OR Spectrum manuscript No.
(will be inserted by the editor)

Critical Level Rationing In Inventory Systems With Continuously Distributed Demand

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Received: date / Accepted: date

Abstract This paper analyzes the use of a constant critical level policy for fast-moving items where rationing is used to provide differentiated service levels to two demand classes (high and low priority). Previous work on critical level models, with either a continuous or periodic review policy, has only considered slow-moving items with Poisson demand. In this work we consider a continuous review (Q,r,C) policy with two demand classes that are modeled through continuous distributions and the service levels are measured by the probability of satisfying the entire demand of each class during the lead time. We formulate a service level problem as an non-linear problem with chance constraints for which we optimally solve a relaxation obtaining a closed form solution that can be computed easily. For instances we tested, computational results show that our solution approach provide good-quality solutions that are on average 0.3% from the optimal solution.

Keywords Inventory system · critical level rationing · fast-moving items · service level · continuous distributed demand · two demand classes

1 Introduction

In the last decades the distribution channels of fast-moving consumer goods (FMCG) have been concentrated on large retails chains, which demand a large quantity of items and are therefore in a position to request high service level in terms of product availability at the supplier's expense. In this paper, we consider a supplier of

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fast moving items that serves several customers, including large retails chains, and is likely to face a stock-out. Given this situation, the supplier would likely prefer to meet the higher service level requested by the large retail chain to ensure a good relationship with the businesses that most impact the bottom line. This makes a natural situation where the supplier decides to meet demand with differentiated service levels and segment operationally their customers based on service levels. The simplest segmentation is to classify customers into two demand classes: (i) High-priority class that would correspond to large retail chains that require high levels of service and, (ii) Low-priority class that would represent small retailers that have to settle for a lower level of service.

An efficient way of providing differentiated service levels is through a *critical level policy*. This policy is an inventory control model for rationing inventory between different classes of customers, where in addition to using the advantage of the pooling effect (Eppen (1979)), it has the flexibility of providing different service levels to different customer classes without having to maintain a large inventory for classes that require less service level than the maximum. This policy can be implemented for several ordering and review policies. For example, a traditional (Q,r) model is extended using a critical level policy to a $(Q,r,\mathbf{C})$ inventory model, where Q is the fixed batch size, r is the reorder point and $\mathbf{C} := \{C_1,...,C_{n-1}\}$ denote a set of critical levels for rationing n classes of demand (Nahmias and Demmy (1981); Melchiors et al (2000); Deshpande et al (2003); Isotupa (2006); Arslan et al (2007); Wang et al (2013a)), and $(S-1,S)$ policies are extended to a $(S-1,S,\mathbf{C})$, where S denotes the base stock level (Ha (1997a,b, 2000); De Vericourt et al (2000, 2002); Bulut and Fadiloğlu (2011); Piplani and Liu (2014) for make-to-stock production system, and Dekker et al (2002, 1998); Möllering and Thonemann (2010); Fadıloğlu and Bulut (2010); Wang et al (2013b) for lot-for-lot inventory systems).

Let us now consider the implementation of a critical level policy for a supplier of fast-moving consumer goods. Previous work on critical level policy has only considered the case of discrete demand, in particular Poisson distributed demand, which is used to model demand for slow-moving items. For example, items with low inventory turnover such as spare parts whose demand is less frequent. For fast-moving items it is usually more representative and efficient to model the demand over a time period with a continuous distribution, e.g., normal or gamma distributions (Peterson and Silver (1979), Axsäter (2006), Ramaekers and Janssens (2008)). Furthermore, to the best of our knowledge, there does not exist previous work that implements a critical level policy when a continuous distribution is used to model the demand of fast-moving items.

The objective of this paper is to determine the optimal parameters of a constant critical level (Q,r,C) inventory policy for fast-moving items, where the rationing is used to provide differentiated service levels to two classes of demand (high and low priority). Although Bulut and Fadiloğlu (2011) show that a (Q,r,C) policy with constant threshold value C is not a optimal rationing policy, it is easy to implement and is still an active area of research.

In this paper, the service level for each demand class is measured by service level type 1 and to determine the operational characteristics of the inventory system we use a *hitting time* approach. We also consider the *threshold clearing mechanism* of Deshpande et al (2003) to allocate backorders when multiple outstanding orders exist. The problem assumes a fast-moving item which makes it reasonable to model the demand over a time period by a continuous distribution with positive support. Given the inventory control strategy, we formulate a non-linear problem with chance constraints, denoted (**SLP**), to determine the parameters of the critical level policy. We propose to solve a relaxation of (**SLP**) which is able to provide good bounds. For strictly increasing non-negative demand, we characterize the optimal solution of this relaxation through a system of equations. We further extend this solution, under mild assumptions, when the normal distribution is used as an approximation of the non-negative demand.

The main contributions in this paper can be summarized as follows: (i) we present a new critical level inventory model and solution technique when demand is modeled through continuous distributions, (ii) we develop expressions for the service level type 1 under rationing and (iii) we provide exact expressions for the steady state backorders under rationing.

The rest of this paper is structured as follows. A review of related work is discussed in the next section. In section 3 we describe the context in which the inventory system operates and present the service level problem analyzed in this work. In section 4 we derive expressions for the approximation using a normal distributed demand. In section 5, we derive the structural properties of the service level constraints that allow us to find the optimal solution for the relaxation of (**SLP**). We present our numerical experiments to evaluate the quality of the proposed solutions in section 6. Section 7 presents our conclusions and future extensions to this work.

2 Related work

A comprehensive review of inventory rationing can be found at Kleijn and Dekker (1999) and a classification at Teunter and Haneveld (2008). In particular, Kleijn and Dekker (1999) classified inventory systems subject to multiple classes of demand based on the *review policy* (continuous and periodic) and the *number of classes* (2 or n classes). The above classification is extended by Teunter and Haneveld (2008), incorporating *shortage treatment* (backorder or lost sale), *rationing policy* (no-rationing, static, dynamic), the *ordering policy* and the way that the time is modeled (discrete or continuous).

Our model corresponds to a constant critical level (Q,r,C) policy of continuous review and demand. In this sense, Nahmias and Demmy (1981) were the first ones that studied the continuous review policy with two demand classes. They assumed a (Q,r,C) policy, Poisson demand, full-backorders and deterministic lead time. This work does not determine the optimal parameters of the critical level policy, but develops an approximate expression for the expected backorder per cycle for both demand classes when there is at most one outstanding order and uses the *hitting time* to model the inventory behavior. Melchiors et al (2000) also analyze a (Q,r,C) inventory model, deterministic lead time and two demands classes, but unlike Nahmias and Demmy (1981), they consider a lost sales environment. In order to determine the optimal parameters of the critical level policy these authors propose a cost optimization problem and present a numeric procedure for its resolution. They assumed Poisson demand and used the *hitting time* and renewal theory to operationally characterize the inventory system. Isotupa (2006) presents a model with the same assumptions as Melchiors et al (2000) but with exponentially distributed lead time.

When implementing a continuous review (Q,r,C) critical level policy with full-backorder, it may happen that the incoming replenishment batch is not large enough to cover the backorders. Therefore, it is important how the backorders of the different classes are satisfied. According to Möllering and Thonemann (2010) it is optimal to fill backorders from high priority classes first when dealing with penalty costs. This form of clearing the backorders is called *priority clearing mechanism*. This policy is difficult to analyze mathematically and given its complexity the literature has focused on manageable but sub-optimal rules, e.g., the *threshold clearing mechanism* from Deshpande et al (2003) and the *FCFS type clearing scheme* from Arslan et al (2007). Deshpande et al (2003) analyzed the same rationing model as Nahmias and Demmy (1981), but without restricting the number of outstanding orders. They derived expressions for the average backorders per cycle and for the expected steady-state on-hand inventory and backorder using a state-dependent demand approach. Based on these expressions, Deshpande et al (2003) proposed a cost optimization model and developed algorithms to compute the optimal parameters of the critical level policy. Arslan et al (2007) presents a service level model to obtain the optimal parameters of a critical level policy with multiple demand classes under the assumptions of Poisson demand, deterministic lead time, and a continuous-review (Q,r) policy. Wang et al (2013a) analyzed the rationing policy under the same operational conditions than Deshpande et al (2003), but considered a mixed service criteria with penalty costs and service level constraints (fill-rate). In that work, they show numerically that the priority clearing mechanism does not always outperform the threshold clearing mechanism when dealing with service levels constraints.

In certain situations a dynamic rationing policy, which allows the critical level to change based on the number and ages of outstanding orders, can outperform a constant critical level policy (Q,r,C). Fadiloglu and Bulut (2010) examine a dynamic rationing policy, in a continuous review (Q,r) inventory model with Poisson demand and deterministic lead time. The authors use simulation-based approaches to find efficient solutions for the cases with backordering and lost sales.

From the literature review conducted only Dekker et al (2002), Arslan et al (2007), Wang et al (2013b) and Möllering and Thonemann (2010) use a service level problem approach to determine the optimal parameters of the critical level policy. These four articles consider the same service level problem: to minimize the expected on-hand inventory subject to having the service level provided to each class exceed its preset level. Depending on the operating conditions defined for the inventory system, what varies is the formulation of the inventory on-hand value and the service level provided to each class. Dekker et al (2002) analyzed the critical level policy when the inventory system works under a continuous review lot-for-lot policy, lost sales and Poisson demand. These authors derive expressions for fill-rate and present an efficient method to obtain optimal solutions. Möllering and Thonemann (2010) analyze a periodic review base-stock policy with two demand classes, deterministic lead time, discrete demand distribution and full backorder. That work models the inventory system as a multidimensional Markov chain and optimally solves a service level problem, based on a service level of type 1 and another on fill-rate. Wang et al (2013b) analyzed the same model as Möllering and Thonemann (2010), but considered an anticipated rationing policy. This policy reserves inventory for the high priority classes considering a constant critical level and the coming replenishment of the next period.

In summary, previous research on inventory rationing solved periodic or continuous review problems with discrete demand. Therefore, to the best of our knowledge, there is no constant critical level model for the case of continuous demand distribution considered in this paper.

3 Service level problem for strictly increasing non-negative demand

Consider a facility that holds inventory of a single type of product to serve two demand classes $i = 1,2$, where class 1 is high priority and class 2 is low priority. Let $D_i(t,t+\tau)$ be the total demand of class i in the interval $(t,t+\tau]$, and $D(t,t+\tau) = D_1(t,t+\tau)+D_2(t,t+\tau)$ the total demand of both classes in the interval $(t,t+\tau]$. We denote by $F_{D_i(\tau)}(x)$ the cumulative distribution function of the total demand of class i in $[0,\tau]$ and $F_{D(\tau)}(x)$ the cumulative distribution function of the total demand of both classes in $[0,\tau]$.

In this paper we consider fast-moving items for which is more representative and efficient to model the demand over a time period by a continuous distribution. Following Zheng (1992) we assume that the total demand of each class are represented by a nondecreasing stochastic process with continuous sample paths, and stationary and independent increments. For simplicity, we will refer to this as strictly increasing non-negative demand. This is a common assumption in stochastic inventory models (Axsäter (2006)) and is implicitly assumed in most (elementary) textbooks on inventory management. However, the assumptions of independence and continuity are conflicting; therefore, rigorously speaking, the assumption is approximate (Browne and Zipkin (1991)). Note that under stationary and independent increments, $D_i(\tau) := D_i(0,\tau) = D_i(t,t+\tau)$ for any $t \geq 0$, $i = 1,2$.

Inventory is replenished according to a continuous review (Q,r,C) policy that operates as follows. When the inventory position falls below a reorder level r, a replenishment order for Q units is placed and arrives a fixed $L > 0$ time units later. Demand from both classes are filled as long as the on-hand inventory level is greater than the critical level C, otherwise only high priority demand is satisfied from inventory on-hand and low priority demand is backordered. If on-hand inventory level reaches zero both demands are backordered. To clear backlogged orders, we consider the threshold clearing mechanism of Deshpande et al (2003).

Given the inventory control strategy, our objective is to find the parameters of the critical level policy that minimize the sum of ordering and holding costs per unit time subject to satisfying the required service level for each class. In this paper, the service level is measured by the probability of satisfying the entire demand of each class during the lead time from on-hand inventory (service level type 1), which does not depend of the replenishment batch quantity. Let $\alpha_i(r,C)$ be the provided service level to class i and $\overline{\alpha}_i$ the preset service level for class i, where $\overline{\alpha}_1 > \overline{\alpha}_2 > 0$. Then, the service level problem is:

$$\min_{Q,r,C} \quad AC(Q,r,C) \tag{1}$$

$$\text{s.t:} \quad \alpha_i(r,C) \geq \overline{\alpha}_i \qquad \forall i = 1,2 \tag{2}$$

$$r,C \geq 0. \tag{3}$$

where $AC(Q,r,C)$ is the average cost per unit time, i.e., the sum of ordering and holding costs per unit time. To develop expressions for $\alpha_i(r,C)$, $i = 1,2$, and $AC(Q,r,C)$ we use a hitting time approach as in Nahmias and Demmy (1981) and the threshold clearing mechanism of Deshpande et al (2003) to allocate backorders when multiple outstanding orders exist.

The hitting time $\tau^x_{H,D}$ is defined as the amount of time that elapses until the demand D reaches x for the first time, i.e.,

$$\tau^x_{H,D} = \inf\{\tau > 0 \mid D(\tau) > x\}. \tag{4}$$

Since we assume strictly increasing non-negative demand, we have $\mathbb{P}(\tau^x_{H,D} \leq \tau) = \mathbb{P}(D(\tau) \geq x)$. Therefore, the distribution function of the hitting time $\tau^x_{H,D}$, for a fixed $x > 0$ is $F^x_{H,D}(\tau) = 1 - F_{D(\tau)}(x)$, and its density distribution is:

$$f^x_{H,D}(\tau) = -\frac{\partial F_{D(\tau)}(x)}{\partial \tau}. \tag{5}$$

Many authors have discussed the hitting time process for strictly increasing non-negative demand. However, an explicit expression for the density of hitting time is not possible in many cases. Meerschaert and Scheffler (2008) develop a density formula for the hitting time of any strictly increasing non-negative demand based on the Laplace transform of the hitting time. Park and Padgett (2005) derived a exact density distribution of hitting time for a gamma process using the same procedure described by equation (5).

3.1 Average cost per unit time.

Let μ be the total average demand per unit of time, h be the holding cost per unit and unit time and S the ordering cost. Then the average cost per unit time is $AC(Q,r,C) = S\frac{\mu}{Q} + h\mathbb{E}(OH(\infty))$, where $OH(\infty)$ is the steady-state on-hand inventory (Axsäter (2006)).

In a (Q,r,C) policy with full-backorders and deterministic lead time, the inventory level is the on-hand inventory net of all backorders, i.e., $IL(t+L) = OH(t+L) - B_1(t+L) - B_2(t+L)$, where $IL(t+L)$ denotes the inventory level, $OH(t+L)$ denotes on-hand inventory and $B_i(t+L)$ denotes class i backorders, $i = 1,2$, all at time $t+L$. Furthermore, for a (Q,r,C) policy with full-backorders and deterministic lead time it is still valid that $IL(t+L) = IP(t) - D(L)$, where $IP(t)$ denotes the inventory position at time t (Deshpande et al (2003)). Under strictly increasing non-negative demand, $IP(t)$ will be uniformly distributed on $(r, r+Q]$ in steady state and independent of lead time demand (Zheng (1992) refers to Serfozo and Stidham (1978) and Browne and Zipkin (1991) for a detailed discussion of this assumption). Then, the on-hand inventory at time $t+L$ is $OH(t+L) = IP(t) - D(L) + B_1(t+L) + B_2(t+L)$. Taking expected value and limit $t \to \infty$, the expected on-hand inventory at steady-state is $\mathbb{E}(OH(\infty)) = \frac{Q}{2} + r - \mu L + \mathbb{E}(B_1^\infty(Q,r,C)) + \mathbb{E}(B_2^\infty(Q,r,C))$, where $\mathbb{E}(B_i^\infty(Q,r,C))$ is the class i steady-state backorder, $i = 1,2$. Then, the average cost per unit time is:

$$AC(Q,r,C) = S\frac{\mu}{Q} + h\left(\frac{Q}{2} + r - \mu L + \mathbb{E}(B_1^\infty(Q,r,C)) + \mathbb{E}(B_2^\infty(Q,r,C))\right) \quad (6)$$

We now develop expressions for the backorders of the low and high priority class in steady state using a hitting time approach, the inventory position and the threshold clearing mechanism. We first describe how the inventory system behaves under rationing, and the threshold clearing mechanism of Deshpande et al (2003).

Consider an arbitrary time $t+L$. By definition, there is rationing at time $t+L$ when $C > OH(t+L) \geq IL(t+L) = IP(t) - D(t,t+L)$. Using the hitting time $\tau_{H,D}^{IP(t)-C}$ defined in equation (4), this last condition states that if there is rationing at $t+L$ then $\tau_{H,D}^{IP(t)-C} < L$. Note that $\tau_{H,D}^{IP(t)-C}$ corresponds to the time required for $IP(t) - C$ demands.

Define t_c as the first time after t when $IP(t) - C$ demand is observed, that is, $t_c = t + \tau_{H,D}^{IP(t)-C}$. If rationing ocurrs at $t+L$ then we have that $\tau_{H,D}^{IP(t)-C} < L$. The threshold clearing mechanism of Deshpande et al (2003) only comes into play when backorders exist on arrival of a replenishment order and uses t_c to separate which backorders need to be cleared once the replenishment order arrives. The general rules to clear the backorders when the replenishment order arrives are:

1. If the entering replenishment batch is large enough to clear all the backorders and leave the on-hand inventory level above C, then clear all backorders,
2. Otherwise:
   2.1 Clear all backlogged demand that arrived before t_c in the order of arrival (FCFS),
   2.2 Clear any remaining backlogged class 1 demands using FCFS until either all class 1 backorders are filled, or no on-hand inventory remains,
   2.3 Carry over (i.e. continue backlog) all class 2 demands that arrive after t_c.

Note that rule 1 ensures that $OH(t) = IL(t)$ when $OH(t) \geq C$. Rule 2.2 and 2.3 mean that all remaining backorders that cannot be fulfilled by the entering replenishment batch, are carried over to be satisfied in the following replenishment arrivals.

Using the hitting time definition it easy to show that the backorders of both low- and high-priority at time $t+L$, deduced by Deshpande et al (2003), are respectively:

$$B_2(t+L) = \begin{cases} D_2(L - \tau_{H,D}^{IP(t)-C}) & \text{if } \tau_{H,D}^{IP(t)-C} < L \\ 0 & \sim \end{cases} \quad (7)$$

$$B_1(t+L) = \begin{cases} D_1(L - \tau_{H,D}^{IP(t)-C} - \tau_{H,D_1}^{C}) & \text{if } \tau_{H,D}^{IP(t)-C} + \tau_{H,D_1}^{C} < L\,, \\ 0 & \sim \end{cases} \quad (8)$$

where $\tau^C_{H,D_1} = \inf\{\tau > 0 \mid D_1(\tau) > C\}$ corresponds to the time required for C demands of class 1. The equivalence between equations (7), (8) and the expressions developed by Deshpande et al (2003), are given by the fact that: $D_2(t_c, t+L) = D_2(L - \tau^{IP(t)-C}_{H,D})$ and $[D_1(t_c, t+L) - C]^+ = D_1(L - \tau^{IP(t)-C}_{H,D} - \tau^C_{H,D_1})$.

Taking expectation of equations (7) and (8) and conditioning on the inventory position $IP(t)$, the expected backorders at steady state of class 1 and 2 are respectively:

$$\mathbb{E}(B^\infty_2(Q,r,C)) = \frac{\mu_2}{Q}\int_r^{r+Q}\int_0^L (L-\tau) f^{y-C}_{H,D}(\tau)\, d\tau dy\ , \tag{9}$$

$$\mathbb{E}(B^\infty_1(Q,r,C)) = \frac{\mu_1}{Q}\int_r^{r+Q}\int_0^L (L-\tau)(f^{y-C}_{H,D} * f^C_{H,D_1})(\tau)\, d\tau dy\ , \tag{10}$$

where: $f^{y-C}_{H,D}(\tau) = -\frac{\partial F_{D(\tau)}(y-C)}{\partial \tau}$; $f^C_{H,D_1}(\tau) = -\frac{\partial F_{D_1(\tau)}(C)}{\partial \tau}$; and we denote by $f^{y-C}_{H,D} * f^C_{H,D_1}(\tau) = \int_0^\tau f^{y-C}_{H,D}(\tau - t) f^C_{H,D_1}(t)dt$ the convolution of $f^{y-C}_{H,D}(\tau)$ and $f^C_{H,D_1}(\tau)$.

3.2 Service level type I under rationing policy

We now develop expressions for $\alpha_i(r,C)$ of class $i = 1,2$ using the hitting time approach. We first describe the events to fully meet the demand of each class during the lead time under strictly increasing non-negative demand.

The conditions to fully meet the demand for class 2 in the lead time, under non-negative demand, are that: (i) there does not exist rationing, i.e., $\tau^{r-C}_{H,D} > L$, where $\tau^{r-C}_{H,D}$ is defined in equation (4) and corresponds to the time required for $r-C$ demands, or (ii) rationing occurs and there is no class 2 demand, i.e., $\tau^{r-C}_{H,D} < L$ and $D_2(\tau) = 0\ ,\ \forall \tau \in [\tau^{r-C}_{H,D}, L]$. Since $D_2(\tau)$ is defined as strictly increasing non-negative demand, the probability that rationing occurs and there is no demand of the class 2 during this period is zero. Therefore, the service level provided to the low priority class is:

$$\alpha_2(r,C) = \ \mathbb{P}(D(L) \le r - C) = \ F_{D(L)}(r-C). \tag{11}$$

The conditions to fully meet the demand of class 1 in the lead time, under non-negative demand, are that: (i) rationing does not exist or (ii) rationing occurs and the class 1 demand during this period not reach the critical level C, i.e., $\tau^{r-C}_{H,D} < L$ and $\tau^C_{H,D_1} \ge L - \tau^{r-C}_{H,D}$. Therefore, the service level provided to the high priority class is:

$$\alpha_1(r,C) = \ \mathbb{P}(D(L) \le r-C) + \mathbb{P}(D_1(L - \tau^{r-C}_{H,D}) \le C\ \cap\ \tau^{r-C}_{H,D} < L), \tag{12}$$

because, $\mathbb{P}(\tau^C_{H,D_1} > L - \tau^{r-C}_{H,D}\ \cap\ \tau^{r-C}_{H,D} < L) = \mathbb{P}(D_1(L - \tau^{r-C}_{H,D}) \le C\ \cap\ \tau^{r-C}_{H,D} < L)$. Conditioning on the hitting time $\tau^{r-C}_{H,D}$, the service level provided to the high priority class can be expressed as:

$$\begin{aligned}\alpha_1(r,C) &= \int_0^L \mathbb{P}(D_1(L-\tau) \le C)\ f^{r-C}_{H,D}(\tau)\ d\tau + \mathbb{P}(D(L) \le r-C)\\ &= \int_0^L \mathbb{P}(D_1(L-\tau) \le C)\ f^{r-C}_{H,D}(\tau)\ d\tau + \alpha_2(r,C).\end{aligned} \tag{13}$$

Note that equation (13) verifies that $\alpha_1(r,C) \geq \alpha_2(r,C)$.

Under strictly increasing non-negative demand, the definition of the hitting time $\tau_{H,D}^{r-C}$ implies that reorder point is strictly greater than the critical level, i.e., $r > C \geq 0$. Otherwise the (Q,r,C) policy is not interesting because the provided service level to low priority class is zero. For example, if $r = C \geq 0$ in every lead time exist rationing and the only possibility to fully meet the demand for class 2 is that there is no class 2 demand during the lead time (in this case the lead time is equal to rationing period for class 2). Then, under strictly increasing non-negative demand and $r = C \geq 0$, $\alpha_2(r,r) = \mathbb{P}(D_2(L) \leq 0) = 0$. In the same way, we can conclude that under strictly increasing non-negative demand, for any $C > r \geq 0$, $\alpha_2(r,C) = 0$. Therefore, in this paper we will study only the case where $r > C \geq 0$.

3.3 Problem formulation

Using equation (11) and (13) we can write the service level problem for strictly increasing non-negative demand, denoted (**SLP**), as the following optimization problem.

Problem (**SLP**):

$$\min_{Q,r,C} \quad S\frac{\mu}{Q} + h\left(\frac{Q}{2} + r - \mu L + \mathbb{E}(B_1^\infty(Q,r,C)) + \mathbb{E}(B_2^\infty(Q,r,C))\right) \tag{14}$$

$$\text{s.t:} \quad \int_0^L \mathbb{P}(D_1(L-\tau) \leq C)\, f_{H,D}^{r-C}(\tau)\, d\tau + \mathbb{P}(D(L) \leq r - C) \geq \overline{\alpha}_1 \tag{15}$$

$$\mathbb{P}(D(L) \leq r - C) \geq \overline{\alpha}_2 \tag{16}$$

$$r > C \geq 0, \tag{17}$$

where $\mathbb{E}(B_1^\infty(Q,r,C))$ and $\mathbb{E}(B_2^\infty(Q,r,C))$ are given by equations (9) and (10) respectively. We note that the constraint $r > C$ in (17) is implied by constraint (16) for demand with positive support, as the probability of that demand being less than zero equals zero and cannot be bigger or equal to $\overline{\alpha}_2 > 0$. We express this strict inequality here to remind us of what the feasible region looks like.

4 SLP using normal distribution as approximation of non-negative demand

A common practice in stochastic inventory models is to use the normal distribution as an approximation of the non-negative demand, i.e., the stochastic inventory models are formulated based on the characteristics of the non-negative demand and then are implemented using normal distribution as an approximation. The problem with the normal distribution is that there is always a small probability of negative demand. The normal distribution is a good approximation of non negative demand when the coefficient of variation is less than or equal to 0.5, i.e., $CV \leq 0.5$ (Peterson and Silver (1979)) in which case the probability of being less than 0 is less than 0.0228.

To solve (**SLP**) using normal distribution as approximation of the non-negative demand, the expressions that characterize the hitting time $\tau_{H,D}^{r-C}$ and τ_{H,D_1}^{C}, and the backorders, under normally distributed demand, are required. For this, consider that each class i has identical and independent normally distributed demand per unit time, with mean $\mu_i > 0$ and variance $\sigma_i^2 > 0$, $D_i(\tau) \sim N(\mu_i \tau, \sigma_i^2 \tau)$, and $D(\tau) \sim N(\mu\tau, \sigma^2\tau)$, where $\mu = \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$. Following equation (5), the density distribution of the hitting time $\tau_{H,D}^{r-C}$ under normally distributed demand is:

$$f_{H,D}^{r-C}(\tau) = \left(\frac{r-C+\mu\tau}{2\tau}\right) \frac{1}{\sigma\sqrt{\tau}}\, \varphi\left(\frac{r-C-\mu\tau}{\sigma\sqrt{\tau}}\right), \tag{18}$$

where $\varphi(x)$ is the density function of the standard normal distribution. In the same way, the density distribution of the hitting time τ_{H,D_1}^{C} under normally distributed demand is:

$$f_{H,D_1}^{C}(\tau) = \left(\frac{C+\mu_1\tau}{2\tau}\right) \frac{1}{\sigma_1\sqrt{\tau}}\, \varphi\left(\frac{C-\mu_1\tau}{\sigma_1\sqrt{\tau}}\right).$$

Then, the expected backorders in steady state given by equations (9) and (10) under normally distributed demand become:

$$\mathbb{E}(B_2^\infty(Q,r,C)) = \frac{\mu_2}{Q}\int_0^L \left(G\left(\frac{r-C-\mu\tau}{\sigma\sqrt{\tau}}\right) - G\left(\frac{r+Q-C-\mu\tau}{\sigma\sqrt{\tau}}\right)\right)\sigma\sqrt{\tau}d\tau\ , \tag{19}$$

$$\mathbb{E}(B_1^\infty(Q,r,C)) = \frac{\mu_1}{Q}\int_0^L\int_t^L f_{H,D_1}^{C}(t)\left(G\left(\frac{r-C-\mu(\tau-t)}{\sigma\sqrt{\tau-t}}\right) - G\left(\frac{r+Q-C-\mu(\tau-t)}{\sigma\sqrt{\tau-t}}\right)\right)\sigma\sqrt{\tau-t}\ d\tau dt \tag{20}$$

where $G(x) = \int_x^\infty (v-x)\varphi(v)dv = \varphi(x) - x(1-\Phi(x)))$ is the *loss function* (Axsäter (2006)) and $\Phi(x)$ is the distribution function of the standard normal distribution.

5 Solution approach

Consider the following relaxation of (**SLP**), obtained by dropping the expected backorder expressions:

Problem (**RSLP**):

$$\begin{aligned} \min_{Q,r,C} \quad & S\frac{\mu}{Q} + h\left(\frac{Q}{2} + r - \mu L\right) \\ \text{s.t:} \quad & (15),(16),(17). \end{aligned} \tag{21}$$

It is easy to show that (**RSLP**) is a relaxation of (**SLP**) because the objective function of the (**RSLP**) is less than or equal to the objective function of (**SLP**) and the feasible region is the same. Therefore, the optimal solution of the problem (**RSLP**) is a lower bound (LB) of problem (**SLP**). Also, if we solve (**RSLP**), and then use the resulting parameters (Q,r,C) to evaluate the objective function of (**SLP**) we obtain a feasible solution and, hence, an upper bound (UB) for the problem (**SLP**). Thus, we have a method that gives a lower bound and an upper bound of the original problem.

Note that (**RSLP**) is separable in two sub-problems. The first sub problem minimizes $S\frac{\mu}{Q} + h\frac{Q}{2}$ without constraints on Q and gives the replenishment batch $Q =$

$\sqrt{\frac{2\mu S}{h}}$ that corresponds to the EOQ problem and the second sub problem, denoted (**SLP0**), is

Problem (**SLP0**):

$$\begin{aligned} \min_{r,C} \quad & r \\ \text{s.t:} \quad & (15), (16), (17). \end{aligned} \tag{22}$$

Therefore, the service level problem reduces to determining the optimal reorder point and critical level (r, C) that minimize the reorder point r subject to satisfying the required service levels.

To determine the optimal parameters of **(SLP0)** we take advantage of the structure of the constraints (15), (16) and (17). From these constraints we derive structural properties that are necessary to obtain the exact solution to the **(SLP0)** problem.

Proposition 1 *$\alpha_2(r,C)$ is increasing in r and decreasing in C and only depends on $(r-C)$.*

Proof From equation (11) we have $\alpha_2(r,C) = F_{D(L)}(r-C)$. The result follows since the distribution function is a monotonically increasing function. □

The main consequence of proposition 1 is that, given a reorder point r, the maximum service level provided to the low priority class is $\alpha_2(r,0)$.

5.1 Solution characterization for (**SLP0**): increasing non-negative demand

For any strictly increasing non-negative demand that represent the total demand of class i, with $i = 1,2$, we obtain the following structural properties.

Proposition 2 *If $D_1(\tau)$ is a strictly increasing non-negative demand, then $\alpha_1(r,C)$ is increasing in r and C. (Proof. A)*

The main consequence of proposition 2 is that, given a reorder point r, the minimum service level provided to high priority class is $\alpha_1(r,0)$.

Let r_i^0 be the minimum reorder point r such that the service level provided to the class i, given a critical level $C = 0$, is greater than or equal to his preset service level $\overline{\alpha}_i$, i.e., $r_i^0 = \min\{r \mid \alpha_i(r,0) \geq \overline{\alpha}_i\}$, with $i = 1,2$. Since functions $\alpha_i(r,0)$ for $i = 1,2$ are increasing in r, from propositions 1 and 2, we have that r_i^0 solves $\alpha_i(r,0) = \overline{\alpha}_i$ for $i = 1,2$. In particular this gives

$$r_2^0 = F_{D(L)}^{-1}(\overline{\alpha}_2). \tag{23}$$

Furthermore, from equation (13) we have that $\alpha_1(r,0) = \alpha_2(r,0)$ for any $r \geq 0$, because $F_{D_1(\tau)}(0) = 0$ for any $\tau > 0$. Then, since $\alpha_1(r_1^0,0) = \overline{\alpha}_1 > \overline{\alpha}_2 = \alpha_2(r_2^0,0) = \alpha_1(r_2^0,0)$, and $\alpha_1(r,0)$ is increasing in r we conclude that $0 < r_2^0 < r_1^0$ for any $\overline{\alpha}_2 > 0$.

Using proposition 1 and 2 we propose the following general solution for the **(SLP0)** problem.

Proposition 3 *If $D_i(\tau)$, with $i = 1,2$, are strictly increasing non-negative demand and $\overline{\alpha}_2 > 0$, then the optimal parameters of the critical level policy are obtained from the equation system formed by $\alpha_1(r,C) = \overline{\alpha}_1$ and $\alpha_2(r,C) = \overline{\alpha}_2$, i.e.,*

$$r^* - C^* = F^{-1}_{D(L)}(\overline{\alpha}_2), \tag{24}$$

$$\int_0^L \mathbb{P}(D_1(L-\tau) \leq C^*)\ f^{r^*-C^*}_{H,D}(\tau)\ d\tau = \overline{\alpha}_1 - \overline{\alpha}_2, \tag{25}$$

and the service levels provided to each class are equal to their preset levels, i.e., $\alpha_i(r^*,C^*) = \overline{\alpha}_i$, $i = 1,2$.

Proof Let $C_2(r)$ be the maximum critical level, given a reorder point r, that ensures a service level $\overline{\alpha}_2$, i.e., $C_2(r) = \max\{C \mid \alpha_2(r,C) \geq \overline{\alpha}_2\}$. From proposition 1 we can derive that $C_2(r)$ is solution of $\alpha_2(r,C) = \overline{\alpha}_2$. Then, $C_2(r) = r - F^{-1}_{D(L)}(\overline{\alpha}_2) = r - r^0_2$, i.e., $C_2(r)$ is increasing and linear in r. Note that, $C_2(r) < r$ for any $\overline{\alpha}_2 > 0$. In the same way we define $C_1(r)$ as the minimum critical level, given a reorder point r, that ensures a service level $\overline{\alpha}_1$, i.e., $C_1(r) = \min\{C \mid \alpha_1(r,C) \geq \overline{\alpha}_1\}$. From proposition 2 we obtain that $C_1(r)$ is solution of $\alpha_1(r,C) = \overline{\alpha}_1$ and that $C_1(r)$ is strictly decreasing in r. Once $C_1(r)$ and $C_2(r)$ are defined, the feasible region of **(SLP0)** problem where all (r,C) satisfy that $\alpha_1(r,C) \geq \overline{\alpha}_1$, $\alpha_2(r,C) \geq \overline{\alpha}_2 > 0$, and $r > C \geq 0$, is the intersection of the areas above $C_1(r)$ and below $C_2(r)$. The feasible region is shown in figure 1.

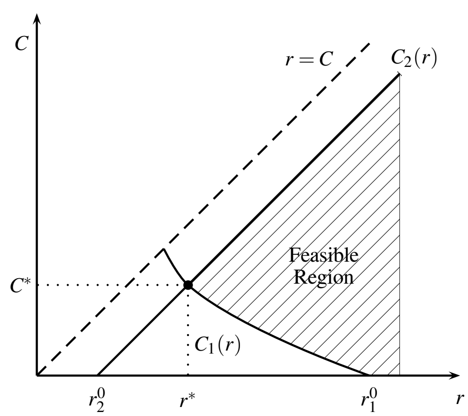


Fig. 1: Feasible region of **SLP0** problem and $\overline{\alpha}_2 > 0$.

The figure 1 shows that the optimal reorder point r^* of **(SLP0)** problem occurs when $C_2(r) = C_1(r) = C^*$. Therefore, the optimal parameters of the critical level policy are obtained from the equation system formed by $\alpha_1(r,C) = \overline{\alpha}_1$ and $\alpha_2(r,C) = \overline{\alpha}_2$, and the presets service levels are satisfied exactly. Note that the existence of an r such that $C^* = C_2(r) = C_1(r)$ is guaranteed, because $0 < r^0_2 < r^0_1$ as shown above, $C_1(r)$ is strictly decreasing and continuous in r, $C_1(r^0_1) = 0$, and from equation (12) we obtain that there exists an $r > 0$ such that $C_1(r) = r > 0$ for any $\overline{\alpha}_1 > 0$. The argument is complete noting that $C_2(r) = r - r^0_2 < r$ is linear and increasing in r. □

Some consequences of the above proof are: (i) the optimal reorder point r^* is strictly greater than the optimal critical level C^* because $r_2^0 > 0$ when $\overline{\alpha}_2 > 0$, therefore, the constraint (17) may be replaced by: $r, C \geq 0$; and (ii) the optimal critical level is strictly greater than zero, i.e., $C^* > 0$, because $r_2^0 < r_1^0$.

Proposition 3 provides a general solution for (**SLP0**) when $D_i(\tau)$ of class $i = 1, 2$, are represented with strictly increasing non-negative demand, and $\overline{\alpha}_2 > 0$. Solving for the optimal solution remains challenging in general, as equations (24)-(25) have to be solved numerically and include the distribution function of $D(L)$ and the density function of $\tau_{H,D}^{r-C}$ which have to be derived from the input.

5.2 Solution characterization for (**SLP0**): normally distributed demand

Recall that we use the normal distribution as approximation of the non-negative demand. Under normally distributed demand we obtain the following structural properties.

Proposition 4 *Under normally distributed demand, the function* $\alpha_1(r, C)$ *is strictly increasing in* r *for any* $0 \leq C < r$.

Proof Using equation (13), the service level provided to the high priority class using normal distribution can be write as:

$$\alpha_1(r,C) = \int_0^L \left\{ \int_{-\infty}^{\frac{C-\mu_1(L-\tau)}{\sigma_1\sqrt{L-\tau}}} \varphi(x)dx \right\} f_{H,D}^{r-C}(\tau)\, d\tau + \mathbb{P}(D(L) \leq r - C)$$

and changing the order of integration we have:

$$\begin{aligned}
\alpha_1(r,C) &= \int_{\frac{C-\mu_1 L}{\sigma_1\sqrt{L}}}^{\infty} \left\{ \int_{\tau(x)}^{L} f_{H,D}^{r-C}(\tau)d\tau \right\} \varphi(x)dx + \int_{-\infty}^{\frac{C-\mu_1 L}{\sigma_1\sqrt{L}}} \left\{ \int_0^L f_{H,D}^{r-C}(\tau)d\tau \right\} \varphi(x)dx + \mathbb{P}(D(L) \leq r-C) \\
&= \int_{\frac{C-\mu_1 L}{\sigma_1\sqrt{L}}}^{\infty} \left\{ \mathbb{P}(D(\tau(x)) \leq r-C) - \mathbb{P}(D(L) \leq r-C) \right\} \varphi(x)dx \\
&\quad + \int_{-\infty}^{\frac{C-\mu_1 L}{\sigma_1\sqrt{L}}} \left\{ 1 - \mathbb{P}(D(L) \leq r-C) \right\} \varphi(x)dx + \mathbb{P}(D(L) \leq r-C) \\
&= \int_{\frac{C-\mu_1 L}{\sigma_1\sqrt{L}}}^{\infty} \mathbb{P}(D(\tau(x)) \leq r-C)\varphi(x)dx + \int_{-\infty}^{\frac{C-\mu_1 L}{\sigma_1\sqrt{L}}} \varphi(x)dx \\
&= \int_{\frac{C-\mu_1 L}{\sigma_1\sqrt{L}}}^{\infty} \mathbb{P}(D(\tau(x)) \leq r-C)\varphi(x)dx + \mathbb{P}(D_1(L) \leq C),
\end{aligned}$$

where $\tau(x)$ is obtained from: $x\sigma_1\sqrt{L-\tau} = C - \mu_1(L-\tau)$. Although $\tau(x)$ is the result of a quadratic equation, the proof remains valid. $\square$

Under normally distributed demand there is always a probability for negative demand. This fact makes it difficult to prove that $\alpha_1(r,C)$ is increasing in C for any $r > C \geq 0$, as we have for strictly increasing non-negative demand. We provide the expression for $\frac{\partial \alpha_1(r,C)}{\partial C}$ in (31) in the appendix. Our numerical computations however have shown that such monotonicity of $\alpha_1(r,C)$ with respect to C exists for large values of the reorder point r. We therefore make this monotonicity an assumption, which we validate with computational results in section 6.

Assumption 1 *Assume normally distributed demand and let $\hat{r}_1$ be the solution of $\alpha_1(\hat{r}_1, 0) = 0.5$. Then, for any $r \geq \hat{r}_1$ the function $\alpha_1(r,C)$ is an increasing function of C in the interval $C \in [0,r)$.*

The main consequence of assumption 1 is that, given a reorder point $r \geq \hat{r}_1$, the minimum service level provided to high priority class is $\alpha_1(r,0)$.

From proposition 4 we derived that r_1^0 is solution of $\alpha_1(r,0) = \overline{\alpha}_1$ and under normally distributed demand we can obtain from equation (13) that $\alpha_1(r,0) > \alpha_2(r,0)$ for any finite $r \geq 0$. Then, as $\alpha_i(r,0)$ is increasing in r, with $i = 1,2$, we infer that the relationship between r_2^0 and r_1^0 depends on the difference $\overline{\alpha}_1 - \overline{\alpha}_2$. Thus, we have two cases: (1) $r_2^0 < r_1^0$ if $\overline{\alpha}_1 - \overline{\alpha}_2$ is large enough; or (2) $r_2^0 > r_1^0$ if $\overline{\alpha}_1 - \overline{\alpha}_2$ is small. A simple way to discriminate if we are in case 1 or 2 is to evaluate numerically $\alpha_1(r_2^0,\ 0)$. Then, if $\alpha_1(r_2^0,\ 0) < \overline{\alpha}_1$, we are in case 1, otherwise, we are in case 2. The method proposed in this paper to solve the **(SLP0)** problem using normally distributed demand, depends on which case occurs. Note that under normally distributed demand $r_2^0 = F_{D(L)}^{-1}(\overline{\alpha}_2) = \mu L + z_{\overline{\alpha}_2}\sigma\sqrt{L} \geq \mu L > 0$ if $\overline{\alpha}_2 \geq 0.5$, where $z_{\overline{\alpha}_2}$ is the inverse standard normal distribution for a preset service level $\overline{\alpha}_2$.

Using proposition 4 and assumption 1 we propose the following solution for the **(SLP0)** problem using normally distributed demand as approximation of non-negative demand.

Proposition 5 *Under normally distributed demand, the assumption 1 and $\overline{\alpha}_2 \in [0.5, 1)$, the optimal parameters of the critical level policy are obtained from the following system of equations:*

(a) If $\alpha_1(r_2^0, 0) < \overline{\alpha}_1$:

$$r^* - C^* = \mu L + z_{\overline{\alpha}_2}\ \sigma\sqrt{L}, \tag{26}$$

$$\int_0^L \mathbb{P}(D_1(L-\tau) \leq C^*)\ f_{H,D}^{r^*-C^*}(\tau)\ d\tau = \overline{\alpha}_1 - \overline{\alpha}_2, \tag{27}$$

and the service levels provided to each class are equal to their preset levels, i.e., $\alpha_i(r^*, C^*) = \overline{\alpha}_i,\ i = 1,2$.

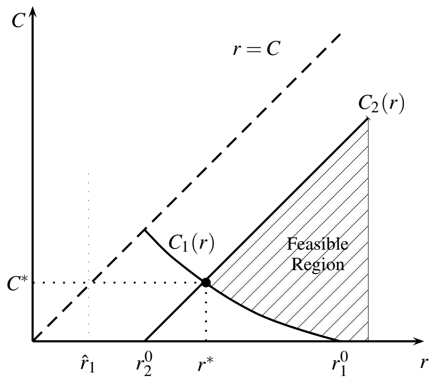
(b) If $\alpha_1(r_2^0, 0) \geq \overline{\alpha}_1$:

$$C^* = 0, \tag{28}$$

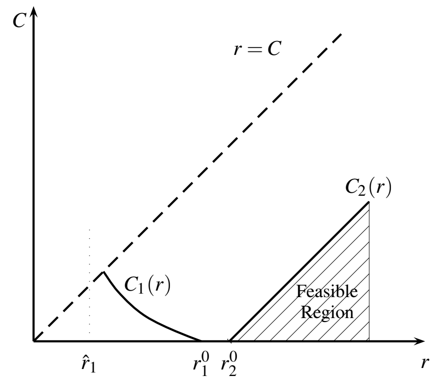
$$r^* = \mu L + z_{\overline{\alpha}_2}\ \sigma\sqrt{L}, \tag{29}$$

and service levels provided to each class are: $\alpha_1(r^*, 0) \geq \overline{\alpha}_1$ and $\alpha_2(r^*, 0) = \overline{\alpha}_2$ for high and low priority class respectively.

Proof Under normally distributed demand, $C_2(r) = r - r_2^0 = r - \mu L - z_{\overline{\alpha}_2}\, \sigma\sqrt{L}$, and continues to be increasing and linear in r. On the other hand, from proposition 4 we can obtain that $C_1(r)$ is solution of $\alpha_1(r,C) = \overline{\alpha}_1$ and under assumption 1 we can conclude that $C_1(r)$ is strictly decreasing in r at least from some $r > C \geq \hat{r}_1$ until $r \leq r_1^0$. Then, under normally distributed demand, the feasible region of **(SLP0)** problem using normally distributed demand where all (r,C) satisfy that $\alpha_1(r,C) \geq \overline{\alpha}_1$, $\alpha_2(r,C) \geq \overline{\alpha}_2$, with $\overline{\alpha}_2 \in [0.5,1)$, and $r > C \geq 0$, is the same as defined for the proof of proposition 3, but in this case, it can take two different forms, shown in figure 2. If $\alpha_1(r_2^0,0) < \overline{\alpha}_1$, then $r_2^0 < r_1^0$ which induces the first feasible regions shown in figure (2a) when $\overline{\alpha}_2 \in [0.5,1)$. If $\alpha_1(r_2^0,0) \geq \overline{\alpha}_1$, then $r_2^0 > r_1^0$, which induces a second feasible region shown in figure (2b) when $\overline{\alpha}_2 \in [0.5,1)$. Note that $r_2^0 \geq \mu L > \hat{r}_1$ when $\overline{\alpha}_2 \in [0.5,1)$.



(a) $\alpha_1(r_2^0,\, 0) < \overline{\alpha}_1$ and $\overline{\alpha}_2 \in [0.5,1)$



(b) $\alpha_1(r_2^0,\, 0) \geq \overline{\alpha}_1$ and $\overline{\alpha}_2 \in [0.5,1)$

Fig. 2: Feasible regions for **SLP0** problem using normally distributed demand and $\overline{\alpha}_2 \in [0.5,1)$

The figure (2a) shows that the optimal reorder point r^* of the **(SLP0)** problem using normally distributed demand occurs when $C_2(r) = C_1(r) = C^*$. Therefore, the optimal parameters of the critical level policy are obtained from the equation system formed by $\alpha_1(r,C) = \overline{\alpha}_1$ and $\alpha_2(r,C) = \overline{\alpha}_2$. From figure (2b) we conclude that the minimum reorder point that guarantees a service level $\overline{\alpha}_1$ provided to high priority class and a service level $\overline{\alpha}_2$ provided to the low priority class is r_2^0. Therefore, $r^* = r_2^0 = \mu L + z_{\overline{\alpha}_2}\sigma\sqrt{L}$ and $C^* = 0$. □

Some consequences of the above proof are: (i) given equation (26), the equation (27) only depends on C^*; (ii) if $\alpha_1(r_2^0,0) < \overline{\alpha}_1$, then r_2^0 is a lower bound of **(SLP0)** problem using normally distributed demand when $\overline{\alpha}_2 \in [0.5,1)$; (iii) if $\alpha_1(r_2^0,0) < \overline{\alpha}_1$ and $\overline{\alpha}_2 \in [0.5,1)$, then $r^* > C^*$, because $r_2^0 \geq \mu L > 0$, therefore, constraint (17) may be replaced by: $r,C \geq 0$; and (iv) if $\alpha_1(r_2^0,0) < \overline{\alpha}_1$ and $\overline{\alpha}_2 \in [0.5,1)$, then $C^* > 0$, because $0 < r_2^0 < r_1^0$ when $\overline{\alpha}_2 \in [0.5,1)$.

The proposition 5 is a general solution for the **(SLP0)** problem using normally distributed demand under assumption 1 and $\overline{\alpha}_2 \in [0.5,1)$. On the other hand, similar to the general solution in the case with demands with non-negative support, it can be difficult to compute the critical level C^* from equation (27).

The following results compare the reorder point induced by the critical level policy with the reorder point induced by the round-up policy and separate stock policy. Let r_u be the reorder point induce by the round-up policy and r_s be the reorder point induced by the separate stock policy. The reorder point of the round-up policy is obtained from $F_{D(L)}(r_u) = \overline{\alpha}_1$ and the reorder point of the separate-stock policy is obtained from $r_s = r_1^s + r_2^s$, where r_1^s is solution of $F_{D_1(L)}(r_1^s) = \overline{\alpha}_1$ and r_2^s is solution of $F_{D_2(L)}(r_2^s) = \overline{\alpha}_2$. Under normally distributed demand, $r_u = \mu L + z_{\overline{\alpha}_1}\sigma\sqrt{L}$, and $r_s = \mu L + z_{\overline{\alpha}_1}\sigma_1\sqrt{L} + z_{\overline{\alpha}_2}\sigma_2\sqrt{L}$, where $z_{\overline{\alpha}_1}$ is the inverse standard normal distribution for a preset service level $\overline{\alpha}_1$. Note that, under normal distributed demand, $r_u \leq r_s$ if $z_{\overline{\alpha}_1} \leq z_{\overline{\alpha}_2}\frac{\sigma_2}{\sigma-\sigma_1}$ and that $\frac{\sigma_2}{\sigma-\sigma_1} > 1$.

Proposition 6 *Under normally distributed demand, the assumption 1 and $\overline{\alpha}_2 \in [0.5,1)$, the optimal reorder point of the critical level policy is strictly less than the reorder point induced by the round-up policy, i.e., $r^* < r_u$, and strictly less than the reorder point induced by the separate stock policy, i.e., $r^* < r_s$, when $\alpha_1(r_2^0,\, 0) \geq \overline{\alpha}_1$,*

Proof From equation (13) we note that $\alpha_1(r_u,0) > \overline{\alpha}_1$ because we assume that the lead time and the parameters of the demand per unit time of both classes are finite and $\overline{\alpha}_1 < 1$. From propositions 5(a) we obtained that $\alpha_1(r^*,C^*) = \overline{\alpha}_1$ and $r^* > C^*$, and from assumption 1 we derive that $\alpha_1(r^*,C^*) = \overline{\alpha}_1 \geq \alpha_1(r^*,0)$. Therefore, it holds that $\alpha_1(r_u,0) > \overline{\alpha}_1 \geq \alpha_1(r^*,0)$ and from proposition 4 we conclude that $r_u > r^*$. On the other hand, from proposition 5(b) we obtained that $\alpha_2(r^*,0) = F_{D(L)}(r^*) = \overline{\alpha}_2$. By definition, $\overline{\alpha}_1 > \overline{\alpha}_2$, then $F_{D(L)}(r_u) > F_{D(L)}(r^*)$, and we conclude that $r_u > r^*$.

Following similar logic to compare the optimal reorder point of the critical level policy with respect to the reorder point induced by the separate stock policy, from proposition 5(b) we obtain that $r^* = \mu L + z_{\overline{\alpha}_2}\sigma\sqrt{L}$. Since $\overline{\alpha}_1 > \overline{\alpha}_2$, we conclude that $r^* < r_s$ from the triangle inequality. $\square$

Unfortunately, we have not found a simple proof that the reorder point induced by the critical level policy is strictly less than the reorder point induced by separate stock policy when $\alpha_1(r_2^0,\, 0) < \overline{\alpha}_1$ and $\overline{\alpha}_2 \in [0.5,1)$. The derivation requires checking that

$$\int_0^L \mathbb{P}\left(D_1(L-\tau) \leq z_{\overline{\alpha}_1}\sigma_1\sqrt{L} + z_{\overline{\alpha}_2}\sigma_2\sqrt{L} - z_{\overline{\alpha}_2}\sigma\sqrt{L}\right)\; f_{H,D}^{r_2^0-0}(\tau)\;\partial\tau > \overline{\alpha}_1 - \overline{\alpha}_2\;, \tag{30}$$

which is not much different from solving the system of equations (26)-(27) and seeing if $r^* < r_s$.

6 Computational study

In this section, we present our numerical study and its results. The main objective of the computational study is to show how good is the performance of our solution approach, compared the critical level policy with the separate stock and round-up policies and provide numerical evidence to validate assumption 1.

For simplicity, we use normally distributed demand as an approximation to the non-negative demand, solving **RSLP**, from which we obtain a lower bound of **SLP**. Let (Q^*, r^*, C^*) be the optimal critical level policy controls of **RSLP**; $AC_{RSLP}(Q^*, r^*, C^*)$ be the objective function of **RSLP**; and $AC_{SLP}(Q^*, r^*, C^*)$ be the objective function of **SLP** given the optimal critical level policy controls of **RSLP**. Note that $AC_{RSLP}(Q^*, r^*, C^*) = LB$, $AC_{SLP}(Q^*, r^*, C^*) = UB$ and $UB > LB$. In order to evaluate the performance of our solution, we carried out several test problems and computed the percentage of optimality gap, $Gap(\%)$, expressed as $100 \times (UB - LB)/LB$.

Recall that **RSLP** is separable in the **EOQ** and **SLP0** problems. Then, for each test problem we determine the replenishment batch solving the EOQ model, i.e., $Q^* = \sqrt{2\mu S/h}$, and the reorder point r^* and the optimal critical level C^* solving the system of non-linear equations given in proposition 5.

The equation systems of proposition 5 were programmed by a C code using Brent-Dekker method. Backorders in the steady state given by equations (19) and (20) were also programmed in C code, like the numerical experiments to validate assumption 1. All test were carried on a PC with Intel Core i7 2.3 GHz processor and 16 GB RAM. The time to compute the parameters of the critical level policy are on average 0.0011 seconds and in the worst case 0.0019 seconds.

6.1 Experimental result for (SLP) problem using normally distributed demand

In order to cover a wide range of data, we design a set of 10 experiments to evaluate the performance of our solution approach and to compare the critical level policy with the separate stock and round-up policies. In each experiment we fix the preset service levels $\overline{\alpha}_1$ and $\overline{\alpha}_2$, and consider a base case with the following parameters: normal demand distributions with mean $\mu_1 = \mu_2 = 25$ and coefficient of variation $CV_1 = CV_2 = 0.2$ ($\sigma_1^2 = \sigma_2^2 = 25$), lead time $L = 5$, ordering cost $S = 300$ and holding cost per unit and unit time $h = 0.75$. We conduct experiments studying the sensitivity of the solutions to changing parameters CV_i, μ_i, S, and h. This gives a total of 135 experiments for each setting of the preset service levels.

Our numerical results show that our solution approach is able to provide good-quality solutions that are on average 0.3% and at worst 7.8% from the optimal solution. Table 1 show the average and maximum relative gap over 45 instances for the ten settings of preset service levels and different values of S.

Table 1 shows that the maximum relative gap occurs when there is maximum difference between the preset service levels and the ordering cost is minimal ($S = 100$). As an example, table 2 shows the relative optimality gap for the 135 problems of the experiment: $\overline{\alpha}_1 = 0.975$ and $\overline{\alpha}_2 = 0.75$.

The pattern of behavior of the relative optimality gap observed in table 2 is repeated for all ten experiments, i.e., the relative gap is decreasing in S and increasing in h. The maximum gap occurs when class 2 dominates on mean and variance ($\mu_2 = 100$ and $CV_2 = 0.6$), the ordering cost is minimal ($S = 100$) and the holding cost per unit and unit time is maximum ($h = 1.25$). Note that we obtain Q from EOQ problem, therefore, for a low ordering cost and high holding cost per unit and unit time, we hope a low batch size. Then, for a low batch size and domain of low priority class in

		Gap(%)					
		$S=100$		$S=300$		$S=500$	
$\overline{\alpha}_1$	$\overline{\alpha}_2$	Average	Max	Average	Max	Average	Max
0.975	0.55	1.37	7.81	0.55	3.36	0.35	2.15
0.975	0.65	0.81	4.39	0.33	1.95	0.21	1.26
0.975	0.75	0.44	2.26	0.18	1.02	0.12	0.67
0.975	0.85	0.19	0.93	0.08	0.43	0.05	0.28
0.975	0.95	0.05	0.18	0.02	0.09	0.01	0.06
0.800	0.75	0.87	3.03	0.34	1.32	0.22	0.85
0.850	0.75	0.70	2.80	0.28	1.22	0.18	0.79
0.900	0.75	0.58	2.59	0.23	1.14	0.15	0.74
0.950	0.75	0.48	2.38	0.20	1.06	0.13	0.69
0.999	0.75	0.38	2.04	0.16	0.93	0.10	0.61

Table 1: Optimality Gap(%) between lower and upper bounds

			Gap(%)								
			$\mu_1=100, \mu_2=25$			$\mu_1=\mu_2=25$			$\mu_1=25, \mu_2=100$		
CV_1	CV_2	h	$S=100$	$S=300$	$S=500$	$S=100$	$S=300$	$S=500$	$S=100$	$S=300$	$S=500$
0.2	0.2	0.25	0.03	0.01	0.01	0.02	0.01	0.00	0.11	0.04	0.02
		0.75	0.07	0.03	0.02	0.06	0.02	0.01	0.29	0.11	0.06
		1.25	0.11	0.04	0.03	0.09	0.03	0.02	0.46	0.17	0.11
0.4	0.4	0.25	0.09	0.03	0.02	0.07	0.03	0.02	0.35	0.13	0.08
		0.75	0.21	0.09	0.06	0.19	0.07	0.04	0.91	0.35	0.22
		1.25	0.30	0.13	0.09	0.29	0.11	0.07	1.36	0.55	0.35
0.6	0.6	0.25	0.16	0.06	0.04	0.14	0.05	0.03	0.67	0.25	0.16
		0.75	0.35	0.16	0.10	0.36	0.14	0.09	1.60	0.67	0.43
		1.25	0.48	0.23	0.16	0.53	0.22	0.14	2.26	1.02	0.67
0.6	0.2	0.25	0.15	0.06	0.04	0.09	0.03	0.02	0.15	0.05	0.03
		0.75	0.34	0.15	0.10	0.22	0.09	0.05	0.40	0.15	0.09
		1.25	0.46	0.22	0.15	0.34	0.14	0.09	0.62	0.24	0.15
0.2	0.6	0.25	0.04	0.01	0.01	0.09	0.03	0.02	0.65	0.24	0.15
		0.75	0.10	0.04	0.02	0.23	0.09	0.05	1.57	0.65	0.41
		1.25	0.15	0.06	0.04	0.35	0.14	0.09	2.23	0.99	0.65

Table 2: Optimality Gap(%) when $\overline{\alpha}_1 = 0.975$ and $\overline{\alpha}_2 = 0.75$

mean and variance, we expected high backorders class 2 for the critical level policy. Since our solution approach is based on a relaxation which despises backorders of class 2, we expect a high gap between lower and upper bound when the parameters induce high backorders of class 2.

In the next set of results we compare the efficiency of the critical level policy obtained with the proposed approach against the separate stock and round-up policies. For every one of the 135 problem parameters considered above and ten preset service levels settings, we determine the three different policies and compute for each the operational costs. As we expected, the critical level policy outperformed both the separate stock and round-up policies in the 1350 problems considered. Table 3 shows the average and maximum relative benefit of the critical level policy with respect to the round-up and separate stock for the 10 settings of preset service levels and different values of S.

Table 3 shows that in all experiments, the average relative benefit is greater with respect to the separate stock policy, but the maximum relative benefit is reached when comparing against the round-up policy. We also note that the relative benefit to the round-up is more sensitive and, by contrast, using two separate lot sizes and two separate reorder points causes a more homogeneous benefit. The maximum relative benefit, with respect to round-up or separate stock policies, occurs when there is maximum difference between the preset service levels and the ordering cost is minimal

		Benefit (%) vs Round-up					
		$S = 100$		$S = 300$		$S = 500$	
$\overline{\alpha}_1$	$\overline{\alpha}_2$	Average	Max	Average	Max	Average	Max
0.975	0.55	18.27	46.88	13.45	38.43	11.34	34.07
0.975	0.65	16.01	41.10	11.73	33.44	9.87	29.57
0.975	0.75	13.38	34.16	9.76	27.62	8.20	24.36
0.975	0.85	9.96	24.95	7.24	20.06	6.08	17.65
0.975	0.95	4.04	8.97	2.93	7.19	2.46	6.33
0.800	0.75	3.74	8.08	2.52	5.90	2.04	4.96
0.850	0.75	5.89	14.07	4.04	10.63	3.30	9.07
0.900	0.75	8.24	20.45	5.77	15.84	4.76	13.68
0.950	0.75	11.21	28.41	8.04	22.53	6.70	19.68
0.999	0.75	19.63	48.15	15.02	40.94	13.03	37.06

		Benefit (%) vs Separate stock					
		$S = 100$		$S = 300$		$S = 500$	
$\overline{\alpha}_1$	$\overline{\alpha}_2$	Average	Max	Average	Max	Average	Max
0.975	0.55	24.56	32.43	25.27	31.08	25.53	30.79
0.975	0.65	24.55	31.65	25.24	30.62	25.51	30.40
0.975	0.75	24.55	30.79	25.22	30.09	25.49	29.96
0.975	0.85	24.56	29.77	25.21	29.43	25.47	29.41
0.975	0.95	24.61	30.10	25.20	29.88	25.45	29.78
0.800	0.75	26.28	31.21	26.43	30.52	26.49	30.28
0.850	0.75	25.99	30.82	26.23	30.29	26.32	30.09
0.900	0.75	25.59	30.38	25.96	30.04	26.10	29.91
0.950	0.75	25.02	30.45	25.56	30.11	25.77	29.97
0.999	0.75	22.97	32.14	24.05	30.18	23.67	24.67

Table 3: Benefit of the critical level vs. Round-up and Separate stock policies

($S = 100$). As an example, table 4 shows the relative benefit regarding round-up and separate stock for the 135 problems of the experiment: $\overline{\alpha}_1 = 0.975$ and $\overline{\alpha}_2 = 0.75$.

The pattern of the maximum relative benefit regarding round-up policy, observed in table 4, is repeated for all ten experiments, i.e., the maximum benefit occurs when the class 2 dominates on mean and variance ($\mu_2 = 100$, $CV_2 = 0.6$), the ordering cost is minimal ($S = 100$) and the holding cost per unit and unit time is maximum ($h = 1.25$). Clearly, the round-up policy is highly inefficient when the class 2 dominates mean and variance, because under this situation, this policy provides too much inventory to the low priority class causing a high reorder point and therefore a high cost. On the other hand, when ordering cost is low and holding cost per unit and unit time is high, batch sizes are small and the expected backorder increases. We observe that the expected backorders induced by the critical level are greater than those induced by the round-up policy, but its effect on cost is relatively low compared with the effect of the reorder point. Note that, as Deshpande et al (2003) observed, the relative benefit regarding Round-up is decreasing in S.

Finally, we analyze how the preset service levels impact the total cost. From equations (26) and (27) we conclude that increasing the preset service level of the high priority class causes the optimal critical level C^* and reorder point r^* to increase. Consequently, we expect an increase in the holding and total costs. In the same way, we conclude from equations (26) and (27) that increasing the preset service level of the low priority class causes the optimal reorder point to increase and the optimal critical level to decrease. Therefor, we expect an increase in the holding cost per unit time, but smaller than when $\overline{\alpha}_1$ increases. Table 5 shows how $AC_{SLP}(Q^*, r^*, C^*)$ in-

			Benefit(%) vs Round-up								
			$\mu_1 = 100, \mu_2 = 25$			$\mu_1 = \mu_2 = 25$			$\mu_1 = 25, \mu_2 = 100$		
CV_1	CV_2	h	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$
0.2	0.2	0.25	2.97	1.90	1.52	4.33	2.65	2.10	11.59	7.41	5.93
		0.75	4.40	2.97	2.43	6.80	4.33	3.46	17.17	11.59	9.47
		1.25	5.16	3.59	2.97	8.26	5.37	4.33	20.14	14.02	11.59
0.4	0.4	0.25	4.93	3.38	2.78	7.53	4.84	3.89	18.87	12.95	10.64
		0.75	6.67	4.93	4.17	11.04	7.53	6.17	25.54	18.87	15.97
		1.25	7.48	5.73	4.93	12.89	9.07	7.53	28.60	21.93	18.87
0.6	0.6	0.25	6.32	4.57	3.84	9.98	6.69	5.45	23.83	17.24	14.47
		0.75	8.09	6.32	5.49	13.91	9.98	8.35	30.43	23.83	20.70
		1.25	8.83	7.16	6.32	15.80	11.75	9.98	33.21	26.97	23.83
0.6	0.2	0.25	6.04	4.35	3.65	7.36	4.79	3.86	12.48	8.12	6.54
		0.75	7.75	6.04	5.24	10.65	7.36	6.07	18.06	12.48	10.28
		1.25	8.48	6.85	6.04	12.34	8.82	7.36	20.91	14.95	12.48
0.2	0.6	0.25	3.76	2.44	1.97	9.15	5.95	4.79	24.36	17.55	14.71
		0.75	5.45	3.76	3.10	13.25	9.15	7.53	31.24	24.36	21.12
		1.25	6.31	4.50	3.76	15.37	10.96	9.15	34.16	27.62	24.36

			Benefit(%) vs Separate stock								
			$\mu_1 = 100, \mu_2 = 25$			$\mu_1 = \mu_2 = 25$			$\mu_1 = 25, \mu_2 = 100$		
CV_1	CV_2	h	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$
0.2	0.2	0.25	24.06	24.59	24.77	29.25	29.27	29.27	24.62	24.95	25.06
		0.75	23.32	24.06	24.33	29.23	29.25	29.26	24.12	24.62	24.79
		1.25	22.89	23.75	24.06	29.21	29.24	29.25	23.81	24.41	24.62
0.4	0.4	0.25	23.17	23.94	24.23	29.16	29.21	29.23	23.99	24.53	24.71
		0.75	22.20	23.17	23.56	29.10	29.16	29.19	23.28	23.99	24.26
		1.25	21.71	22.74	23.17	29.07	29.14	29.16	22.90	23.68	23.99
0.6	0.6	0.25	22.56	23.45	23.80	29.06	29.14	29.17	23.54	24.18	24.42
		0.75	21.56	22.56	23.00	28.98	29.06	29.10	22.79	23.54	23.86
		1.25	21.10	22.10	22.56	28.94	29.02	29.06	22.42	23.20	23.54
0.6	0.2	0.25	21.56	22.79	23.26	29.96	29.73	29.64	28.43	27.33	26.95
		0.75	20.12	21.56	22.16	30.25	29.96	29.84	29.95	28.43	27.87
		1.25	19.43	20.90	21.56	30.39	30.09	29.96	30.79	29.09	28.43
0.2	0.6	0.25	23.42	24.16	24.42	25.40	26.89	27.39	18.93	21.27	22.10
		0.75	22.46	23.42	23.79	23.25	25.40	26.16	15.88	18.93	20.11
		1.25	21.96	23.00	23.42	22.03	24.48	25.40	14.27	17.59	18.93

Table 4: Benefit(%) vs. Round-up and Separate stock when $\overline{\alpha}_1 = 0.975$ and $\overline{\alpha}_2 = 0.75$

creases, for different input parameters, when $\overline{\alpha}_2 = 0.75$ and the preset service level of the high priority class increase from $\overline{\alpha}_1 = 0.975$ to $\overline{\alpha}_1 = 0.999$.

			Increase (%) the total cost								
			$\mu_1 = 100, \mu_2 = 25$			$\mu_1 = \mu_2 = 25$			$\mu_1 = 25, \mu_2 = 100$		
CV_1	CV_2	h	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$
0.2	0.2	0.25	10.06	6.59	5.32	4.28	2.62	2.07	2.98	1.83	1.45
		0.75	14.47	10.06	8.32	6.75	4.28	3.42	4.65	2.98	2.39
		1.25	16.72	12.01	10.06	8.20	5.32	4.28	5.62	3.69	2.98
0.4	0.4	0.25	16.22	11.51	9.59	7.96	5.12	4.11	5.37	3.48	2.80
		0.75	21.24	16.22	13.95	11.71	7.96	6.52	7.79	5.37	4.42
		1.25	23.47	18.56	16.22	13.70	9.60	7.96	9.04	6.44	5.37
0.6	0.6	0.25	20.36	15.26	13.02	11.03	7.39	6.02	7.29	4.94	4.04
		0.75	25.21	20.36	17.98	15.41	11.03	9.22	10.02	7.29	6.13
		1.25	27.19	22.69	20.36	17.54	13.00	11.03	11.30	8.53	7.29
0.6	0.2	0.25	20.31	15.19	12.94	10.13	6.64	5.36	5.53	3.49	2.78
		0.75	25.21	20.31	17.91	14.55	10.13	8.38	8.35	5.53	4.48
		1.25	27.22	22.66	20.31	16.80	12.09	10.13	9.89	6.74	5.53
0.2	0.6	0.25	11.00	7.32	5.95	6.57	4.22	3.38	4.77	3.18	2.58
		0.75	15.48	11.00	9.17	9.70	6.57	5.38	6.67	4.77	3.98
		1.25	17.71	13.01	11.00	11.35	7.94	6.57	7.59	5.62	4.77

Table 5: $AC_{SLP}(Q^*, r^*, C^*)$ increase (%) between using $\bar{\alpha}_1 = 0.999$ and $\bar{\alpha}_1 = 0.975$.

From table 5 we observe that increase the service level of the high priority class from $\overline{\alpha}_1 = 0.975$ to $\overline{\alpha}_1 = 0.999$, increases the total cost in 9.8% average, and as expected, the maximum increase of $AC_{SLP}(Q^*,r^*,C^*)$ occurs when the class 1 dominates on mean and variance ($\mu_1 = 100$ and $CV_1 = 0.6$), the ordering cost is minimal ($S = 100$) and the holding cost per unit and unit time is maximum ($h = 1.25$). This is because, when class 1 is larger in mean and variance, more items are reserved for the high priority class and the threshold level C increases. On the other hand, a low ordering cost and high holding cost per unit and unit time produce a small batch size and high backorders for class 2. Then, the high class 2 backorders cause the holding cost per unit time to increase. The result is a higher holding cost and thus a higher total cost. Table 6 highlights how $AC_{SLP}(Q^*,r^*,C^*)$ increases for different imput parameters, when $\overline{\alpha}_1 = 0.975$ and the preset service level of low priority class increase from $\overline{\alpha}_2 = 0.75$ to $\overline{\alpha}_2 = 0.85$.

			Increase (%) the total cost								
			$\mu_1 = 100, \mu_2 = 25$			$\mu_1 = \mu_2 = 25$			$\mu_1 = 25, \mu_2 = 100$		
CV_1	CV_2	h	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$	$S = 100$	$S = 300$	$S = 500$
0.2	0.2	0.25	0.73	0.46	0.37	1.20	0.73	0.57	3.50	2.17	1.72
		0.75	1.08	0.73	0.59	1.92	1.20	0.95	5.40	3.50	2.82
		1.25	1.27	0.88	0.73	2.35	1.50	1.20	6.47	4.31	3.50
0.4	0.4	0.25	1.07	0.73	0.60	2.06	1.30	1.04	5.97	3.93	3.17
		0.75	1.44	1.07	0.90	3.09	2.06	1.68	8.47	5.97	4.95
		1.25	1.61	1.24	1.07	3.64	2.51	2.06	9.69	7.09	5.97
0.6	0.6	0.25	1.23	0.89	0.74	2.70	1.77	1.43	7.78	5.37	4.41
		0.75	1.55	1.23	1.07	3.85	2.70	2.24	10.41	7.78	6.61
		1.25	1.68	1.38	1.23	4.41	3.21	2.70	11.58	9.00	7.78
0.6	0.2	0.25	1.14	0.82	0.69	1.88	1.21	0.97	3.67	2.32	1.85
		0.75	1.44	1.14	0.99	2.77	1.88	1.54	5.52	3.67	2.98
		1.25	1.56	1.29	1.14	3.24	2.27	1.88	6.52	4.47	3.67
0.2	0.6	0.25	0.96	0.62	0.50	2.69	1.71	1.37	8.15	5.58	4.58
		0.75	1.40	0.96	0.79	4.00	2.69	2.19	10.99	8.15	6.90
		1.25	1.62	1.15	0.96	4.70	3.26	2.69	12.27	9.46	8.15

Table 6: $AC_{SLP}(Q^*,r^*,C^*)$ increase (%) between using $\overline{\alpha}_2 = 0.85$ and $\overline{\alpha}_2 = 0.75$

From table 6 we observe that increase the service level of the low priority class from $\overline{\alpha}_2 = 0.75$ to $\overline{\alpha}_2 = 0.85$, increases the total cost in 3% average, and as expected, the maximum increase of $AC_{SLP}(Q^*,r^*,C^*)$ occurs when the class 2 dominates on mean and variance ($\mu_2 = 100$ and $CV_2 = 0.6$), the ordering cost is minimal ($S = 100$) and the holding cost per unit and unit time is maximum ($h = 1.25$). This is because, high class 2 backorders are produced in the critical level policy when class 2 dominates and the batch size is small.

The results obtained by using normally distributed demand are similar to those obtained by using a Poisson process (obtained from Deshpande et al (2003)). However, using normally distributed demand as an approximation of strictly increasing non-negative demand allows us to observe the effect of changes in variance on the critical level policy, i.e., how changing the coefficient of variation CV (ratio of standard deviation to mean) affects our results. For any parameters setting that we tested, we observe that increasing σ_i^2, equivalent to increasing CV_i for a fixed μ_i, causes an increase in $AC_{SLP}(Q^*,r^*,C^*)$. This is because, we expect high backorders and large reorder point r and critical level C, when variance of class i increases for $i = 1$ or 2. Furthermore, we observe from table 4 that:

– benefit with respect to round-up is increasing in CV_i for $i = 1,2$,

– benefit with respect to separate stock is decreasing in CV_1 when μ_1 dominates and increasing when $\mu_1 = \mu_2$ or μ_2 dominates,
– benefit with respect to separate stock is decreasing in CV_2.

6.2 Numerical evidence: assumption 1

Proposition 5 that characterizes the optimal critical level solution in the case of normally distributed demand makes the assumption that $\alpha_1(r,C)$ is increasing in $C \in [0,r)$ for a large enough r (assumption 1). In order to numerically validate this assumption and cover a wide range of data, a set of 8 experiments were designed. In each experiment we generate 100000 random sets of $\{L,\mu_1,CV_1,\mu_2,CV_2\}$, within predefined limits that appear in table 7. For each randomly generated set of parameters, we also generate a random reorder point in the interval $[\mu L, \mu L + z_{0.9999}\sigma\sqrt{L}]$ and a random critical level in the interval $[0,r)$. Then, for each random set $\{L,\mu_1,CV_1,\mu_2,CV_2,r,C\}$ we evaluate $\frac{\partial\alpha_1(r,C)}{\partial C}$ and $\alpha_1(r,0)$. We provide the expression for $\frac{\partial\alpha_1(r,C)}{\partial C}$ in (31) in the appendix. For each experiment we obtain the minimum $\alpha_1(r,0)$ such that the service level provided to the high priority class is increasing in C, i.e., $\min\{\alpha_1(r,0) \mid \partial\alpha_1(r,C)/\partial C \geq 0\}$. The first experiment randomly vary the parameters within the limits of the base case. Then, the limits of these parameters are varied. Table 7 shows the parameters limits at each experiment and the result obtained.

Exp.	L	μ_1	CV_1	μ_2	CV_2	$\{\min\{\alpha_1(r,0) \mid \partial\alpha_1(r,C)/\partial C \geq 0\}\}$
1	[1,5]	(0,25]	(0,0.2]	(0,25]	(0,0.2]	0.4268
2	[1,25]	**(25,100]**	(0,0.2]	(0,25]	(0,0.2]	0.4206
3	[1,25]	(0,25]	**(0.2,2]**	(0,25]	(0,0.2]	0.3334
4	[1,25]	(0,25]	(0,0.2]	**(25,100]**	(0,0.2]	0.3715
5	[1,25]	(0,25]	(0,0.2]	(0,25]	**(0.2,2]**	0.4813
6	[1,25]	**(25,100]**	(0,0.2]	**(25,100]**	(0,0.2]	0.3642
7	[1,25]	(0,25]	**(0.2,2]**	(0,25]	**(0.2,2]**	0.4346
8	[1,25]	**(25,100]**	**(0.2,2]**	**(25,100]**	**(0.2,2]**	0.4535

Table 7: Numerical evidence for assumption 1

Based on the results shown in Table 7, we infer that for any $r \geq \hat{r}_1$, with $\hat{r}_1$ solution of $\alpha_1(\hat{r}_1,0) = 0.5$, the function $\alpha_1(r,C)$ is an increasing function of C in the interval $C \in [0,r)$, i.e., we infer that assumption 1 seems to be valid at least within the limits of the experiments of Table 7.

7 Conclusions

In this paper we analyzed the constant critical level policy for fast-moving items when rationing is used to provide differentiated service levels to two demand classes (high and low priority). The inventory system operates under continuous review (Q,r) policy, with a critical threshold value C, full-backorder, deterministic lead time, and the service level provided to each class is measured by service level type 1.

Using the hitting time approach and the threshold clearing mechanism to satisfy backorders when multiple outstanding orders exist, we develop expressions for service levels type I under rationing and expected backorders of high and low priority classes. We formulate a service level problem as a nonlinear problem with chance constraints (service level constraints) to determine the optimal parameters of the critical level policy. We propose to optimally solve a relaxation, which allows us to obtain good-quality bounds. For strictly increasing non-negative demand, we characterize the optimal solution of the relaxed service level problem through a system of equations and, under mild assumptions, when normally distributed demand is used as approximation of the non-negative demand.

The computational results show that our solution approach can find good-quality solutions that are on average 0.3% and at worst 7.8% from the optimal solution. Given the nature of our relaxation, the maximum gap(%) occurs when the class 2 dominates on mean and variance, the ordering cost is minimal, the holding cost per unit and unit time is maximum and difference between the preset service levels is maximum.

As expected, the critical level policy outperformed both the separate stock and round-up policies in terms of total cost. Using normally distributed demand as an approximation of strictly increasing non-negative demand allows us to observe the effect of varying the coefficient of variation of the demand distribution, situation that is not possible with Poisson demand process. We observe that the benefit of critical level policy with respect round-up is increasing in the variance of the demand distributions for both classes, and the benefit with respect separate stock is decreasing in the variance of class 2 demand, decreasing in the variance of class 1 demand when the mean of class 1 demand is larger and increasing when the mean of class 2 dominates. In addition, we observe the following managerial insights:

- *the average savings induced by the critical level policy are greater with respect to separate stock, but the maximum savings are achieved when comparing to round up policy.*
- *critical level policy leads to significant savings with respect to round-up when class 2 dominates on mean and variance, the ordering cost is minimal, holding cost per unit and unit time is maximum and difference between preset service levels is maximum.*
- *critical level policy leads to significant savings with respect to separate stock when class 2 dominates on mean, class 1 dominates on variance, the ordering cost is minimal, holding cost per unit and unit time is maximum and difference between preset service levels is maximum.*
- *the cost of increasing the service level of the high priority class is significantly greater than the cost of increasing the service level of the low priority class.*

There are a number of questions and issues left for future research. The first one, is to solve exactly the SLP problem or solve a relaxation that does not drop backorders. Second is to expand the results to more than two classes. Third is to broaden the measures of service level. For instance we could use the fill-rate as service level measures, leading to different problems and therefore different solutions. In particular, the fill-rate or ready rate depend of the replenishment batch quantity, therefore, although we consider a relaxation, the service level problem is not separable as in our case. Therefore, the problem becomes more difficult to solve because the replenishment batch quantity, the reorder point and the critical level must be optimized jointly in the same service level problem. Another line of future work is to propose a cost optimization problem where backorders of each class are penalized with different cost.

References

Arslan H, Graves SC, Roemer T (2007) A single-product inventory model for multiple demand classes. Management Science 53:1486–1500

Axsäter S (2006) Inventory Control. Springer's, USA

Browne S, Zipkin P (1991) Inventory models with continuous, stochastic demands. The Annals of Applied Probability pp 419–435

Bulut Ö, Fadiloğlu MM (2011) Production control and stock rationing for a make-to-stock system with parallel production channels. IIE Transactions 43(6):432–450

De Vericourt F, Karaesmen F, Dallery Y (2000) Dynamic scheduling in a make-to-stock system: A partial characterization of optimal policies. Operations Research 48(5):811–819

De Vericourt F, Karaesmen F, Dallery Y (2002) Optimal stock allocation for a capacitated supply system. Management Science 48(11):1486–1501

Dekker R, Kleijn MJ, De Rooij P (1998) A spare parts stocking policy based on equipment criticality. International Journal of Production Economics 56:69–77

Dekker R, Hill RM, Kleijn MJ, Teunter RH (2002) On the (s-1,s) lost sales inventory model with priority demand classes. Naval Research Logistics 49:593–610

Deshpande V, Cohen MA, Donohue K (2003) A threshold inventory rationing policy for service-differentiated demand classes. Management Science 49:683–703

Eppen G (1979) Effect of centralization on expected cost in a multi-location newsboy problem. Magement Science 25:498–501

Fadiloglu M, Bulut O (2010) A dynamic rationing for continuous-review inventory systems. European Journal of Operational Research 202:675–685

Fadıloğlu MM, Bulut Ö (2010) An embedded markov chain approach to stock rationing. Operations Research Letters 38(6):510–515

Ha AY (1997a) Inventory rationing in a make-to-stock production system with several demand classes and lost sales. Management Science 43(8):1093–1103

Ha AY (1997b) Stock-rationing policy for a make-to-stock production system with two priority classes and backordering. Naval Research Logistics (NRL) 44(5):457–472

Ha AY (2000) Stock rationing in an m/ek/1 make-to-stock queue. Management Science 46(1):77–87

Isotupa KS (2006) An (s,q) markovian inventory system with lost sales and two demand classes. Mathematical and Computer Modelling 43:687–694

Kleijn M, Dekker R (1999) An Overview of Inventory Systems with Several Demand Classes. Lecture Notes in Economics and Mathematical Systems, Springer Berlin Heidelberg

Meerschaert MM, Scheffler HP (2008) Triangular array limits for continuous time random walks. Stochastic Processes and their Applications 118(9):1606 – 1633

Melchiors P, Dekker R, Kleijn M (2000) Inventory rationing in an (s,q) inventory model with lost sales and two demand classes. Journal of Operational Research Society 51:111–122

Möllering KT, Thonemann UW (2010) An optimal constant level rationing policy under service level constraints. OR Spectrum 32:319–341

Nahmias S, Demmy WS (1981) Operating characteristics of an inventory system with rationing. Management Science 27:1236–1245

Park C, Padgett WJ (2005) Accelerated degradation models for failure based on geometric brownian motion and gamma processes. Lifetime Data Analysis 11:511–527

Peterson R, Silver EA (1979) Decision systems for inventory management and production planning. Wiley New York

Piplani R, Liu S (2014) Comparison of capacity rationing policies for a make-to-order production system with two customer classes. International Journal of Industrial and Systems Engineering 16(2):223–237

Ramaekers K, Janssens GK (2008) On the choice of a demand distribution for inventory management models. European Journal of Industrial Engineering 2(4):479–491

Serfozo R, Stidham S (1978) Semi-stationary clearing processes. Stochastic Processes and their Applications 6(2):165–178

Teunter RH, Haneveld WKK (2008) Dynamic inventory rationing strategies for inventory systems with two demand classes, poisson demand and backordering. European Journal of Operational Research 190:156–178

Wang D, Tang O, Huo J (2013a) A heuristic for rationing inventory in two demand classes with backlog cost and service constraint. Computers and Operations Resarch

Wang Y, Zhang S, Sun L (2013b) Anticipated rationing policy for two demand classes under service level constraints. Computers and Industrial Enginnering 23:1200–1212

Zheng YS (1992) On properties of stochastic inventory systems. Management science 38(1):87–103

A Proof of proposition 2

Lemma 1 *Let X, Y be two univariate continuous random variables, where Y has positive support. Then, for any C we have*

$$\mathbb{P}(X+Y>C) \geq \mathbb{P}(X>C) \ .$$

Proof We note that the set of realizations $\{\omega \mid X(\omega) > C\} \subset \{\omega \mid X(\omega)+Y(\omega) > C\}$, which gives the inequality $\mathbb{P}(X+Y>C) \geq \mathbb{P}(X>C)$. $\square$

Given the lemma 1, the demonstration of proposition 2 is:

Proof Let $\tau_{R,D}^{r-C} = \min\{\tau_{H,D}^{r-C}, L\}$ be the *time to rationing*, which corresponds to the amount of time that elapses from the moment an order is placed until the critical level C is reached if this event occurs during the lead time. If the hitting time $\tau_{H,D}^{r-C}$ does not occur during lead time then the time to rationing is defined as $\tau_{R,D}^{r-C} = L$. In this case, rationing coincides with the reception of the replenishment batch, and therefore, to be precise, rationing is not produced.

Given a $k > 0$ we have that, for every demand realization ω, the hitting time satisfies $\tau_{H,D}^{r-C}(\omega) < \tau_{H,D}^{r+k-C}(\omega)$. This is because exactly k additional units of demand are necessary to reach C, and the demand is a strictly increasing non-negative demand. This implies that for any $k > 0$ we have that

$$\tau_{R,D}^{r-C}(\omega) < \tau_{R,D}^{r+k-C}(\omega) \quad \forall \omega \text{ s.t. } \tau_{H,D}^{r-C}(\omega) < L,$$
$$L = \tau_{R,D}^{r-C}(\omega) = \tau_{R,D}^{r+k-C}(\omega) \quad \forall \omega \text{ s.t. } \tau_{H,D}^{r-C}(\omega) \geq L\ .$$

From these relations we have that $\tau_{R,D}^{r+k-C} - \tau_{R,D}^{r-C} \geq 0$ with probability 1, which combined with the assumptions on the demand gives us $D_1(L - \tau_{R,D}^{r-C}) = D_1(L - \tau_{R,D}^{r+k-C}) + D_1(\tau_{R,D}^{r+k-C} - \tau_{R,D}^{r-C})$, where the last term is a positive support random variable when $\tau_{H,D}^{r-C} < L$. We therefore have

$$\begin{aligned}
\alpha_1(r,C) &= \mathbb{P}(D_1(L-\tau_{R,D}^{r-C}) \leq C) \\
&= \mathbb{P}(D_1(L-\tau_{R,D}^{r-C}) \leq C \mid \tau_{H,D}^{r-C} \geq L)\mathbb{P}(\tau_{H,D}^{r-C} \geq L) + \mathbb{P}(D_1(L-\tau_{R,D}^{r-C}) \leq C \mid \tau_{H,D}^{r-C} < L)\mathbb{P}(\tau_{H,D}^{r-C} < L) \\
&= \mathbb{P}(D_1(L-\tau_{R,D}^{r+k-C}) \leq C \mid \tau_{H,D}^{r-C} \geq L)\mathbb{P}(\tau_{H,D}^{r-C} \geq L) \\
&\quad + \mathbb{P}(D_1(L-\tau_{R,D}^{r+k-C}) + D_1(\tau_{R,D}^{r+k-C} - \tau_{R,D}^{r-C}) \leq C \mid \tau_{H,D}^{r-C} < L)\mathbb{P}(\tau_{H,D}^{r-C} < L) \\
&\leq \mathbb{P}(D_1(L-\tau_{R,D}^{r+k-C}) \leq C \mid \tau_{H,D}^{r-C} \geq L)\mathbb{P}(\tau_{H,D}^{r-C} \geq L) + \mathbb{P}(D_1(L-\tau_{R,D}^{r+k-C}) \leq C \mid \tau_{H,D}^{r-C} < L)\mathbb{P}(\tau_{H,D}^{r-C} < L) \\
&= \mathbb{P}(D_1(L-\tau_{R,D}^{r+k-C}) \leq C) = \alpha_1(r+k,C)\ .
\end{aligned}$$

Here the inequality uses lemma 1 with $X = D_1(L - \tau_{R,D}^{r+k-C})$ and $Y = D_1(\tau_{R,D}^{r+k-C} - \tau_{R,D}^{r-C})$.

We repeat the argument to show the tendency of $\alpha_1(r,C)$ with respect to C. Given any $k > 0$ we have that $\tau_{H,D}^{r-(C+k)}(\omega) < \tau_{H,D}^{r-C}(\omega)$ for any demand realization ω. Similarly, for any $k > 0$, we now have

$$\tau_{R,D}^{r-(C+k)}(\omega) < \tau_{R,D}^{r-C}(\omega) \quad \forall \omega \text{ s.t. } \tau_{H,D}^{r-(C+k)}(\omega) < L$$
$$L = \tau_{R,D}^{r-(C+k)}(\omega) = \tau_{R,D}^{r-C}(\omega) \quad \forall \omega \text{ s.t. } \tau_{H,D}^{r-(C+k)}(\omega) \geq L\ .$$

The demand can be now separated $D_1(L - \tau_{R,D}^{r-(C+k)}) = D_1(L - \tau_{R,D}^{r-C}) + D_1(\tau_{R,D}^{r-C} - \tau_{R,D}^{r-(C+k)})$, where for every demand realization ω this last term satisfies $D_1(\tau_{R,D}^{r-C} - \tau_{R,D}^{r-(C+k)})(\omega) \leq k$. This because $\tau_{R,D}^{r-C}(\omega) - \tau_{R,D}^{r-(C+k)}(\omega) \leq \tau_{H,D}^{r-C}(\omega) - \tau_{H,D}^{r-(C+k)}(\omega)$ and $D_1(\tau_{H,D}^{r-C} - \tau_{H,D}^{r-(C+k)}) \leq D(\tau_{H,D}^{r-C} - \tau_{H,D}^{r-(C+k)}) = k$ by definition of hitting time. This gives

$$\begin{aligned}
\alpha_1(r,C) &= \mathbb{P}(D_1(L-\tau_{R,D}^{r-C}) \leq C) = \mathbb{P}(D_1(L-\tau_{R,D}^{r-C}) \leq C \mid \tau_{H,D}^{r-(C+k)} \geq L)\mathbb{P}(\tau_{H,D}^{r-(C+k)} \geq L) \\
&\quad + \mathbb{P}(D_1(L-\tau_{R,D}^{r-C}) \leq C \mid \tau_{H,D}^{r-(C+k)} < L)\mathbb{P}(\tau_{H,D}^{r-(C+k)} < L) \\
&= \mathbb{P}(D_1(L-\tau_{R,D}^{r-(C+k)}) \leq C+k \mid \tau_{H,D}^{r-(C+k)} \geq L)\mathbb{P}(\tau_{H,D}^{r-(C+k)} \geq L) \\
&\quad + \mathbb{P}(D_1(L-\tau_{R,D}^{r-(C+k)}) \leq C + D_1(\tau_{R,D}^{r-C} - \tau_{R,D}^{r-(C+k)}) \mid \tau_{H,D}^{r-(C+k)} < L)\mathbb{P}(\tau_{H,D}^{r-(C+k)} < L) \\
&\leq \mathbb{P}(D_1(L-\tau_{R,D}^{r-(C+k)}) \leq C+k \mid \tau_{H,D}^{r-(C+k)} \geq L)\mathbb{P}(\tau_{H,D}^{r-(C+k)} \geq L) \\
&\quad + \mathbb{P}(D_1(L-\tau_{R,D}^{r-(C+k)}) \leq C+k \mid \tau_{H,D}^{r-(C+k)} < L)\mathbb{P}(\tau_{H,D}^{r-(C+k)} < L) \\
&= \mathbb{P}(D_1(L-\tau_{R,D}^{r-(C+k)}) \leq C) = \alpha_1(r,C+k)\ .
\end{aligned}$$

Here we add a k in the first term of the second equality because $D_1(L - \tau_{R,D}^{r-(C+k)}) = D_1(0)$ when $\tau_{H,D}^{r-(C+k)} \geq L$, so that first probability equals 1. The inequality comes from the fact that $\mathbb{P}(D_1(\tau_{R,D}^{r-C} - \tau_{R,D}^{r-(C+k)}) \leq k) = 1$. $\square$

B Partial derivative of $\alpha_1(r,C)$ with respect to C

Here we give the expression of $\frac{\partial \alpha_1(r,C)}{\partial C}$ in the case when the demands for both classes are normally distributed and the density function of the hitting time $\tau_{H,D}^{r-C}$ is given by equation (18). We denote by $\bar{\varphi}_{\mu,\sigma^2}(x)$ the density function of a normal random variable with mean μ and variance σ^2. The partial derivative then can be expressed as:

$$\frac{\partial \alpha_1(r,C)}{\partial C} = \int_0^L \left(\frac{r-C+\mu\tau}{2\tau} - \frac{C+\mu_1(L-\tau)}{2(L-\tau)} \right) \bar{\varphi}_{\mu_1(L-\tau),\sigma_1^2(L-\tau)}(C)\bar{\varphi}_{\mu\tau,\sigma^2\tau}(r-C)d\tau \,. \tag{31}$$