paper we use robust optimization for the VRPSD. We follow the robust optimization methodology as introduced by Ben-Tal and Nemirovski (1998, 1999) and El-Ghaoui *et al.* (1998) for linear quadratic and general convex programs, that were subsequently extended to integer programming by Bertsimas and Sim (2003). The general approach of robust optimization is to optimize against the worst instance that might arise due to data uncertainty by using a min-max objective. This typically results in solutions that exhibit little sensitivity to data variations and are said to be *immunized* to this uncertainty. Robust solutions have the potential to be efficient solutions in practice, since they tend not to be far from the optimal solution of the deterministic problem and significantly outperform the deterministic optimal solution in the worst case (Goldfarb and Iyengar, 2003; Bertsimas and Sim, 2004).

The robust optimization methodology assumes that the uncertain parameters belong to a given bounded uncertainty set. For fairly general uncertainty sets, the resulting robust counterpart can have a comparable complexity to the original problem. For example, a Linear Program (LP) with uncertain parameters belonging to a polyhedral uncertainty set has a robust problem which is an LP whose size is polynomial in the size of the original problem (Ben-Tal and Nemirovski, 1999). This nice complexity result, however, does not carry over to robust models of problems with recourse, where LPs with polyhedral uncertainty can result in NP-hard problems, see Ben-Tal *et al.* (2004). An important question therefore is how to formulate a robust problem that is not more difficult to solve than its deterministic counterpart.

3. RVRP formulations

There exist a number of different VRP formulations and since each could lead to a different RVRP, it is important to identify a VRP formulation that leads to a RVRP that is not too difficult to solve. In addition to the VRP formulation, the form of the uncertainty sets considered also influences the resulting RVRP and the level of difficulty associated with its solution.

In this section, we first identify the deterministic VRP formulation and demand uncertainty sets that will be used and then we present the derivation for the RVRP.

3.1. *Identifying the VRP formulation*

In addition to the problem size, the difficulty in solving a problem is influenced by three aspects: (i) the problem data; (ii) the problem formulation; and (iii) the solution procedure. For instance, the observed runtimes of a fixed Integer Program (IP) solver show different behavior as we vary the VRP formulations (Ordóñez *et al.*, 2007). In addition, the fixed general IP solver was more efficient in solving the MTZ formulation than other arc-based VRP formulations considered in that study for a wide range of problem parameters.

Another important criterion in identifying a suitable formulation for our robust optimization framework is the nature of the formulation with respect to uncertain parameters. Since we are interested in introducing uncertainty in demand, when we consider the parts of the formulation related to demand, the MTZ formulation has constraints in the form of inequalities. In the robust optimization methodology, it is preferable to have inequality constraints involving uncertain parameters than equality constraints, since it is more difficult to satisfy equalities for all values of the uncertainty. In fact Ben-Tal *et al.* (2004) show that even for simple LPs, if there are uncertain parameters in equality constraints the robust counterpart problem can be NP-hard.

The MTZ formulation of the CVRP follows: it considers the problem of routing at minimum cost a uniform fleet of K vehicles, each with capacity C, to service geographically dispersed customers, each with a deterministic demand that must be serviced by a single vehicle. Let V be the set of n demand nodes and a single depot, denoted as node 0. Let d_i be the demand at each node i. We consider the fully connected network, and denote the deterministic travel time between node i and node j by c_{ij}. The arc-based model considers integer variables x_{ij} which indicate whether or not a vehicle goes from node i to node j. In addition, the MTZ formulation includes continuous variables u_i for every $i \in V \setminus \{0\}$ that represent the flow in the vehicle after it visits customer i. The constraints (1)–(5) are routing constraints and the constraints (6) and (7) impose both the capacity and connectivity of the feasible routes.

$$\text{(CVRP)} \quad \min \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij}, \tag{1}$$

$$s.t. \sum_{i \in V} x_{ij} = 1 \qquad \text{for } j \in V \setminus \{0\}, \tag{2}$$

$$\sum_{j \in V} x_{ij} = 1 \qquad \text{for } i \in V \setminus \{0\}, \tag{3}$$

$$\sum_{i \in V} x_{i0} = K, \tag{4}$$

$$\sum_{j \in V} x_{0j} = K, \tag{5}$$

$$u_j - u_i + C(1 - x_{ij}) \geq d_j \quad \text{if } i, j \in V \setminus \{0\}, \\ i \neq j, \tag{6}$$

$$d_i \leq u_i \leq C \qquad \text{if } i \in V \setminus \{0\}, \tag{7}$$

$$x_{ij} \in \{0, 1\} \qquad \text{if } i, j \in V. \tag{8}$$

Notice that the uncertain demand d_i appears by itself and only on constraints (6) and (7). However, the lower bound on constraint (7) is implied from constraint (6), the fact that every node is visited, and that $u_i \geq 0$ for all $i \in V \setminus \{0\}$. We will therefore only consider the uncertainty in constraint (6) and replace all d_i with zero in constraint (7).

Although the nature of the MTZ formulation is the preferred one with respect to uncertain parameters for our robust optimization framework, there is a caveat. This formulation can have large initial LP gaps, which lead to large solution times. However, it is possible to improve this gap by adopting the lifting techniques proposed by Desrochers and Laporte (1991).

3.2. *Uncertainty in demand*

We consider that the demand parameter d is uncertain and belongs to a bounded set U. We consider uncertainty sets which are constructed as deviations around an expected demand value $\mathbf{d}^0$. The possible deviation directions from these nominal values are fixed and identified by scenario vectors, $\mathbf{d}^k \in \Re^n$, where n is the number of nodes. The scenario vectors are allowed to have negative deviation values. For a given number of scenario vectors, s, the general uncertainty set U is a linear combination of the scenario vectors with weights $\mathbf{y} \in \Re^s$ that must belong to a bounded set $\mathbf{y} \in Y$:

$$U_D = \left\{ \mathbf{d} \mid \mathbf{d}^0 + \sum_{k=1}^{s} y_k \mathbf{d}^k, \ \mathbf{y} \in Y \right\},$$

In particular, we consider the following three sets for Y:

$$\text{convex hull} \quad \mathbf{Y}_1 = \left\{ \mathbf{y} \in \Re^s \mid \mathbf{y} \geq 0, \ \sum_{k=1}^{s} y_k \leq 1 \right\},$$

$$\text{box} \quad \mathbf{Y}_2 = \left\{ \mathbf{y} \in \Re^s \mid \|\mathbf{y}\|_\infty \leq 1 \right\},$$

$$\text{ellipsoidal} \quad \mathbf{Y}_3 = \left\{ \mathbf{y} \in \Re^s \mid \mathbf{y}^\mathrm{T} \mathbf{Q} \mathbf{y} \leq 1 \right\},$$

where the ellipsoidal set is defined for some given positive definite matrix $\mathbf{Q}$, for example $\mathbf{Q} = \mathbf{I}$. We refer to the uncertainty set formed by considering the combination set $\mathbf{Y}_i$ as U_i for $i = 1, 2, 3$. Note that if $s = n$ and the scenario vectors $\mathbf{d}^k$ correspond to the coordinate axis, then $\mathbf{Y}_2$ leads to $U_D = \mathbf{d}^0 + \{\mathbf{d} \mid \|\mathbf{d}\|_\infty \leq 1\}$ and $\mathbf{Y}_3$ to $U_D = \mathbf{d}^0 + \{\mathbf{d} \mid \mathbf{d}^\mathrm{T} \mathbf{Q} \mathbf{d} \leq 1\}$ the full dimensional box and ellipse centered at $\mathbf{d}^0$, respectively. We will show that for these three sets U_i the resulting RVRP problem is an instance of the CVRP.

3.3. *Robust VRP formulation*

We now propose the robust counterpart problem RVRP for CVRP with demand belonging to an uncertainty set U. Recall that we consider the problem only with uncertainty in constraint (6) with constraint (7) equal to $0 \leq u \leq C$.

The robust VRP finds the optimal route that satisfies all possible demand outcomes, in other words the problem has to identify routes x_{ij} and vehicle usage u_i such that:

$$u_j - u_i + C(1 - x_{ij}) \geq d_j \quad \forall \mathbf{d} \in U, \quad i, j \in V \setminus \{0\}, i \neq j. \tag{9}$$

We can therefore state the RVRP. This problem minimizes objective (1), subject to constraints (2), (3), (4), (5), (7), (8), (9). If we substitute in the definition of the uncertainty set U, we can write the robust constraint (9) as the following inequality:

$$u_j - u_i + C(1 - x_{ij}) - d_j^0 \geq \sum_{k=1}^{s} y_k d_j^k \quad \forall y \in Y, \\ i, j \in V \setminus \{0\}, i \neq j. \tag{10}$$

For given decision variables x and u we refer to the left-hand side of the above inequality as $\phi_{ij}(x, u) = u_j - u_i + C(1 - x_{ij}) - d_j^0$ for $i, j \in V \setminus \{0\}$, $i \neq j$. Then, to enforce that the above inequality holds for all $\mathbf{y} \in Y$ it suffices to enforce it for $\sup_{\mathbf{y} \in Y} \sum_{k=1}^{s} y_k d_j^k = \sup_{\mathbf{y} \in Y} \mathbf{y}^\mathrm{T} \mathbf{D}_{j\bullet}$. Here we denote by $\mathbf{D} = [\mathbf{d}^1 \dots \mathbf{d}^s] \in \Re^{n \times s}$ the matrix of scenario vectors and $\mathbf{D}_{j\bullet} = (d_j^1, \dots, d_j^s)^\mathrm{T}$ the jth row of $\mathbf{D}$ as a column vector. Let us also denote $\mathbf{e}$ as the column vector of all ones of appropriate dimension.

Proposition 1. *Under uncertainty set U_1, the robust counterpart is obtained by replacing constraint* (6) *in CVRP with constraint* (11). *We refer to the resulting RVRP as RVRP$_1$:*

$$u_j - u_i + C(1 - x_{ij}) \geq d_j^0 + \max \left\{ \max_k d_j^k, 0 \right\},$$
$$i, j \in V \setminus \{0\}, i \neq j. \tag{11}$$

Proof. Using the definition of Y_1 we can write $\sup_{\mathbf{y} \in Y_1} \mathbf{y}^T \mathbf{D}_{j\bullet}$ and its dual as the following pair of LPs:

(Primal) max $\mathbf{y}^T \mathbf{D}_{j\bullet}$ (Dual) min θ
s.t. $\mathbf{e}^T \mathbf{y} \leq 1$ s.t. $\theta \mathbf{e} \geq \mathbf{D}_{j\bullet}$
      $y \geq 0$. $\theta \geq 0$.

From weak duality, the condition $\phi_{ij}(x, u) \geq \sup_{\mathbf{y} \in Y_1} \mathbf{y}^T \mathbf{D}_{j\bullet}$ is equivalent to having $\phi_{ij}(x, u) \geq \theta$ for some dual feasible θ. This means that $\phi_{ij}(x, u) \geq 0$ and $\phi_{ij}(x, u) \geq d_j^k$ for $k = 1, \ldots, s$. Combining these conditions together for all $\phi_{ij}(x, u)$ gives constraint (11). ∎

Proposition 2. *Under uncertainty set U_2, the robust counterpart is obtained by replacing constraint* (6) *in CVRP with constraint* (12). *We refer to the resulting RVRP as RVRP$_2$:*

$$u_j - u_i + C(1 - x_{ij}) \geq d_j^0 + \sum_k \left| d_j^k \right|, \quad i, j \in V \setminus \{0\}, i \neq j. \tag{12}$$

Proof. Using the definition of Y_2 we can write $\sup_{\mathbf{y} \in Y_2} \mathbf{y}^T \mathbf{D}_{j\bullet}$ and its dual as the following pair of LPs:

(Primal) max $\mathbf{y}^T \mathbf{D}_{j\bullet}$,   (Dual) min $\mathbf{e}^T (\boldsymbol{\alpha} + \boldsymbol{\beta})$,
s.t. $\mathbf{y} \leq \mathbf{e}$,   s.t. $\boldsymbol{\alpha} - \boldsymbol{\beta} = \mathbf{D}_{j\bullet}$,
      $\mathbf{y} \geq -\mathbf{e}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \geq 0$.

It is simple to verify that the optimal solution to the dual problem will satisfy $\alpha_k^* + \beta_k^* = |d_j^k|$ for every $k = 1, \ldots, s$. Therefore, the dual optimal objective value is $\sum_{k=1}^s |d_j^k|$. Enforcing the robust feasibility condition on $\phi_{ij}(x, u)$ with the above optimal dual objective value we obtain constraint (12). ∎

Proposition 3. *Under uncertainty set U_3, the robust counterpart is obtained by replacing constraint* (6) *in CVRP with constraint* (13). *We refer to the resulting RVRP as RVRP$_3$:*

$$u_j - u_i + C(1 - x_{ij}) \geq d_j^0 + \sqrt{\mathbf{D}_{j\bullet}^T \mathbf{Q}^{-1} \mathbf{D}_{j\bullet}}, \quad i, j \in V \setminus \{0\}, \quad i \neq j. \tag{13}$$

Proof. Using the definition of Y_3 we have that $\sup_{\mathbf{y} \in Y_3} \mathbf{y}^T \mathbf{D}_{j\bullet} = \max \mathbf{y}^T \mathbf{D}_{j\bullet} : \mathbf{y}^T \mathbf{Q} \mathbf{y} \leq 1$. From the Karush–Kuhn–Tucker optimality conditions we have that the optimal solution to this problem is

$$\mathbf{y}^* = \frac{1}{\sqrt{\mathbf{D}_{j\bullet}^T \mathbf{Q}^{-1} \mathbf{D}_{j\bullet}}} \mathbf{Q}^{-1} \mathbf{D}_{j\bullet}.$$

When we plug this optimal solution into the robust feasibility condition $\phi_{ij}(x, u) \geq (\mathbf{y}^*)^T \mathbf{D}_{j\bullet}$, we obtain constraint (13). ∎

For the three RVRPs with demand uncertainty studied, the only change from the original CVRP formulation is an increase in the demands that appear in constraint (6). Since the deviation vectors, $\mathbf{d}^k$, are fixed, each of the RVRPs is an instance of the CVRP. We can therefore make use of the efficient exact algorithms in the literature to solve the robust problems. We note that the demand parameters used in the RVRPs are at least as big as the deterministic demand parameters, thus the RVRPs are typically more capacity constrained than the corresponding deterministic problem. We note this because it has been observed in practice that solving a CVRP becomes harder as the problem is more capacity constrained. Thus, although the RVRPs are instances of a CVRP, in practice solving RVRPs is likely to be more difficult than solving the deterministic versions. Lastly note that, depending on the nature of the scenario vectors, RVRPs may result in infeasible problems even though the deterministic CVRP is feasible.

For different types of demand uncertainty sets $\mathbf{d} \in U$, the key step in the derivation of the RVRP is to compute $\sup_{\mathbf{d} \in U} d_j$ and substitute this value for the right-hand side of Equation (9). This can be done for different uncertainty sets than those considered in this work. We do not pursue these formulations here for simplicity, since many require additional constraints and variables making the resulting robust problem not a CVRP that may necessitate a specialized solution procedure.

Since the robust formulations with uncertainty in demand, RVRP$_1$, RVRP$_2$ and RVRP$_3$ are instances of a CVRP, it is possible to introduce uncertainty in travel time in addition to the uncertainty in demand by using the approach proposed by Bertsimas and Sim (2003) for IPs with uncertain cost coefficients. The authors consider a box uncertainty set for the cost coefficients with an additional restriction on the number of cost coefficients that vary. They show that the optimum solution of the robust counterpart can be obtained by solving a polynomial number of nominal problems with modified cost coefficients. Since the RVRPs for our proposed uncertainty sets are instances of a general IP, formulating and solving the robust counterparts with independent uncertainty in both demand and travel time is a straightforward application of this methodology.

4. Experimental analysis

In this section, we first present performance measures that will be used to compare robust and deterministic solutions. We then present computational results on instances from the literature and analyze the trade-offs of robust solutions on families of clustered instances. We also compare the robust solution with alternative methods to address demand uncertainty in VRPs.

Our computational results compare the solution values for the different problem formulations of the VRPSD. To solve the deterministic, robust and chance constrained models, which are instances of CVRP, we use the branch-and-cut-based VRP solver in the open source SYM-PHONY library due to Ralphs *et al.* (2003), available online at `http://branchandcut.org/VRP`. We solve the stochastic models with recourse using CPLEX 9.0. All experiments were carried out with a runtime limit of an hour on a Dell Precision 670 computer with a 3.2 GHz Intel Xeon Processor and 2 GB RAM running Red Hat Linux 9.0.

4.1. *Performance measures*

The first performance measure, the ratio κ, quantifies the relative extra cost of the robust solution with respect to the cost of the deterministic solution. It is given by $\kappa = (z_r - z_d)/z_d$ where z_d is the optimal objective function value of the deterministic CVRP (with expected demand) and z_r is the optimal objective value of the robust approach counterpart (with worst-case demand). This ratio gives information on how much extra cost we will incur if we want to implement the robust approach to protect against the worst-case realization of the uncertainty, instead of implementing the deterministic approach. Note that the calculation of the ratio requires solving two instances of CVRP.

The second performance measure considers the effect of the solutions on the demand when it is subject to demand uncertainty. The ratio δ is the relative unsatisfied demand for the deterministic solution when it faces its worst-case demand. It is given by $\delta = (\gamma_d / \sum_{i \in V} d_i^0)$ where the numerator γ_d is the maximum unsatisfied demand that can occur if the optimal deterministic solution is used. The denominator is the total demand of the deterministic case and it is assumed that the deterministic problem is feasible. To obtain γ_d, we fix the routing variables to the deterministic optimal solution and maximize the unmet demand by varying the demand outcome within the demand uncertainty set. Note that the calculation of this ratio requires solving only one instance of the CVRP.

By definition, the robust counterpart gives a solution with zero unmet demand that may have a larger cost than the optimal solution for the expected demand. This optimal solution in turn may have scenarios with unmet demand. Therefore, these two measures, unmet demand and cost, represent the trade-offs that routing solutions must balance in a VRPSD. Depending on the specific problem more importance should be given to the unmet demand or the cost to decide which solution provides the best combination of these two competing objectives. In this work we arbitrarily select the relative measures defined above and decide which solution is best by a simple majority. For different choices of criteria and performance measures the results would vary, however, the tendencies should remain the same.

4.2. *Robust and deterministic approaches to standard problems*

Our first set of experiments address problem set A (random instances), set B (clustered instances) and set P (modified instances from the literature) of the CVRP suite of problems by Augerat *et al.* (1995). The instances range from 15 to 100 customers. We modified these instances to include demand uncertainty. We allow each demand parameter to further increase up to a fixed percentage of the deterministic value. We randomly generate a total of five scenarios within the allowed percent deviation for the demand uncertainty set. More specifically, we use the following values of percent deviation in demand parameters: 5, 10, 15 and 20. We first present computational results only for the convex hull uncertainty set. We then discuss how the results change for the box and ellipsoidal uncertainty sets.

Table 1 shows the results based on the performance measures δ for the percent unmet demand ratio and κ for the percent cost ratio of the solutions, where "No" indicates the number of the instance, "T" indicates the percent tightness ratio of the instance which is defined as the ratio of the total expected demand to total vehicle capacity, "IN" indicates infeasible instance and "NA" indicates that an optimal solution could not be found within the 1 hour runtime limit.

The first observation is that since the original instances are already tight (with a percent tightness ratio between 81 and 99%), the robust counterparts run quickly into infeasibility as the uncertainty increases even though the deterministic CVRPs are feasible and could be solved to optimality. In almost all of the instances with a tightness ratio greater than 90%, the cost ratio could not be calculated for the percent deviation values of 15 and 20% since either the robust counterpart became infeasible or the runtime limit was reached. Recall that the robust instances are more capacity constrained due to increased demand in the data and empirical observations have shown that CVRPs become more difficult to solve in practice as problems are more capacity constrained.

We note that it is unclear when robust solutions are preferable to their deterministic counterparts since this involves trading off route cost to meet potential demand. Whether the additional coverage justifies the extra cost depends on each application. A general observation from Table 1 is that more robust instances could be solved for the clustered problems, set B, and with moderate cost ratios. A close analysis of the solutions shows that the robust solution performs well on clustered instances when it can redistribute efficiently the unused vehicle capacity. If a planned route does not have enough capacity for a possible demand, then the robust solution would route the vehicles differently. In this case, if there is a vehicle nearby with enough slack, then it could service this demand with a small cost increase. However, if a vehicle with enough slack is far from the extra

Table 1. Augerat *et al.* (1995) sets

		Set																								
		A									*B*									*P*						
		Unmet demand				Cost ratio					Unmet demand				Cost ratio					Unmet demand				Cost ratio			
No	T	5	10	15	20	5	10	15	20	T	5	10	15	20	5	10	15	20	T	5	10	15	20	5	10	15	20
1	82	0.5	2.5	4.8	6.9	0	1.5	4.2	9.3	82	0	0.7	2.6	4.5	0	1.3	1.6	3.3	87	0.5	1.2	2.2	3.3	0	2.2	5.1	IN
2	89	0.2	1.7	3.5	5.9	0	2.1	6.5	IN	91	1	2.5	4.4	6.5	0	0.1	5.5	IN	96	0.5	2.3	5	8.2	0	IN	IN	IN
3	90	0.3	1.8	4	6.5	0	2.7	7.1	IN	87	0.7	2.3	4.4	6.7	0.1	2.8	3.2	4.4	96	0.8	2.2	4.8	7.5	0	IN	IN	IN
4	92	0	0.3	1.6	3.2	0	1.5	4.6	IN	85	0.4	2	4.2	7.3	0	0.1	1.7	1.9	93	0	0	0.9	3.4	0	0	0	IN
5	88	0.8	2.1	4	6.3	0	1.8	4.5	6.3	88	1.2	2.7	4.6	6.7	0	2.2	2.6	6.4	96	0.9	2.4	3.8	5.9	0	3.2	IN	IN
6	81	0.2	1.6	3.3	4.8	0	2.7	4.8	5.4	94	0.9	2.4	4.3	6.7	0	4.2	IN	IN	93	0.7	1.9	3.3	4.9	14.9	IN	IN	IN
7	95	0.7	2.4	4.5	6.8	0.1	NA	IN	IN	86	1.1	3.5	6.4	9.4	0	1.5	4.9	5.5	97	1.2	3.4	5.8	8.2	2.8	IN	IN	IN
8	96	1.4	3.8	6.4	9	0	IN	IN	IN	91	1.2	3.6	6	8.5	0	4.6	8.5	IN	88	0.3	1.2	2	3.1	0	0.7	2.4	IN
9	95	1	3.1	5.2	7.9	0	1.5	IN	IN	97	0.9	3.1	5.3	8.1	0	IN	IN	IN	92	1.2	3.3	5.5	7.6	0.6	2	IN	IN
10	87	1.3	4.2	7.3	10	0	0.4	2.3	5.1	98	1.8	4.4	6.9	9.3	0	IN	IN	IN	90	0.5	2	4.1	6.1	0.5	2	NA	IN
11	95	0.9	2.2	3.8	5.8	0	1.5	IN	IN	87	0.6	1.5	2.8	4.2	0	0.5	1.3	1.9	99	1.5	3.9	6.7	9.3	IN	IN	IN	IN
12	98	1	3.7	6.4	9.1	0	IN	IN	IN	91	1.2	3	5.3	7.4	NA	2.7	NA	IN	95	0.7	2.1	4.3	6.7	NA	IN	IN	IN
13	90	0.8	2.4	4	6	NA	NA	NA	IN	97	1.2	3.2	5.8	8.7	0	IN	IN	IN	97	1.1	3.2	5.3	7.7	NA	IN	IN	IN
14	86	1.4	3.4	5.7	8.1	0	4.2	6.8	8	86	0.7	2	3.7	5.7	0	0.7	1.5	3.2	87	0.4	2	4.3	6.9	NA	NA	NA	NA
15	89	1	2.6	4.5	6.6	0	NA	NA	IN	88	1.1	3	4.9	6.9	0	3	3.4	5.8	81	0.5	1.9	3.6	5.3	NA	NA	NA	NA
16	94	1.1	3.3	5.7	8.2	0.5	3.7	IN	IN	99	2	5	8.1	11	IN	IN	IN	IN	90	0.8	2.9	5.1	7.7	NA	NA	NA	IN
17	95	0.4	1.9	4.5	7	NA	NA	IN	IN	89	1.2	3.1	5.3	7.5	NA	NA	NA	NA	99	1.6	4.4	7.2	10	NA	IN	IN	IN
18	93	1.3	3.4	5.8	8.5	0.1	2.8	IN	IN	92	1.1	3	5.1	7.4	NA	NA	NA	IN	94	0.5	1.7	3.7	5.9	NA	NA	IN	IN
19	92	0.9	2.1	3.8	5.8	NA	NA	NA	IN	97	1.2	3.3	5.6	8	0	IN	IN	IN	94	0.8	2.3	4	5.9	NA	NA	IN	IN
20	98	1.7	4.4	7.2	10	NA	IN	IN	IN	95	1.4	3.6	5.9	8.4	NA	IN	IN	IN	93	0.6	1.8	3.3	5.3	NA	NA	IN	IN
21	91	1.3	3.6	6.1	8.6	NA	4.7	NA	IN	90	1	2.5	4.4	6.5	NA	NA	NA	IN	97	0.5	2.3	4.4	6.6	NA	IN	IN	IN
22	97	1.2	3.1	5.4	7.7	NA	IN	IN	IN	93	1.6	3.9	6.4	9.1	NA	NA	IN	IN	97	0.8	2.4	5	7.7	0.5	IN	IN	IN
23	93	1.1	3.3	5.6	8.1	NA	NA	IN	IN	93	1.4	3.7	6.2	8.9	NA	NA	IN	IN	97	0.9	2.5	5.1	7.9	NA	IN	IN	IN
24	94	1.2	3.5	5.9	8.4	NA	NA	IN	IN										91	0	0.9	2.9	5	0	0.4	0.6	IN
25	97	1.3	3.8	6.3	8.9	NA	IN	IN	IN																		
26	93	0.8	2.5	4.5	6.8	NA	NA	IN	IN																		
27	94	0.7	2.5	4.4	6.6	NA	NA	IN	IN																		

demand, then the robust solution for this demand can be significantly more expensive. Therefore, the distribution of the demand across the network plays a key role in how the unused capacity is distributed in an optimal solution and its impact on the success of a robust solution. The instances in the Augerat *et al.* (1995) suite of problems suggest that when the network is clustered, optimal solutions can have vehicles close by which could share unused capacity at a low cost.

4.3. *Robust and deterministic approaches to a family of clustered instances*

To validate our findings and to generalize them with respect to the structure of the network, we randomly generate instances with four vehicles of capacity 1500 and 49 customers with a uniform demand of 100, in three different problem sets. In each set, there are four clusters of customers. First of all, we consider points which are on the circle of a given radius R, centered at a depot, and we randomly select a point on that circle to be the center of a cluster (see Fig. 1). Then we generate customers for that cluster within the circle with a given radius r. We also use a measure for clustering for

our instances, θ, which is given by: $\theta = (R/r)$. We fix the value of $r = 20$ and consider the following values for R: 0, $2r$, $4r$, $6r$ and $8r$.

When $\theta = 0$ all the clusters are centered at the depot and the instance becomes random with no clustering effect; however, as θ increases, clusters separate from each other.
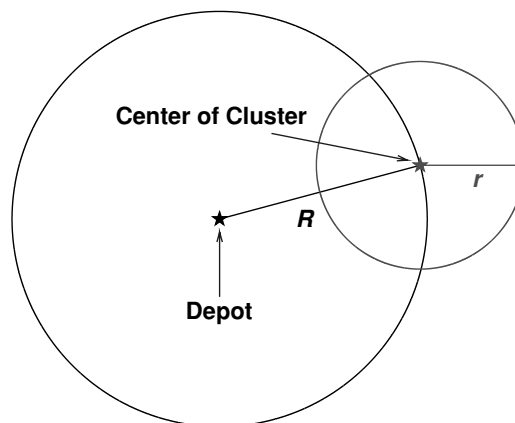

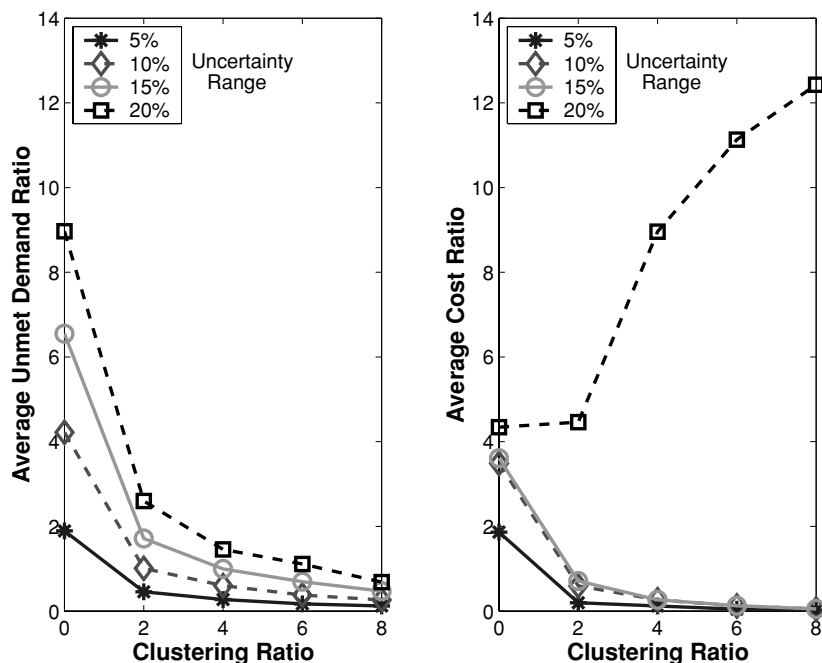**Fig. 1.** Cluster generation in random sets.

Fig. 2. Comparison of deterministic and robust solutions for random set 1.

In problem sets 1 and 2, the three clusters have 13 customers and the fourth one has ten. Note that a vehicle can service up to 15 customers. The reason for this selection is that, in clustered instances where each cluster will be serviced by only one vehicle, there will be one vehicle with relatively more slack, namely the one servicing the fourth cluster. The only difference between sets 1 and 2 is that in the latter as we increase θ, we always keep the fourth cluster centered at the depot. This serves the purpose of having a random zone around the depot and some clusters far from the depot. In set 3, we make the random zone denser by increasing the number of customers in the fourth cluster to 25 and decreasing the one for the others to eight.

Figures 2, 3 and 4 display the results of the three sets for percent unmet demand ratio δ and percent cost ratio κ as a
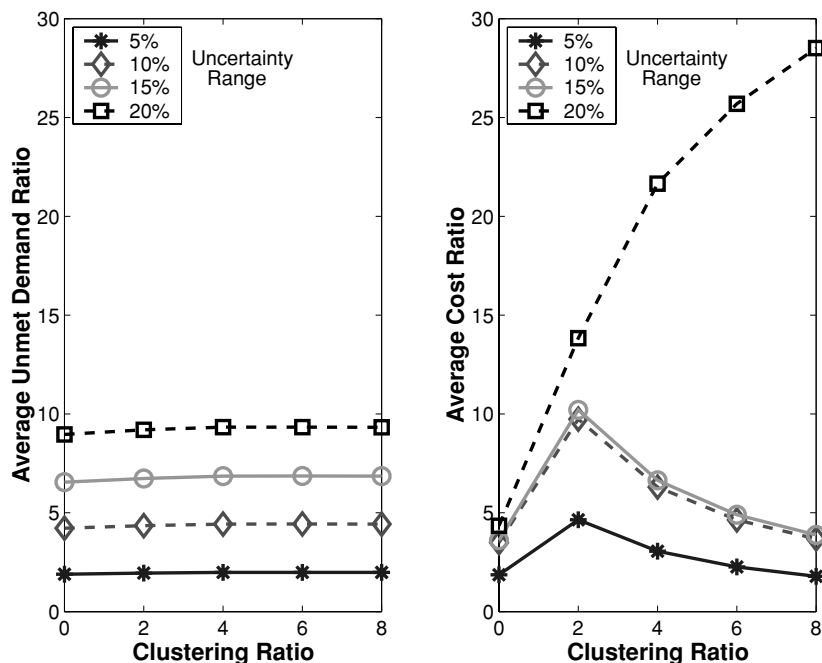

Fig. 3. Comparison of deterministic and robust solutions for random set 2.
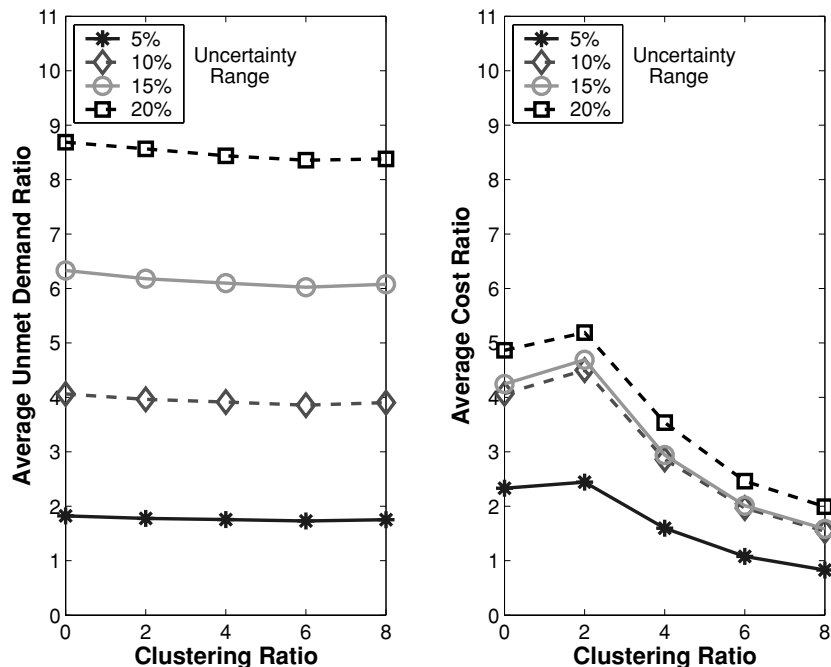
Fig. 4. Comparison of deterministic and robust solutions for random set 3.

function of percent clustering ratio θ for different values of percent deviations of the uncertainty set. Each data point on the figures is an average of 30 instances. We also noticed that for these three sets the robust solutions are significantly different from the deterministic solution in most cases. The average fraction of arcs that are different between the robust and deterministic solutions varies from 3 to 33% for set 1, from 20 to 34% for set 2 and from 24 to 33% for set 3, depending on the range of the demand uncertainty.

The results of set 1 suggest that both the deterministic and the robust solutions benefit from clustering. For percent deviation of 5 and 10%, both results are comparable, for 15% the robust is better, and for 20% the robust is worse when $\theta \geq 2$. In fact as the uncertainty increases, we would expect the robust approach to outperform the deterministic approach. The reason for this odd behavior is the distribution of the slack in the network. When the instances are clustered for the bigger values of θ, each cluster is serviced by only one vehicle in the deterministic approach. In the case of a high uncertainty such as a 20% deviation, if the total demand of a cluster exceeds the vehicle capacity then another vehicle has to be routed to this cluster by the robust approach. When these vehicles are not close, the robust approach results in a large travel cost. The network structures with pure clusters as in set 1 therefore do not allow a good distribution of slack on the average and the robust approach is not convenient for high uncertainties.

When we look at the results of set 2, as before we see the same phenomenon in the increase of the cost ratio for the robust approach with a 20% deviation. However, clustering helps only after $\theta > 2$. The reason for this behavior is due to the random zone around the depot. When $\theta \leq 2$, the circles of clusters intersect and the vehicles do not necessarily service only customers in the same cluster. This interaction of customers keeps the instance as random until $\theta > 2$ since from that point onwards the three clusters become more distinct than the fourth one around the depot and the effect of clustering gets more pronounced in this instance. Increasing θ until it reaches a value of two only makes the size of the network enclosing all the customers bigger, and therefore the cost of the robust approach increases for these bigger random instances. When it comes to the amount of unmet demand of the deterministic approach, the effect of the random zone is more drastic. No matter how much the network is clustered, the unmet demand is always constant and much worse compared to set 1. The vehicles in the deterministic approach service customers in the random zone on their way to the clusters and usually three out of four vehicles are filled to capacity, which is not the case in the deterministic approach in set 1. These vehicles with full capacity are the minimum cost solutions but they have a significant potential to incur unmet demand under uncertainty. The network structures with a scattered random demand zone around the depot, as in set 2, therefore have a very negative effect on the deterministic solution.

When the random zone is denser around the depot, the results of set 3 are similar to the results of set 2. The deterministic approach results in high unmet demand values and is outperformed by the robust approach in almost all cases. Having more customers in the random zone helps the robust approach even more. The reason why the phenomenon with 20% uncertainty disappears is due to the fact that the
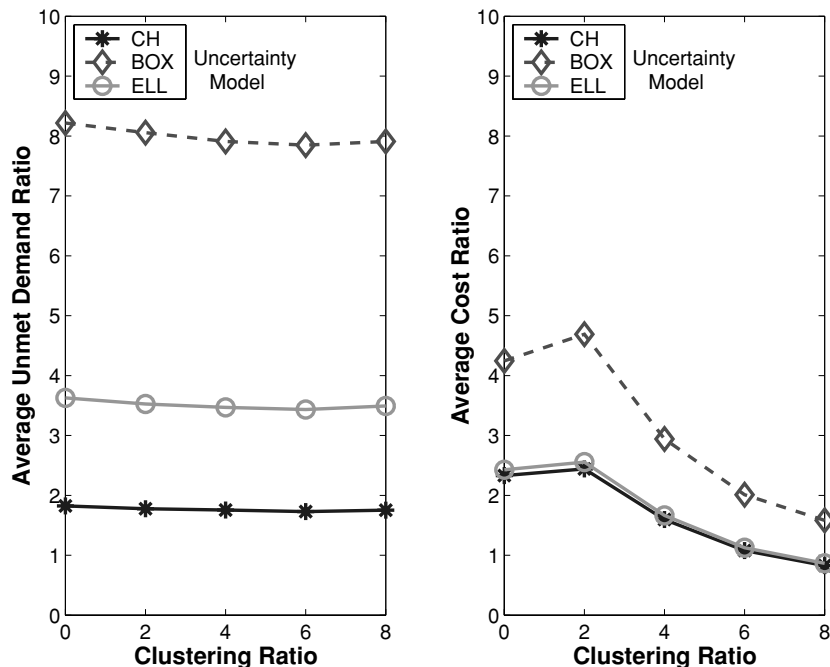
Fig. 5. Comparison of three uncertainty models for random set 3.

vehicles are close now since the number of customers they service in the random zone on their way to the clusters is significantly larger compared to set 2. Therefore, when the slack in one vehicle needs to be distributed to the network, this can be achieved through customers in the random zone. The network structures with clusters and a dense random zone around the depot, as in set 3, therefore allow a good distribution of slack on the average and the robust solution benefits from this at little extra cost.

Lastly, Fig. 5 depicts results for the three uncertainty models (convex hull, box and ellipsoidal) on random set 3 with a 5% deviation in demand. We observe that box uncertainty results in the most unmet demand and cost whereas the convex hull uncertainty results in the lowest values. In addition all models behave with similar trends. Indeed the results obtained from the ellipsoidal and box uncertainty sets follow the same trends as the results presented for the convex hull uncertainty, although with higher values of unmet demand and cost. We therefore omit additional results on these other two uncertainty sets in the interest of space conservation.

We conclude this experimental subsection by emphasizing that our findings in the instances by Augerat *et al.* (1995) are confirmed by a larger class of random instances from the population of instances with the same characteristics. In particular, we showed that both the existence of enough slack in the solution and its distribution over the network are very important factors affecting the quality of the robust. Our experiments reveal that clustered network structures with a dense random zone around the depot favor the robust. For this scenario, we showed that the deterministic

could result in a large amount of unmet demand and the extra cost of the robust approach is relatively small.

4.4. *Robust versus recourse and chance constrained models*

In this section, we compare the robust solution with solutions obtained from different stochastic VRP models: chance constrained and stochastic with recourse models. This comparison will highlight the different assumptions and uncertainty models of each approach. Which is the most suitable model for a given application is a complicated question that has to be addressed in the context of the application. Here, we develop the stochastic models based on the uncertainty model considered for the robust problem. To measure the quality of the solutions, we adapt the percent unmet demand ratio δ and the percent cost ratio κ to $\delta' = \gamma_s / \sum_{i \in V} d_i^0$ and $\kappa' = (z_r - z_s)/z_d$, where γ_s is the unmet demand and z_s the optimal objective value of the stochastic model, either chance constrained or recourse.

The chance constrained formulation replaces constraint (6) in Problem (CVRP) with a probabilistic constraint that can be violated with probability α:

$$P\{u_j - u_i + C(1 - x_{ij}) \geq d_j\} \geq 1 - \alpha, \quad i, j \in V \setminus \{0\}, i \neq j.$$

For a general distribution $G(\mu_j, \sigma_j)$ for demand parameter d_j, this is equivalent to

$$u_j - u_i + C(1 - x_{ij}) \geq k_j^\alpha, \qquad i, j \in V \setminus \{0\}, i \neq j,$$

where the constant $k_j^\alpha = \mu_j + z_\alpha \sigma_j$ in which z_α is the αth percentile of the cumulative distribution of d_j. Thus, this chance constrained model is an instance of a CVRP with

modified demand data similar to the robust model. The difference is in how the demand value is determined. We assume that each demand parameter d_j follows the same uniform distribution $U(a, b)$ that was used to generate the scenarios for the robust model. This gives:

$$k_j^\alpha = \frac{a+b}{2} + (b-a)\left(\frac{1}{2} - \alpha\right).$$

We note that the distribution assumptions and confidence level α for the chance constrained model and the uncertainty set assumptions for the robust model, influence which model considers higher levels of demand and how it is distributed among the nodes. Therefore, depending on the uncertainty assumptions used either of these two models can be the most difficult to satisfy.

In the recourse formulation, we use s scenarios to model the demand uncertainty as in the robust model. In each scenario we consider the recourse action of replenishing at the depot to resume the route. This recourse action is executed by visiting the depot after visiting customer i and before visiting customer j by following arcs $(i, 0)$ and $(0, j)$ instead of following the arc (i, j) in the pre-planned route. We assume that the demand data become available at the beginning of each day (scenario) and that the demands of a customer must be delivered as a single batch. As a result, a vehicle can replenish at the optimum location, not necessarily at the location the capacity is exceeded, and the recourse action does not allow visiting a customer more than once. A vehicle can replenish as many times as necessary along the route. The recourse model is obtained by introducing into Problem (CVRP) additional flow vari-

ables u_i^k for each scenario k and binary recourse variables r_{ij}^k to indicate the arc (i, j) in the pre-planned route for which the recourse action is taken in scenario k. These variables are integrated into Problem (CVRP) with the constraints:

$$d_j^k + u_i^k \le u_j^k + C\left(1 + r_{ij}^k - x_{ij}\right), \ i, j \in V \setminus \{0\}, i \ne j,$$
$$k = 1 \ldots s,$$
$$r_{ij}^k \le x_{ij}, \qquad\qquad i, j \in V, i \ne j, k = 1 \ldots s,$$
$$d_i^k \le u_i^k \le C, \qquad\qquad i \in V \setminus \{0\}, k = 1 \ldots s,$$

and the objective function:

$$\min s \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij} + \sum_{i \in V \setminus \{0\}} \sum_{j \in V \setminus \{0\}} \sum_{k=1}^s r_{ij}^k (c_{i0} + c_{0j} - c_{ij}).$$

Note that the value of z_s in κ' for this recourse model is obtained by dividing this objective value with s to make it comparable to the cost of the robust solution.

Thus, this recourse model results in zero unmet demand when the problem is feasible, just as in the robust model. Moreover, its cost (average cost per scenario) is no worse than the cost of the robust solution since the robust solution is feasible for the recourse model. As a result, this particular recourse model is no worse than the robust model in terms of unmet demand and average cost per scenario; however, the size of the problem increases with the number of scenarios, requiring specialized solution procedures. The results in Fig. 6 are for the convex hull uncertainty set with different ranges of demand uncertainty on random set 1. We use five scenarios as before for the recourse model and we use 10% for the value of alpha in the chance model,
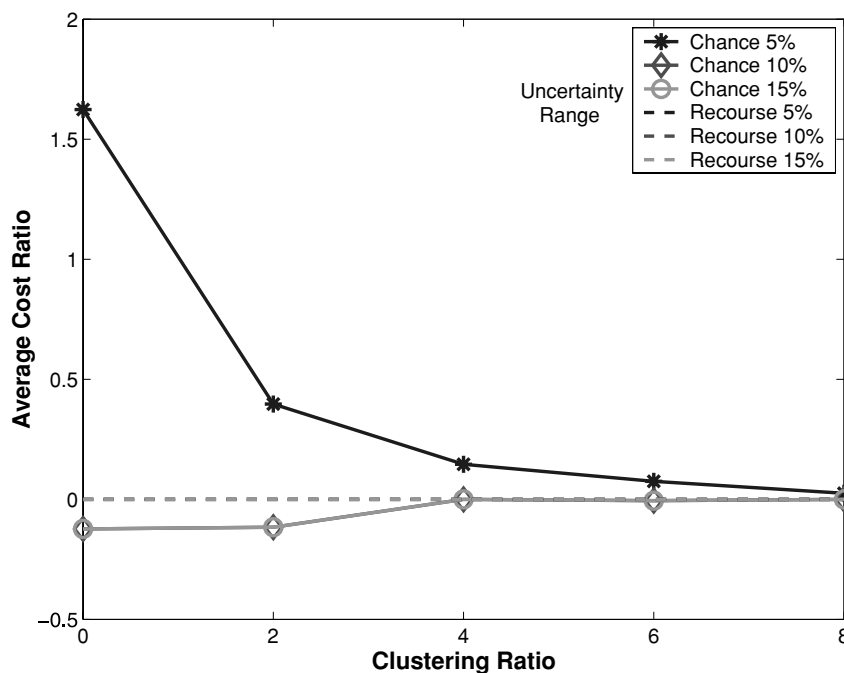

Fig. 6. Comparison of stochastic solutions (chance and recourse) and robust solution.

i.e., $\alpha = 0.1$. Both models result in no unmet demand in all of the cases; therefore the graph for the percent unmet demand ratio is omitted. We were able to obtain optimal solutions for the chance model using SYMPHONY since this model is an instance of CVRP. However, the recourse model is not a standard CVRP and because of its size a special solution procedure is necessary to solve large instances. Instead, we used CPLEX to solve this model but reduced the size of the instances to be able to obtain optimal solutions within an hour. The instances used in the comparison with recourse are small versions of the random set 1 problems. They consider a total of 12 customers in four clusters (three customers in each cluster) with a uniform demand of 100; and there is a total of four vehicles with an identical capacity of 355. Note that the recourse model has sn^2 additional integer variables, sn additional continuous variables and sn^2 additional constraints.

Figure 6 shows that for purely clustered instances the chance constrained model with $\alpha = 0.1$ performs better than the robust model when uncertainty is low (5%), as it can satisfy all the demand at a smaller cost. However, as the uncertainty becomes more pronounced (10 and 15%), the chance constrained model becomes more costly than the robust model. In fact for 20% uncertainty, the chance constrained model considers such high demand values that it results in infeasible instances. The recourse model satisfies all the demand at the same cost as the robust model for 5, 10 and 15% uncertainty. However, the recourse model is much more efficient for the 20% uncertainty, because the robust routing sends vehicles between distant clusters to satisfy all possible demands. We omit the results for 20% uncertainty in Fig. 6 because the chance constrained model could not be solved and the recourse comparison shows the same trend as in Fig. 2 because the robust solution is inefficient. This positive result for recourse models should be tempered with the fact that here we can only solve instances with 13 nodes, as opposed to 50 for the other models. We also observed that the number of recourse actions taken in a solution decreases as the clustering ratio increases, because it is less costly to replenish at the depot when the clusters are not far from the depot.

4.5. *Robust versus distributions of excess vehicle capacity*

The robust solution distributes the excess vehicle capacity in the expected demand case aiming to obtain routes at minimum cost that satisfy all demand outcomes from the uncertainty set. In this section we explore how this compares to two simple strategies of distributing this excess capacity among all the vehicles: uniformly and non-uniformly.

We randomly generate instances with four vehicles of capacity 2100 and 68 customers with a uniform expected demand of 100. Therefore, there is a total of 1600 units of excess vehicle capacity to be used to address the demand uncertainty. We generate these instances according to the three sets as before. The uniform distribution of excess capacity
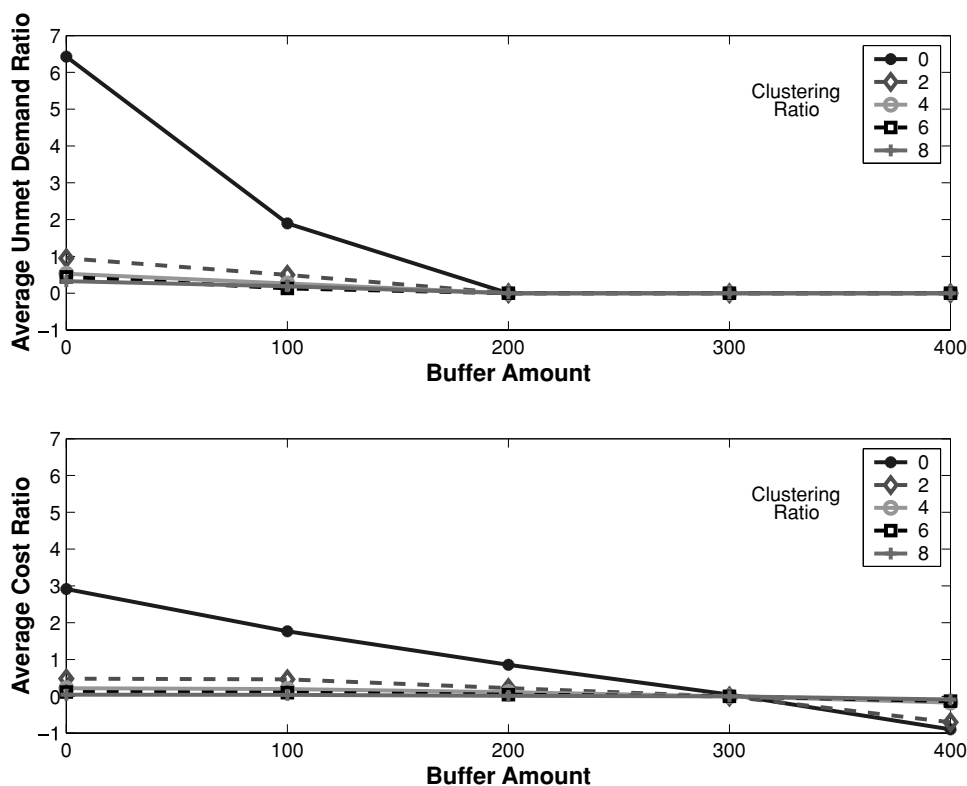

Fig. 7. Comparison of uniform buffer capacity and robust solutions for random set 1.

reserves a buffer of zero, 100, 200, 300 and 400 units of capacity in each vehicle. The distribution of any remaining buffer capacity is automatically determined by the solution procedure to minimize the total cost of the deterministic solution based on the particular instance. That is, the uniform buffer amount (excess capacity) is removed from consideration to compute the optimal deterministic solution, however, it is considered when determining the amount of unmet demand that this optimal solution can face in its worst case. On the other hand, the non-uniform distribution of excess capacity uses the same settings but allows assigning the predetermined reserve of one vehicle to another, that is the uniform buffer amount of one vehicle can be doubled at the expense of not having a predetermined reserve in another vehicle. Since there are four vehicles in total, at most two of them can have a doubled predetermined reserve. This assignment, as well as the distribution of the remaining amount of buffer, are again automatically determined by the solution procedure that minimizes the cost.

We compare the solutions with the generalized percent unmet demand ratio $\delta'' = \gamma_b / \sum_{i \in V} d_i^0$ and the generalized percent cost ratio $\kappa'' = (z_r - z_b)/z_d$ where z_b is the optimal objective value for the deterministic solution with reduced vehicle capacity (buffer capacity) and γ_b is its unmet demand under the worst case (when the buffer capacity is removed). Figures 7 and 8 display respectively for the uniform and non-uniform distributions the average results over 30 instances for different values of percent clustering ratio θ

for set 1 with 15% deviation in the uncertainty scenarios. Similar trends are observed in the other randomly generated sets and percent deviation values for our three types of uncertainty sets. In Fig. 7, for a given value of percent clustering ratio, it is clear that increasing the buffer amount makes a uniform distribution of slack have less unmet demand but with an increased cost which may exceed the cost of the robust solution in some cases, giving negative values for the percent cost ratio κ''. When we compare the quality of the two solutions, we see that when the buffer amount is smaller than 200, this reserve capacity is insufficient to handle the uncertain demand. When the buffer amount is equal to 200, the uniform distribution of slack leads to a less costly solution than the robust model with the same zero unmet demand. When the buffer amount is equal to 300, the two methods have the same cost with the same zero unmet demand. After this transition point (when the buffer amount is greater than 300), if we increase the buffer capacity unnecessarily, the resulting solution is more costly than the robust approach and is not preferable. These trends become less pronounced as the percent clustering ratio increases. That is, clustering is good for a uniform distribution of slack, which makes sense since such an even distribution of slack benefits by having each vehicle assigned to distinct, far away clusters with the same demand and uncertainty, as in the case of $\theta = 8$.

In Fig. 8, the first observation is that there is a minimal increase in the cost ratios compared to the uniform case,
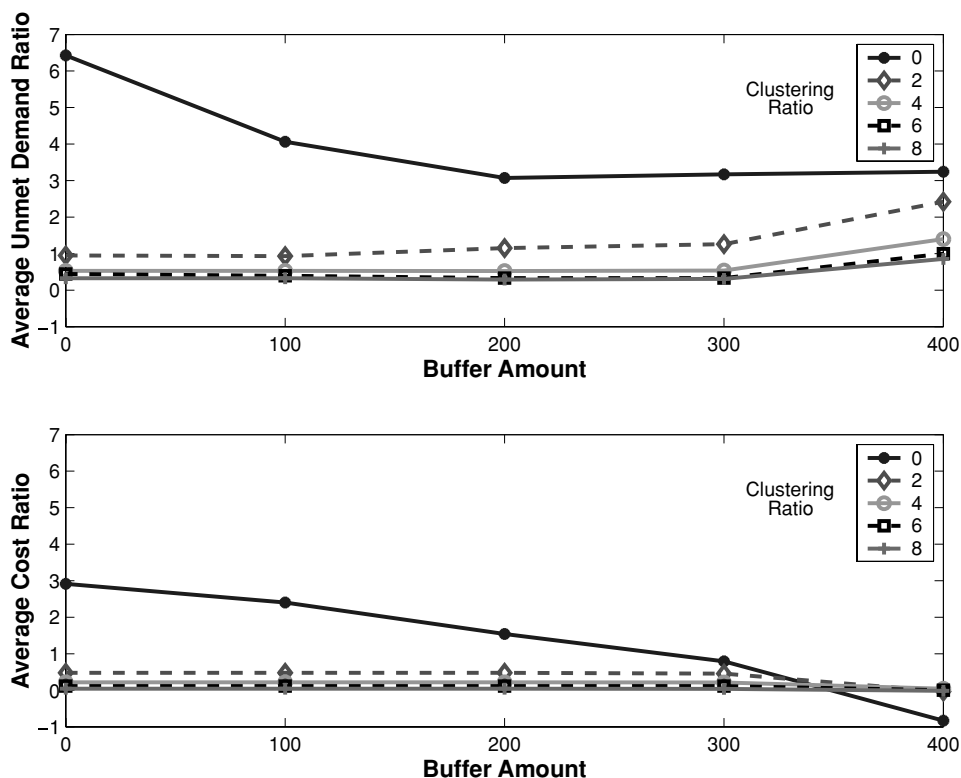

Fig. 8. Comparison of non-uniform buffer capacity and robust solutions for random set 1.

which is expected since a non-uniform distribution explores a wider set of feasible solutions by allowing non-uniform patterns in the solution. The second observation is that the demand ratios are also higher than the uniform case. This is in fact the price of having a less costly deterministic solution for the non-uniform case. Due to particular assignments of the predetermined buffer amount among vehicles, non-uniform distribution creates less costly routes where some of the vehicles have a doubled predetermined buffer capacity whereas some others have no predetermined buffer capacity which in return may result in unmet demand. This effect of having an uneven distribution of predetermined buffer capacity is much more pronounced in the case of 400 units of buffer capacity since all the available 1600 units of extra capacity are allocated to vehicles by this non-uniform pattern. That is, when the solution procedure chooses to double the buffer capacity of a vehicle, another vehicle will have exactly zero slack in the solution. Overall, the non-uniform strategy does not outperform the robust or the uniform distribution although it has a lower cost than the uniform distribution.

5. Conclusions

In this study, we propose the use of robust optimization to obtain efficient routing solutions for problems under uncertainty. Our work has shown that robust optimization is an attractive alternative for formulating routing problems under uncertainty since it does not require distribution assumptions on the uncertainty or a cumbersome representation through scenarios.

We derived a robust counterpart for the VRPSD which requires the solution of a single CVRP with modified data. This theoretical result was based on using the MTZ formulation of the VRP, however, the resulting problem can be solved with any algorithm for the CVRP. This RVRP model assumes that a solution that is feasible for every possible demand outcome exists. Therefore, a natural extension of this work is to allow for problems with infeasible worst cases.

Our computational results showed that if the network structure allows a strategic distribution of the slack in the vehicles throughout the network in such a way that the vehicles can easily collaborate by sharing their slacks in the case of uncertainty, then the robust solution is favorable on average compared to the deterministic solution. Such a network structure appears in a problem with clustered zones far from the depot with a dense random zone near the depot. In comparing the robust solution against the other stochastic models, we observed that the recourse model satisfies all the demand with the same cost or better on small instances, however, these models are significantly more difficult to solve because of their large problem size and require specialized algorithms. The RVRP is more closely related to the chance constrained model. Indeed, both lead to CVRP problems with modified demand values, which depend on the uncertainty representation. Our computational results show that the chance constrained model can be more or less efficient than the robust model depending on the problem parameters and uncertainty assumptions. We also verified that the robust solution is superior to simple strategies of distributing the excess capacity among all vehicles. We notice that such strategies compete better with the robust solution as the network structure is more clustered. However, future work is still needed to identify the best distribution of the excess capacity in general.

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