A traffic assignment model for a ridesharing transportation market

Huayu Xu[1], Fernando Ordóñez[2] and Maged Dessouky[1]*

[1]*Daniel J. Epstein Department of Industrial & Systems Engineering, University of Southern California, 3715 McClintock Avenue, GER 240, Los Angeles CA 90089, U.S.A.*
[2]*Industrial Engineering Department, Universidad de Chile, Republica 701, Santiago 8370439, Chile*

SUMMARY

A nascent ridesharing industry is being enabled by new communication technologies and motivated by the many possible benefits, such as reduction in travel cost, pollution, and congestion. Understanding the complex relations between ridesharing and traffic congestion is a critical step in the evaluation of a ridesharing enterprise or of the convenience of regulatory policies or incentives to promote ridesharing. In this work, we propose a new traffic assignment model that explicitly represents ridesharing as a mode of transportation. The objective is to analyze how ridesharing impacts traffic congestion, how people can be motivated to participate in ridesharing, and, conversely, how congestion influences ridesharing, including ridesharing prices and the number of drivers and passengers. This model is built by combining a ridesharing market model with a classic elastic demand Wardrop traffic equilibrium model. Our computational results show that (i) the ridesharing base price influences the congestion level, (ii) within a certain price range, an increase in price may reduce the traffic congestion, and (iii) the utilization of ridesharing increases as the congestion increases. Copyright © 2014 John Wiley & Sons, Ltd.

KEY WORDS: flow; transport and traffic systems analysis; travel time; ridesharing; traffic assignment; elastic demand; equilibrium

1. INTRODUCTION

With rapid population growth and city development, traffic congestion has become an important issue, especially in large cities. The 2012 Annual Urban Mobility Report developed by the Texas Transportation Institute [1] estimates that (i) the amount of delay endured by the average commuter was 38 hours, up from 16 hours in 1982, and (ii) the cost of congestion is more than $120bn, nearly $820 for every commuter in the USA. At the same time, there is no public support for increased spending on infrastructure capacity expansion. Thus, there is a need for innovative transportation modes that can be implemented to improve transportation conditions in a cost-efficient manner. Ridesharing appears as one such innovative transportation mode that could at least help mitigate the congestion increase, as it can tap into the significant amount of unused capacity in transportation networks.

Benefits of ridesharing include travel cost savings, reducing travel time, mitigating traffic congestion, conserving fuel, and reducing air pollution [2–5]. However, ridesharing is still not a regular transportation alternative and is considered an informal and disorganized activity. The lack of efficient methods to coordinate itineraries and schedules is an important factor that inhibits the wide adoption of ridesharing. Recently, technological advances including global positioning systems and mobile devices have greatly enhanced the communication capabilities of travelers, facilitating the creation of ridesharing in real time. Taking advantage of this opportunity, a number of companies, such as Avego (Carma), SideCar, flinc, and CarpoolWorld, have emerged to develop systems where travelers

---

(including both drivers and passengers) can be matched in real time via Web browsers and mobile apps [6]. In a sense, these companies are establishing a marketplace for drivers to offer up their empty seats to other travelers. In these newly developed systems, the drivers receive a compensation for participating, which can be in the form of smaller travel times, reduced tolls, or direct payment that help mitigate the travel costs. The essential difference of such ridesharing systems from traditional public transit systems is that they do not hire professional drivers and they function as a matching agency that pairs passengers with "citizen" drivers.

In this paper, we study a transportation system where ridesharing has the ability of capturing a significant portion of travel demand via a real-time matching agency. In this system, we assume that the passengers will pay the drivers for the ridesharing services to share the travel cost. The ridesharing price is an abstraction to represent compensation that drivers take into account in their decision to participate in ridesharing, such as a reduction in travel time or toll costs that will occur by being able to use high-occupancy vehicle (HOV) lanes. We assume that the system operates as an open marketplace and thus the ridesharing price will be determined by the market as well as congestion conditions.

The purpose of this paper is to determine how attractive ridesharing will be to travelers in a given city where the ridesharing market exists. The key decision factors would include the complex interaction between traffic congestion, the ridesharing price, and its adoption, that is, the number of drivers and passengers that participate in ridesharing. For instance, to decide whether to participate in ridesharing, drivers may weigh the inconvenience, such as loss of privacy, against the compensation they may earn for taking on passengers. In turn, passengers would trade off the inconvenience, such as security concerns and loss of freedom, against the travel time and cost of a shared ride. These trade-offs would balance in an equilibrium that determines congestion, the ridesharing price, and the number of drivers and passengers that participate in ridesharing. For example, an increase in ridesharing could lead to a reduction in congestion and possibly an increase in ridesharing price, which in turn would make driving more attractive. On the other hand, an increase in ridesharing price would reduce the number of potential passengers, leading to an increase in potential drivers and congestion. Understanding how ridesharing would influence traffic congestion is fundamental in the evaluation of a ridesharing enterprise or in assessing the convenience of regulatory policies or incentives to promote ridesharing.

To achieve this goal, we propose a new static traffic assignment model based on a user-equilibrium (UE) assumption that would take into consideration the unique characteristics of ridesharing. Such a model could allow us to analyze how ridesharing and traffic congestion would interact with each other and also to determine the impact that different regulatory interventions could have on ridesharing, and hence on traffic. Therefore, the goal of this paper is to establish a static traffic assignment model that can determine (i) the ridesharing price, (ii) how ridesharing will impact the traffic congestion conditions, and (iii) the number of travelers that participate in ridesharing. Existing traffic assignment models have to be extended to consider the specific characteristics of ridesharing, where (i) the cost/price of ridesharing is determined by the number of people participating and (ii) the offer for shared rides (capacity of the transportation mode) varies with congestion and price. The approach to accomplish this is to combine two equilibrium models: a market pricing model that we refer to as the economic equilibrium system and a traditional traffic equilibrium system. Through common price and congestion parameters, the two systems interact with each other and thus become one integrated system that determines the prices and the congestion levels simultaneously.

Note that in this paper, we assume there exist drivers and passengers, and we do not model each individual's choice between these transportation alternatives. We assume that there are separate utility functions for drivers and passengers used to represent an elastic demand and that both drivers and passengers decide independently whether to travel or not. The number of drivers in the network will be captured by the driver utility function, and the number of passengers will be determined by the passenger utility function. A potential driver can only decide whether to drive or not travel at all. A potential passenger can only decide to take a shared ride or not (they may choose other public transit, but they will not drive and thus will not impact traffic). In any case, if a driver determines not to drive or a passenger determines not to take a ride, they will not be considered in the scope of this paper and will not influence traffic congestion or the balance in ridesharing trips. We are only interested in how many drivers and passengers are in our system.

In addition, the drivers in a ridesharing setting include both solo drivers (who drive alone) and ridesharing drivers (who take on passengers). We treat them the same because we are evaluating the ridesharing market from a system point of view and our model does not make specific assignments of passengers to vehicles. Therefore, the only quantity of interest for congestion and balance in the ridesharing market is the number of drivers.

The paper is organized as follows. The literature review is presented in Section 2 where we discuss existing ridesharing systems and traffic assignment problems that help determine the traffic congestion cost. Section 3 describes the integrated model in detail and gives its mathematical formulation. Section 4 briefly describes our solution approach and presents the computational results and analysis. We finish the paper with conclusions in Section 5.

2. LITERATURE REVIEW

Ridesharing is a joint trip of at least two participants that share a vehicle and requires coordination with respect to itineraries [6]. Ridesharing has drawn much interest in both industry and academic fields in recent years. According to Furuhata *et al.* [6], ridesharing activities can be divided into three main types: (i) unorganized ridesharing; (ii) semi-organized ridesharing; and (iii) organized ridesharing.

Unorganized ridesharing, which involves family, colleagues, neighbors, and friends, has a long history, yet is limited scaled because of inefficient communication methods. Semi-organized ridesharing services occur spontaneously among individual travelers motivated by access to faster HOV lanes or reduced toll. Examples of this type of ridesharing service are casual carpooling [4, 7] and slugging, which formed in the Washington D.C. area free of charge to the participants [8, 9]. These services run on their own momentum; they are not started or run by a public or private entity [10]. Therefore, they are limited to specific locations or circumstances and are difficult to replicate elsewhere.

Organized ridesharing is operated by agencies that provide ride-matching opportunities for participants without regard to any previous historical involvement [11]. With innovative technologies, inhibitors of ridesharing can be overcome, and a number of private matching agencies have emerged during the last decade [2, 11–14]. By introducing mobile technologies such as smart phones as well as global positioning systems, ridesharing systems can be implemented in a real-time fashion, and its degree of adoption will increase when agencies can help match any two participants with their travel itineraries and current locations.

However, there are still a limited number of papers that deal with issues of dynamic/real-time ridesharing services [15]. Although most of the existing papers focus on the matching mechanisms of real-time ridesharing services, the pricing problem has received less attention in the literature. Pricing specifies the amount of money transferred between the involved parties (drivers and passengers), including how to share the costs of gas, toll, and parking and how to charge transaction fees by the matching agencies [6].

One type of pricing for dynamic ridesharing is based on the auction mechanism, where drivers or passengers specify their least or highest preferred price, respectively [16]. A related example can be found in eBay, where multiple sellers offer the same commodity with different deadlines and the clearing prices are not identical. Another type of pricing is called cost sharing, where a price is determined by a cost calculation formula specified by a matching agency. Several different cost-sharing mechanisms have been designed, and they are applicable to share the transportation cost in a static setting [17, 18]. This type of pricing is difficult to implement in dynamic ridesharing because it should involve the consideration of fairness when splitting the cost with multiple passengers who may be picked up or dropped off at any time during the trip. A third type of pricing is more negotiable, where the matching agencies are not involved in pricing. The price is negotiated between the potential participants (either drivers or passengers) while they determine whether or not to participate in the ridesharing activity [6].

There are few papers discussing the relationship or the interaction between rideshare pricing and traffic congestion. Yang and Huang [19] discussed the carpooling behavior and the optimal congestion pricing in a multilane highway with or without HOV lanes where the first-best pricing and the second-best pricing models were formulated and compared. The models, however, were limited to identical commuters (single origin and single destination), and the number of passengers in each carpooling vehicle is fixed to one. Later, Qian and Zhang [20] studied the morning commute problem with three

modes: transit, driving alone, and carpool. They analyzed the interactions among the three modes and how different factors affect their mode shares and network performance. Again, the model is limited to a single origin and single destination network and does not consider the tremendous interactions of rideshare pricing among different origin-destination (OD) pairs.

To study the effects of multiple OD pairs, one classic model is the traffic assignment problems (TAPs), which evaluates the distribution of travelers among different routes and OD pairs. There are many methods to assign traffic to paths, a standard assumption in TAP is the UE assumption, also known as Wardrop's user traffic equilibrium law [21]. According to this assumption, the travel times (congestion costs) in all the used paths are equal and less than those which would be experienced by a single vehicle on any unused path. In other words, no traveler can reduce their travel cost/congestion by switching to another route at the equilibrium [22]. However, by traveling, every individual causes congestion and helps determine the travel cost for everyone else. Therefore, the traffic assignment problem with the UE assumption establishes a model that predicts how travelers choose their routes given a road network. In this paper, the TAP refers to the TAP with UE assumption. There could exist several fastest routes in this assumption, as long as they have the same travel (time) cost.

One of the most important variations of TAP is elastic demand. By introducing a "utility" (or "disutility") function, the problem decides not only how people will choose their paths but also how many people would travel given certain congestion conditions. This kind of model serves to illustrate those cases where people might not travel when the traffic is highly congested. The problem can also be formulated as a convex optimization problem with a decreasing convex utility function for a trade-off between congestion and demand. The elastic demand TAP is well studied in the literature [23–30].

Another variation of the traffic assignment is the multi-mode model, where two or more transit modes, say private vehicle and public transit, are being studied simultaneously. The multi-mode variant also includes another characteristic of traffic equilibrium: multiclass model, where travelers are divided into several classes, say high and low income. In the work of Aashtiani and Magnanti [31], the total demand of travelers in all modes is a constant, and the well-known logit model is applied in deciding the demand for each mode. They also proved the existence and uniqueness of the solution to their model. Boyce and Bar-Gera [32] described the formulation, estimation, and validation of combined models for making detailed urban travel forecasts. Their model was based on a large-scale, multiclass model of peak period urban travel (Chicago region). Other models and methods of multi-mode traffic assignment problems can be found in [28, 33, 34].

There exist many solution approaches to the traffic assignment problems. One of them is to model it as a convex optimization problem. The model can be formulated using either arc-based variables [30, 35, 36] or path-based variables [22, 29, 37, 38], which have been shown to be equivalent [22, 39, 40]. One of the most widely used methods to solve the TAP in convex optimization is the Frank–Wolfe method [41, 42], which works as a reduced gradient method. It generally makes good progress toward the optimum during the first iterations, but convergence often slows down substantially when close to the optimal solution. Another well-used solution approach is the Analytic Center Cutting Plane Method (ACCPM) [43], which is related to the Dantzig–Wolfe decomposition method. It is based on a column generation technique defining a sequence of primal linear programming (LP) maximization problems. This method is efficient for solving UE models with fixed demands. Later, Babonneau and Vial [29] extended ACCPM to solve UE models with elastic demands. They showed that ACCPM is capable of solving large instances at a high level of accuracy.

A more generalized form of convex optimization is the complementarity problem. The Karush–Kuhn–Tucker conditions of the convex optimization problem include complementary slackness conditions, which together with variable nonnegativity form a complementarity problem. Facchinei and Pang in their book "Finite-Dimensional Variational Inequalities and Complementarity Problems, Volume I," pages 41–46, gave the detailed formulation of the UE as a complementarity problem. The benefit of formulating the UE as a complementarity problem instead of a convex optimization is that it does not require an objective function. In some cases, it is hard for some types of TAPs to have an explicit objective function. Agdeppa *et al*. [44] studied the traffic equilibrium problem with nonadditive costs, that is, the cost of a path *does not equal* the sum of the costs of the arcs belonging to the path. The paper formulates the problem as a monotone mixed complementarity problem under appropriate conditions, and hence, the existence and uniqueness of the solution can be proved.

With rapid urban development and dramatic information explosion, the efficiency and the scalability have both become significant issues and stimulate researches to find better algorithms. Another type of solution approach for TAPs is based on the network features of the problem. Researchers have been focusing on specialized algorithms making use of the actual paths of the traffic network. For example, Jayakrishnan *et al.* [45] proposed a path-based algorithm, and their results show that their method converges in 1/10 iterations than the conventional Frank–Wolfe algorithm. Later, Bar-Gera [46] utilized the acyclic subnetworks rooted at origins and designed a new approach called the origin-based algorithm. In a following paper, Bar-Gera [38] introduced an efficient solution method called the traffic assignment by paired alternative segments algorithm. Instead of comparing two entire paths, the traffic assignment by paired alternative segments algorithm focuses on pairs of alternative segments. These algorithms are efficient but are only designed to solve the basic traffic assignment model with fixed demands. To the best of our knowledge, they have not been applied to models with elastic demands or multiple modes.

Although there is rich literature on elastic demand TAPs and multi-modal TAPs, these models cannot easily be adapted to ridesharing because of the existence of an endogenous modal price. Although vehicle tolls and transit fares have been considered in the prior literature [32], these costs are treated as exogenous, known ahead of time and independent of how users occupy the transportation network. Ridesharing prices on the other hand could conceivably be determined by the amount of ridesharing that occurs because of traffic conditions. For instance, in the presence of high congestion (or high transportation costs due to high gas prices), it is plausible that more drivers are willing to participate in ridesharing, thus lowering the ridesharing price. A low ridesharing price would increase the number of people willing to be passengers reducing congestion. The presence of this endogenous transportation mode price, whose dependence on congestion is not straightforward, is a novel feature of the TAP studied here. Finally, we note that few papers have considered the impact that, even an exogenous, ridesharing cost would have on traffic congestion.

3. MATHEMATICAL MODEL

To describe the interactions between traffic congestion and ridesharing activities in the market, we consider the elastic demand traffic assignment model under the UE assumption. Suppose drivers and passengers are traveling according to their own decisions. Drivers can decide to travel or not depending on both traffic congestion and ridesharing conditions (prices, number of travelers, etc.). Passengers may also decide to take a shared ride or not according to the traffic congestion and the ridesharing price.

The following model captures the aforementioned decision activities and the interactive impacts among traffic congestion, the number of travelers, and the price paid for ridesharing services. Such interactions include, as shown in Figure 1, (i) traffic congestion would impact the number of travelers and the ridesharing price, (ii) the number of travelers (or more specifically, the number of drivers) would determine traffic congestion and also will influence the price, and (iii) the ridesharing price would impact the number of travelers and also the traffic congestion.

3.1. Problem description

Consider a transportation network represented by a graph with nodes and arcs, where nodes could be origins, destinations or intermediate stops, and arcs are direct roads that connect two nodes. Each
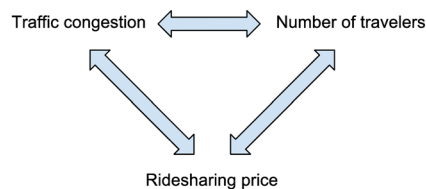

Figure 1. Interaction among congestion, number of travelers, and ridesharing price.

individual travels from an origin to a destination, which is called an OD pair. For each OD pair, there exist multiple paths that start from the same origin and end at the same destination. The congestion cost of each path is evaluated by the travel time along that path, which is a summation of travel times on each arc that builds up the path. *The travel time of each arc is determined by the number of vehicles (drivers) traveling on that link/arc.* Therefore, each arc may be shared by multiple paths (may or may not from the same OD pair), and conversely, each path of a certain OD pair may encounter drivers from other OD pairs. This fact is essential because a slight change in the number of drivers on one arc may impact the travel times or congestion costs of many paths. A UE is a state where for each OD pair, the travel times of all used paths are equal or less than those which would be experienced by a single vehicle/driver on any unused path [21].

In addition to UE, to include the specific features of ridesharing, we also assume that

- *Elastic demand*: the number of drivers (respectively passengers) of each OD pair is not fixed. It is determined by drivers' (respectively passengers') willingness to travel, that is, the utility function. Drivers and passengers have different utility functions.
- *Unified driver utility function*: the utility function of drivers is identical for all drivers across OD pairs. For this model, one utility function is employed for all drivers, including both solo and ridesharing drivers. It is determined by traffic congestion and ridesharing prices. Note that we may define two different utility functions for both solo and ridesharing drivers, yet the two can be combined as one generalized function (APPENDIX A).
- *Same OD pair*: ridesharing drivers would only be willing to take on passengers that travel between the same OD pair, that is, drivers would not pick up or drop off any passenger in the middle of their route, even if no detour is required. Not restricting ridesharing to the same OD pair would require more variables in order to keep track of how drivers may pick up or drop off a passenger in the middle of their trips. Ridesharing among travelers of the same OD pair can represent situations where each node in the graph represents a neighborhood (area) and there still exist a number of people traveling between the same two neighborhoods (areas). Such simplification helps us grasp the most essential features of ridesharing activities.
- *Congestion cost*: the travel time or the congestion cost is calculated only by the total number of vehicles/drivers (including both solo and ridesharing drivers). That is to say, the number of passengers in each vehicle does not contribute to the congestion cost.
- *Inconvenience cost*: the passenger pick-up and drop-off times will be treated as part of the inconvenience cost of drivers.
- *The ridesharing prices*: passengers would pay drivers some fee for the trip. The price is determined by the availability of vehicles and requests of passengers for each OD pair. The fee can be seen as a form of compensation to the drivers for the additional cost and inconvenience and turns out to limit the number of passengers in the network.
- *Unlimited capacity*: the vehicle capacity, that is, the number of passengers per vehicle is unlimited. Under this assumption, we do not distinguish different types of drivers in the model, because all passengers can be squeezed into one vehicle or may be distributed evenly among all available vehicles. It is reasonable from the perspective of the total travel cost: a driver can be either driving alone or sharing a ride only when the costs of the two are equal. In other words, suppose the travel cost of a solo driver is only the congestion cost, whereas the travel cost of a ridesharing driver is the congestion cost plus the inconvenience cost minus the ridesharing income. In this case, the inconvenience cost of ridesharing will be canceled by the income (or profit) of sharing a ride. Otherwise, all drivers would prefer the lower total cost: either driving alone or taking on passengers. In this work, we ignore how passengers are assigned to vehicles/drivers because other behavioral considerations (security, environmental conscience, and economics) influence these decisions.
- *Aggregate form*: based on the aforementioned assumptions, we are not treating drivers or passengers individually. We are considering an aggregate model per OD pair, that is, all the drivers of each OD pair are collaborating and so do all the passengers. Such simplification helps us understand how ridesharing activities would impact the traffic congestion at a system level.

Given the aforementioned assumptions, the main interactions between travelers, ridesharing prices, and traffic congestion can be summarized as follows (also see Figure 1).

- The traffic congestion cost is calculated by the total number of drivers.
- The total number of drivers is determined by traffic congestion and the ridesharing prices according to the drivers' willingness to travel on the road. Similarly, the number of passengers is also determined by traffic congestion and the ridesharing prices according to the passengers' willingness to participate in ridesharing.
- The ridesharing prices are determined by the interaction between ridesharing drivers and passengers and are also dependent on the congestion cost.

Our goal is to derive a model that reflects the aforementioned interactions and provides the number of travelers (both drivers and passengers), ridesharing prices, and congestion cost (travel time) for each OD pair at the equilibrium.

In the succeeding text is a list of notations that we use to formulate an elastic demand TAP model that includes ridesharing. Note that we use an arc-based formulation of the traffic assignment UE model such as in [29] or [36].

$\mathcal{N}$	set of nodes.
$\mathcal{A}$	set of arcs.
$\mathcal{O} \subseteq \mathcal{N}$	set of origins.
$\mathcal{D} \subseteq \mathcal{N}$	set of destinations.
$\mathcal{K}$	set of OD pairs, $\mathcal{K} \subseteq \mathcal{O} \times \mathcal{D}$
$k \in \mathcal{K}$	OD pair, where $k = (o_k, d_k), o_k \in \mathcal{O}, d_k \in \mathcal{D}$.
$a \in \mathcal{A}$	arc.
p_k	ridesharing price for each passenger of OD pair $k \in \mathcal{K}$.
q_k	number of passengers for OD pair $k \in \mathcal{K}$.
$\lambda_k^{(0)}$	free-flow time (fixed) for OD pair $k \in \mathcal{K}$.
λ_k	congestion cost for OD pair $k \in \mathcal{K}, \lambda_k \geq \lambda_k^{(0)}$.
δ_k	total number of drivers for OD pair k.
x_a^k	amount of flow (number of drivers) for OD pair $k \in \mathcal{K}$ on arc $a \in \mathcal{A}$.
y_a	total amount of flow on arc $a \in \mathcal{A}, y_a = \Sigma_{k \in \mathcal{K}} x_a^k$.
δ	vector with components $\delta_k, \boldsymbol{\delta} \in \mathbb{R}^{|\mathcal{K}|}$.
x^k	vector with components x_a^k with respect to $k, x^k \in \mathbb{R}^{|\mathcal{A}|}$.
x	vector with components $x_a^k, x \in \mathbb{R}^{|\mathcal{K}| \times |\mathcal{A}|}$.
y	vector with components $y_a, y \in \mathbb{R}^{|\mathcal{A}|}$.

3.2. Adding ridesharing prices to an elastic demand traffic assignment problem

In the elastic demand traffic assignment problem, the objective function consists of two components: the sum of the integrals of the congestion cost over all arcs and the sum of the integrals of the utility function over all OD pairs.

We consider that each arc a has a congestion function $tt_a(y_a)$ to represent the travel cost/time of traversing arc a when there is a flow of y_a on that arc. We assume that this is a strictly increasing function of y_a. The classic Bureau of Public Roads function [47] is one of the most widely used congestion functions.

The utility function of drivers is denoted by $\Lambda_k(\delta_k, p_k)$, which is a function of the total number of drivers δ_k and ridesharing price p_k for OD pair k. It provides the worst congestion cost the drivers could endure given the number of vehicles and the ridesharing price. Therefore, it represents an aggregate utility for all drivers (including both solo drivers and ridesharing drivers) according to certain traffic congestion and ridesharing prices. The aforementioned utility function looks like one in the standard elastic demand model, except that it includes the ridesharing price as a second variable. In elastic demand models, this driver utility function decreases with δ_k as more drivers (more congestion) make a trip less appealing. The dependence on the ridesharing price will capture the fact that a payment for taking on passengers can be a form of compensation to drivers. We therefore assume that the utility function increases with the ridesharing price, that is, drivers may accept worse traffic condition if there is an increase in their compensation of providing ridesharing services, while the number of drivers stays the same.
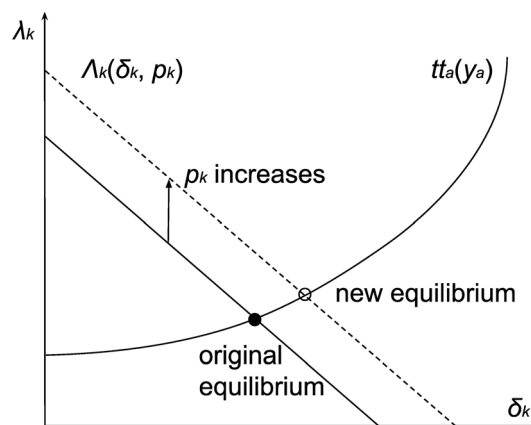
Figure 2. Traffic equilibrium.

From the aforementioned definitions, we have the following relationship between the number of drivers and the traffic congestion (Figure 2): when the number of drivers (or the amount of traffic flow) δ_k (or y_a) increases, the congestion cost $tt_a(y_a)$ would increase, whereas the utility $\Lambda_k(\delta_k, p_k)$ would decrease. Also, the utility $\Lambda_k(\delta_k, p_k)$ would increase if the ridesharing price p_k goes up. The traffic equilibrium is attained balancing these two relationships.

The ridesharing price for every OD pair should be determined by the balance between supply and demand for shared rides in the market. The economic equilibrium system [48] is adopted to describe the trade-off between the ridesharing price and the number of drivers and passengers, where drivers are the supply and passengers are the demand. Let us denote by $\mathcal{S}_k(q_k)$ the aggregate supply function and by $\mathcal{D}_k(q_k)$ the aggregate demand function for each OD pair k. These functions represent for the supply (respectively demand) the price at which drivers are willing to offer (respectively passengers are willing to pay) given the number of available ridesharing seats (the number of passengers) q_k. The price p_k and the number of passengers q_k is determined when these two functions are equal, that is, $\mathcal{S}_k(q_k) = \mathcal{D}_k(q_k) = p_k$ for each OD pair k.

In the next subsection, we will give an explicit form to these supply and demand functions. In particular, we include the fact [4, 7, 8] that the willingness of drivers to participate in ridesharing is increasing with the congestion cost, making it possible for drivers to ask for a lower price. Therefore, the supply function is decreasing in the congestion level λ_k, giving a supply function of the form $\mathcal{S}_k(q_k, \lambda_k)$. We note that these aggregate supply and demand functions should be such that ridesharing is only possible when there are drivers traveling on that OD pair. We show in Section 4.2 how this is enforced for specific supply and demand functions.

In sum, we are trying to evaluate the ridesharing market by modifying an elastic demand traffic assignment model and incorporating the balance between supply and demand for shared rides in every OD pair. This can be expressed as the following model:

$$\min_{x,y} \sum_{a \in \mathcal{A}} \int_0^{y_a} tt_a(s)\mathrm{d}s - \sum_{k \in \mathcal{K}} \int_0^{\delta_k} \Lambda_k(r, p_k)\mathrm{d}r \qquad (1)$$

$$\text{s.t.} \quad Nx^k - \Delta^k \delta_k = 0, \quad \forall k \in \mathcal{K} \qquad (2)$$

$$\sum_{k \in \mathcal{K}} x_a^k - y_a = 0, \quad \forall a \in \mathcal{A} \qquad (3)$$

$$\mathcal{S}_k(q_k, \lambda_k) = \mathcal{D}_k(q_k) = p_k, \quad \forall k \in \mathcal{K} \tag{4}$$

$$x_a^k \geq 0, \quad \forall a \in \mathcal{A}, \forall k \in \mathcal{K} \tag{5}$$

Constraint (3) represents the flow decomposition constraint, that is, the total amount of flow on each arc equals the sum of flows over all OD pairs on that arc. Constraint (4) describes the demand–supply balancing constraint from the succeeding text, where λ_k equals the congestion that δ_k drivers create. Constraint (5) is the nonnegativity constraint of variables.

Constraint (2) depicts the flow conservation constraint in a compact form, where N and Δ^k are coefficient matrix and vector. $N = [(i, a)]_{i \in \mathcal{N}, a \in \mathcal{A}}$ is an $|\mathcal{N}| \times |\mathcal{A}|$ matrix with element

$$(i, a) = \begin{cases} 1, & \text{node } i \in \mathcal{N} \text{ is the head of arc } a \in \mathcal{A}, \text{i.e., } a = (j, i) \\ -1, & \text{node } i \in \mathcal{N} \text{ is the head of arc } a \in \mathcal{A}, \text{i.e., } a = (i, j) \\ 0, & \text{otherwise} \end{cases}$$

and $\Delta^k = \left(\Delta_i^k\right)_{i \in \mathcal{N}}$ is a vector in $\mathbb{R}^{|\mathcal{N}|}$ for any $k \in \mathcal{K}$, with element

$$\Delta_i^k = \begin{cases} 1, & i = d_k \in \mathcal{D} \\ -1, & i = o_k \in \mathcal{O} \\ 0, & \text{otherwise} \end{cases}$$

In other words, for each OD pair k, constraint (2) is a compact form of $|\mathcal{N}|$ constraints, each of which is a flow conservation constraint at each node: (i) if it is a demand (destination) node, all incoming flows minus all outgoing flows should be equal to the demand of OD pair k; (ii) if it is a supply (origin) node, the difference should be the negative value of the demand; or (iii) if otherwise, the difference must be zero.

Note that in this model, we are focusing on the congestion cost caused by the drivers *only*; therefore, the objective in expression (1) is defined *only* with respect to the drivers' total congestion cost and total utility cost. The passengers' utility will be taken into account when determining the ridesharing prices, that is, in constraint (4). Models (1)–(5) show how the ridesharing price influences the drivers' willingness to travel and the traffic congestion. Next, we illustrate how the traffic congestion impacts the ridesharing prices, where the passengers' utility would be introduced.

3.3. Determining ridesharing prices including traffic congestion

The ridesharing prices, as a type of compensation to the drivers, are determined by the number of drivers and passengers participating in the ridesharing activities.

For a given OD pair k, suppose p_k is the price for each passenger and q_k is the number of passengers in total. We assume that the drivers will have a joint utility given by $U(p_k, q_k, \lambda_k) = p_k q_k - W(q_k, \lambda_k)$, which represents a revenue of $p_k q_k$ minus an additional inconvenience cost $W(q_k, \lambda_k)$ for providing q_k ridesharing seats in total when the congestion cost is λ_k. Assume that $W(q_k, \lambda_k)$ is convex and quadratic in terms of q_k. By setting $\frac{\partial U}{\partial q_k}(p_k, q_k, \lambda_k) = 0$, the maximum utility would be attained at $p_k = \frac{\partial W}{\partial q_k}(q_k, \lambda_k) = \mathcal{S}_k(q_k, \lambda_k)$, which gives us the supply function. We consider that $\mathcal{S}_k(q_k, \lambda_k)$ is increasing in q_k and decreasing in λ_k, given the fact that, from the perspective of drivers, the price should increase with the number of passengers, q_k, but should decrease with the traffic congestion (in order to mitigate the congestion for faster travel). Therefore, the inconvenience cost $W(q_k, \lambda_k)$ is also decreasing with respect to the congestion cost λ_k. This is applicable, because the drivers are more likely to take on passengers under higher congestion cost. Suppose picking up passengers takes a driver 5 minutes, no matter how much the congestion cost is. Therefore, the inconvenience cost would become less

when the congestion cost is 100 minutes, compared with a cost of 10 minutes, because both drivers and passengers are traveling on the same OD pair. As a result, it is reasonable that both the price and the inconvenience cost should decrease with congestion, from the perspective of drivers.

On the other hand, we wish to maximize the benefit to the passengers as well. Suppose $u(q_k)$ is the utility function of passengers, that is, the benefit one can obtain being a passenger. With a total cost $p_k q_k$, the profit of all passengers is $u(q_k) - p_k q_k$. Hence, assuming that $u(q_k)$ is concave and quadratic with respect to q_k, the maximal utility of passengers will be received at $p_k = \frac{du}{dq_k}(q_k) = \mathcal{D}_k(q_k)$, which gives us the demand function in the economic equilibrium model. In addition, we consider that $\mathcal{D}(q_k)$ is decreasing in q_k, because as the price increases, the number of passengers should decrease.

Given the aforementioned assumptions, we should have the following patterns (Figure 3): when the number of passengers q_k increases, drivers (supply) would increase the price p_k, whereas passengers (demand) would decrease it. Also, the price from the drivers would drop if the traffic congestion λ_k increases. The economic equilibrium that determines the ridesharing price is obtained at the intersection. The preceding text is true for all OD pairs.

3.4. Combining the pricing model with elastic demand traffic assignment problem

Combining the two equilibria together (Figures 2 and 3), we can see the following changing pattern depicted in Figure 4: when increasing the congestion cost λ_k, the ridesharing price p_k would decrease and the number of passengers q_k would increase (from the economic equilibrium, Figure 3). When decreasing the price p_k, both the congestion cost λ_k and the number of drivers δ_k would decrease (from the traffic equilibrium, Figure 2), and vice versa. Therefore, the two equilibrium models influence each other, and they will try to balance these interactions in a common equilibrium.

To formulate a tractable optimization problem that represents the combined equilibria described earlier and stated in general terms in models (1)–(5), we consider specific functional forms for the
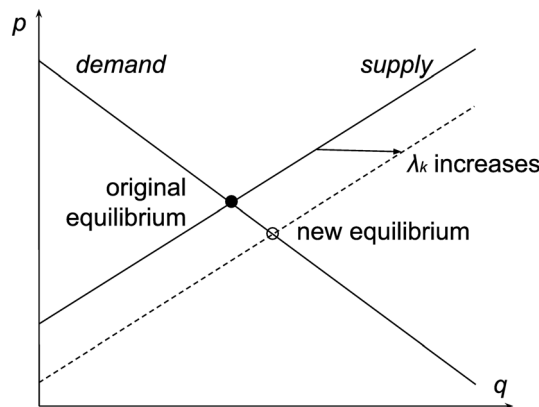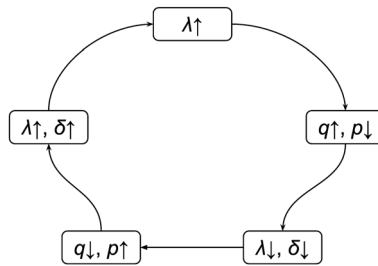

Figure 3. Economic equilibrium.

Figure 4. Changing pattern.

supply and demand functions of the economic equilibrium, as well as specific driver utility and congestion functions of the traffic equilibrium. The following result presents a tractable formulation for the combined equilibrium model using generic quadratic functions for the economic utilities and linear function for the driver's utility function.

3.4.1. Model formulation

Consider the following definition of the cost/utility functions:

- Congestion cost: $tt_a(y_a)$.
- Driver utility function: $\Lambda_k(\delta_k, p_k) = \alpha_k p_k - \beta_k \delta_k$.
- Driver inconvenience cost: $W(q_k, \lambda_k) = b_k q_k^2 + \frac{d_k}{\lambda_k} q_k + e_k, \quad b_k > 0, d_k \geq 0, \lambda_k > 0$.
- Passenger utility function: $u(q_k) = -f_k q_k^2 + g_k q_k + h_k, \quad f_k > 0$.

Then, the combined equilibrium model generated from Equations (1) to (5) can be formulated as follows.

$$\min_{\mathbf{x},\mathbf{y}} \quad \sum_{a\in\mathcal{A}}\int_0^{y_a} tt_a(s)ds - \sum_{k\in\mathcal{K}}\int_0^{\delta_k}\Lambda_k(r)dr \tag{6}$$

$$\text{s.t.} \quad Nx^k - \Delta_k\delta_k = 0, \quad \forall k \in \mathcal{K} \tag{7}$$

$$\sum_{k\in\mathcal{K}} x_a^k - y_a = 0 \quad \forall a \in \mathcal{A} \tag{8}$$

$$0 \leq \delta_k \leq \frac{\alpha_k b_k g_k}{\beta_k(b_k+f_k)} + \frac{\alpha_k d_k f_k}{\beta_k(b_k+f_k)}\frac{1}{\lambda_k^{(0)}} - \frac{\lambda_k^{(0)}}{\beta_k}, \quad \forall k \in \mathcal{K} \tag{9}$$

$$x_a^k \geq 0, \quad \forall a \in \mathcal{A}, \forall k \in \mathcal{K} \tag{10}$$

where

$$\Lambda_k(\delta_k) \triangleq \frac{\beta_k}{2}\delta_k + \frac{\Gamma_k(\delta_k)}{2(b_k+f_k)} + \frac{\alpha_k b_k g_k}{2(b_k+f_k)} \tag{11}$$

$$\Gamma_k(\delta_k) \triangleq \sqrt{\beta_k^2(b_k+f_k)^2\delta_k^2 - 2\alpha_k\beta_k b_k g_k(b_k+f_k)\delta_k + \left[4\alpha_k d_k f_k(b_k+f_k) + a_k^2 b_k^2 g_k^2\right]} \tag{12}$$

for all $k \in \mathcal{K}$. Note that p_k can also be expressed as a function of δ_k, thus $\Lambda_k(\delta_k, p_k)$ is replaced by $\Lambda_k(\delta_k)$, that is, from a two-variable function to a one-variable function.

3.4.2. Model analysis

Proposition 1. *Models* (6)–(10) *are generated from models* (1)–(5), *substituting p_k (and its corresponding constraint* (4)) *by δ_k from the result of the economic equilibrium.*

Proof. See APPENDIX B.

Proposition 2. *If the parameters satisfy the following,*

- $b_k, f_k > 0$ and $d_k \geq 0$,

- $g_k \geq d_k/\lambda_k^{(0)}, \quad \forall k \in \mathcal{K}$,

- $\frac{a_k d_k f_k}{b_k + f_k} \leq \left[\lambda_k^{(0)} \right]^2, \quad \forall k \in \mathcal{K}$

then model (6)–(10) are convex optimization problems.

Proof. See APPENDIX C.

Proposition 3. *Under the assumptions of Proposition 2, there exists an optimal solution to model (6)–(10) because of convexity. Moreover, the optimal solution is a UE.*

Proof. See APPENDIX D.

4. COMPUTATIONAL RESULTS

As introduced in Section 3, Equations (6)–(10) are convex optimization problems with a convex objective function and linear constraints. In our computational experiments, we solved this problem using the Frank–Wolfe algorithm.

4.1. Frank–Wolfe algorithm

Let z denote the solution triplet $(x, y\delta)$. Let $F(z)$ and $\mathcal{Z}$ denote, respectively, the objective function and the feasible space defined by constraints (7)–(10), that is,

$$F(z) = F(\boldsymbol{x}, \boldsymbol{y}, \delta) \triangleq \sum_{a \in \mathcal{A}} \int_0^{y_a} tt_a(s) \mathrm{d}s - \sum_{k \in \mathcal{K}} \int_0^{\delta_k} \Lambda_k(r) \mathrm{d}r,$$

$$\mathcal{Z} \triangleq \{z = (\boldsymbol{x}, \boldsymbol{y}, \delta) | (\boldsymbol{x}, \boldsymbol{y}, \delta) \text{ satisfying } (7) - (10)\}$$

The core idea of the Frank–Wolfe algorithm is to approximate the objective function by its first-order Taylor expansion, that is,

$$F(z') \approx F(z) + \nabla F(z)^T (z' - z),$$

where $z, z' \in \mathcal{Z}$. Therefore, minimizing $F(z')$ can be approximated by minimizing $F(z) + \nabla F(z)^T(z' - z)$, where z is fixed. Hence, it is sufficient to minimize $\nabla F(z)^T z'$ over $z' \in \mathcal{Z}$. The problem is simplified as a LP.

The Frank–Wolfe algorithm for problems (6)–(10) works as follows.

Step 1. *Initialization*

Set $\delta_k^{(0)}$ as the given initial demand $D_k^{(0)}$, for all $k \in \mathcal{K}$. Find the initial feasible solution $\boldsymbol{x}^{(0)}$ and $\boldsymbol{y}^{(0)}$ using all or nothing assignment (i.e., for each OD pair k, send all demand $\delta_k^{(0)}$ to the least cost path, which is calculated by free-flow times). Set $z_0 = (\boldsymbol{x}^{(0)}, \boldsymbol{y}^{(0)}, \delta^{(0)})$ and $t = 0$.

Step 2. *Finding descending direction*

Solve the LP

$$\min_{z \in \mathcal{Z}} \nabla F(z_t)^T z \tag{13}$$

The solution to Equation (13) is denoted by $\mathbf{z}_t^* \triangleq \mathrm{argmin}_{z \in \mathcal{Z}} \nabla F(z_t)^T \mathbf{z}$.

Step 3. *Finding step size*

Apply a line search to find the step size, that is, $\omega_t^* \triangleq \mathrm{argmin}_{\omega \in [0,1]} F((1 - \omega)z_t + \omega z_t^*)$.

Step 4. *Update solution*

Let $z_{t+1} \triangleq (1 - \omega_t^*)z_t + \omega_t^* z_t^*$. Set $t = t+1$ and go to step 2.

The algorithm stops after a given fixed number of total steps or until the solution stays at the same level after a certain number of iterations.

4.2. Computation settings

The goal of the computational experiments is to analyze how the parameter settings will impact the numerical outcome so that we can choose appropriate settings to obtain a reasonable range of the congestion level and the amount of passengers and drivers for ridesharing.

The network tested in this proposal is the classic Sioux Falls [29, 38]. The test data can be downloaded from the website called "Transportation Network Test Problems" (http://www.bgu.ac.il/~bargera/tntp/, accessible on April 16, 2014). All the parameter settings in this paper are listed in Table I, where D_k is the fixed demand for each OD pair k provided in the benchmark input files.

According to the aforementioned settings, we have

$$p_k = \frac{b_k g_k + d_k f_k / \lambda_k}{b_k + f_k} = \frac{1}{2}\left(g_k + \frac{d_k}{\lambda_k}\right) \in \left(\frac{1}{2} \in \lambda_k^{(0)}, \frac{1}{2}\left(\in \lambda_k^{(0)} + \sigma\right)\right] \tag{14}$$

$$q_k = \frac{g_k - d_k/\lambda_k}{2(b_k + f_k)} = \frac{D_k}{4}\left(g_k - \frac{d_k}{\lambda_k}\right) \in \left[\frac{D_k}{4}\left(\in \lambda_k^{(0)} - \sigma\right), \frac{D_k}{4} \in \lambda_k^{(0)}\right) \tag{15}$$

So the range of price p_k for OD pair k is determined by the free-flow time $\lambda_k^{(0)}$ and coefficients $\in$ and σ; it is reasonable that the ridesharing price for each OD pair is proportional to the free-flow time and can fluctuate in a small range because of congestion levels. Also, the range of the number of passengers q_k is proportional to the given demand D_k and has slight fluctuations with the congestion.

Also in Equation (15), because $\lambda_k \geq \lambda_k^{(0)}$ and $g_k \geq d_k/\lambda_k^{(0)}$, it guarantees that the number of passengers is always nonnegative. Furthermore, if we select parameters such that $g_k = d_k/\lambda_k^{(0)}$, then we shall have $q_k = 0$ if and only if $\lambda_k = \lambda_k^{(0)}$, that is, when there is no vehicle (driver) on the road, there is no passenger in the market. When there are few vehicles on the road (less than the road capacity c_a), we shall have (according to the Bureau of Public Roads function) $\lambda_k = \lambda_k^{(0)} + e$, where e is very small. Therefore, we also have very small q_k, that is, the number of passengers is close to zero, meaning that few people will take a shared ride when the congestion cost is very low.

The computation was conducted on a PC (Windows OS, CPU 3.10 GHz, RAM 4 GB), and all the codes were written in C++, using an LP solver, CPLEX (IBM ILOG CPLEX Optimization Studio, Armonk, New York, U.S.), for finding the descending direction in the Frank–Wolfe algorithm. The program is terminated after 100 iterations. The following subsections illustrate an overall summary of the computational results, a sensitivity analysis of congestion, price, and the number of passengers and drivers.

Table I. Constant settings.

Constant	Value
α_k	D_k
β	1 or 10
b_k, f_k	$1/D_k$
d_k	$\sigma\lambda_k^{(0)}, \sigma = 1, 2, 4$
g_k	$\epsilon\lambda_k^{(0)}, \epsilon = 1, 2, 4$

4.3. Optimality check

Before any further analysis, we first check the optimality of our computational results. Figure 5 gives the sorted average excess cost for all OD pairs. We can see that about 60% of the OD pairs have very low average excess cost.

Figure 6 shows the convergence of the objective functions with parameter settings $\beta = \epsilon = \sigma = 1$. It indicates briefly that the algorithm evolves efficiently at the first few iterations and it converges well within 100 iterations.

4.4. Overall results

Table II summarizes the outputs of all the 18 combinations of β, d_k, and g_k settings. Note that we change the values of d_k and g_k by changing and σ, according to Table I. The fourth to the sixth columns list the averages over all OD pairs. $\bar{p} \triangleq \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} p_k$, $\bar{q} \triangleq \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} q_k$, and $\bar{\delta} \triangleq \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} \delta_k$ are the average price, the average number of passengers, and the average number of drivers, respectively. The last two columns are the two parts of the objective function (6), where $\mathcal{F}_1 = \sum_{a \in \mathcal{A}} \int_0^{y_a} tt_a(s) \mathrm{d}s$ is the sum of the integrated congestion costs and $\mathcal{F}_2 = -\sum_{k \in \mathcal{K}} \int_0^{\delta_k} \Lambda_k(r) \mathrm{d}r$ shows the sum of the integrated disutility (negative utility). $\bar{e}$ is the average excess cost for the entire solution.

From Table II, we can draw the following conclusions:

(1) The base price, that is, g_k or ϵ, influences the number of drivers as a higher price makes it more attractive to become a driver.
(2) Within a certain price range, that is, fixing g_k or ϵ, an increase in price (d_k or σ) may reduce the traffic congestion.
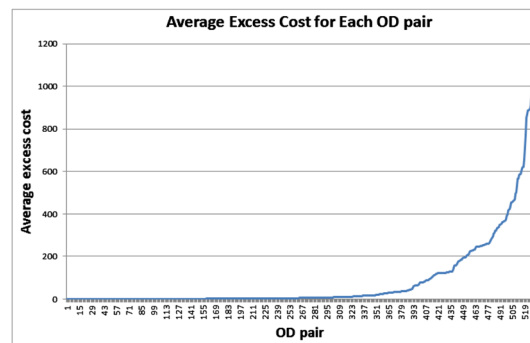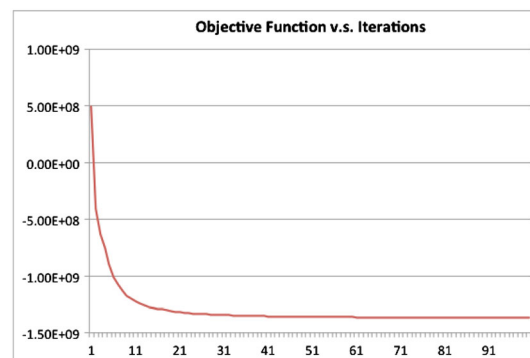

Figure 5. Optimality check.

Figure 6. Algorithm convergence.

Table II. Overall results.

| β	ϵ	σ	$\overline{p}$	$\overline{q}$	$\overline{\delta}$	$\mathcal{F}_1$	$\mathcal{F}_2$	$\overline{e}$
1	1	1	5.55	1934.63	1790.38	1.73×10^8	-1.46×10^9	5.33
1	1	2	5.57	1930.04	1795.37	1.63×10^8	-1.46×10^9	4.42
1	1	4	5.59	1920.67	1799.67	1.59×10^8	-1.45×10^9	5.24
1	2	1	11.08	3876.44	2557.01	8.11×10^8	-3.68×10^9	12.41
1	2	2	11.09	3874.95	2621.43	7.09×10^8	-3.79×10^9	22.45
1	2	4	11.10	3872.08	2595.07	7.92×10^8	-3.69×10^9	14.18
1	4	1	22.16	7755.17	3614.55	3.11×10^9	-5.46×10^9	69.55
1	4	2	22.16	7754.56	3631.62	2.85×10^9	-5.54×10^9	66.15
1	4	4	22.17	7753.50	3607.73	2.95×10^9	-5.27×10^9	52.47
10	1	1	5.96	1790.77	302.56	1.87×10^6	-1.98×10^8	37.38
10	1	2	6.37	1649.38	315.08	1.93×10^6	-1.98×10^8	27.29
10	1	4	7.18	1365.41	321.57	1.97×10^6	-1.98×10^8	32.34
10	2	1	11.33	3791.73	602.27	5.55×10^6	-7.92×10^8	146.81
10	2	2	11.53	3718.84	603.07	5.29×10^6	-7.92×10^8	155.22
10	2	4	11.96	3568.80	593.07	5.17×10^6	-7.92×10^8	829.07
10	4	1	22.20	7739.90	1151.04	3.65×10^7	-3.16×10^9	290.44
10	4	2	22.24	7725.99	1150.35	3.72×10^7	-3.16×10^9	394.31
10	4	4	22.32	7699.12	1150.81	3.88×10^7	-3.16×10^9	304.10
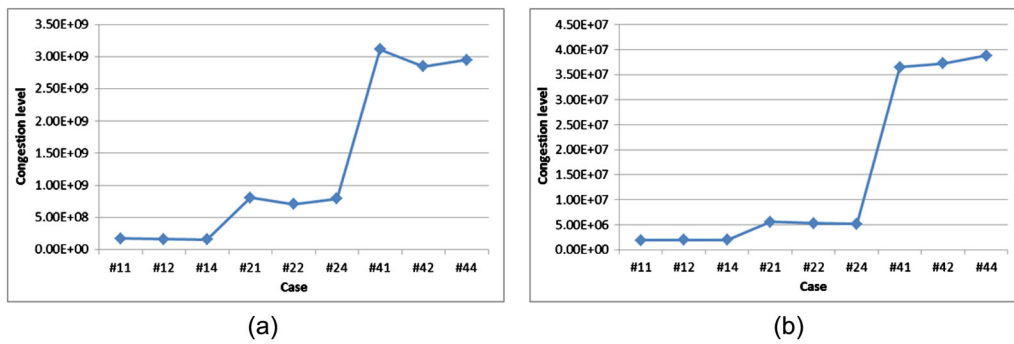
Figure 7. Congestion levels.

The following subsections provide further examination and analysis on the important system parameters.

4.5. Congestion and disutility levels

It can be seen from Table II that the setting of β has a major impact on the congestion levels. That is, column $\mathcal{F}_1$ shows that $\beta = 1$ has much higher congestion than $\beta = 10$. From Figure 7 (where the horizontal axis labels "#xy" represent the cases where $\epsilon = x$ and $\sigma = y$. For example, case "12" means ϵ is set to 1 and σ is set to 2. This same labeling is used in the following figures.), it can also be seen that $g_k = \lambda_k^{(0)}$ has a significant impact on the congestion level when β is fixed. The congestion increases when the price parameters g_k increase. This explains that when the prices go up, more people are attracted to become drivers.

However, the impact of the parameters d_k on the congestion between the different β values is quite the opposite. At $\beta = 1$ (Figure 7(a)), where the congestion scale is high, d_k has a negative impact on the congestion. When both the price and the congestion are high, a slight change in the price contributes more to the people's inconvenience (more congestion) than the drivers' benefits (i.e., the prices are not high enough to compensate for their congestion cost). Therefore, the number of drivers decreases, and thus, the congestion level goes down. At $\beta = 10$ (Figure 7(b)), where the congestion scale is not so high, increasing d_k will have a similar outcome to increasing g_k. Both of these are price parameters, and increasing the price attracts more drivers to the traffic network system.

The aforementioned analysis can also be verified when observing the changes to the disutility level. In Figure 8, it can be concluded that both β and g_k have a significant impact on the disutility levels. At $\beta = 1$, when the congestion scale is high, the disutility levels are lower than for $\beta = 10$, when the congestion scale is much smaller. In other words, the utility levels (negative values of disutility values) are higher when the congestion levels are higher because the traffic equilibrium is a balance between the congestion and the utility functions. Consequently, there must be enough of a benefit to attract travelers using this mode of transportation; otherwise, people will choose alternative modes because of high congestion.

Similarly, a higher g_k leads to smaller disutility levels (or higher utility levels), because g_k dominates the price values (see the next section) and a higher price gives larger utility values. d_k has little impact on the disutility levels, especially when $\beta = 10$ and the congestion scale is low. Increasing prices in a small range is not enough to draw people out under low congestion because of lack of passengers. When the congestion scale is high, increasing prices a little bit, especially when the price base is high (a larger g_k), does not show enough of an attraction to the drivers but causes more inconvenience. Therefore, the disutility levels would increase as d_k increases. It is consistent with the impact on congestion levels.

In summary, (i) β mainly decides the scale of the congestion levels and a smaller β leads to higher congestion levels and lower disutility levels, (ii) g_k (base price parameters) has a significant influence on congestion and disutility levels and a larger g_k results in higher congestion levels and lower disutility levels, and (iii) d_k has a small impact on the congestion and disutility levels.

4.6. Ridesharing prices

From Equation (A1) (APPENDIX B), it is known that the ridesharing price p_k is mainly determined by the parameters in the economic equilibrium model, which can be influenced by the government agencies or the marketplace. Furthermore, the price may decrease in a reasonable range when the congestion goes up so as to encourage people to share their vehicles.

Figure 9 shows one example of the distribution of price among all the 528 OD pairs and their corresponding free-flow time, where $\beta = 10$, $d_k = 4\lambda_k^{(0)}$, and $g_k = \lambda_k^{(0)}$. According to Equation (14), $p_k = \frac{1}{2}\left(g_k + \frac{d_k}{\lambda_k}\right) = \frac{1}{2}\left(\lambda_k^{(0)} + \frac{4\lambda_k^{(0)}}{\lambda_k}\right)$, that is, the ridesharing price is slightly above half the free-flow time. It can be seen in Figure 9 that the ridesharing price is close to half of the free-flow time for each OD pair. Although the free-flow times are sorted in ascending order, the price is not strictly increasing with them. This serves to illustrate the impact of congestion on the prices. The closer the congestion λ_k is to the free-flow time $\lambda_k^{(0)}$, the more difference the price p_k is above the half free-flow time.

Figure 10 compares the average prices of all the 18 combinations of β, d_k, and g_k. From Figure 10, it can be seen that there is not as much as a difference between cases $\beta = 1$ and $\beta = 10$ because β does not have a direct impact on prices but mainly on the congestion. Figure 10 shows that prices are mainly determined by g_k, with slight changes in prices by varying d_k. Also from Section 4.5, we know that $\beta = 1$ has much higher congestion than $\beta = 10$, making it hard to have an impact on prices.
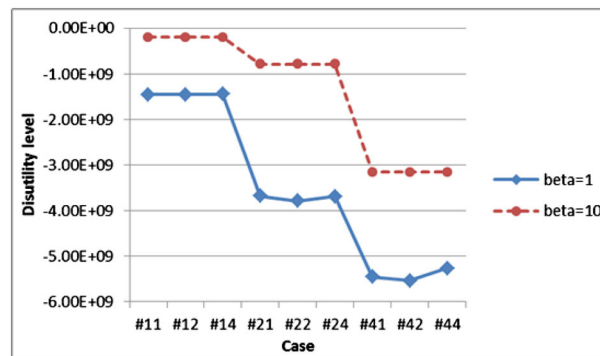

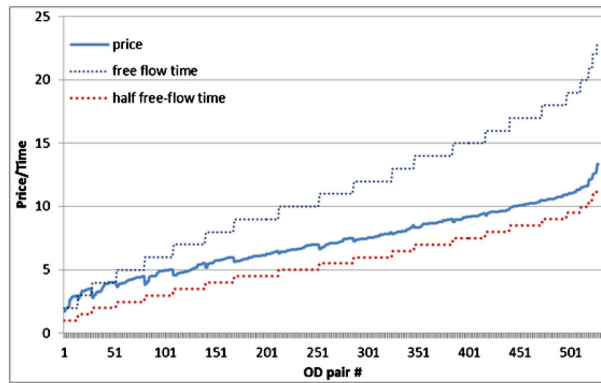Figure 8. Sensitivity analysis of disutilities.
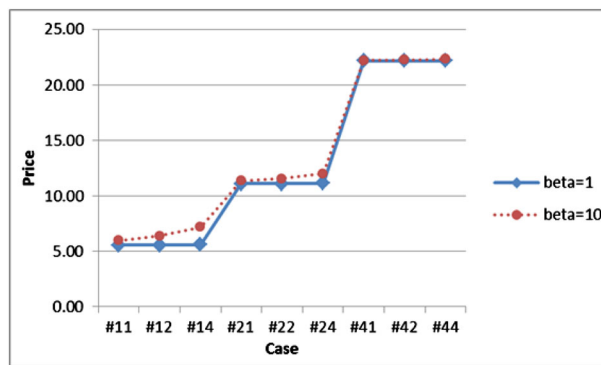
Figure 9. Price distribution.

Figure 10. Sensitivity analysis of average prices.

4.7. Number of travelers

Figure 11 plots the number of drivers for all 18 cases. The number of drivers is consistent with the congestion levels discussed in Section 5, because the congestion is a quartic function of arc flows, which are linear combinations of the number of drivers. From Figure 11, β mainly decides the scale of the number of drivers. A smaller β corresponds to more drivers, and a larger g_k also increases the number of drivers. d_k has little impact on the number of drivers.

Figure 12 plots the number of passengers for all 18 cases. The number of passengers behaves similarly to the ridesharing prices, because both of them are primarily determined by the economic
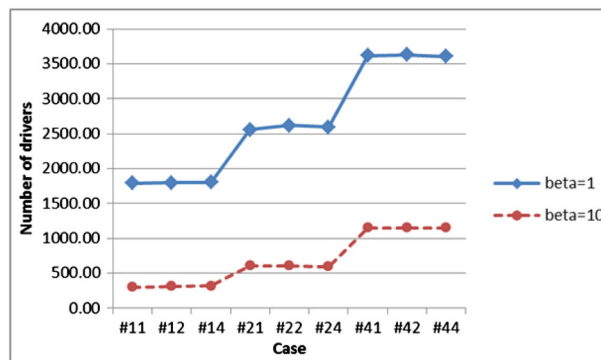

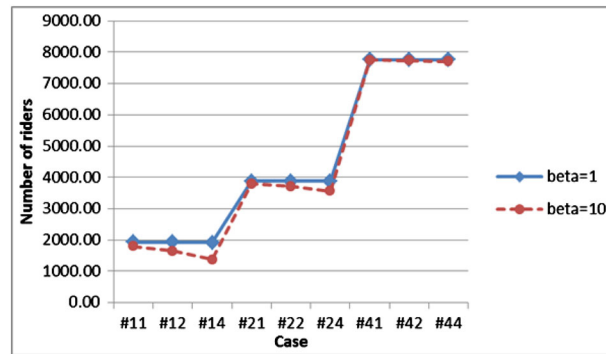Figure 11. Sensitivity analysis of the number of drivers.

Figure 12. Sensitivity analysis of the number of passengers.

equilibrium model. From Figure 12, it can be concluded that there is no significant difference between $\beta = 1$ and $\beta = 10$. As we can also see from Equation (17) (APPENDIX B), the number of passengers increases with g_k but decreases with d_k. When the congestion cost λ_k is high ($\beta = 1$), the impact of d_k can be neglected.

5. CONCLUSIONS

This paper proposes an integrated model that solves the traffic assignment problem with ridesharing. The model determines (i) the most appropriate price for ridesharing, (ii) how ridesharing impacts the traffic congestion, (iii) how many people are encouraged to take or provide ridesharing offers, and (iv) how traffic circumstances influence the ridesharing activities. By introducing two common variables, the ridesharing price and the congestion cost, the economic and traffic equilibrium models interact with each other and reach a common equilibrium of the entire system. The combined model is formulated as a convex optimization problem, and thus, the Frank–Wolfe algorithm is applied to solve this problem. The computational analysis shows the changing patterns of the ridesharing price, congestion levels, and the degree of adoption of ridesharing under various system parameters. The sensitivity analysis can be used to guide transit planners on how to encourage people to take shared rides so as to reduce the increasing traffic pressure.

The analysis of the model and its results shows that the ridesharing price can either decrease or increase as the congestion cost increases because of a number of factors. On one hand, a decrease in price occurs directly from our definition of price (Equation (13)). This price is defined for each passenger. Hence, this does not mean that each driver will earn less (a driver's income equals to the price times the number of passengers in the vehicle). The rationale is with more congestion, the more a driver wants to take on passengers to further compensate his or her loss because of congestion. Hence, a lower price would attract more passengers, whereby perhaps increasing the compensation to the driver. The latter situation cannot be easily seen from any definition of our model but from some of the computational results. In this case, travel is more expensive, and the ridesharing market does not have enough volume to decrease the price (other factors/parameters may impact the willingness to participate in ridesharing).

An important challenge for future work is to enhance the current model by removing some of the less realistic assumptions made. For example, in this model, we focused more on an aggregate level for each OD pair and did not treat each traveler as an individual agent. Only the drivers' congestion cost and utility are considered in the objective function. In future work, we will move our focus to the individual level and take into account each traveler's decision. Also, it would be interesting yet challenging to include the fact that ridesharing may not necessarily take place between travelers of the same OD pair. Drivers may pick up or drop off a passenger at any time and even detour from their original routes. In this case, more variables are needed to capture such diversity, making the scale of the problem significantly larger. Also, constraints can be added to represent vehicle capacity.

6. LIST OF ABBREVIATIONS

ACCPM   Analytic Center Cutting Plane Method
HOV   High-Occupancy Vehicle
KKT   Karush-Kuhn-Tucker
OD   Origin-Destination
TAP   Traffic Assignment Problem
UE   User Equilibrium

ACKNOWLEDGEMENT

REFERENCES

1. Schrank D, Eisele B, Lomax T. 2012 urban mobility report. Texas A&M Transportation Institute, 2012.
2. Chan N, Shaheen S. Ridesharing in North America: past, present, and future. In Transportation Research Board 90th Annual Meeting. Washington, D. C., 2011.
3. Ferguson E. The rise and fall of the American carpool: 1970–1990. *Transportation* 1997; **24**(4):349–376.
4. Kelley K. Casual carpooling—enhanced. *Journal of Public Transportation* 2007; **10**(4):119–130.
5. Morency C. The ambivalence of ridesharing. *Transportation* 2007; **34**(2):239–253.
6. Furuhata M, Dessouky M, Ordóñez F, Brunet M, Wang X, Koenig S. Ridesharing: the state-of-the-art and future directions. *Transportation Research Part B: Methodological* 2013; **57**:28–46.
7. Burris M, Winn J. Slugging in Houston—casual carpool passenger characteristics. *Journal of Public Transportation* 2006; **9**(5):23–40.
8. LeBlanc D. *Slugging: The Commuting Alternative for Washington DC*. Forel Publishing, 1999.
9. Spielberg F, Shapiro P. Mating habits of slugs: dynamic carpool formation in the i-95/i-395 corridor of Northern Virginia. *Transportation Research Board* 2000; **1711**:31–38.
10. Levofsky A Greenberg A. Organized dynamic ride sharing: the potential environmental benefits and the opportunity for advancing the concept. In *Transportation Research Board 2001 Annual Meeting*, Washington, DC, 2001.
11. Dailey DJ, Loseff D, Meyers D. Seattle smart traveler: dynamic ridematching on the World Wide Web. *Transportation Research Part C: Emerging Technologies* 1999; **7**(1):17–32.
12. Heinrich S. Implementing real-time ridesharing in the San Francisco Bay Area. Master's thesis, San Jose State University, 2010.
13. Amey A. Real-time ridesharing: exploring the opportunities and challenges of designing a technology-based rideshare trial for the MIT community. Master's thesis. Massachusetts Institute of Technology, 2010.
14. Ghoseiri K, Haghani A, Hamedi M. Real-time rideshare matching problem. Technical report, University of Maryland, 2011.
15. Agatz N, Erera A, Savelsbergh M, Wang X. Optimization for dynamic ride-sharing: a review. *European Journal of Operational Research* 2012; **223**(2):295–303.
16. Kleiner A, Nebel B, Ziparo V. A mechanism for dynamic ride sharing based on parallel auctions. In *The Twenty-Second international joint conference on Artificial Intelligence,* Palo Alto, California, 2011; 266–272.
17. Winter S, Nittel S. Ad-hoc shared-ride trip planning by mobile geosensor networks ad-hoc shared-ride trip planning by mobile geosensor networks. *International Journal of Geographical Information Science* 2006; **20**:899–916.
18. Frisk M, Gothe-Lundgren M, Jornsten K, Ronnqvist M. Cost allocation in collaborative forest transportation. *European Journal of Operational Research* 2010; **205**(2):448–458.
19. Yang H, Huang H. Carpooling and congestion pricing in a multilane highway with high-occupancy-vehicle lanes. *Transportation Research Part A: Policy and Practice* 1999; **33**(2):139–155.
20. Qian Z, Zhang H. Modeling multi-modal morning commute in a one-to-one corridor network. *Transportation Research Part C: Emerging Technologies* 2011; **19**(2):254–269.
21. Wardrop J. Some theoretical aspects of road traffic research. *ICE Proceedings: Engineering Division* 1952; **1**(3):325–362.
22. Patriksson M. *The Traffic Assignment Problem: Models and Methods*. CRC Press, 1994.
23. Gartner N. Optimal traffic assignment with elastic demands: a review part I. Analysis framework. *Transportation Science* 1980a; **14**(2):174–191.
24. Gartner N. Optimal traffic assignment with elastic demands: a review part II. Algorithmic approaches. *Transportation Science* 1980b; **14**(2):192–208.
25. Florian M, Nguyen S. A method for computing network equilibrium with elastic demands. *Transportation Science* 1974; **8**(4):321–332.
26. Fukushima M. A modified Frank–Wolfe algorithm for solving the traffic assignment problem. *Transportation Research Part B: Methodological* 1984; **18**(2):169–177.

27. Hearn D, Yildirim M. A toll pricing framework for traffic assignment problems with elastic demand. *Applied Optimization* 2002; **63**:135–145.
28. LeBlanc L, Farhangian K. Efficient algorithms for solving elastic demand traffic assignment problems and mode split-assignment problems. *Transportation Science* 1981; **15**(4):306–317.
29. Babonneau F, Vial J-P. An efficient method to compute traffic assignment problems with elastic demands. *Transportation Science* 2008; **42**(2):249–260.
30. Yildirim M, Hearn D. A first best toll pricing framework for variable demand traffic assignment problems. *Transportation Research Part B: Methodological* 2005; **39**(8):659–678.
31. Aashtiani H, Magnanti T. Equilibria on a congested transportation network. *SIAM Journal on Algebraic Discrete Methods* 1981; **2**(3):213–226.
32. Boyce D, Bar-Gera H. Validation of multiclass urban travel forecasting models combining origin-destination, mode, and route choices. *Journal of Regional Science* 2003; **43**(3):517–540.
33. Beckmann M, McGuire C, Winsten C. Studies in the economics of transportation. Technical report, RAND Corporation. 1955.
34. Abdulaal M, LeBlanc L. Methods for combining modal split and equilibrium assignment models. *Transportation Science* 1979; **13**(4):292–314.
35. LeBlanc L, Abdulaal M. Combined mode split-assignment and distribution-model split-assignment models with multiple groups of travelers. *Transportation Science* 1982; **16**(4):430–442.
36. Nie YM. A note on Bar-Gera's algorithm for the origin-based traffic assignment problem. *Transportation Science* 2012; **46**(1):27–38.
37. Bar-Gera H. Primal method for determining the most likely route flows in large road networks. *Transportation Science* 2006; **40**(3):269–286.
38. Bar-Gera H. Traffic assignment by paired alternative segments. *Transportation Research Part B: Methodological* 2010; **44**(8-9):1022–1046.
39. Jones KL, Lustig IJ, Farvolden JM, Powell WB. Multicommodity network flows: the impact of formulation on decomposition. *Mathematical Programming* 1993; **62**(1-3):95–117.
40. Facchinei F, Pang J-S. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer, 2003.
41. Frank M, Wolfe P. An algorithm for quadratic programming. *Naval Research Logistics Quarterly* 1956; **3**(1-2):95–110.
42. LeBlanc L. An efficient approach to solving the road network equilibrium traffic assignment problem. *Transportation Research* 1975; **9**(5):309–318.
43. Goffin J, Haurie A, Vial J-P. Decomposition and nondifferentiable optimization with the projective algorithm. *Management Science* 1992; **38**(2):284–302.
44. Agdeppa RP, Yamashita N, Fukushima M. The traffic equilibrium problem with nonadditive costs and its monotone mixed complementarity problem formulation. *Transportation Research Part B: Methodological* 2007; **41**:862–874.
45. Jayakrishnan R, Tsai WK, Prashker JN, Rajadhyaksha S. A faster path-based algorithm for traffic assignment. *Transportation Research Record* 1994; **1443**:75–83.
46. Bar-Gera H. Origin-based algorithm for the traffic assignment problem. *Transportation Science* 2002; **36**(4):398–417.
47. Dafermos SC, Sparrow FT. The traffic assignment problem for a general network. *Journal of Research of the National Bureau of Standards* 1969; **73B**(2):91–118.
48. Hubbard RG, O'Brien AP. *Economics* (4th). Prentice Hall, 2012.

APPENDIX A

UNIFIED DRIVER UTILITY FUNCTION

Claim. Suppose there are two different utility functions for solo and ridesharing drivers. We claim that the two can be combined as one generalized function.

Proof. Suppose for each OD pair k, the number of ridesharing drivers is denoted by δ_k^1 and the number of solo drivers is denoted by δ_k^2. The utility function of ridesharing drivers is given by

$$\Lambda_k^1(\delta_k, p_k) = \alpha_1 p_k - \beta_1 \delta_k^1$$

and the utility function of solo drivers is given by

$$\Lambda_k^2(\delta_k) = \alpha_2 - \beta_2 \delta_k^2$$

The utility functions depict that given the number of drivers (either solo or ridesharing drivers), how much congestion cost can they endure. The inverse with respect to the number of drivers indicates that under some certain traffic congestion, how many drivers are willing to travel, that is,

$$\delta_k^1 = \frac{\alpha_1}{\beta_1} p_k - \frac{\Lambda_k^1}{\beta_1},$$

$$\delta_k^2 = \frac{\alpha_2}{\beta_2} - \frac{\Lambda_k^2}{\beta_2}$$

Therefore, under the same traffic congestion $\Lambda_k^1 = \Lambda_k^2 = \Lambda_k$, the total number of drivers will be

$$\delta_k \triangleq \delta_k^1 + \delta_k^2 = \left(\frac{\alpha_1}{\beta_1} p_k + \frac{\alpha_2}{\beta_2} \right) - \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \Lambda_k$$

Note that there is only one constant term different from the utility function of ridesharing drivers. Therefore, without loss of generality, we may use one unified function to describe the utilities of both solo and ridesharing drivers.

B. PROOF OF PROPOSITION 1.

Proposition 1. *Models (6)–(10) are generated from models (1) to (5), substituting p_k (and its corresponding constraint (4)) by δ_k from the result of the economic equilibrium.*

Proof. According to the definition, the driver's ridesharing utility function, that is, the driver's profit by taking on passengers, is given by $U(p_k, q_k, \lambda_k) = p_k q_k - W(q_k, \lambda_k)$. First-order optimality condition gives us $p_k = \mathcal{S}_k(q_k, \lambda_k) = \frac{\partial W}{\partial q_k}(q_k, \lambda_k) = 2b_k q_k + \frac{d_k}{\lambda_k}$. Hence, $q_k = \frac{1}{2b_k}\left(p_k - \frac{d_k}{\lambda_k}\right)$.

Similarly, by optimizing the passenger's profit $u(q_k) - p_k q_k$, we have $p_k = D(q_k) = \frac{du}{dq_k}(q_k) = -2f_k q_k + g_k$. Hence, $q_k = \frac{1}{2f_k}(g_k - p_k)$. According to constraint (4), we have the system

$$p_k = 2b_k q_k + \frac{d_k}{\lambda_k} = -2f_k q_k + g_k,$$

$$q_k = \frac{1}{2b_k}\left(p_k - \frac{d_k}{\lambda_k}\right) = \frac{1}{2f_k}(g_k - p_k)$$

we have

$$p_k = \frac{b_k g_k + d_k f_k / \lambda_k}{b_k + f_k} \tag{A1}$$

$$q_k = \frac{g_k - d_k / \lambda_k}{2(b_k + f_k)} \tag{A2}$$

According to the result at the equilibrium of an elastic demand TAP, the congestion cost equals the utility function of drivers. We define the parameter $\lambda_k = \Lambda_k(\delta_k, p_k) = \alpha_k p_k - \beta_k \delta_k$. It can then be substituted into Equation (A1), and thus

$$p_k = \frac{b_k g_k + d_k f_k / \lambda_k}{b_k + f_k} = \frac{b_k g_k + d_k f_k / (\alpha_k p_k - \beta_k \delta_k)}{b_k + f_k},$$

$$\Rightarrow \quad \alpha_k(b_k + f_k)p_k^2 - [\beta_k(b_k + f_k)\delta_k + \alpha_k b_k g_k]p_k + (\beta_k b_k g_k \delta_k - d_k f_k) = 0$$

Therefore, by solving the aforementioned quadratic equation in terms of p_k, we have

$$\Pi(\delta_k) \triangleq p_k = \frac{\beta_k}{2\alpha_k}\delta_k + \frac{b_k g_k}{2(b_k + f_k)} \pm \frac{\Gamma(\delta_k)}{2\alpha_k(b_k + f_k)} \tag{A3}$$

where

$$\Gamma(\delta_k) = \sqrt{[\beta_k(b_k+f_k)]^2\delta_k^2 - 2\alpha_k\beta_k b_k g_k(b_k+f_k)\delta_k + \left[4\alpha_k d_k f_k(b_k+f_k) + (\alpha_k b_k g_k)^2\right]}$$

Thus, p_k can be expressed as a function of δ_k.

Moreover, the negative root can be discarded. The plus sign is chosen to guarantee the convexity of the objective function, that is, making $\Lambda_k(\delta_k)$ a decreasing function (Proposition 2). In sum, we have

$$p_k = \frac{\beta_k}{2\alpha_k}\delta_k + \frac{b_k g_k}{2(b_k+f_k)} + \frac{\Gamma(\delta_k)}{2\alpha_k(b_k+f_k)}$$

Substituting the aforementioned utility function $\Lambda_k(\delta_k, p_k)$, we have

$$\Lambda_k(\delta_k) = -\frac{\beta_k}{2}\delta_k + \frac{\alpha_k b_k g_k}{2(b_k+f_k)} + \frac{\Gamma(\delta_k)}{2(b_k+f_k)}$$

C. PROOF OF PROPOSITION 2.

Proposition 2. *If the parameters satisfy the following,*

- $b_k, f_k > 0$ and $d_k \geq 0$,

- $g_k \geq d_k/\lambda_k^{(0)}, \quad \forall k \in \mathcal{K}$,

- $\frac{\alpha_k d_k f_k}{b_k+f_k} \leq \left[\lambda_k^{(0)}\right]^2, \quad \forall k \in \mathcal{K}$

then models (6)–(10) are convex optimization problems.

Proof. Constraints (7)–(10) are all linear. It is sufficient to show that the objective function (6) is a convex function.

The first part of the objective function is convex because it is an increasing function with respect to y_a (or equivalently δ_k). To show the convexity of the second part, it is sufficient to prove that $\Lambda_k(\delta_k)$ is a decreasing function of δ_k.

First of all, consider the fact that $\Lambda_k(\delta_k, p_k)$ can never be negative (it is treated as the congestion cost that δ_k drivers could bear and thus should always be positive; see [29]). Hence, we must have $a_k = max\left\{0, \frac{1}{\beta_k}(\alpha_k p_k - \lambda_k)\right\}$. Substituting Equation (A1),

$$0 \leq \delta_k \leq \frac{\alpha_k b_k g_k + d_k f_k/\lambda_k}{\beta_k} \frac{\lambda_k}{b_k+f_k} - \frac{\lambda_k}{\beta_k}$$

$$= \frac{\alpha_k b_k g_k}{\beta_k(b_k+f_k)} + \frac{\alpha_k d_k f_k}{\beta_k(b_k+f_k)\lambda_k}\frac{1}{\lambda_k} - \frac{\lambda_k}{\beta_k}$$

$$\leq \frac{\alpha_k b_k g_k}{\beta_k(b_k+f_k)} + \frac{\alpha_k d_k f_k}{\beta_k(b_k+f_k)\lambda_k^{(0)}}\frac{1}{\lambda_k^{(0)}} - \frac{\lambda_k^{(0)}}{\beta_k}$$

that is, constraint (9). The last inequality holds because $\lambda_k \geq \lambda_k^{(0)}$, for all $k \in \mathcal{K}$.

On the other hand, substituting Equation (A3) into $\Lambda_k(\delta_k, p_k)$, we have

$$\Lambda_k(\delta_k) \triangleq \Lambda_k(\delta_k, p_k) = \alpha_k \Pi(\delta_k) - \beta_k \delta_k = -\frac{\beta_k}{2}\delta_k + \frac{\alpha_k b_k g_k}{2(b_k + f_k)} \pm \frac{\Gamma(\delta_k)}{2(b_k + f_k)}$$

Note that the symmetric axis of $\Gamma(\delta_k)$ is $\delta_k = \frac{\alpha_k b_k g_k}{\beta_k(b_k + f_k)}$. Therefore, $\Lambda_k(\delta_k)$ is decreasing only when

(a) $\quad \Lambda_k(\delta_k) = -\frac{\beta_k}{2}\delta_k + \frac{\alpha_k b_k g_k}{2(b_k + f_k)} + \frac{\Gamma(\delta_k)}{2(b_k + f_k)} \quad$ and $\quad \delta_k \leq \frac{\alpha_k b_k g_k}{\beta_k(b_k + f_k)}$, or

(b) $\quad \Lambda_k(\delta_k) = -\frac{\beta_k}{2}\delta_k + \frac{\alpha_k b_k g_k}{2(b_k + f_k)} - \frac{\Gamma(\delta_k)}{2(b_k + f_k)} \quad$ and $\quad \delta_k \geq \frac{\alpha_k b_k g_k}{\beta_k(b_k + f_k)}$

Combining constraint (9), we must have

$$\frac{\alpha_k b_k g_k}{\beta_k(b_k + f_k)} + \frac{\alpha_k d_k f_k}{\beta_k(b_k + f_k)}\frac{1}{\lambda_k^{(0)}} - \frac{\lambda_k^{(0)}}{\beta_k} \leq \frac{\alpha_k b_k g_k}{\beta_k(b_k + f_k)}, \quad \text{or} \quad \frac{\alpha_k b_k g_k}{\beta_k(b_k + f_k)} \leq 0$$

Because all the parameters are nonnegative and some are strictly positive, the latter will never hold. Therefore, we must have $\Lambda_k(\delta_k) = -\frac{\beta_k}{2}\delta_k + \frac{\alpha_k b_k g_k}{2(b_k+f_k)} + \frac{\Gamma(\delta_k)}{2(b_k+f_k)}$ and $\frac{\alpha_k b_k g_k}{\beta_k(b_k+f_k)} + \frac{\alpha_k d_k f_k}{\beta_k(b_k+f_k)}\frac{1}{\lambda_k^{(0)}} - \frac{\lambda_k^{(0)}}{\beta_k} \leq \frac{\alpha_k b_k g_k}{\beta_k(b_k+f_k)}$, that is, $\frac{\alpha_k d_k f_k}{b_k+f_k} \leq \left[\lambda_k^{(0)}\right]^2$ for all OD pair $k \in \mathcal{K}$.

Moreover, to assure the nonnegativity of the number of passengers in Equation (A2), we must have $g_k \geq d_k/\lambda_k^{(0)}$, for all $k \in \mathcal{K}$.

In sum, given the aforementioned conditions for parameter settings, $\Lambda_k(\delta_k)$ is a decreasing function of δ_k in domain (9). And thus models (6)–(10) are convex optimization problems.

D. PROOF OF PROPOSITION 3.

Proposition 3. *Under the assumptions of Proposition 2, there exists an optimal solution to problems (6)–(10) because of convexity. Moreover, the optimal solution is a UE.*

Proof. Here, we show that the optimal solution to models (6)–(10), which exists because the problem is convex, is a UE.

We do so by showing that the cost of any path between the origin and destination of an OD pair that has positive flow equals the minimum path cost for that OD pair. Consider the Karush–Kuhn–Tucke conditions for problems (6)–(10). Here, μ^k, λ_a, $\gamma_k \geq 0$, $\pi_k \geq 0$, and $\theta_a^k \geq 0$ are the multipliers to constraints (7)–(10), and μ^k is a vector, that is, $\mu^k = \left(u_i^k\right)_{i \in \mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}$.

$$tt_a(y_a) - \lambda_a = 0, \qquad \forall a \in \mathcal{A} \tag{A4}$$

$$\mu_j^k - \mu_i^k + \lambda_a - \theta_a^k = 0, \qquad \forall a \in \mathcal{A}, \forall k \in \mathcal{K} \tag{A5}$$

$$-\Lambda_k(\delta_k) + \mu_{o_k}^k - \mu_{d_k}^k - \gamma_k + \pi_k = 0, \qquad \forall k \in \mathcal{K} \tag{A6}$$

$$\gamma_k \delta_k = 0, \qquad \forall k \in \mathcal{K} \tag{A7}$$

$$\pi_k\left(\delta_k - \frac{\alpha_k b_k g_k}{\beta_k(b_k+f_k)} + \frac{\alpha_k d_k f_k}{\beta_k(b_k+f_k)}\frac{1}{\lambda_k^{(0)}} - \frac{\lambda_k^{(0)}}{\beta_k}\right) = 0, \qquad \forall k \in \mathcal{K} \tag{A8}$$

$$\theta_a^k x_a^k = 0, \qquad \forall a \in \mathcal{A}, \forall k \in \mathcal{K} \tag{A9}$$

plus primal constraints (7)–(10) and all nonnegativities.

In Equation (A5), i and j correspond to arc a, that is, $a=(i,j)$; similarly in Equation (A6), o_k and d_k correspond to the origin and the destination of k, that is, $k=(o_k,d_k)$. Also note that Equation (A4) and (A5) gives us

$$tt_a(y_a) = \mu_i^k - \mu_j^k + \theta_a^k$$

Consider now any path l of a given OD pair k, (where l consists of a sequence of consecutive arcs that starts at o_k and ends at d_k) the travel time (congestion cost) on the path is the summation of costs on consecutive arcs that consist of path l, that is,

$$\sum_{a \in l} tt_a(y_a) = \sum_{a \in l}\left(\mu_i^k - \mu_j^k + \theta_a^k\right)$$
$$= \mu_{o_k}^k - \mu_{d_k}^k + \sum_{a \in l}\theta_a^k$$
$$\geq \Lambda_k(\delta_k) + \gamma_k - \pi_k$$

The second equality holds because the arcs are consecutive, and thus, μ_i^k's are canceled out except for the origin and the destination. The third inequality holds because of Equation (A6) and the fact that $\theta_a^k \geq 0$. Therefore, the quantity $\Lambda_k(\delta_k) + \gamma_k - \pi_k$ is a lower bound on the cost of any path of OD pair k.

Note that if path l has positive flow, which means that $x_a^k > 0$ for any $a \in l$, from Equation (A9) we have that $\theta_a^k = 0$ for all $a \in l$, and thus $\sum_{a \in l}\theta_a^k = 0$. This shows that the cost of any path of OD pair k with positive flow is the same, and it equals $\Lambda_k(\delta_k) + \gamma_k - \pi_k$.

Note that $\Lambda_k(\delta_k)$ represents the least travel congestion cost a driver could bear given δ_k vehicles on the road.

- If Equation (9) strictly holds, that is, $0 < \delta_k < \frac{\alpha_k b_k g_k}{\beta_k(b_k+f_k)} + \frac{\alpha_k d_k f_k}{\beta_k(b_k+f_k)}\frac{1}{\lambda_k^{(0)}} - \frac{\lambda_k^{(0)}}{\beta_k}$, we must have $\gamma_k = \pi_k = 0$, because of the complementary slackness (A7) and (A8). This indicates that all vehicles (drivers) are traveling with the same congestion cost, that is,

$$\lambda_k = \sum_{a \in l} tt_a(y_a) = \Lambda_k(\delta_k), \qquad \forall l \tag{A10}$$

- If $\delta_k = 0$, we must have $\pi_k = 0$ and $\gamma_k \geq 0$. Therefore, $\sum_{a \in l}tt_a(y_a) \geq \Lambda_k(\delta_k)$, indicating that the congestion cost is larger than what drivers could bear, and thus, there is no driver for OD pair k, that is, $\delta_k = 0$.
- If $\delta_k = \frac{\alpha_k b_k g_k}{\beta_k(b_k+f_k)} + \frac{\alpha_k d_k f_k}{\beta_k(b_k+f_k)}\frac{1}{\lambda_k^{(0)}} - \frac{\lambda_k^{(0)}}{\beta_k}$, we shall have $\gamma_k = 0$ and $\pi_k \geq 0$. This illustrates the congestion cost is less than drivers could bear, and thus, all potential drivers are traveling on the road.