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CONVEX BACKORDERS OF A RATIONING INVENTORY POLICY WITH TWO DIFFERENT DEMAND CLASSES

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Abstract. We study the constant critical level policy for fast-moving items of an inventory system facing random demands from two customer classes (high and low priority). We consider a continuous review (Q, r, C) policy with continuously distributed demands. Using the properties of the nondecreasing stationary stochastic demand and the *threshold clearing mechanism* we formulate a convex cost minimization problem to determine the optimal parameters of the critical level policy, which can be optimally solved through KKT conditions. For instances we tested, computational results show that the critical level policy induce a benefit on average 5.9% and 33.5% against the round-up and separate stock policies respectively.

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1. Introduction

Fast moving-items are products that have either a high demand volume, or items with high inventory turnover. Examples include non-perishable food, toiletries, over-the-counter drugs, cleaning supplies, building supplies and office supplies. The distribution channels of these products have been concentrated to large retails chains that require a high level of service in terms of product availability at the supplier's expense. Therefore, many wholesalers segment their customers based on service levels, where the simplest segmentation is to classify customers into two demand classes: (i) high-priority class that corresponds to large retail chains that require high levels of service and; (ii) low-priority class that corresponds to smaller retailers that can be provided with a lower level of service.

An efficient way of providing differentiated service levels is through a *critical level policy*. This policy is an inventory control model for rationing inventory between different classes of customers, where its main application is in inventory systems that must provide differentiated service levels to two or more classes of demand. This policy can be implemented for several ordering and review policies. For example, a traditional (Q, r) model is extended using a critical level policy to a $(Q, r, \mathbf{C})$ inventory model, where Q is the fixed lot size, r is the reorder point and $\mathbf{C} := \{C_1, \ldots, C_{n-1}\}$ denotes a set of critical levels for rationing n classes of demand, *i.e.*, when the on-hand inventory reaches the level C_i fails to satisfy the demand for class i with $i = 1, \ldots, n-1$ [1,5,15,10,12,18].

Keywords. Inventory system, critical level policy, shortage penalty cost, fast moving items, two demand classes.

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Let us now consider the implementation of a critical level policy for fast moving items. For these items, it is usually more convenient and efficient to model the demand over a time period with a continuous distribution, *e.g.*, normal or gamma distributions [2, 13, 14]. To the best of our knowledge, only Escalona and Ordóñez [7] have analyzed the constant critical level policy when the demand volume is large, while previous works only considered the case of discrete demand, in particular Poisson distributed demand, which is the appropriate form in which to model the demand of slow-moving items. Our work is different to Escalona and Ordóñez [7] because we consider a cost optimization problem with differentiated shortage costs.

The objective of this paper is determine the optimal parameters of a continuous review (Q, r, C) policy for fast-moving items when an inventory system faces random demands of two customer classes (high and low priority), and where backorders have different penalty costs for each demand class.

Assuming the critical level policy as the inventory control strategy, and using the properties of the non-decreasing stationary stochastic demand and the *threshold clearing mechanism* to allocate backorders when multiple outstanding orders exist, we obtain convex backorders for each class in steady state. This approach allows us to formulate a convex cost minimization problem to determine the optimal parameters of the critical level policy, which can be solved through a system of equations derived from Karush−Kuhn−Tucker (KKT) conditions (see [3]).

The main contribution of this paper is that we model and solve a critical level model when demand is modeled through continuous distributions and backorders are penalized with differentiated costs. Furthermore, we develop expressions for backorders of each class and we demonstrate its convexity.

The remainder of this paper is structured as follows. A review of related work is discussed in the following section. In Section 3 we describe the context in which the inventory system operates and present the cost optimization problem analyzed in this work. In Section 4 we prove the convexity of the backorders of both class demand. In Section 5 we propose a system of equations to determine the parameters of the critical level policy. We present our numerical experiments to evaluate the performance of the critical level policy with respect to the separate stock and round-up policies in Section 6. Section 7 presents our conclusions and future extensions to this work.

2. Related work

A comprehensive review of inventory rationing can be found in Kleijn and Dekker [9]. These authors classified inventory systems that were subjected to multiple classes of demand based on *review policies* (continuous and periodic) and the *number of classes* (2 or n classes). The above classification was extended by Teunter and Haneveld [17], who incorporated *shortage treatment* (backorder or lost sale), *rationing policies* (no-rationing, static, dynamic), *ordering policies* and the way that time is modeled (discrete or continuous).

In this paper, we classify the inventory rationing problem with a continuous review policy based on the problem type for determining the optimal parameters of the critical level policy, *i.e.*, a cost optimization problem or a service level problem. The first group minimizes the sum of the ordering cost, the holding cost, and depending on the shortage treatment, the backorders cost and/or lost sales costs. The second group minimizes the expected on-hand inventory subject to service level constraints. Depending on the operating conditions defined for the inventory system, what varies is the formulation of the on-hand inventory value and the service level provided to each class. Clearly, some papers may not be assigned to this classification because their aim was not to determine the optimal parameters of the critical level policy, or because they had a hybrid formulation. In this sense, Nahmias and Demmy [12], that were the first to study the continuous review policy with two demand classes, did not determine the optimal parameters of the critical level policy, but instead developed an approximate expression for the expected amount of backorders per cycle for both demand classes when there was at most one outstanding order. They assumed a continuous review (Q, r, C) policy, Poisson demand, full-backorders and a deterministic lead time.

From the literature review conducted by Melchiors *et al.* [10], Isotupa [15], Deshpande *et al.* [5] and Fadiloglu and Bulut [8], these authors used a cost optimization approach to determine the optimal parameters of the critical level policy. Melchiors *et al.* [10] analyzed a (Q, r, C) inventory model that had a deterministic lead time and two demands classes, but unlike Nahmias and Demmy [12], they considered a lost sales environment. They assumed a Poisson demand and used the *hitting time* and renewal theory to operationally characterize the inventory system. Isotupa [15] presented a model with the same assumptions as Melchiors *et al.* [10] but with an exponentially distributed lead time. Deshpande *et al.* [5] analyzed the same rationing model as Nahmias and Demmy [12], but without restricting the number of outstanding orders. They derived expressions for the average backorders per cycle, and for the expected steady-state on-hand inventory and backorders. Based on these expressions, Deshpande *et al.* [5] proposed a cost optimization model and developed algorithms to compute the optimal parameters of the critical level policy. Fadiloglu and Bulut [8] examined a dynamic rationing policy with a continuous review (Q, r) inventory model that had a Poisson demand and a deterministic lead time. The authors used simulation-based approaches to find efficient solutions to cases with backordering and lost sales.

From the literature review conducted by Dekker *et al.* [4], Arslan *et al.* [1],Wang *et al.* [19], Möllering and Thonemann [11] and Escalona and Ordóñez [7], these authors used a service level problem approach to determine the optimal parameters of the critical level policy. Dekker *et al.* [4] analyzed the critical level policy when the inventory system worked under a continuous review of lot-for-lot policy, lost sales and Poisson demand. These authors derived expressions for the fill-rate, and presented an efficient method to obtain optimal solutions. Möllering and Thonemann [11] analyzed a periodic review base-stock policy with two demand classes, a deterministic lead time, discrete demand distribution and full backorders. Their work modeled the inventory system as a multidimensional Markov chain and optimally solved a service level problem based on a service level of type 1, and another on the fill-rate. Wang *et al.* [19] analyzed the same model as Möllering and Thonemann [11], but considered an anticipated rationing policy. This policy reserved inventory for the high-priority classes by considering a constant critical level, and incoming replenishment for the next period. Arslan *et al.* [1] presented a service level model to obtain the optimal parameters of a critical level policy with multiple demand classes under the assumptions of Poisson demand, deterministic lead times, and a continuous-review (Q, r) policy. Escalona and Ordóñez [7] analyzed the constant critical level policy for fast-moving items and two demand classes. The inventory system operates under a continuous review (Q, r) policy with a type I service level, full-backordering, deterministic lead times and a continuous demand distribution. They formulated the service-level problem as a nonlinear problem with chance constraints for which they optimally solved a relaxation, and obtained a closed form solution that can be easily computed.

A hybrid formulation is the work of Wang *et al.* [18], who analyzed the rationing policy under the same operational conditions as Deshpande *et al.* [5], but they considered a mixed service criteria with penalty costs and service level constraints (fill-rate).

When implementing a continuous review (Q, r, C) policy with full-backordering, it may happen that the incoming replenishment batch is not large enough to cover the backorders. Therefore, it is important how the backorders of the different classes are satisfied. This policy is difficult to analyze mathematically and given its complexity the literature has focused on manageable, but sub-optimal, rules, *e.g.*, the *threshold clearing mechanism* from Deshpande *et al.* [5] and the *FCFS type clearing scheme* from Arslan *et al.* [1].

In summary, only Escalona and Ordóñez [7] analyzed the constant critical level policy for fast-moving demand, but unlike them, we consider a cost optimization problem with differentiated shortage costs for each demand class.

3. Model framework

Consider a facility that holds inventory of a single type of product to serve two demand classes $i = 1, 2$, where class 1 is high priority and class 2 is low priority. Let $D_i(t, t + \tau)$ be the total demand of class i in the interval $(t, t + \tau]$, and $D(t, t + \tau) = D_1(t, t + \tau) + D_2(t, t + \tau)$ the total demand of both classes in the interval $(t, t + \tau]$. We denote by $F_{D_i(\tau)}(x)$ the cumulative distribution function of the total demand of class i in $[0, \tau]$ and $F_{D(\tau)}(x)$ the cumulative distribution function of the total demand of both classes in $[0, \tau]$.

In this paper we consider fast-moving items for which is more representative and efficient to model the demand over a time period by a continuous distribution. Following Zheng [20] we assume that the total demand of each class are represented by a nondecreasing stochastic process with stationary increments and continuous sample paths. For simplicity, we will refer to this as *strictly increasing non-negative demand*. Note that under stationary increments, $D_i(\tau) := D_i(0,\tau) = D_i(t, t+\tau)$ for any $t \geq 0$, $i = 1, 2$.

Inventory is replenished according to a continuous review (Q, r, C) policy that operates as follows. When the inventory position (=inventory on hand + outstanding orders − backorders, [2]) falls below a reorder level r, a replenishment order for Q units is placed and arrives a fixed $L > 0$ time units later. Demand from both classes are filled as long as the on-hand inventory level is greater than the critical level C, otherwise only high priority demand is satisfied from inventory on-hand and low priority demand is backordered. If on-hand inventory level reaches zero both demands are backordered. To clear backlogged orders, we consider the threshold clearing mechanism of Deshpande *et al.* [5].

Given the inventory control strategy, our objective is to find the parameters of the critical level policy that minimize the sum of the ordering cost, the holding cost and shortage costs. Let $AC(Q, r, C)$ be the average cost per unit time:

$$AC(Q,r,C) = S\frac{\mu}{Q} + h\mathbb{E}(OH^{\infty}(Q,r,C)) + b_1\mathbb{E}(B_1^{\infty}(Q,r,C)) + b_2\mathbb{E}(B_2^{\infty}(Q,r,C)), \tag{3.1}$$

where μ is the average demand per unit of time; b_i is the shortage cost per unit and per unit time of class i with $b_1 > b_2 > 0$; h is the holding cost per unit and per unit time; S is the ordering cost; $\mathbb{E}(OH^{\infty}(Q, r, C))$ is the steady-state on-hand inventory; and $\mathbb{E}(B_i^{\infty}(Q, r, C))$ is the class i steady-state backorder, $i = 1, 2$.

In a (Q, r, C) policy with full-backorders and deterministic lead time, the inventory level is the on-hand inventory net of all backorders (Deshpande *et al.* [5]), *i.e.*, $IL(t + L) = OH(t + L) - B_1(t + L) - B_2(t + L)$, where $IL(t+L)$ denotes the inventory level, $OH(t+L)$ denotes on-hand inventory and $B_i(t+L)$ denotes class i backorders, $i = 1, 2$, all at time $t + L$. Furthermore, for a (Q, r, C) policy with full-backorders and deterministic lead time it is still valid that $IL(t + L) = IP(t) - D(L)$, where $IP(t)$ denotes the inventory position at time t (Deshpande *et al.* [5]). Then, the on-hand inventory at time $t + L$ is $OH(t + L) = IP(t) - D(L) + B_1(t + L) + B_2(t + L)$, and taking the expected value and considering the limit $t \to \infty$, the expected on-hand steady-state inventory is:

$$\mathbb{E}(OH^{\infty}(Q,r,C)) = \frac{Q}{2} + r - \mu L + \mathbb{E}(B_1^{\infty}(Q,r,C)) + \mathbb{E}(B_2^{\infty}(Q,r,C)). \tag{3.2}$$

Equation (3.2) is valid as long as the inventory position in steady state is uniformly distributed on $(r, r + Q]$ and independent of the lead time demand $D(L)$. These conditions are met for strictly increasing non-negative demand [20, 16].

3.1. Steady state backorders

In this section we develop expressions for backorders of the low and high priority classes in the steady state by using properties of the *strictly increasing non-negative demand,* the inventory position and the threshold clearing mechanism. We first describe how the inventory system behaves under rationing, and the threshold clearing mechanism of Deshpande *et al.* [5].

Consider an arbitrary time $t + L$. By definition, there is rationing at time $t + L$ when $C > OH(t + L) \geq IL(t + L) = IP(t) - D(t, t + L)$. Under rationing conditions at $t + L$, let t_c be the first time after t when $IP(t) - C$ demand is observed. The threshold clearing mechanism of Deshpande *et al.* [5] only comes into play when backorders exist on arrival of a replenishment order and uses t_c to separate which backorders need to be cleared once the replenishment order arrives. The general rules to clear the backorders when the replenishment order arrives are:

1. If the entering replenishment batch is large enough to clear all the backorders and leave the on-hand inventory level above C, then clear all backorders.

2. Otherwise:
   2.1. Clear all backlogged demand that arrived before t_c in the order of arrival (FCFS);
   2.2. Clear any remaining backlogged class 1 demands using FCFS until either all class 1 backorders are filled, or no on-hand inventory remains;
   2.3. Carry over (*i.e.* continue backlog) all class 2 demands that arrive after t_c.

Note that rule 1 ensures that $OH(t) = IL(t)$ when $OH(t) \geq C$. Rules 2.2 and 2.3 mean that all remaining backorders that cannot be fulfilled by the entering replenishment batch, are carried over to be satisfied in the following replenishment arrivals. Then, using the threshold clearing mechanism, the backorders of the low priority class at time $t+L$ is the total demand of class 2 in the interval $(t_c, t+L]$ if the inventory level at $t+L$ is below the critical level C, *i.e.*,

$$B_2(t+L) = \begin{cases} D_2(t_c, t+L) & \text{if} \quad IP(t) - D(t, t+L) < C \\ 0 \quad \sim & , \end{cases} \tag{3.3}$$

where $IL(t+L) = IP(t) - D(t, t+L)$; and the backorders of the high priority class at time $t+L$ is the positive part of the total demand of class 1 in the interval $(t_c, t+L]$ less C, *i.e.*,

$$B_1(t+L) = \left[D_1(t_c, t+L) - C\right]^+ . \tag{3.4}$$

The inventory position $IP(t)$ does not provide enough information to determine the backorders in the steady state [5]. To address this lack of information, Escalona and Ordóñez [7] proposed to use the *hitting time* approach to characterize the inventory system. They obtain exact expressions of backorders of both low- and high-priority class at steady state under the threshold clearing mechanism and strictly increasing non negative demand. In our paper, we propose to use an intuitive relationship to address the lack of information and leads to defining convex expressions of the backorders at steady state. Our relationship is intuitive, based on the fact that demand class i during $[t_c, t+L]$ is proportional to the total demand for both classes during this period, *i.e.*,

$$D_i(t_c, t+L) = k_i(D(t, t+L) - IP(t) + C), \quad \forall\ i = 1, 2, \tag{3.5}$$

where $D(t_c, t+L) = D(t, t+L) - IP(t) + C$, and k_i is the proportionality factor for class i. Note that k_i is a random variable and for any proportionality factor, it must be satisfied that $k_1 + k_2 = 1$. We assume that the proportionality factor k_i is constant for $i = 1, 2$.

Replacing equation (3.5) in equations (3.3)–(3.4) and taking expected value, the expected backorders of class 1 and 2 at time $t+L$ are respectively:

$$\mathbb{E}[B_2(t+L)] = k_2 \mathbb{E}\left[D(L) - IP(t) + C\right]^+ , \tag{3.6}$$

$$\mathbb{E}[B_1(t+L)] = k_1 \mathbb{E}\left[D(L) - IP(t) - C^{'}\right]^+ , \tag{3.7}$$

where $C^{'} = C\left(\frac{1}{k_1} - 1\right)$. Then, conditioning on the inventory position and changing the order of integration, the expected backorders in the steady state of class 1 and 2 are respectively:

$$\mathbb{E}[B_2^{\infty}(Q, r, C)] = \frac{k_2}{Q}\left[\beta(r - C) - \beta(r + Q - C)\right], \tag{3.8}$$

$$\mathbb{E}[B_1^{\infty}(Q, r, C)] = \frac{k_1}{Q}\left[\beta(r + C^{'}) - \beta(r + Q + C^{'})\right], \tag{3.9}$$

where: $\beta(v) = \int_v^{\infty}(x - v)(1 - F_{D(L)}(x))\mathrm{d}x$. Note that equations (3.8)–(3.9) are a standard form to express the backorders used by Zipkin [21].

Proposition 3.1. *Under strictly increasing non-negative demand and the threshold clearing mechanism, equation* (3.8) *and* (3.9) *are exact expression for the expected backorders of class* 2 *and* 1 *at steady state when* $k_2 = \frac{\mu_2}{\mu}$ *and* $k_1 = \frac{\mu_1}{\mu}$ *respectively, where* μ_i *is the average demand per unit time of class* i, $i = 1, 2$ *and* $\mu = \mu_1 + \mu_2$.

Proof. Under strictly increasing non-negative demand and the threshold clearing mechanism, Escalona and Ordóñez [7] developed the following exact expressions for backorders.

$$B_2(t+L) = D_2\left((L - \tau_{H,D}^{IP(t)-C})^+\right)\ , \tag{3.10}$$

$$B_1(t+L) = D_1\left((L - \tau_{H,D}^{IP(t)-C} - \tau_{H,D_1}^{C})^+\right), \tag{3.11}$$

where $\tau_{H,D}^{IP(t)-C} = \inf\{\tau > 0 | D(\tau) > IP(t) - C\}$ is the time (hitting time) required for $IP(t) - C$ demands, and $\tau_{H,D_1}^{C} = \inf\{\tau > 0 \mid D_1(\tau) > C\}$ corresponds to the time required for C demands of class 1.

The equivalence between equations (3.10)–(3.11) and (3.3)–(3.4) are given by the fact that $t_c = t + \tau_{H,D}^{IP(t)-C}$; and $IP(t) - D(L) < C$ is equivalent to $\tau_{H,D}^{IP(t)-C} < L$.

Taking expectation of equation (3.10) we have that:

$$\begin{aligned}
\mathbb{E}(B_2(t+L)) &= \mu_2 \mathbb{E}\left(\max\{L - \tau_{H,D}^{IP(t)-C}, 0\}\right) \\
&= \frac{\mu_2}{Q}\int_r^{r+Q} \mathbb{E}\left(\max\{L - \tau_{H,D}^{y-C}, 0\}\right) \mathrm{d}y \\
&= \frac{1}{Q}\frac{\mu_2}{\mu}\int_r^{r+Q} \mathbb{E}\left(D(\max\{L - \tau_{H,D}^{y-C}, 0\})\right) \mathrm{d}y \\
&= \frac{1}{Q}\frac{\mu_2}{\mu}\int_r^{r+Q} \mathbb{E}\left(D(L - \tau_{H,D}^{y-C}) | \tau_{H,D}^{y-C} < L\right) \mathbb{P}\left(\tau_{H,D}^{y-C} < L\right) \mathrm{d}y \\
&= \frac{1}{Q}\frac{\mu_2}{\mu}\int_r^{r+Q} \mathbb{E}\left(D(L) - y + C | D(L) > y - C\right) \mathbb{P}(D(L) > y - C) \mathrm{d}y \\
&= \frac{\mu_2}{\mu}\mathbb{E}\left(D(L) - IP(t) + C | D(L) > IP(t) - C\right) \mathbb{P}(D(L) > IP(t) - C) \\
&= \frac{\mu_2}{\mu}\mathbb{E}\left(\max\{D(L) - IP(t) + C, 0\}\right) \\
&= \frac{\mu_2}{\mu}\mathbb{E}\left(D(L) - IP(t) + C\right)^+ .
\end{aligned}$$

In the same way, it is easy to show that equation (3.7) is equivalent to (3.11) when $k_1 = \frac{\mu_1}{\mu}$. ☐

Therefore, in what follows, we consider that the proportion between the demand of class i during the interval $[t_c, t + L]$ and total demand in the same interval is constant and equal to: $k_i = \frac{\mu_i}{\mu}$, $i = 1, 2$.

3.2. Cost optimization problem: strictly increasing non-negative demand

Replacing equation (3.2), (3.8) and (3.9) in equation (3.1) we can express the average cost per unit time as:

$$\begin{aligned}
AC(Q, r, C) = {} & S\frac{\mu}{Q} + h\left(\frac{Q}{2} + r - \mu L\right) + (b_1 + h)\frac{k_1}{Q}\left(\beta(r + C') - \beta(r + Q + C')\right) \\
& + (b_2 + h)\frac{k_2}{Q}\left(\beta(r + C') - \beta(r + Q + C')\right).
\end{aligned} \tag{3.12}$$

Our objective is to determine the optimal parameters of the (Q, r, C) policy that minimizes the total cost. Then, our problem for a strictly increasing non-negative demand can be written as a nonlinear optimization problem, denoted (**P0**), as follows.

Problem (**P0**):

$$\underset{Q,r,C}{\text{Min}} \quad AC(Q,r,C) \tag{3.13}$$

$$s.t \quad r \geq C \geq 0 \tag{3.14}$$

$$Q \geq 0 \ , \tag{3.15}$$

where $AC(Q,r,C)$ is given by equation (3.12). Constraint (3.14) ensures that the replenishment order is placed before the lower priority class is no longer served.

3.3. P0 using normal distribution as approximation of non-negative demand

A common practice in stochastic inventory models is to use the normal distribution as an approximation of non-negative demand, *i.e.*, stochastic inventory models are formulated based on the characteristics of non-negative demand and are then implemented using a normal distribution. The problem with the normal distribution is that there is always a small probability for negative demand. The normal distribution is a good approximation of non negative demand when the coefficient of variation is less than or equal to 0.5, *i.e.*, $CV \leq 0.5$ [13].

To solve (**P0**) using normal distribution as approximation of the non-negative demand, the expressions of backorders are required. For this, consider that each class i has identical and independent normally distributed demand per unit time, with mean $\mu_i > 0$ and variance $\sigma_i^2 > 0$, $D_i(\tau) \sim N(\mu_i\tau, \sigma_i^2\tau)$, and $D(\tau) \sim N(\mu\tau, \sigma^2\tau)$, where $\mu = \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$. The expected backorders in the steady state of class 1 and 2 using a normally distributed demand are, respectively:

$$\mathbb{E}(B_2^{\infty}(Q,r,C)) = \frac{\sigma^{'2}}{Q}\ k_2 \left[H\left(\frac{r-C-\mu^{'}}{\sigma^{'}}\right) - H\left(\frac{r+Q-C-\mu^{'}}{\sigma^{'}}\right)\right], \tag{3.16}$$

$$\mathbb{E}(B_1^{\infty}(Q,r,C)) = \frac{\sigma^{'2}}{Q}\ k_1\ \left[H\left(\frac{r+C^{'}-\mu^{'}}{\sigma^{'}}\right) - H\left(\frac{r+Q+C^{'}-\mu^{'}}{\sigma^{'}}\right)\right], \tag{3.17}$$

where: $\mu^{'} = \mu L,\ \ \sigma^{'} = \sigma\sqrt{L}$,

$$H(x) = \int_x^{\infty} G(v)\mathrm{d}v = \frac{1}{2}\left[(x^2+1)(1-\Phi(x)) - x\varphi(x)\right],$$

and

$$G(x) = \int_x^{\infty} (v-x)\varphi(v)\mathrm{d}v = \varphi(x) - x(1-\Phi(x)),$$

is the so-called loss function, $\Phi(x)$ is the distribution function of the standard normal distribution and $\varphi(x)$ is the density function. It is easy to show that equations (3.16)–(3.17) are equivalent to (3.8)–(3.9) because, for normally distributed demand: $\beta(v) = \sigma^{'2}H\left(\frac{v-\mu^{'}}{\sigma^{'}}\right)$ [2].

As $H(x)$ is decreasing and convex [2], it is easy to note that $\mathbb{E}(B_1^{\infty}(Q,r,C))$ is decreasing in r and C, while $\mathbb{E}(B_2^{\infty}(Q,r,C))$ is increasing in C and decreasing in r. The same behavior is described by Deshpande *et al.* [5] for Poisson demand.

Then, the problem (**P0**) using normal distribution as approximation of non-negative demand can be written as follows.

Problem $(\mathbf{P0}+\mathbf{N})$:

$$\begin{aligned}\underset{Q,r,C}{\text{Min}}\quad & S\frac{\mu}{Q}+h\left(\frac{Q}{2}+r-\mu L\right)\\ & +(b_1+h)\frac{\sigma'^2}{Q}\;k_1\;\left[H\left(\frac{r+C'-\mu'}{\sigma'}\right)-H\left(\frac{r+Q+C'-\mu'}{\sigma'}\right)\right]\\ & +(b_2+h)\frac{\sigma'^2}{Q}\;k_2\left[H\left(\frac{r-C-\mu'}{\sigma'}\right)-H\left(\frac{r+Q-C-\mu'}{\sigma'}\right)\right] \qquad (3.18)\\ s.t\quad & (3.14),(3.15).\end{aligned}$$

4. Convexity of the average cost per unit time

Consider the objective function of $(\mathbf{P0})$. Since it is a nonlinear function, finding the optimal parameters of the (Q,r,C) policy is difficult, unless the objective function is convex. Clearly the first and second term of (3.12) are convex, hence the convexity of $AC(Q,r,C)$ will depend on whether the backorders are convex or not.

Proposition 4.1. *Backorders of class* 1 *and class* 2 *defined by equations* (3.8) *and* (3.9) *are strictly convex in* Q,r *and* C*.*

Proof. Consider the backorder in the steady-state for the continuous review (Q,r) policy:

$$\mathbb{E}(B^{\infty}(Q,r))=\frac{1}{Q}\left(\beta(r)-\beta(r+Q)\right). \qquad (4.1)$$

Zipkin [21] has already proved that (4.1) is jointly convex in Q and r when $f_{D(L)}>0$ for any $t>0$, where $f_{D(L)}$ is the density function of the lead time demand.

Equations (3.8)–(3.9) describe the steady-state backorders in terms of Q and a linear combination of r and C. Note that class 1 backorders depend on $r+C'$, and class 2 backorders depend on $r-C$. It is a fact that the composition of a convex function with an affine mapping preserves convexity (see [3]). Thus, $AC(Q,r,C)$ given by (3.12) is jointly convex in Q, r and C and $(\mathbf{P0})$ is a nonlinear convex problem. □

The proposition 4.1 applies also to the case where $F_{D(L)}(x)$ is normal. Therefore, $AC(Q,r,C)$ given by (3.18) is jointly convex in Q, r and C and $(\mathbf{P0}+\mathbf{N})$ is a nonlinear convex problem.

5. Solution approach

For any convex optimization problem with differentiable objective and constraint functions, any points that satisfy KKT conditions are primal and dually optimal, and have zero duality gap [3]. Since $(\mathbf{P0})$ and $(\mathbf{P0}+\mathbf{N})$ are strictly convex optimization problems, our approach to finding a solution will be based on solving the KKT conditions.

Let us consider the optimization problem $(\mathbf{P0})$. A simple analysis of the objective function allows us to relax Q, $C\geq 0$ from $(\mathbf{P0})$ because (i) when $Q\to 0^+$, then $AC(Q,r,C)\to\infty$, so it is not possible that $Q\geq 0$ is active in the optimum; and (ii) $\mathbb{E}(B_1^{\infty}(Q,r,C))$ decreases in C whereas $\mathbb{E}(B_2^{\infty}(Q,r,C))$ increases in C, therefore non-positive values of C would be obtained only if $b_2\geq b_1$ which is a contradiction according to the model framework. The above relaxation applies also to $(\mathbf{P0}+\mathbf{N})$.

Regarding the reorder point, although a negative value of r does not have any practical sense, we cannot disregard the non-negativity constraint of this variable, mostly because there are some situations when this constraint is active, *e.g.*, if the ordering cost S is too high, the optimal lot size will be so large that the optimal reorder point will have to be zero. Thus, we will solve $(\mathbf{P0})$ and $(\mathbf{P0}+\mathbf{N})$ by only considering the $r\geq C$ and $r\geq 0$ constraints, denoted as $(\mathbf{P1})$ and $(\mathbf{P1}+\mathbf{N})$, respectively.

The Lagrangian function of (**P1**) is $\mathcal{L}(Q,r,C,\lambda) = AC(Q,r,C) + \lambda_1(C-r) - \lambda_2 r$, where λ_i is a Lagrangian multiplier. Then, the KKT conditions are defined as follows:

$$\begin{aligned}
\frac{\partial}{\partial r}\mathcal{L}(Q,r,C,\lambda_i) &= \frac{\partial}{\partial r}AC(Q,r,C) - \lambda_1 + \lambda_2 &&= 0\\
\frac{\partial}{\partial C}\mathcal{L}(Q,r,C,\lambda_i) &= \frac{\partial}{\partial C}AC(Q,r,C) + \lambda_1 &&= 0\\
\frac{\partial}{\partial Q}\mathcal{L}(Q,r,C,\lambda_i) &= \frac{\partial}{\partial Q}AC(Q,r,C) &&= 0\\
&& \lambda_1(C-r) &= 0\\
&& \lambda_2(-r) &= 0\\
&& \lambda_i &\geq 0.
\end{aligned}$$

Note that the domain of (**P1**) is defined by linear constraints which make the KKT conditions easier to solve compared to a problem that is subjected to nonlinear constraints (*e.g.*, service level type 1, or the fill-rate). Since these constraints are few and easy to deal with, it is straightforward to define an algorithm in terms of the activation/deactivation of them.

Let (Q^*, r^*, C^*) be the optimal solution of (**P1**) and let (Q^u, r^u, C^u) be the solution of (**P1**), which is unrestricted, *i.e.*, when $\lambda_1 = \lambda_2 = 0$. The solution set is obtained from the following algorithm:

Algorithm 1. Active-constraints algorithm.

Solve the KKT conditions of (**P1**) (unrestricted)
if $r^u > C^u$ **then**
  The optimal solution is (Q^u, r^u, C^u)
else if $r^u > 0$ **then**
  (Q^*, r^*, C^*) is obtained from the KKT conditions by considering $r^* = C^*$ ($\lambda_1 > 0$ and $\lambda_2 = 0$)
else
  (Q^*, r^*, C^*) is obtained from the KKT conditions by considering $r^* = C^* = 0$ ($\lambda_i > 0$)
end if

Algorithm 1 ensures that the optimal solution will be found, however, the complexity of the equation systems derived from the KKT conditions will depend on how the demand process is modeled.

Let us now consider the (**P1 + N**) problem. Before applying Algorithm 1 to find the optimal solution, it is convenient to define:

$$f_i(Q,r,C) = 1 - \frac{\sigma'}{Q}\left[G\left(\frac{a_i-\mu'}{\sigma'}\right) - G\left(\frac{a_i+Q-\mu'}{\sigma'}\right)\right], \tag{5.1}$$

where a_i is a linear combination of r and C, and $r + C'$ and $r - C$ for class 1 and class 2 demands, respectively. We can express the partial derivatives of $AC(Q,r,C)$ with respect to r and C in terms of f_i:

$$\frac{\partial}{\partial r}AC(Q,r,C) = h + k_1(b_1+h)(f_1-1) + k_2(b_2+h)(f_2-1), \tag{5.2}$$

$$\frac{\partial}{\partial C}AC(Q,r,C) = k_2(b_1+h)(f_1-1) + k_2(b_2+h)(1-f_2), \tag{5.3}$$

and the partial derivative of $AC(Q,r,C)$ with respect to Q is given by the following equation:

$$\frac{\partial}{\partial Q}AC(Q,r,C) = \frac{h}{2} - S\frac{\mu}{Q^2} - (b_1+h)\frac{\sigma'^2}{Q^2}k_1\left[H\left(\frac{r+C'-\mu'}{\sigma'}\right) - H\left(\frac{r+C'+Q-\mu'}{\sigma'}\right) - \frac{Q}{\sigma'}G\left(\frac{r+C'+Q-\mu'}{\sigma'}\right)\right] - (b_2+h)\frac{\sigma'^2}{Q^2}k_2\left[H\left(\frac{r-C-\mu'}{\sigma'}\right) - H\left(\frac{r-C+Q-\mu'}{\sigma'}\right) - \frac{Q}{\sigma'}G\left(\frac{r-C+Q-\mu'}{\sigma'}\right)\right]. \tag{5.4}$$

Let (Q^u, r^u, C^u) be the optimal solution of $(\mathbf{P1}+\mathbf{N})$, which is unrestricted. This solution set can be obtained from the following system of equations:

$$f_1(Q,r,C) = \frac{b_1}{b_1+h}, \tag{5.5}$$

$$f_2(Q,r,C) = \frac{b_2}{b_2+h}, \tag{5.6}$$

$$\frac{\partial}{\partial Q}AC(Q,r,C) = 0. \tag{5.7}$$

Then (Q^u, r^u, C^u) is the optimal solution of $(\mathbf{P1}+\mathbf{N})$ only if it belongs to the domain of the problem, otherwise $AC(Q^u, r^u, C^u)$ is a lower bound and it is necessary to solve another system of equations to determine the optimal solution. Then, if r^u is non-negative but smaller than C, the optimal solution of $(\mathbf{P1}+\mathbf{N})$ is: $r^* = C^*$, and C^*, Q^* are obtained from equations (5.5) and (5.7), respectively. Otherwise, the optimal solution of $(\mathbf{P1}+\mathbf{N})$ is $r^* = C^* = 0$, and Q^* is obtained from equation (5.7). The above procedure can be carried out *via* the following algorithm.

Algorithm 2. Iterative technique to solve $(\mathbf{P1}+\mathbf{N})$.

```
Q^(0) = sqrt(2μS / h)
Obtain (r + C')^(k) from (5.5)
Obtain (r − C)^(k) from (5.6)
Obtain r^(k) and C^(k)
if r^(k) > C^(k) > 0 then
    Given (r^(k), C^(k)) obtain Q^(k+1) from (5.7)
else if 0 < r^(k) ⩽ C^(k) then
    Recalculate (r^(k), C^(k)) from r^(k) = C^(k) and (5.5)
    Given (r^(k), C^(k)) obtain Q^(k+1) from (5.7)
else
    Redefine r^(k) = C^(k) = 0
    Given (r^(k), C^(k)) obtain Q^(k+1) from (5.7)
end if
if AC(Q^(k), r^(k−1), C^(k−1)) − AC(Q^(k+1), r^(k), C^(k)) ≤ ε then
    Stop
else
    go to 2
end if
```

We initialize the Algorithm 2 with the EOQ solution because it is a lower bound of the problem $(\mathbf{P0})$. Note that, $(\mathbf{P0})$ and $(\mathbf{P0}+\mathbf{N})$ can also be solved using a nonlinear convex solver.

6. Computational study

In this section, we present our numerical study and its results. The main objective of the computational study is to compare the critical level policy with the separate stock and round-up policies.

For simplicity, we use normally distributed demand as an approximation to the non-negative demand, solving (**P0**) by using Algorithm 2. Let (Q^*, r^*, C^*) be the optimal critical level policy controls of (**P0**), and let $AC(Q^*, r^*, C^*)$ be the average cost per unit time of evaluating the optimal critical controls of (**P0**).

In order to cover a wide range of data, we design a set of 10 experiments to measure how the optimal parameters (Q^*, r^*, C^*) change when the parameters of the inventory system are modified; and compare the critical level policy with the separate stock and round-up policies. In each experiment we fix the shortage costs per unit and unit time b_1 and b_2, and consider a base case with the following parameters: normal demand distributions with mean $\mu_1 = \mu_2 = 25$ and coefficient of variation $CV_1 = CV_2 = 0.2$ ($\sigma_1^2 = \sigma_2^2 = 25$), lead time $L = 5$, ordering cost $S = 300$ and holding cost per unit and unit time $h = 0.75$. We conduct experiments studying the sensitivity of the solutions to changing parameters $CV_i = \{0.2, 0.4, 0.6\}$, $\mu_i = \{25, 100\}$, $S = \{100, 300, 500\}$, and $h = \{0.25, 0.75, 1.25\}$. This gives a total of 135 experiments for each setting of the shortage costs. The fixed setting for the shortage costs are: $(b_1, b_2) = \{(30, 5), (30, 10), (30, 15), (30, 20), (30, 25), (10, 5), (15, 5), (20, 5), (25, 5), (35, 5)\}$. This give a total of $135 \times 10 = 1350$ experiments. We denote these instances as *test sets*.

To illustrate the industrial applicability of the critical level policy we also consider a illustrative example of a company that manufactures products derived from fruits and vegetables presented by Escalona *et al.* [6].

The equation systems of Algorithm 2 was programmed by a C code using Brent−Dekker method. All test were carried on a PC with Intel Core i7 2.3 GHz processor and 16 GB RAM. The time to compute the parameters of the critical level policy are on average 8.8E-05 s and in the worst case 8.8E-4 s.

6.1. Test sets

To evaluate the performance of the critical level police respect the round-up and separate stock policies, we computed the benefit of the critical level policy obtained with the proposed approach against the round-up and separate stock policies at each of the 1350 experiments. Let AC_u and AC_s be the average cost per unit time induced by the round-up and separate stock policies respectively and $100 \times (AC_u - AC)/AC$ and $100 \times (AC_s - AC)/AC$ the benefit of the critical level policy against the separate stock and round-up policies respectively

Our numerical results show that the benefit of the critical level policy obtained with the proposed approach against the round-up and separate stock policies is on average 5.9% and 33.5% respectively. Table 1 shows the average and maximum relative benefit of the critical level policy with respect to the round-up and separate stock for the 10 settings of shortage costs and different values of S.

Table 1 shows that in all experiments, the average and maximum relative benefit is greater with respect to the separate stock policy. We also note that the relative benefit to the round-up is more sensitive and, by contrast, using two separate lot sizes and two separate reorder points causes a more homogeneous benefit. The maximum relative benefit, with respect to round-up and separate stock, occurs when there is maximum difference between the shortage costs and the ordering cost is minimal ($S = 100$). As an example, Table 2 shows the relative benefit regarding round-up and separate stock for the 135 problems of the experiment: $b_1 = 30$ and $b_2 = 5$.

The pattern of the maximum relative benefit regarding round-up policy, observed in Table 2, is repeated for all ten experiments, *i.e.*, the maximum benefit occurs when the class 2 dominates on mean and variance ($\mu_2 = 100$, $CV_2 = 0.6$), the ordering cost is minimal ($S = 100$) and the holding cost per unit and unit time is maximum ($h = 1.25$). Clearly, the round-up policy is highly inefficient when the class 2 dominates mean and variance, because under this situation, this policy provides too much inventory to the low priority class causing a high reorder point and therefore a high cost. On the other hand, when ordering cost is low and holding cost per unit and unit time is high, batch sizes are small and the expected backorder increases. We observe that the expected backorders induced by the critical level are greater than those induced by the round-up policy,

TABLE 1. Benefit of the critical level *vs.* Round-up and Separate stock policies.

		Benefit (%) *vs.* round-up					
		$S = 100$		$S = 300$		$S = 500$	
b_1	b_2	Average	Max	Average	Max	Average	Max
30	5	11.73	38.18	9.26	27.82	8.17	25.33
30	10	6.41	21.89	5.05	14.71	4.44	13.44
30	15	3.80	15.01	2.99	8.57	2.63	7.84
30	20	2.13	10.77	1.68	4.75	1.47	4.35
30	25	0.93	7.80	0.73	2.05	0.64	1.88
10	5	5.16	20.74	4.10	11.95	3.64	10.92
15	5	7.79	27.01	6.17	18.22	5.46	16.62
20	5	9.50	31.70	7.52	22.37	6.64	20.39
25	5	10.75	35.34	8.50	25.42	7.50	23.16
35	5	12.53	40.51	9.89	29.77	8.71	27.10

		Benefit *vs.* separate stock (%)					
		$S = 100$		$S = 300$		$S = 500$	
b_1	b_2	Average	Max	Average	Max	Average	Max
30	5	32.79	45.43	33.87	44.26	34.47	44.04
30	10	31.79	41.57	33.67	41.90	34.29	42.16
30	15	31.30	41.42	33.56	41.42	34.18	41.42
30	20	30.99	41.42	33.48	41.42	34.10	41.42
30	25	30.77	41.42	33.42	41.42	34.05	41.42
10	5	32.04	41.42	34.23	42.07	34.81	42.23
15	5	32.33	42.33	34.09	42.93	34.68	42.93
20	5	32.53	43.66	33.99	43.50	34.59	43.41
25	5	32.68	44.65	33.93	43.92	34.53	43.76
35	5	32.89	46.07	33.83	44.53	34.43	44.27

but its effect on cost is relatively low compared with the effect of the reorder point. Note that, as Deshpande *et al.* [5] observed, the relative benefit regarding Round-up is decreasing in S. We did not find a pattern for the maximum relative benefit regarding separate stock policy.

6.2. Ilustrative example for industrial application

Consider the case of a company that manufactures products derived from fruits and vegetables. The supply chain consisting of one plant, one distribution center, and 38 customers. The company segments its customers by volume of annual demand. Thus, customers who demand more than the average annual demand are classified as high priority, with $b_1 = 0.5(\text{US\$/kg} - \text{day})$, and customers who require less than the average annual demand are classified as low priority with $b_2 = 0.025(\text{US\$/kg} - \text{day})$.

The products manufactured by the company are derivative of fruits and vegetables, with holding cost per unit and unit time $h = 0.005(\text{US\$/kg} - \text{day})$; ordering cost $S = 250(\text{US\$/order})$; lead time $L = 4$ (day); normal demand distributions with mean $\mu_1 = 17680(\text{kg/day})$ and coefficient of variation $CV_1 = 0.28$; $\mu_2 = 6534$ and coefficient of variation $CV_2 = 0.12$.

We analyze the inventory problem with two demand classes using critical level, round-up and separate stock policies. Table 3 shows the objective function and the cost components (OC: ordering cost; HC: holding cost; SC: shortage cost).

TABLE 2. Benefit(%) *vs.* Round-up and Separate stock when $b_1 = 30$ and $b_2 = 5$.

			Benefit(%) *vs.* Round-up								
			$\mu_1 = 100, \mu_2 = 25$			$\mu_1 = \mu_2 = 25$			$\mu_1 = 25, \mu_2 = 100$		
CV_1	CV_2	h	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$
0.2	0.2	0.25	2.13	1.52	1.29	3.73	2.60	2.21	9.12	6.37	5.36
		0.75	3.72	2.82	2.46	7.13	5.25	4.61	16.83	12.38	10.67
		1.25	4.74	3.72	3.29	9.58	7.29	6.50	22.31	16.90	14.74
0.4	0.4	0.25	3.06	2.32	1.99	5.78	4.10	3.46	13.50	9.97	8.50
		0.75	4.86	3.97	3.53	10.26	7.72	6.69	22.88	18.10	15.86
		1.25	5.94	5.01	4.53	13.26	10.30	9.06	29.12	23.80	21.16
0.6	0.6	0.25	3.58	2.86	2.51	7.25	5.30	4.51	16.10	12.55	10.90
		0.75	5.39	4.64	4.22	12.21	9.57	8.38	25.88	21.68	19.44
		1.25	6.46	5.71	5.28	15.38	12.47	11.09	32.29	27.82	25.33
0.6	0.2	0.25	3.55	2.83	2.48	6.17	4.40	3.73	10.27	7.24	6.11
		0.75	5.36	4.60	4.18	10.81	8.20	7.13	18.54	13.84	11.94
		1.25	6.43	5.67	5.23	13.86	10.88	9.58	24.30	18.71	16.34
0.2	0.6	0.25	2.38	1.72	1.46	6.17	4.40	3.73	15.94	12.37	10.73
		0.75	4.05	3.12	2.73	10.81	8.20	7.13	25.70	21.45	19.20
		1.25	5.10	4.07	3.61	13.86	10.88	9.58	32.10	27.56	25.06

			Benefit(%) *vs.* separate stock								
			$\mu_1 = 100, \mu_2 = 25$			$\mu_1 = \mu_2 = 25$			$\mu_1 = 25, \mu_2 = 100$		
CV_1	CV_2	h	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$
0.2	0.2	0.25	31.19	32.40	32.80	41.41	41.42	41.42	32.41	33.42	33.72
		0.75	29.76	31.37	31.91	41.38	41.39	41.39	32.02	33.50	33.95
		1.25	28.89	30.67	31.26	41.33	41.33	41.33	31.88	33.67	34.23
0.4	0.4	0.25	28.89	30.78	31.48	41.41	41.41	41.42	30.28	32.05	32.66
		0.75	27.04	29.24	30.14	41.37	41.38	41.38	29.16	31.51	32.38
		1.25	26.10	28.34	29.30	41.31	41.33	41.33	28.59	31.27	32.31
0.6	0.6	0.25	27.30	29.43	30.31	41.41	41.41	41.41	28.65	30.80	31.63
		0.75	25.43	27.63	28.66	41.35	41.37	41.38	27.23	29.83	30.93
		1.25	24.60	26.68	27.73	41.29	41.31	41.32	26.55	29.33	30.59
0.6	0.2	0.25	23.95	27.58	29.00	41.91	42.20	42.23	39.44	37.61	36.96
		0.75	21.80	25.68	27.35	43.25	43.42	43.31	43.03	40.26	39.20
		1.25	20.93	24.76	26.49	44.20	44.26	44.04	45.43	42.12	40.79
0.2	0.6	0.25	32.37	32.70	32.86	35.33	37.74	38.56	15.94	26.10	27.92
		0.75	30.64	31.20	31.48	32.07	35.57	36.85	25.70	23.10	25.54
		1.25	29.44	30.13	30.49	30.00	34.18	35.74	32.10	21.39	24.19

TABLE 3. Ilustrative example: results.

		Cost component (US$/day)		
Policy	AC (US$/day)	OC	HC	SC
Critical level	307.6	110.6	171.3	24.6
Round-up	329.4	112.9	195.6	20.9
Separate stock	413.4	153.1	229.2	31.1

Table 3 indicates that the lower cost is achieved with the critical level policy with a benefit of 7.1% and 34.4% per day against the round-up and separate stock policies respectively. Note that this benefit is produced by lower ordering and holding costs induced by the critical level policy.

7. Conclusions

In this paper we analyzed the constant critical level policy for fast-moving items when the inventory system faced random demands from two customer classes (high and low priority). The inventory system operated under a continuous review (Q, r) policy, with a critical threshold value C, full-backorders and a deterministic lead time. Penalty cost of backorders of the high-priority class were greater than the low-priority class, and the demand of each class was characterized by a nondecreasing stochastic process with stationary increments and continuous sample paths. We also characterized the demand of each class with a normal distribution, which acted as an approximation of non-negative demand.

Using the properties of the nondecreasing stationary stochastic demand and the *threshold clearing mechanism*, we obtained convex backorders for each class demand in steady state. We then proposed a nonlinear cost optimization problem with convex objective function to determine the optimal parameters of the critical level policy. Given the convexity of the cost optimization problem and based on Karush−Kuhn−Tucker (KKT) conditions, we proposed a system of equations to solve it.

Our numerical results show that our approach is able to provide good-quality solutions because the benefit of the critical level policy obtained with the proposed approach against the separate stock and round-up policies is on average 5.9% and 33.5% respectively. In addition, we observe the following managerial insights:

- *The average and maximum relative benefit induced by the critical level policy are greater with respect to the separate stock policy.*
- *The benefits induced by the critical level policy are higher for large difference between the shortage costs and small for low difference between the shortage costs.*

There are a number of questions and issues left for future research. The first one is to expand our results to more than two classes and second one is to investigate the equivalence between shortage costs and fill-rate service levels. Another possible extensions are: (i) investigate the effect of correlated demand and (ii) extend this paper to the design of distribution networks.

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