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Robust solutions for network design under transportation cost and demand uncertainty

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In many applications, the network design problem (NDP) faces significant uncertainty in transportation costs and demand, as it can be difficult to estimate current (and future values) of these quantities. In this paper, we present a robust optimization-based formulation for the NDP under transportation cost and demand uncertainty. We show that solving an approximation to this robust formulation of the NDP can be done efficiently for a network with single origin and destination per commodity and general uncertainty in transportation costs and demand that are independent of each other. For a network with path constraints, we propose an efficient column generation procedure to solve the linear programming relaxation. We also present computational results that show that the approximate robust solution found provides significant savings in the worst case while incurring only minor sub-optimality for specific instances of the uncertainty.

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Introduction

The network design problem (NDP) is an important problem in the planning and operation of many applications, such as inventory and distribution systems. Network design makes decisions on where to increase arc capacities to reduce the overall network routing/transmission cost. There exists substantial research on network design (or capacity planning) problems in different domains such as manufacturing (Eppen *et al*, 1989; Barahona *et al*, 2004, Zhang *et al*, 2004), electric utilities (Murphy and Weiss, 1990; Malcolm and Zenios, 1994), telecommunications (Balakrishnan *et al*, 1995; Laguna, 1998; Riis and Andersen, 2004), inventory management (Hsu, 2002), and transportation (Magnanti and Wong, 1984; Minoux, 1989).

In this work, we are interested in NDPs with both transportation cost and demand uncertainty, a reasonable modelling assumption in many settings as it can be difficult to estimate current (and future) costs and demand. Most prior work on network design under uncertainty addresses the uncertainty through scenario-based stochastic programming (Birge and Louveaux, 1997) or its robust optimization approach introduced in Mulvey *et al* (1995). These scenario-based methods face the following difficulties: (1) they assume a known discrete description of the uncertainty, which can be a crude approximation of reality; (2) the large number of scenarios used in accurately representing the uncertainty can lead to large, computationally challenging problems; and (3) the solution obtained can be sensitive to possible uncertainty outcomes, a difficulty already addressed by Mulvey *et al* (1995).

These drawbacks of scenario-based approaches are addressed via the *Robust Optimization* methodology (Ben-Tal and Nemirovski, 1998; El-Ghaoui *et al*, 1998), which aims for a solution that is robust or insensitive to the uncertainty considered and thus is an efficient solution in practice. In addition, this robust optimization approach can obtain robust solutions by solving a problem that is no harder than the deterministic problem. Although initially developed for continuous convex optimization, there are extensions of robust optimization to integer programming (Atamtürk, 2003; Bertsimas and Sim, 2003) and in particular to the NDP (Atamtürk and Zhang, 2004; Ordóñez and Zhao, 2004). Alternative robust optimization methods for integer programming problems (such as Averbakh and Berman, 2000; Kouvelis and Yu, 1997; Yaman *et al*, 2001) typically rely on combinatorial arguments making them difficult to generalize, in addition they can lead to problems that are significantly more difficult to solve than the deterministic version of each problem.

In this paper, we extend the prior work on robust optimization for NDPs (Atamtürk and Zhang, 2004; Ordóñez and Zhao, 2004) by generalizing some key demand uncertainty assumptions. We consider the uncertainty on demand and transportation cost represented by independent closed convex sets for network problems with single origin and destination per commodity. We show that, for general uncertainty sets, obtaining an approximate robust solution is as difficult as solving

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the deterministic problem and the solution provides significant protection against the worst case at a modest loss of optimality for a given scenario. We also show that for problems with single source and sink per commodity and non-negative costs, the tractable robust approximation corresponds exactly to the robust counterpart (RC) of the path formulation of the NDP. Finally, we present a column generation approach, suitable for network design with path constraints that efficiently solves the linear programming (LP) relaxation of the robust problem under polyhedral cost and demand uncertainty.

The structure of the paper is as follows: in the next section, we present the NDP formulation, describe the robust optimization methodology as it pertains to our problem, and introduce the general form of the robust network design problem (RNDP). We then present the approximate RNDP that can be solved efficiently and discuss its equivalence to RNDP for the path formulation of the problem. In the Path Constrained Network Design section, we introduce a column generation method suitable for the LP relaxation of RNDP with path constraints under polyhedral uncertainty. We present computational results for the approximate RNDP method and the column generation method in the Computational Experiments section. We finish the paper with some concluding remarks.

Problem formulation

In this section, we present the NDP considered in this paper and its robust optimization formulation. We also discuss related robust optimization literature.

Network design problem

We consider in this work a classic multi-commodity NDP (Minoux, 1989). In other words, the problem decides, for a network with n nodes and m arcs, whether or not to increase the arc capacity $y \in \{0, 1\}^m$ and the arc flows $x^k \in \Re^m_+$ to route $k \in \{1, \ldots, K\}$ commodities at a minimum total transportation and investment cost. We denote by A the set of directed arcs and by $b^k \in \Re^n$ the vector of node demands and supplies for commodity $k \in \{1, \ldots, K\}$. For each arc $(i, j) \in A$, we let u_{ij} represent the existing arc capacity, c_{ij} represent the linear transportation cost coefficient, and f_{ij} be the linear investment cost of increasing the capacity of arc $(i, j) \in A$ by v_{ij}. Hence, we formulate the NDP as the following mixed integer programming (MIP) problem:

$$\begin{aligned}
\min \quad & \sum_{k=1}^{K} \sum_{(i,j)\in A} c_{ij} x_{ij}^k + \sum_{(i,j)\in A} f_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{j:(i,j)\in A} x_{ij}^k - \sum_{j:(j,i)\in A} x_{ji}^k = b^k \qquad \forall i, k = 1, \ldots, K \\
& \sum_{k=1}^{K} x_{ij}^k \leqslant u_{ij} + v_{ij} y_{ij} \qquad \forall (i,j) \in A \\
& x_{ij}^k \geqslant 0 \qquad \forall (i,j) \in A,\ k = 1, \ldots, K \\
& y_{ij} \in \{0, 1\} \qquad \forall (i,j) \in A
\end{aligned} \tag{1}$$

The objective function equals the total transportation cost plus the total investment cost. The first constraint ensures that all flow demand is routed for every commodity $k \in \{1, \ldots, K\}$, while the second constraint ensures that the total flow on every arc is less than or equal to the available capacity at that arc. The last two constraints ensure that flow is non-negative and that the investment variables are integer. In matrix notation the NDP is

$$\begin{aligned}
\min \quad & \sum_{k=1}^{K} c^{\mathrm{T}} x^k + f^{\mathrm{T}} y \\
\text{s.t.} \quad & N x^k = b^k \qquad \forall k = 1, \ldots, K \\
& \sum_{k=1}^{K} x^k \leqslant u + V y \\
& x^k \geqslant 0 \qquad \forall k = 1, \ldots, K \\
& y \in \{0, 1\}^m
\end{aligned}$$

where N is the network's node-arc incidence matrix and the matrix V is a $m \times m$ diagonal matrix with v_{ij} entries on the diagonal.

Without loss of generality, we assume that the network flow problem is always feasible, even for $y=0$, by introducing uncapacitated, high cost, artificial arcs between all source and sink nodes for each commodity. These arcs allow to route the flow regardless of the capacity expansion solution. As in a big M procedure, the large transportation cost on the artificial arcs ensures they would not be part of the optimal routing solution if the original problem is feasible, thus providing a method to detect infeasible instances.

Robust optimization methodology

The robust optimization approach was introduced in Ben-Tal and Nemirovski (1998) for convex optimization and in El-Ghaoui *et al* (1998) for semi-definite programming. The robust solution for an optimization problem under uncertainty is defined as the solution that has the best objective value in its worst case uncertainty scenario. Attractive features of a robust solution are that while it is only close to optimal for any specific scenario, it behaves well over all likely uncertainty outcomes. In addition, in many settings finding the robust solution is no harder than solving the deterministic problem. Robust optimization has provided interesting answers to applications on structural design (Ben-Tal and Nemirovski, 1997), least-square optimization (El-Ghaoui and Lebret, 1997), portfolio optimization problems (El-Ghaoui *et al*, 2003; Goldfarb and Iyengar, 2003), supply chain management problems (Ben-Tal *et al*, 2005; Bertsimas and Thiele, 2003), and integer programming and network flows (Bertsimas and Sim, 2003; Atamtürk and Zhang, 2004).

In particular, the work by Bertsimas and Sim (2003) considers robust solutions for network flow problems with box uncertainty in cost coefficients; also the recent work by

Atamtürk and Zhang (2004) investigates robust solutions for the network flow and NDPs and shows that for certain networks and specific demand uncertainty sets the robust problem is tractable. Finally, in Ordóñez and Zhao (2004), we show that the RNDP is tractable for networks with single origin and destination per commodity and under general transportation cost uncertainty and box uncertainty in the demand. Here, we extend these prior results by considering the NDP under more general and realistic uncertainty assumptions. We investigate the NDP on a general network with single origin–destination commodities and general transportation cost and demand uncertainty sets.

To illustrate the robust optimization methodology, consider the following optimization problem under uncertainty:

$$\begin{aligned} \min_{u,v} \quad & f(u,v,w) \\ \text{s.t.} \quad & g(u,v,w) \leqslant 0 \end{aligned}$$

where the uncertainty parameter w belongs to a closed bounded and convex uncertainty set $w \in \mathscr{U}$. The robust solution is obtained by solving the following RC problem:

$$\begin{aligned} z_{\mathrm{RC}} = \min_{u,v,\gamma} \quad & \gamma \\ \text{s.t.} \quad & f(u,v,w) \leqslant \gamma \text{ for all } w \in \mathscr{U} \\ & g(u,v,w) \leqslant 0 \text{ for all } w \in \mathscr{U} \end{aligned} \tag{2}$$

The complexity of solving problem RC has been shown to be the same as the complexity of solving the deterministic problem (fixed $w \in \mathscr{U}$) for various problems and uncertainty sets $\mathscr{U}$. For example, the RC of an LP is equivalent to an LP when $\mathscr{U}$ is a polyhedron and to a quadratically constrained convex program when $\mathscr{U}$ is a bounded ellipsoidal set (Ben-Tal and Nemirovski, 1998). In addition, the size of the resulting RC problem is bounded by a polynomial in the deterministic problem's dimensions.

The RC for a stochastic problem with recourse, dubbed the adjusted robust counterpart problem (ARC), is introduced in Ben-Tal *et al* (2004). In a problem with recourse, some of the decision variables u are decided *a priori* (or here and now), while the rest v *can adjust* to the outcome of the uncertainty (or wait and see). Using variables v to accommodate to the uncertainty outcome leads to the ARC problem:

$$\begin{aligned} z_{\mathrm{ARC}} = \min_{u,\gamma} \quad & \gamma \\ \text{s.t.} \quad & \text{for all } w \in \mathscr{U} \text{ exists } v : \begin{cases} f(u,v,w) \leqslant \gamma \\ g(u,v,w) \leqslant 0 \end{cases} \end{aligned} \tag{3}$$

Clearly $z_{\mathrm{ARC}} \leqslant z_{\mathrm{RC}}$, since selecting one v that is feasible for all $w \in \mathscr{U}$ is possible for ARC. However, we do not retain the nice complexity results. In fact, Theorem 3.5 of Guslitser (2002) shows that the ARC of an LP with polyhedral uncertainty is NP-hard. An approximate solution for ARC (AARC) is given by limiting the recourse variables to some linear function of the uncertainty, say $v = Qw + q$. This yields the following AARC

$$\begin{aligned} z_{\mathrm{AARC}} = \min_{u,\gamma,Q,q} \quad & \gamma \\ \text{s.t.} \quad & f(u, Qw+q, w) \leqslant \gamma \text{ for all } w \in \mathscr{U} \\ & g(u, Qw+q, w) \leqslant 0 \text{ for all } w \in \mathscr{U} \end{aligned} \tag{4}$$

This approximate problem is potentially tractable as it is of the form of the regular RC. Note also that $z_{\mathrm{ARC}} \leqslant z_{\mathrm{AARC}} \leqslant z_{\mathrm{RC}}$ since, on the one hand, we limited the possible recourse variables and, on the other, we can select $Q = 0$ and $q = v$ for some v that is feasible for all $w \in \mathscr{U}$.

Robust Network Design Problem

The NDP has a natural separation between 'here and now' decisions and 'wait and see' decisions: investment decisions must be made before we observe the results of the demand and transportation cost uncertainty, while the routing decisions made by the planner have to route whatever demand occurred and under the transportation conditions that are set forth by the realized transportation costs. Hence, if we dispose of good data management tools, we can assume that the routing decisions are made with the knowledge of the actual traffic conditions. In conclusion, investment variables y are decided before the uncertainty while routing variables x are decided as a recourse to the uncertainty. Denoting the demand uncertainty set $\mathscr{U}_b$ and the transportation cost uncertainty set $\mathscr{U}_c$ we express the adjustable RNDP by

$$\begin{aligned} \min_{y,\gamma} \quad & f^{\mathrm{T}} y + \gamma \\ \text{s.t.} \quad & y \in \{0,1\}^m, \\ & \begin{matrix} \forall\, b \in \mathscr{U}_b,\, c \in \mathscr{U}_c \\ \text{there exists } x \text{ s.t.} \end{matrix} \begin{cases} \sum_{k=1}^{K} c^{\mathrm{T}} x^k \leqslant \gamma \\ N x^k = b^k \quad \forall k = 1, \ldots, K \\ \sum_{k=1}^{K} x^k \leqslant u + Vy \\ x \in \Re_+^{Km} \end{cases} \end{aligned} \tag{5}$$

It is possible that some routing decisions also have to be made prior to the full realization of the transportation network conditions. This can be included in this model by deciding those routing variables jointly with the investment variables y.

Approximate adjusted RNDP

Uncertainty sets

We now briefly discuss the types of sets considered in this work to represent the uncertainty in transportation cost and demand. The uncertainty sets in this work are defined as deviations from an estimated or nominal value of the uncertain parameter. For example, for the uncertain parameter $z \in \Re^k$, we consider sets around the estimated value $\bar{z} \in \Re^k$ and using a scalar value ρ to control the confidence on the estimate, of the following form:

$$\text{Polyhedral:} \quad \mathcal{U} = \{\bar{z} + \rho z \in \Re^k : L_z \leqslant h, z \leqslant 0\} \tag{6}$$

$$\text{Box:} \quad \mathcal{U} = \{z \in \Re^k : |z_k - \bar{z}_k| \leqslant \rho G_k, k = 1, \ldots, K\} \tag{7}$$

$$\text{Ellipsoidal:} \quad \mathcal{U} = \{z \in \Re^k : (z - \bar{z})^t S^{-1}(z - \bar{z}) \leqslant \rho^2\} \tag{8}$$

In particular, sets (6) and (8) are quite general and can represent arbitrary correlation structures in the uncertain parameter z. In this work, we present results where the uncertainty in demand $\mathcal{U}_b$ and in transportation costs $\mathcal{U}_c$ are defined in this form. This allows us to consider correlations between demands of different commodities on the network, which is a reasonable assumption in transportation systems. In such systems, a large demand to a specific location could be due to morning rush hour traffic and hence be related to large demands from other locations/commodities. We do not consider at this time correlations between transportation costs and demands.

The sets above can be interpreted as special cases of sets formed by combining deviation scenarios around an estimated value, as we show in the following proposition.

Proposition 1 *The uncertainty sets* (6), (7), *and* (8) *are special cases of uncertainty sets of the form*

$$\mathcal{U} = \left\{ z \in \Re^k : z = \bar{z} + \sum_{s=1}^{S} \xi_s z^s, \, \xi \in \chi \right\} \tag{9}$$

where the set χ is some easily characterizable closed convex set.

Proof We only show that sets (6) and (8) are special cases of sets of the form (9), since the box uncertainty set (7) is a special case of (6).

First, let $S=k$ and consider $z^s = \rho e_s$, for $s=1, \ldots, k$, where e_s is the vector with a one in the s coordinate and zero in the rest. Setting $\chi = \{\xi \in \Re^k : L\xi \leqslant h, \, \xi \geqslant 0\}$ makes (9) equivalent to (6). Now, to express an ellipsoidal set (8) using scenarios, simply let $\chi = \{\xi \in \Re^k : \xi^{\mathrm{T}} \xi \leqslant \rho^2\}$ and $z = \bar{z} + S^{1/2}\xi$, that is let z^s be the sth column of $S^{1/2}$. □

Tractable approximate adjusted RNDP

As the adjusted problem is hard to solve in general, the solution approach introduced in Ben-Tal *et al* (2004) approximates this problem by limiting the second stage variable to be some linear function of the uncertainty. It is natural to define this function for the flow of the kth commodity only in terms of b^k as follows:

$$x^k = Q^k b^k, \qquad k = 1, \ldots, K \tag{10}$$

This linear relationship between the uncertain demand and arc flows leads to the following routing constraints in matrix form

$$(NQ^k - I)b^k = 0 \qquad \forall k = 1, \ldots, K \quad \forall b \in \mathcal{U}_b$$

This condition constraints Q^k to be such that all rows of $(NQ^k - I)$ are orthogonal to the projection of $\mathcal{U}_b$ onto the kth commodity demand b^k. Handling this type of constraints for arbitrary uncertainty sets in general networks can be difficult; therefore, we assume that each commodity has a single source and sink which leads to an important simplification. A single origin–destination assumption is fairly realistic for many applied problems, such as in transportation or communications where the origin and destination classify the network flow. If each commodity k sends d_k flow from an origin s_k to a destination t_k and if we let $e_i \in \Re^n$ be the vector of all zeros except a 1 in the ith position, we have that $b^k = (e_{s_k} - e_{t_k})d_k$. This simplifies expression (10) to:

$$x_{ij}^k = q_{ij}^k d_k \qquad \forall (i, j) \in A, \quad k = 1, \ldots, K$$

and the network flow constraints to $Nq^k = e_{s_k} - e_{t_k}$.

Representing the second stage flow variables of the adjusted RNDP with this affine function leads to the following problem:

$$\begin{aligned}
\min_{y,\gamma,q} \quad & f^{\mathrm{T}} y + \gamma \\
\text{s.t.} \quad & y \in \{0, 1\}^m \\
& Nq^k = e_{s_k} - e_{t_k} \qquad \forall k = 1, \ldots, K \\
& \sum_{k=1}^{K} c^{\mathrm{T}} q^k d_k \leqslant \gamma \qquad \forall d \in \mathcal{U}_b, \; c \in \mathcal{U}_c \\
& \sum_{k=1}^{K} q^k d_k \leqslant u + Vy \qquad \forall d \in \mathcal{U}_b \\
& q_{ij}^k d_k \geqslant 0 \qquad \forall (i, j) \in A, \; k = 1, \ldots, K, \\
& d \in \mathcal{U}_b.
\end{aligned} \tag{11}$$

Here and in the remainder of the paper, we incur the slight abuse of notation $d \in \mathcal{U}_b$. In the case of single origin–destination per commodity, the demand of the system is described by the vector $d \in \Re^K$. It is this vector over which we define the demand uncertainty set, but keep the notation $\mathcal{U}_b$ for consistency with the prior discussion.

In Guslitser (2002), it is shown that the AARC can be NP-hard when the recourse matrix is stochastic, which is the case in the problem above. However our selection of the linear function of the second stage variables guarantees that we are still able to solve the approximate problem efficiently, as we prove in the next theorem. Here, and in the remainder of the paper, we refer to the kth column and jth row of a matrix $A \in \Re^{J \times K}$ as $A_{\bullet,k} \in \Re^J$ and $A_{j,\bullet} \in \Re^K$, respectively.

Theorem 1 *The approximate adjusted RNDP with uncertain demand and transportation costs given by polyhedral sets* $\mathcal{U}_b = \{\bar{d} + \rho_b d \in \Re^K : Ld \leqslant h, \; d \geqslant 0\}$, *with* $h \in \Re^{l_b}$, *and* $\mathcal{U}_c = \{\bar{c} + \rho_c c \in \Re^m : Mc \leqslant g, \; c \geqslant 0\}$, *with* $g \in \Re^{l_c}$, *is equivalent to the following MIP problem*:

$$\begin{aligned}
\min_{y,\gamma,q,\lambda,\pi,\theta,\kappa} \quad & f^{\mathrm{T}}y + g^{\mathrm{T}}\kappa + \gamma \\
\text{s.t.} \quad & Nq^k = e_{s_k} - e_{t_k}, \qquad k = 1, \ldots, K \\
& \sum_{k=1}^{K} \bar{c}^{\mathrm{T}} q^k \bar{d}_k + \rho_b \lambda^{\mathrm{T}} h \leqslant \gamma \\
& L^T_{\bullet,k}\lambda \geqslant \bar{c}^{\mathrm{T}} q^k, \qquad k = 1, \ldots, K \\
& \sum_{k=1}^{K} q^k_{ij}\bar{d}_k + \rho_b \pi^{\mathrm{T}}_{ij} h \leqslant u_{ij} + v_{ij} y_{ij}, \quad (i, j) \in A \\
& L^{\mathrm{T}}_{\bullet,k}\pi_{ij} \geqslant q^k_{ij}, \qquad (i, j) \in A, \; k = 1, \ldots, K \\
& \sum_{k=1}^{K} \rho_c q^k_{ij}\bar{d}_k + \rho_c \rho_b \theta^{\mathrm{T}}_{ij} h (M^{\mathrm{T}}\kappa)_{ij}, \qquad (i, j) \in A \\
& L^T_{\bullet,k}\theta_{ij} \geqslant q^k_{ij}, \qquad (i, j) \in A, \; k = 1, \ldots, K \\
& q \in \Re^{Km}_{+}, \lambda \in \Re^{l_b}_{+}, \pi \in \Re^{ml_b}_{+}, \theta \in \Re^{ml_b}_{+}, \\
& \kappa \in \Re^{l_c}_{+}, \; y \in \{0, 1\}^m \qquad (12)
\end{aligned}$$

Proof The proof of this theorem is based on the fact that from duality theory we have that the following are equivalent for a polyhedral uncertainty set:

$$\begin{aligned}
\max_{z} \quad & m^{\mathrm{T}}\bar{z} + \rho m^{\mathrm{T}} z \leqslant \Gamma & & \min_{\lambda} \quad m^{\mathrm{T}}\bar{z} + \rho h^{\mathrm{T}}\lambda \leqslant \Gamma \\
\text{s.t.} \quad & Lz \leqslant h & \Leftrightarrow \quad & \text{s.t.} \quad L^T\lambda \geqslant m \\
& z \geqslant 0 & & \qquad \lambda \geqslant 0
\end{aligned}$$

Therefore, this constraint on the optimal objective function value of a linear program is equivalent to having a feasible dual variable, $\lambda \geqslant 0$, $L^{\mathrm{T}}\lambda \geqslant m$ such that $m^{\mathrm{T}}\bar{z} + \rho h^{\mathrm{T}}\lambda \leqslant \Gamma$. If we apply this property to find the most restrictive linear constraint involving $c \in \mathscr{U}_c$ in problem (11) we obtain

$$\begin{aligned}
\min_{y,\gamma,q} \quad & \sum_{(i,j)\in A} f_{ij} y_{ij} + \gamma \\
\text{s.t.} \quad & Nq^k = e_{s_k} - e_{t_k}, \qquad k = 1, \ldots, K \\
& \left.\begin{aligned} & \sum_{k=1}^{K} \bar{c}^{\mathrm{T}} q^k \bar{d}_k + \rho_c g^{\mathrm{T}}\kappa \leqslant \gamma \\ & M^T\kappa \geqslant \sum_{k=1}^{K} q^k d_k \end{aligned}\right\} \qquad \forall d \in \mathscr{U}_b \\
& \sum_{k=1}^{K} q^k d_k \leqslant u + Vy \qquad \forall d \in \mathscr{U}_b \\
& q^k d_k \leqslant 0 \qquad \forall k = 1, \ldots, K, \; d \in \mathscr{U}_b \\
& y \in \{0, 1\}^m, \; \kappa \in \Re^{l_c}_{+}
\end{aligned}$$

Using a change of variables and considering the more constrained problem where the second and third constraints have to satisfy the bounds for independent $d \in \mathscr{U}_b$, we can replace the objective function in the problem above by $f^Ty + g^T\kappa + \gamma$, the second constraint by $\sum_{k=1}^{K}\bar{c}^{\mathrm{T}}q^k d_k \leqslant \gamma$ for all $d \in \mathscr{U}_b$ and the third constraint by $\rho_c\sum_{k=1}^{K} q^k d_k \leqslant M^T\kappa$ for all $d \in \mathscr{U}_b$. We note that since $d \in \mathscr{U}_b$ implies $d \geqslant 0$, the constraint $q^k_{ij} d_k \geqslant 0$ is equivalent to $q_{ij} \geqslant 0$. The proof concludes by applying the duality property for the polyhedral uncertainty sets to identify the $d \in \mathscr{U}_b$ that make the second, third, and fourth constraints most restrictive. □

Below we present similar results for box and ellipsoid uncertainty sets. We omit the proofs of these results as they are analogous to the previous one, relying on the closed form solutions to the optimization of a linear objective over a box

$$\begin{aligned}
\bar{z} + \text{sign }(m)\rho = \text{argmax } & m^T z \\
\text{s.t.} \quad & |z - \bar{z}| \leqslant \rho
\end{aligned}$$

and over an ellipse

$$\begin{aligned}
\bar{z} + \frac{\rho}{\sqrt{m^T Sm}} Sm = \text{argmax } & m^T z \\
\text{s.t.} \quad & (z - \bar{z})S^{-1}(z - \bar{z}) \leqslant \rho^2
\end{aligned}$$

Theorem 2 *The approximate adjusted RNDP with ellipsoidal set demand uncertainty* $\mathscr{U}_b = \{d \in \Re^k : (d - \bar{d})^{\mathrm{T}} S^{-1} (d - \bar{d}) \leqslant \rho_b^2\}$, *such that* $\mathscr{U}_b \subset \Re^K_+$, *and polyhedral set transportation costs uncertainty* $\mathscr{U}_c = \{\bar{c} + \rho_c c \in \Re^m : Mc \leqslant g, \; c \geqslant 0\}$, *with* $g \in \Re^{l_c}$, *is equivalent to the following MIP problem with quadratic constraints.*

$$\begin{aligned}
\min_{y,\gamma,q,\kappa} \quad & f^{\mathrm{T}}y + g^{\mathrm{T}}\kappa + \gamma \\
\text{s.t.} \quad & Nq^k = e_{s_k} - e_{t_k}, \quad k = 1, \ldots, K \\
& \sum_{k=1}^{K} \bar{c}^{\mathrm{T}} q^k \bar{d}_k + \rho_b \sqrt{\bar{c}^{\mathrm{T}} q S q^{\mathrm{T}} \bar{c}} \leqslant \gamma \\
& \sum_{k=1}^{K} q^k_{ij}\bar{d}_k + \rho_b \sqrt{q^{\mathrm{T}}_{ij} S q_{ij}} \leqslant u_{ij} + v_{ij} y_{ij}, \quad (i, j) \in A \\
& \sum_{k=1}^{K} \rho_c q^k_{ij}\bar{d}_k + \rho_c \rho_b \sqrt{q^{\mathrm{T}}_{ij} S q_{ij}} \leqslant (M^{\mathrm{T}}\kappa)_{ij}, \quad (i, j) \in A \\
& q \in \Re^{Km}_{+}, \; k \in \Re^{l_c}_{+}, \; y \in \{0, 1\}^m \qquad (13)
\end{aligned}$$

We note that the alternate combination of a polyhedral uncertainty set in b and an ellipsoidal uncertainty set in c is similar and leads to the same type of problem. The solution to an integer program with quadratic constraints requires specialized solution procedures which are not pursued in this paper.

Theorem 3 *The approximate adjusted RNDP with uncertain demand and transportation costs given by polyhedral sets* $\mathscr{U}_b = \{d \in \Re^K : |d_k - \bar{d}_k| \leqslant \rho_b H_k, \; k = 1, \ldots, K\}$, *such that* $\mathscr{U}_b \subset \Re^K_+$, *and* $\mathscr{U}_c = \{c \in \Re^m : |c_{ij} - \bar{c}_{ij}| \leqslant \rho_c G_{ij}, \; (i, j) \in A\}$ *is equivalent to the following MIP problem*

$$\begin{aligned}
\min_{y,\gamma,q} \quad & f^{\mathrm{T}}y + \gamma \\
\text{s.t.} \quad & Nq^k = e_{s_k} - e_{t_k} \quad k = 1, \ldots, K \\
& \sum_{k=1}^{K} (\bar{c} + \rho_c G)^{\mathrm{T}} q^k (\bar{d}_k + \rho_b H_k) \leqslant \gamma \\
& \sum_{k=1}^{K} q^k (\bar{d}_k + \rho_b H_k) \leqslant u + Vy \\
& q \in \Re^{Km}_{+}, \; y \in \{0, 1\}^m \qquad (14)
\end{aligned}$$

Quality of robust approximation

We now explore the efficiency of the approximation to the ARC problem obtained by limiting the recourse variables to $x_{ij}^k = q_{ij}^k d_k$. Recall that for affine functions of the uncertainty, we have that $z_{\text{RC}} \geqslant z_{\text{AARC}} \geqslant z_{\text{ARC}}$. In this subsection, we show that the approximate adjusted RNDP can be seen as the RC of the path formulation of the NDP, as such it seems it would be a conservative solution.

We begin by introducing the path formulation of the multicommodity NDP with single source and single sink. Let P^k be the set of all paths from a source to a sink that carry an amount d_k of commodity k, let w_P^k be the amount of flow of commodity k that use path P^k, and let $\delta_{ij}^k = \{P \in P^k | (i, j) \in P\}$ represent the set of paths P carrying commodity k that cross arc (i, j), and define $c_P = \sum_{(i,j)\in P} c_{ij}$ the transportation cost of path P. Then the path formulation of the NDP is

$$
\begin{aligned}
\min \quad & \sum_{k=1}^{K} \sum_{P\in P^k} c_P w_P^k + \sum_{(i,j)\in A} f_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{P\in P^k} w_P^k = d_k \qquad k = 1, \ldots, K \\
& \sum_{k=1}^{K} \sum_{P\in\delta_{ij}^k} w_P^k \leqslant u_{ij} + v_{ij} y_{ij} \qquad (i, j) \in A \\
& w \geqslant 0 \\
& y_{ij} \in \{0, 1\} \qquad (i, j) \in A
\end{aligned} \tag{15}
$$

Proposition 2 *For the NDP with single sink and single source for each commodity and non-negative cost for each arc, the AARC of the arc-flow formulation is equivalent to the RC of the path-flow formulation.*

Proof First, we note that in the case of a single origin and destination with a demand of d_k per commodity, we can formulate the path-based NDP in terms variables that quantify the fraction of the total flow on each path by the change of variables $w_P^k = w_P^k / d_k$. The NDP now becomes:

$$
\begin{aligned}
\min_{y,\omega} \quad & \sum_{k=1}^{K} \sum_{P\in P^k} c_P w_P^k d_k + \sum_{(i,j)\in A} f_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{P\in P^k} w_P^k = 1, \qquad k = 1, \ldots, K \\
& \sum_{k=1}^{K} \sum_{P\in\delta_{ij}^k} w_P^k d_k \leqslant u_{ij} + v_{ij} y_{ij}, \qquad (i, j) \in A \\
& \omega \geqslant 0, \ \ y \in \{0, 1\}^m
\end{aligned}
$$

whose RC is

$$
\begin{aligned}
\min_{y,\omega,\gamma} \quad & \sum_{(i,j)\in A} f_{ij} y_{ij} + \gamma \\
\text{s.t.} \quad & \sum_{P\in P^k} w_P^k = 1, \qquad k = 1, \ldots, K \\
& \sum_{k=1}^{K} \sum_{P\in P^k} c_P \omega_P^k d_k \leqslant \gamma \qquad \forall c \in \mathscr{U}_c, \ b \in \mathscr{U}_b \\
& \sum_{k=1}^{K} \sum_{P\in\delta_{ij}^k} w_P^k d_k \leqslant u_{ij} + v_{ij} y_{ij}, \qquad (i, j) \in A, \ \forall b \in \mathscr{U}_b \\
& \omega \geqslant 0, \ \ y \in \{0, 1\}^m
\end{aligned}
$$

The problem above is equivalent to the AARC in Problem (11) via the change of variables $q_{ij}^k = \sum_{P\in\delta_{ij}^k} w_P^k$. □

Path constrained network design

A number of different applications can involve path-based constraints on the solutions of NDPs. For example, in infrastructure investment political considerations might require an even distribution of capacity resources among different routes; or when transporting hazardous material we should design routes where the total population exposed to the material does not exceed a given threshold. These constraints are naturally expressed in a path-based formulation, such as Problem (15), with additional constraints of the form

$$
a_P^k w_P^k \leqslant T_k \quad P \in P^k, \qquad k = 1, \ldots, K \tag{16}
$$

The natural solution algorithms for problems with path constraints are column generation methods. In addition, large multicommodity network flow problems can be efficiently solved using column generation algorithms on path formulations of the problem.

In this section, we present a column generation procedure that is appropriate for the linear relaxation of a path constrained, or large, RNDP. Being able to solve the linear relaxation effectively is the first step toward lower bounds and algorithms for the integer RNDP. A column generation algorithm can solve the LP relaxation of robust path-based NDPs with polyhedral uncertainty. For simplicity, we outline below the column generation algorithm for an NDP with path constraints (16) and polyhedral transportation cost uncertainty, $\mathscr{U}_c = \{\bar{c} + \rho_c c : Md \leqslant g, \ d \geqslant 0\}$. The algorithm for problems with demand polyhedral uncertainty or additional path constraints is analogous but more involved.

Column generation is an iterative procedure that gradually incorporates profitable variables (or columns) to a reduced master problem. As such, the procedure for the RNDP maintains only a subset $J^k \subset P^k$ of the path flow variables ω_P^k, giving a reduced master problem of the form:

$$\min_{y,\omega,\kappa} \quad \sum_{(i,j)\in A} f_{ij}y_{ij} + \sum_{k=1}^{K}\sum_{P\in J^k} \bar{c}_P \omega_P^k d_k + \rho_c g^{\mathrm{T}}\kappa \quad (17a)$$

$$\text{s.t.} \quad \sum_{k=1}^{K}\sum_{P\in\delta_{ij}^k\cap J^k} \omega_P^k d_k \leqslant (M^{\mathrm{T}}\kappa)_{ij}, \quad (i,j)\in A \quad (17b)$$

$$\sum_{k=1}^{K}\sum_{P\in\delta_{ij}^k\cap J^k} \omega_P^k d_k \leqslant u_{ij} + v_{ij}y_{ij}, \quad (i,j)\in A \quad (17c)$$

$$\sum_{P\in J^k} \omega_P^k = 1, \qquad k=1,\ldots,K \quad (17d)$$

$$\omega\geqslant 0, \quad \kappa\geqslant 0, \quad y\in[0,1]^m \quad (17e)$$

Since the path constraints (16) do not relate to different paths, they are not included in the overall master problem, but rather appear as conditions that valid path variables w_P^k must satisfy.

With an optimal primal and dual solution to the reduced master problem above, we can express the reduced cost for any path flow variable ω_P^k. Let $\sigma_{ij}\leqslant 0$, $\zeta_{ij}\leqslant 0$, and θ_k be the optimal dual variables for constraints (17b–17d), respectively. The reduced cost of a path flow variable ω_P^k is given by

$$\bar{c}_P d_k - d_k \sum_{(i,j)\in P} \sigma_{ij} - d_k \sum_{(i,j)\in P} \zeta_{ij} - \theta_k \quad (18)$$

Path flow variables that have a negative reduced cost are likely to improve the solution if brought into the basis. In fact, an optimal solution has all reduced costs non-negative. Therefore, to generate profitable columns to bring into the reduced master, we find the path flow variable with minimum reduced cost. We achieve this by solving a shortest path problem on the same network for all single origin–destination commodities, but with non-negative arc costs given by $-\sigma_{ij} - \zeta_{ij}$. If the optimal path for commodity k is P, then if

$$-\sum_{(i,j)\in P} \sigma_{ij} - \sum_{(i,j)\in P} \zeta_{ij} \geqslant \frac{\theta_k}{d_k} - \bar{c}_P$$

we have that all path flow variables for commodity k have non-negative reduced costs, should not be brought into the reduced master, and the current solution is in fact optimal for the whole problem. If the opposite strict inequality holds, we know that w_P^k should be added to the reduced master. We summarize below the main steps in the column generation algorithm for the NDP with polyhedral transportation cost uncertainty.

Column Generation Algorithm

Step 1:	Initialization. For each $k=1,\ldots,K$ define J^k with paths formed by artificial arcs between source and sink.
Step 2:	Solve reduced master problem. Let σ_{ij}, ζ_{ij}, and θ_k be the optimal dual variables of constraints (17b–17d), respectively.
Step 3:	For each $k=1,\ldots,K$ *3.1:*Solve the constrained shortest path problem on same network, with arc costs $-\sigma_{ij}-\zeta_{ij}\geqslant 0$. Let P be the path of the optimal solution. *3.2:*If $(-\sum_{(i,j)\in P}\sigma_{ij} - \sum_{(i,j)\in P}\zeta_{ij}) < \frac{\theta_k}{d_k} - \bar{c}_P$ then Add P to J^k. Goto 2.
Step 4:	END. The optimal solution to the reduced master solves the problem. All ω_P^k have a non-negative reduced cost.

The column generation approach is similar in the cases with only polyhedral uncertainty in b or in the case with polyhedral uncertainty in both b and c. The differences arise because the initial robust path-based NDP considers different constraints involving variables w_P^k in these alternative uncertainty cases. For example, the arc capacity constraints under polyhedral demand uncertainty yield the following RC constraints

$$\sum_{k=1}^{K}\sum_{P\in\delta_{ij}^k\cap J^k} \omega_P^k \bar{d}_k + \rho_b \pi_{ij}^{\mathrm{T}} h \leqslant u_{ij} + v_{ij}y_{ij}, \quad (i,j)\in A$$

$$L_{\bullet,k}^T \pi_{ij} \geqslant \sum_{k=1}^{K}\sum_{P\in\delta_{ij}^k\cap J^k} \omega_P^k, \quad (i,j)\in A, \ k=1,\ldots,K$$

These constraints require that not only we consider a dual variable ζ_{ij} associated to the first set of constraints, but also a dual variable $\mu_{ij}^k\leqslant 0$ associated to the second set of constraints, which would then also participate in the computation of the reduced cost.

Computational experiments

We now present computational experiments that investigate the relative merit of the robust solution when compared to the deterministic solution. The experiments conducted show both the trade-off limits between these solutions and their performance under simulated uncertainty scenarios.

We present results on three different networks, including a path-constrained problem, for polyhedral and ellipsoidal sets. For each experiment, we compute and contrast the NDP solution and the approximate adjusted RNDP solution, or its linear relaxation when considering ellipsoidal sets or path-constraints on different uncertainty levels. We present results for different uncertainty levels, obtained by varying the value of ρ in the definition of the uncertainty sets (6) or (8).

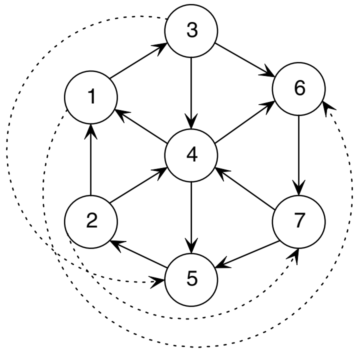

Figure 1 Graph of Network 1, used for Experiments 1 and 2.

Polyhedral uncertainty sets on demand are constructed from independent upper bounds on each $d_k \leqslant \bar{d}_k$ and an additional overall constraint on the sum $\sum_{k=1}^{K} d_k \leqslant \bar{D}$. The polyhedral uncertainty set on c is constructed analogously. The ellipsoidal uncertainty set on demand considers a diagonal matrix, $\sum_{k=1}^{K} (d_k - \bar{d}_k)^2/s_k \leqslant \rho_b^2$. These experiments were coded in AMPL and use either CPLEX 8.1 or LOQO to solve the problem. The models and data used for these experiments are available at http://www-rcf.usc.edu/~fordon/RNDdata.html.

To study the trade-off limits of the robust solution, we compute the following values:

Z_{D} the optimal value of the deterministic solution
Z_{R} the optimal value of the robust solution
Z_{WC} the objective value of the deterministic solution under its worst case scenario
Z_{AC} the objective value of the robust solution under the deterministic scenario

For each network problem below, we present the relative increase of the deterministic solution in its worst case $r_{\mathrm{WC}} = (Z_{\mathrm{WC}} - Z_{\mathrm{R}})/Z_{\mathrm{R}}$ and the relative increase of the robust solution in the deterministic case $r_{\mathrm{AC}} = (Z_{\mathrm{AC}} - Z_{\mathrm{D}})/Z_{\mathrm{D}}$.

For each experiment, the simulation study generates 100 random scenarios uniformly from the appropriate uncertainty sets for the problem. For every simulated value of demand and transportation costs, we compute the minimum cost routing of the demand for the deterministic and robust network design strategies. We report the mean and standard deviations of this minimum total transportation cost for both strategies.

Small network examples

Our first set of experiments consider the network in Figure 1, which consists of seven nodes, 12 arcs, and three commodities which contribute an additional artificial arc to the network each. The parameters of the problem are the design cost, f_{ij}, the existing capacity, u_{ij}, and the additional capacity after the design, v_{ij}. The outside arcs (on the edge of the network) have relatively high transportation cost and existing capacities. The design cost and the additional capacity after design are equal

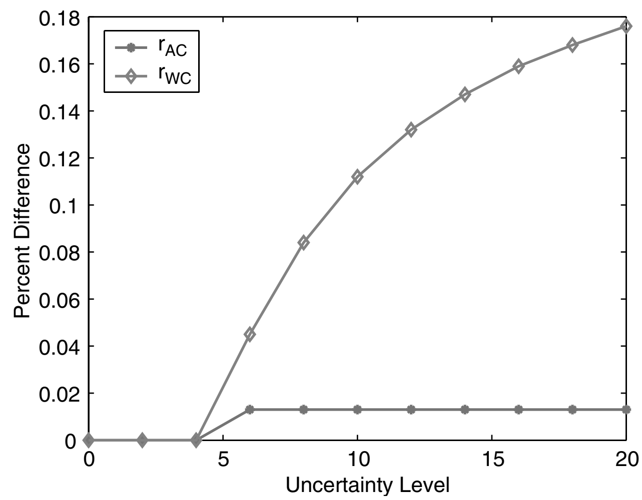

Figure 2 Relative improvement of solutions as a function of the uncertainty level ρ, Network 1, $\mathscr{U}_b$ and $\mathscr{U}_c$ polyhedral.

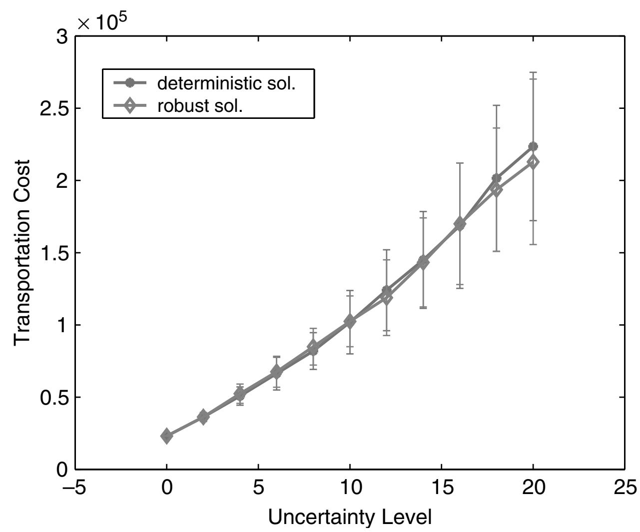

Figure 3 Simulated transportation cost mean and standard deviation as a function of the uncertainty level ρ, Network 1, $\mathscr{U}_b$ and $\mathscr{U}_c$ polyhedral.

to 300 and 700, respectively, for every arc. The deterministic demands for the commodities 1, 2, and 3 are 100, 150, and 120, respectively.

Experiment 1: Consider Network 1 and suppose both demand and transportation cost are under polyhedral uncertainty sets. The comparison of the approximate adjusted RNDP and NDP solutions are shown in Figures 2 and 3.

Experiment 2: Consider Network 1 and an ellipsoidal uncertainty set on demand and polyhedral uncertainty set on transportation cost. Because of the ellipsoidal uncertainty, we solve for the LP relaxation of the approximate adjusted RNDP. The comparisons between the robust and NDP solutions are shown in Figures 4 and 5.

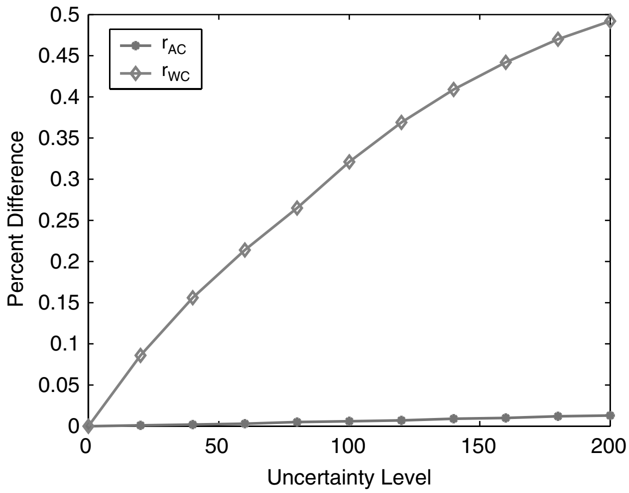

Figure 4 Relative improvement of solutions as a function of the uncertainty level ρ, Network 1, $\mathscr{U}_b$ ellipsoidal and $\mathscr{U}_c$ polyhedral.

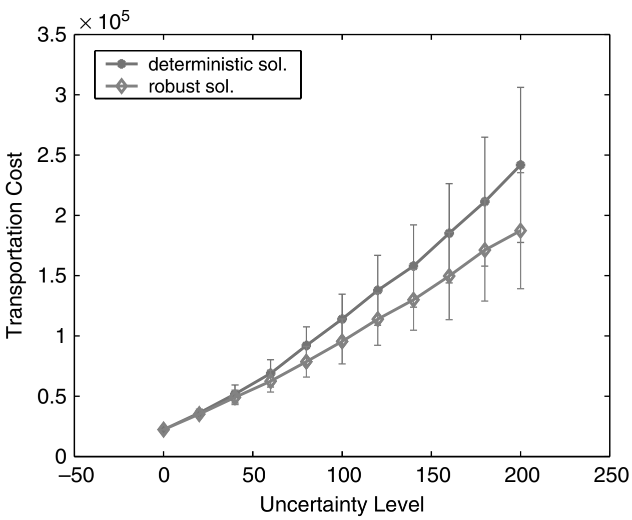

Figure 5 Simulated transportation cost mean and standard deviation as a function of the uncertainty level ρ, Network 1, $\mathscr{U}_b$ ellipsoidal and $\mathscr{U}_c$ polyhedral.

The results from both experiments show that the robust solution is only modestly suboptimal for the deterministic data parameters while significantly reducing the worst case cost, in particular as the uncertainty increases. In the simulation study, however, the robust solution does not seem to provide any benefit. It is practically indistinguishable from the deterministic solution for Experiment 1 and has a slightly improved mean cost in Experiment 2 as the uncertainty increases but with no observable reduction in standard deviation. Our next experiment shows whether these observations hold for slightly larger networks.

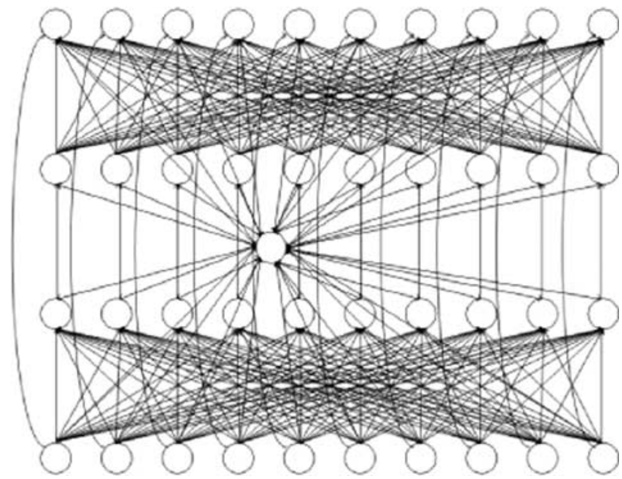

Figure 6 Graph of Network 2, used for Experiment 3.

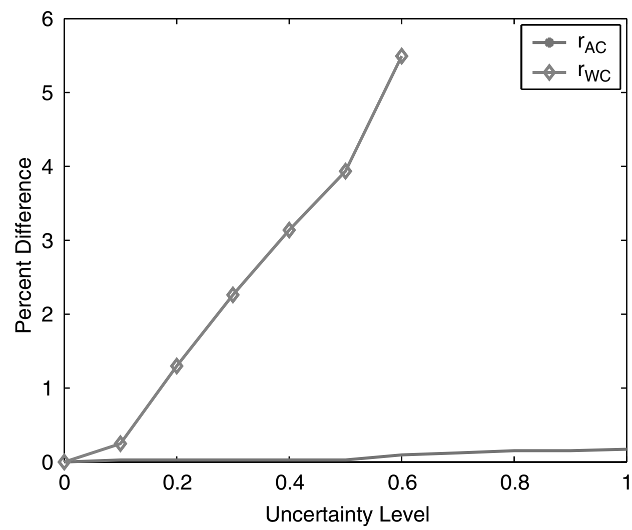

Figure 7 Relative improvement of solutions as a function of the uncertainty level ρ, Network 2, $\mathscr{U}_b$ and $\mathscr{U}_c$ polyhedral.

Increasing the size of the network

To increase the size of our problem, we considered problem cexb2 of the Gunluk suite of problems (available from http://www.di.unipi.it/~frangio/). This problem has 43 nodes, 10 commodities and 330 arcs and is shown in Figure 6. We modified the problem slightly: for every commodity, we add an artificial arc from the source to the sink with large existing capacity and transportation cost. We do not allow design decisions on these additional arcs.

Experiment 3: We consider a larger network to study whether the observations of Experiments 1 and 2 hold. For this, we solve the problem approximate adjusted RNDP with polyhedral uncertainty set on both demand and transportation costs. We show the comparison of the approximate adjusted RNDP and NDP in Figures 7 and 8.

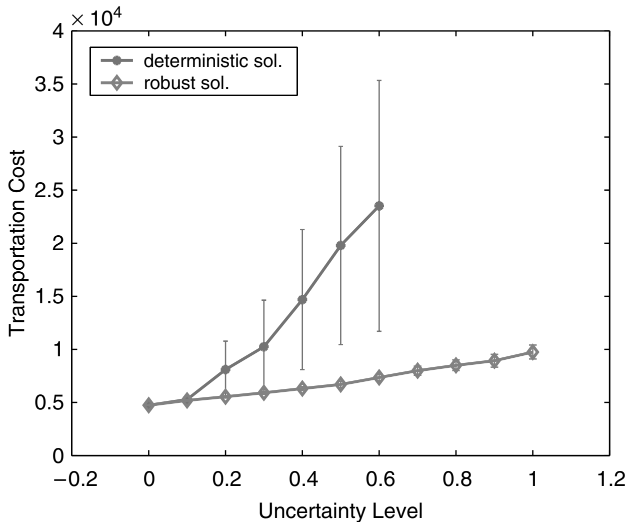

Figure 8 Simulated transportation cost mean and standard deviation as a function of the uncertainty level ρ, Network 2, $\mathcal{U}_b$ and $\mathcal{U}_c$ polyhedral.

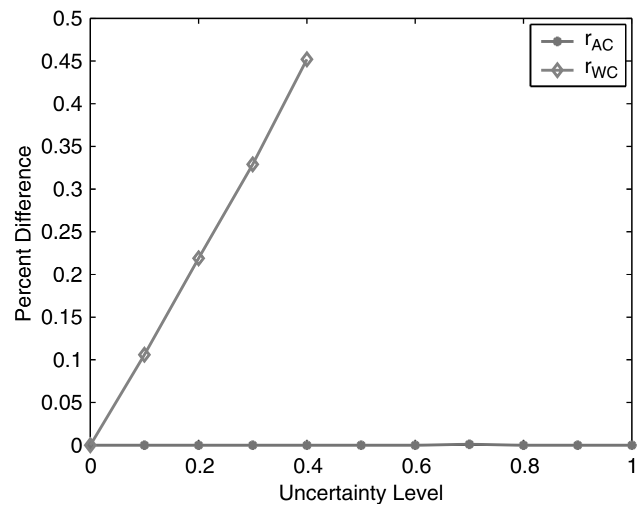

Figure 9 Relative improvement of solutions as a function of the uncertainty level ρ, Sioux Falls Network, $\mathcal{U}_b$ and $\mathcal{U}_c$ polyhedral, path constraints.

Notice that when the uncertainty level is high, the design plan obtained from the deterministic problem is infeasible if the worst case happens, because capacity of the network is not enough to accommodate the flows. We observe that the robust solution again provides protection in the worst case at a modest sub-optimality for the deterministic case; however in this Experiment, the robust solution also has a much smaller simulated mean and standard deviation.

Sioux falls experiment

We compute and compare the robust and deterministic solutions for a path constrained problem based on the classical Sioux Falls multicommodity flow problem, available from http://www.bgu.ac.il/~bargera/tntp/. The Sioux Falls network has 24 nodes, 76 arcs, and 528 commodities, to which we added one artificial node and 48 artificial arcs connecting each node to the artificial node and back to ensure feasibility. We adapted the problem to our current setting by setting the transportation cost equal to the *Free Flow Time* and the existing capacity u_{ij} equal to the *Capacity* in the data. The design cost per arc is set to $f_{ij} = 1000$ and if an arc is expanded the capacity is increased by $v_{ij} = 500\,000$. This experiment considers polyhedral uncertainty sets for demand and transportation cost. This problem also includes a path constraint to ensure that a feasible path P satisfies $|P| \leqslant T_k$, where T_k is a commodity dependent limit on the number of arcs that can be considered in each path. Note that this constraint is enforced by adding the constraint $\sum_{(i,j)\in A} x_{ij}^k \leqslant T_k$ to the column generation shortest path problem with integer x_{ij}^k.

Experiment 4: Consider the Sioux Falls network with polyhedral demand and transportation cost uncertainty and a path constraint on routes. Because of this path constraint, we solve for the LP relaxation of the approximate adjusted RNDP using the column generation method. We present the trade-off limit comparison between the robust and NDP solutions in Figure 9. We did not compute the simulation results for this experiment because it requires for each simulated scenario the solution to a difficult constrained shortest path problem which leads to excessive computation time.

Conclusions

In this paper, we consider the NDP where the demand and the transportation costs are uncertain. We use the robust optimization methodology to find a solution which has a small worst case value, and hence behaves efficiently for all uncertain parameter scenarios. Because of the nature of the NDP, it is natural to consider a problem with recourse, where the investment variables are decided prior to the uncertainty and the routing variables adjust to it. We formulate for general uncertainty sets a tractable problem that computes an approximation to the robust solution for this recourse problem. We also show that this approximate solution is in fact the robust solution without recourse for the path-based formulation of the NDP, which suggests that it might be a conservative solution as it does not adapt to the outcome of the uncertainty. In addition, for path-based and path-constrained NDP, we introduce a column generation approach which can solve the LP relaxation of the RNDP with polyhedral uncertainty sets.

Our computational results show that the approximate robust solution, and even its LP relaxation, have modest sub-optimality on any specific deterministic scenario while significantly reducing the worst case cost, in particular as the uncertainty increases. In addition, the simulation studies show that the approximate robust solution reduces the mean and standard deviation of the total cost, in particular for large problems.

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