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Heuristic approaches for the inventory-routing problem with backlogging

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A B S T R A C T

We study an inventory-routing problem in which multiperiod inventory holding, backlogging, and vehicle routing decisions are to be taken for a set of customers who receive units of a single item from a depot with infinite supply. We consider a case in which the demand at each customer is deterministic and relatively small compared to the vehicle capacity, and the customers are located closely such that a consolidated shipping strategy is appropriate. We develop constructive and improvement heuristics to obtain an approximate solution for this NP-hard problem and demonstrate their effectiveness through computational experiments.

1. Introduction

Recent decades have seen fierce competition in local and global markets, forcing manufacturing enterprises to streamline their logistic systems, as they constitute over 30% of the cost of goods sold for many products (Thomas & Griffin, 1996). The major components of logistic costs are transportation costs, representing approximately one third, and inventory costs, representing one fifth (Buffa & Munn, 1989). The transportation and inventory cost reduction problems have been thoroughly studied separately; while, the integrated problem has recently attracted more interest in the research community as new ideas of centralized supply chain management systems, such as vendor managed inventory (VMI), have gained acceptance in many supply chain environments.

The integration of transportation and inventory decisions is represented in the literature by a general class of problems referred to as dynamic routing and inventory (DRAI) problems. As defined by Baita, Ukovich, Pesenti, and Favaretto (1998), this class of problems is "characterized by the simultaneous vehicle routing and inventory decisions that are present in a dynamic framework such that earlier decisions influence later decisions." They classify the approaches used for DRAI problems into two categories. The first category operates in the frequency domain where the decision variables are replenishment frequencies, or headways between shipments. Examples in the literature include the work of Blumenfeld, Burns, Diltz, and Daganzo (1985), Hall (1985), Daganzo (1987), and Ernst and Pyke (1993) (for more references see Daganzo, 1999). Anily and Federgruen (1990) introduced the idea of fixed-partition policies (FPPs) for solving the frequency-domain DRAI problems. FPPs are policies that solve the problem by partitioning the set of customers into a number of regions such that each region is served separately and independently from all other regions. In addition to that, whenever a customer in a partition is visited, all other customers in that partition are visited by the same vehicle. The solution is considered optimal in the set that includes all the FPPs if, with respect to vehicle capacities, it defines regions that minimize the average of the sum of inventory holding costs and transportation costs. Examples in the literature include Anily and Federgruen (1993), Bramel and Simchi Levi (1995). However, Hall (1992) points out that the FPPs approach can not model the case in which deliveries are coordinated. As a consequence, the results it provides are either valid only in the case of independent deliveries, or can be just considered as providing upper bounds for real costs.

The second category, referred to as the time domain approach, determines the schedule of shipments. With discrete time models, quantities and routes are decided at fixed time intervals. Within this category the most famous problem is the inventory routing problem (IRP), which arises in the application of the distribution of industrial gases. The main concern for this kind of application is to maintain an adequate level of inventory for all customers and to avoid any stockout. In the IRP, it is assumed that each customer has a fixed demand rate and the focus is on

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minimizing the total transportation cost; while inventory costs are mostly not of concern. Examples of this application in the literature include Bell, Dalberto, Fisher, and Greenfield (1983), Golden, Assad, and Dahl (1984), Dror, Ball, and Golden (1985), Dror and Ball (1987), Campbell, Clarke, and Savelsbergh (2002) and Adelman (2003).

In this paper, we consider a DRAI problem that addresses the integrated inventory and vehicle routing decisions in the time domain at the operational planning level. This problem, referred to as the inventory-routing problem with backlogging (IRPB), considers multiple planning periods, both inventory and transportation costs, and a situation in which backorders are permitted. The kind of application that permits backorders is, of course, different from the distribution of industrial gases, where stockout is not allowed. The proposed model is suitable to industrial applications in which a manufacturer distributes its product to geographically disbursed factories/retailers which are located in cities close to its warehouse. At the operational planning level, backorder decisions are generally justified in two cases. The first is when there is a transportation cost saving that is higher than the incurred shortage cost by a customer. The second case is when there is insufficient vehicle capacity to deliver to a customer given that renting additional vehicles is not an option due to technological or economic constraints.

In the literature, the integration of vehicle routing and inventory decisions with the consideration of inventory costs in the time domain approaches of the DRAI problems has taken different forms. In a few cases a single period planning problem has been addressed as found in Federgruen and Zipkin (1984) and Chien, Balakrishnan, and Wong (1989). In the multi-period problem, the decisions are conducted for a specific number of planning periods, or the problem is reduced to a single period problem by considering the effect of the long term decisions on the short term ones. Examples include Dror and Ball (1987), Trudeau and Dror (1992), Viswanathan and Mathur (1997), and Herer and Levy (1997).

Other researchers take into consideration various forms such as distributing perishable products (Federgruen, Prastacos, & Zipkin, 1986), and the consideration of the time value of money for long-term planning (Dror & Trudeau, 1996). Some work focused on different structures of the distribution network such as Bard, Huang, Jaillet, and Dror (1998) in the case of satellite facilities, Chan and Simchi-Levi (1998) in the case where warehouses act as transshipment points in a 3-level distribution network, and Hwang (1999and 2000) in the case of a multi-depot problem.

Solution heuristics that have been proposed in the literature for the different variations of the inventory routing problem are either based on subgradient optimization of a Lagrangian relaxation (see Bell et al., 1983 and Chien et al., 1989) or constructive and improvement heuristics. The constructive heuristics are broadly classified into heuristics that allocate customers to service days and then solve a VRP to generate vehicle routes for each day (Dror & Ball, 1987); and heuristics that allocate customers to days and vehicles and then solve a traveling salesman problem for every assignment (Dror et al., 1985). Improvement heuristics found in the literature (Dror & Levy, 1986 and Federgruen & Zipkin, 1984 in the single period case) are generally considered as extensions to the arc-exchange and node-exchange heuristics as found in the vehicle routing literature.

In the literature of the time domain approaches of the DRAI problems, some models in the case of multi-period planning may include shortage or stockout costs; however, backorder decisions are generally not explicitly considered. Instead, the shortage or stock-out cost is treated as the penalty cost that is incurred due to making direct deliveries to customers whose demand is not fulfilled in the regular delivery route in a given period. Examples of such models in the literature include Herer and Levy (1997) and Jaillet, Bard, Huang, and Dror (2002). In this paper, we consider a situation in which backorder decisions are either unavoidable or more economical, and they have to be coordinated with other inventory holding and vehicle routing decisions over a specific planning horizon. We introduce constructive and improvement heuristic approaches for solving the problem with backorders, and benchmark it against lower and upper bounds found by a commercial software package, CPLEX.

The rest of this paper is organized as follows. In Section 2, we formulate the problem as a mixed integer linear program. The motivating ideas and search plan for the developed heuristics are presented in Section 3. Sections 4 and 5 provide descriptions of the constructive and improvement heuristics, respectively. In Section 6, the experimental results are presented followed by the conclusion and directions for future research in Section 7.

2. Problem definition and mixed integer programming model

In the IRPB, we study a distribution system consisting of a depot, denoted 0, and geographically dispersed customers, indexed $1,\ldots,N$. Each customer i faces a different demand d_{it} for a single item per time period t (day/week). As traditionally considered, a single item does not restrict the problem to the case of a single product distribution, as the word 'item' can refer to a unit weight or volume of the distributed products and each customer can be viewed as a consumption center for packages of unit weight or volume (Daganzo, 1999). Accordingly, the proposed model can be applied to the case of multiple products given that the values of the inventory holding and shortage costs per unit volume/weight have small variance among the different products. We consider the case in which the demand of each customer is relatively small compared to the vehicle capacity, and the customers are located closely such that a consolidated shipping strategy is appropriate. Deliveries to customers $1,\ldots,N$ are to be made by a capacitated heterogeneous fleet of V vehicles, each with capacity q_v starting from the depot at the beginning of each period. Vehicles must return to the depot at the end of the period, and no further delivery assignments should be made in the same period. In this model, we consider the case in which renting additional vehicles during the short planning horizon is not an option, and it is assumed that the fleet of vehicles remains unchanged throughout the planning horizon.

Each customer i maintains its own inventory up to capacity C_i and incurs inventory carrying cost of h_i per period per unit and a backorder penalty (shortage cost) of π_i per period per unit on the end of period inventory position. We assume that the depot has sufficient supply of items that can cover all customers' demands throughout the planning horizon. The planning horizon considers T periods. Transportation costs include f_{vt} a fixed usage cost for vehicle v, which depends on the period t, and c_{ij} a direct transportation cost between i and j, which satisfies the triangular inequality. The objective is to minimize the overall transportation, inventory carrying and backlogging costs incurred over a specific planning horizon. We consider an integer variable x^v_{ijt}, which equals 1 if vehicle v travels from i to j in period t, and 0 if it does not. The amount transported on that trip is represented by y^v_{ijt}. At customer i, the inventory and backorder at the end of time t is I_{it} and B_{it}, respectively. The following is a mixed integer programming formulation for the problem.

2.1. [IRPB] – Inventory routing problem with backlogging

$$\min \sum_{t=1}^{T}\left[\sum_{j=1}^{N}\sum_{v=1}^{V} f_{vt}x_{0jt}^{v} + \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{i=0}^{V} C_{ij}x_{ijt}^{v} + \sum_{i=1}^{N}(h_i I_{it} + \pi_i B_{it})\right] \quad (0)$$

subject to

$$\sum_{\substack{j=0\\ j\neq i}}^{N} x_{ijt}^{v} \leqslant 1 \quad i=0,\ldots,N, t=1,\ldots,T \text{ and } v=1,\ldots,V \quad (1)$$

$$\sum_{\substack{k=0\\ k\neq i}}^{N} x_{ikt}^{v} - \sum_{\substack{l=0\\ l\neq i}}^{N} x_{lit}^{v} = 0 \quad i=0,\ldots,N,\ t=1,\ldots,T$$
$$\text{and } v=1,\ldots,V \quad (2)$$

$$Y_{ijt}^{v} - q_v x_{ijt}^{v} \leqslant 0 \quad i=0,\ldots,N, j=0,\ldots,N, i\neq j, t=1,\ldots,T$$
$$\text{and } v=1,\ldots,V \quad (3)$$

$$\sum_{\substack{l=0\\ l\neq i}}^{N} y_{lit}^{v} - \sum_{\substack{k=0\\ k\neq i}}^{N} y_{ikt}^{v} \geq 0 \quad i=1,\ldots,N, t=1,\ldots,T \text{ and } v=1,\ldots,V \quad (4)$$

$$I_{iy-1} - B_{it-1} - I_{it} + B_{it} + \sum_{v=1}^{V}\left(\sum_{\substack{l=0\\ l\neq i}}^{N} y_{lit}^{v} - \sum_{\substack{k=0\\ k\neq i}}^{N} y_{ikt}^{v}\right) = d + it i = 1,\ldots,N$$
$$\text{and } t=1,\ldots,T \quad (5)$$

$$I_{it} \leqslant C_i \quad i=1,\ldots,N \text{ and } t=1,\ldots,T \quad (6)$$

$$I_{it} \geq 0 \quad i=1,\ldots,N \text{ and } t=1,\ldots,T \quad (7)$$

$$B_{it} \geq 0 \quad i=1,\ldots,N \text{ and } t=1,\ldots,T \quad (8)$$

$$Y_{ijt}^{v} \geq 0 \quad i=0,\ldots,N,$$
$$j=0,\ldots,N, i\neq j,\ t=1,\ldots,T \text{ and } v=1,\ldots,V \quad (9)$$

$$Y_{ijt}^{v} \varepsilon_{0,1}, \quad i=0,\ldots,N,\ j=0,\ldots,N, i\neq j,\ t=1,\ldots,T$$
$$\text{and } v=1,\ldots,V \quad (10)$$

The objective function (0) includes transportation costs and inventory carrying and shortage costs on the end of period inventory position. Constraints Eq. (1) make sure that a vehicle will visit a location no more than once in a time period, and constraints Eq. (2) ensure route continuity. Constraints Eq. (3) serve for two purposes. The first one is to ensure that the amount transported between two locations will always be zero whenever there is no vehicle moving between these locations, and the second is to ensure that the amount transported is less than or equal to the vehicle's capacity. Constraints Eq. (4) along with the other elements of the model ensure that efficient solutions will not contain subtours. We illustrate in the Appendix A how this condition is achieved. Constraints Eq. (5) are the inventory balance equations for the customers. Constraints Eq. (6) limit the inventory level of the customers to the corresponding storage capacity. It is assumed that the amount consumed by each customer in a given period is not kept in the customer's storage location; accordingly, it is not accounted for in constraints Eq. (6). Constraints Eqs. (7)–(10) are the domain constraints.

3. Motivating ideas and heuristic design

The IRPB is NP-hard since it includes the capacitated vehicle routing problem (VRP) as a subproblem. In this section, we present the key ideas in the proposed constructive and improvement heuristics for this NP-hard problem.

A key decision in solving the IRPB is the amount delivered to customer i in period t, as this quantity, let us define it by $w_{it} = \sum_{v=1}^{V}\left(\sum_{\substack{l=0\\ l\neq i}}^{N} y_{lit}^{v} - \sum_{\substack{k=0\\ k\neq i}}^{N} y_{ikt}^{v}\right) \geq 0$, effectively separates the routing and inventory problems. In fact, given delivery values w_{it} for all customers and periods, the inventory and backorder values are determined by constraints Eqs. (5)–(8). At the same time, the best routing solution for these w_{it} is obtained by solving T separate capacitated vehicle routing problems. Each VRP computes the optimal transportation costs to deliver $W_t = (w_{it} : i = 1,\ldots,N)$ in period t by solving the following feasible problem whenever the delivery amounts satisfy

$$\sum_{i=1}^{N} w_{it} \leqslant \sum_{v=1}^{v} q_v :$$

$$TC_t(W_t) = \min \sum_{j=1}^{N}\sum_{v=1}^{V} f_{vt}x_{0jt}^{v} + \sum_{i=0}^{N}\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{v=1}^{V} c_{ij}x_{ijt}^{v}$$

Subject to :

$$\sum_{\substack{j=0\\ j\neq i}}^{N} x_{ijt}^{v} \leqslant 1 \qquad i = 0,\ldots N \text{ and } v = 1,\ldots V \quad (1')$$

$$\sum_{\substack{k=0\\ k\neq i}}^{N} x_{ikt}^{v} - \sum_{\substack{l=0\\ l\neq i}}^{N} x_{lit}^{v} = 0 \qquad i = 0,\ldots N \text{ and } v = 1,\ldots V \quad (2')$$

$$Y_{ijt}^{v} - q_v x_{ijt}^{v} \leqslant 0 \qquad i,j = 0,\ldots N,\quad i\neq j \text{ and } v = 1,\ldots V \quad (3')$$

$$\sum_{\substack{l=0\\ l\neq i}}^{N} y_{lit}^{v} - \sum_{\substack{k=0\\ k\neq i}}^{N} y_{ikt}^{v} \geq 0 \qquad i = 1,\ldots N, \text{ and } v = 1,\ldots V \quad (4')$$

$$\sum_{v=1}^{v}\left(\sum_{\substack{l=0\\ l\neq i}}^{N} y_{lit}^{v} - \sum_{\substack{k=0\\ k\neq i}}^{N} y_{ikt}^{v}\right) = w_{it} \qquad i = 1,\ldots N \quad (10)$$

$$Y_{ijt}^{v} \geq 0 \text{ and } x_{ijt}^{v} = 0 \text{ or } 1 \qquad i,j = 0,\ldots N,\quad i\neq j \quad \text{and} \quad v = 1,\ldots V \quad (11)$$

Therefore, the key in solving IRPB is to be able to identify the optimal delivery amounts w_{it} since what is left is a vehicle routing problem for which there exist several efficient algorithms. Our proposed heuristics build on this observation by focusing on how to determine the w_{it} variables efficiently. The procedure used to determine the w_{it} values must take into consideration the tradeoff existing between inventory and transportation costs.

In Section 4, we propose a constructive heuristic that sets the delivery amounts by balancing this tradeoff. The idea of the heuristic is to estimate a transportation cost value for each customer in each period from an approximate routing solution. Actual delivery amounts, w_{it}, are then decided by comparing these transportation cost estimates with the corresponding inventory costs. This process is done sequentially from the first period onward and in each period the comparison of transportation and inventory costs is done in two phases. The first phase looks into backorder decisions that are either imposed by insufficient vehicle capacity or preferred due to savings in transportation costs that are higher than backordering costs. The second phase investigates inventory decisions that would cover demand requirements in future periods in the case that excess vehicle capacity is available at the current period. The heuristic looks into inventory decisions that provide savings in future transportation costs that are higher than inventory carrying costs.

The improvement heuristic introduced in Section 5 investigates possible improvements to the solutions generated by the constructive heuristic by looking into modifications to the delivery quantities that would reduce transportation and/or inventory costs and result in overall cost savings. In particular the improvement heuristic relaxes the requirement made in the constructive heuristic to reduce the search space, that is all demand satisfied in a given period must be satisfied exactly not partially.

A key step in this heuristic is to be able to effectively estimate the transportation cost of each customer. Below we present a result that provides insight into the structure of the total transportation cost in period t as a function of the delivery amount W_t.

Proposition 1. *$TC_t(W_t)$ is a multi-dimensional monotonic increasing step function.*

Proof. Given that the definition of $TC_t(W_t)$ is based on an MIP model for the capacitated vehicle routing problem (VRP) in which triangular inequality holds. Starting from an optimal solution of a specific VRP at an initial $W_t^0 = (w_{it}^0 : i = 1, \ldots, N)$, and by adding $\Delta W_t^+ = (\partial w_{it} : \partial w_{it} \geqslant 0, i = 1, \ldots, N)$ to W_t^0 (i.e. increasing the demand values for a subset of the customers) such that $\sum_{i=1}^{N}(w_{it}^0 + \partial w_{it}) \leqslant \sum_{v=1}^{V} q_v$, one of two possible consequences will occur: (1) new arc or arcs will be added to the current solution to satisfy the vehicle capacity constraints Eq. (3), which will increase $TC_t W_t^0$ by the corresponding cij and/or ft amounts as needed, or 2) the current VRP solution remains optimal. Thus $TC_t(W_t + \Delta W_t^+) \geq TC_t(W_t)$. Since the changes of $TC_t(W_t)$ occur at discrete points according to the vehicle capacities, $TC_t(W_t)$ takes the form of a multidimensional step function. □

As a result of proposition 1, the solution scheme can focus only on those values of the continuous variables, w_{it}, at which changes to the transportation cost occur. We can look at this result from another perspective. Given planned delivery amounts to customers in a period, by reducing the delivery quantity of a specific customer, the transportation costs will be reduced at discrete points and the maximum possible reduction will occur when the delivery to that customer is dropped to zero. Although proposition 1 is proven for optimal solutions to the VRP, this result can still be used for solutions generated by efficient heuristics as an approximation, such as the savings algorithm (Clarke & Wright, 1964).

4. Constructive heuristic

As mentioned earlier, the constructive heuristic is based on the idea of estimating a transportation cost value for each customer in each period, which is necessary to facilitate the comparison between transportation and inventory carrying and shortage costs. We therefore refer to the constructive heuristic as the Estimated Transportation Costs Heuristic (ETCH). In Subsection 4.1, we describe how the transportation cost estimates are evaluated and continuously updated throughout the course of the heuristic. Using these estimates, we show in Subsection 4.2 how the inventory problem in IRPB can be decomposed into two subproblems that are solved by the heuristic in two phases. The solution techniques for these subproblems are illustrated in Subsection 4.3.

4.1. Estimating transportation costs

Let w_{it}^{PL} be the planned delivery amount for customer i in period t. For period τ in which $\sum_{j=1}^{N} w_{j\tau}^{PL} \leqslant \sum_{v=1}^{V} q_v$, let $W_\tau = (w_{j\tau} : w_{j\tau} = w_{j\tau}^{PL}, j = 1, \ldots, N)$. For customer i whose $w_{i\tau}^{PL} > 0$, let $W_\tau^{(i)} = (w_{j\tau} : w_{i\tau} = 0,\ w_{j\tau} = w_{j\tau}^{PL}, j = 1, \ldots, N, j \neq i)$. Then, the transportation cost reduction that would result from reducing customer i's delivery in period τ to zero can be calculated as $TC_\tau(W_\tau) - TC_\tau(W_\tau^{(i)})$. Since the transportation cost function involves the solution of a VRP, which is known to be NP-hard, it may not be possible to calculate its exact value, especially for large problem sizes. Instead, an efficient heuristic can be used to approximate it. In our implementation, the savings algorithm is used for this purpose.

Let $ATC_\tau(W_\tau)$ be an approximation for $TC_\tau(W_\tau)$ when the savings algorithm is used to solve the associated VRP. The transportation cost estimate for customer i in period τ is calculated as $ETC_i(W_\tau) = ATR_\tau(W_\tau) - ATR_\tau(W_\tau^{(i)})$. However, resolving a VRP every time the transportation cost estimate for each customer is calculated is in fact computationally inefficient. Instead, a faster approximation scheme can be constructed by evaluating the transportation cost saving that will result when a customer is removed from its delivery tour assigned to it in a given VRP solution. This means that for given delivery amounts, W_τ, the associated VRP will be solved only once and the resulting vehicle tours will be used for generating transportation cost estimates.

$ATC_t(W_t)$ and $ETC_i(W_t)$ are functions of the planned delivery amounts w_{it}^{PL} which are determined based on the customers' net demand requirements in period t. However, the values of w_{it}^{PL} must be defined such that the vehicle capacity constraint, $\sum_{j=1}^{N} w_{jt}^{PL} \leqslant \sum_{v=1}^{V} q_v$, is satisfied. Given the inventory position at the beginning of period t, $I_{i,t-1}$ - $B_{i,t-1}$, and the demand requirements $d_{i\tau}$ for all periods $\tau \geqslant t$, ETCH evaluates the net demand requirement for each customer, and based on that it estimates $w_{i\tau}^{PL}$. If the vehicle capacity constraint is not satisfied in a given period, the $w_{i\tau}^{PL}$ values are adjusted such that customers with the lowest unit shortage costs, π_i, will have part of their demand requirements postponed to future periods. The following list describes the steps of this approach.

Procedure PLNDLV(t)

1. Let OC = ordered set of all customers in which customers are sorted in a non-increasing order of their π_i values;
2. For every customer $i \in$ OC, let $inv_i = I_{i,t-1} - B_{i,t-1}$;
3. For period $\tau = t$ to T do
4. Let $Q^{\max} = \sum_{v=1}^{V} q_v$;
5. For every customer $i \in$ OC using the order in set OC do
6. $(w_{i\tau}^{PL} = \min(Q^{\max}, \max(d_{i\tau} - inv_i, 0))$;
7. $Q^{\max} = Q^{\max} - w_{i\tau}^{PL}$;
8. $Inv_i = inv_i + w_{i\tau}^{PL} - d_{i\tau}$;
9. End-Loop;

End-Loop;The resultant $w_{i\tau}^{PL}$ values can be safely used in evaluating both functions $ATC_\tau(W_\tau)$ and $ETC_i(W_\tau)$. During the course of the algorithm, if a change in the delivery amounts occurs, a VRP for the period in which the change occurred is instantiated and solved to update the values of the transportation cost estimates.

4.2. Problem decomposition and solution scheme

In the ETCH, the comparison between the transportation cost estimates and inventory carrying and shortage costs is separated into two subproblems that are solved sequentially. This comparison is conducted for every period t starting from the first period onward. The first subproblem is concerned with deciding whether to have backorders on period t and the second subproblem is concerned with deciding whether to use remaining vehicle capacity in period t, if any, to cover future customer demand.

Backorders can be profitable for two reasons; it is either cheaper to pay the backorder cost than the transportation cost, or there is insufficient capacity in the vehicles to satisfy demand. Let $\delta_{i,t}$ = -max($d_{i,t} - I_{i,t-1} + B_{i,t-1}$, 0) be the outstanding demand at customer i at the beginning of period t, and CD be the set of customers that have $\delta_{i,t} > 0$. The following subproblem decides whether to deliver to customer i in period t or not (z_i = 1 or 0, respectively) and the quantity r_i to deliver such that the sum of backorder cost and estimated transportation cost is minimized and vehicle capacity constraints are satisfied.

[SUB1] – Backorder decisions subproblem

$$\text{Min } ATC_t(\Omega_t) + \sum_{i \in CD} \pi_i(\delta_{i,t} - r_i)$$

Subject to :

$$\sum_{i \in CD} r_i \leqslant \sum_{v=1} q_v \tag{12}$$

$$r_i = \delta_{i,t} z_i \quad \forall i \in CD \tag{13}$$

$$\Omega_t = (\omega_{it} : \omega_{it} = r_i, i \in CD) \quad \forall i \in CD \tag{14}$$

$$z_i = 0 \quad or \quad 1 \quad \forall i \in CD \tag{15}$$

In SUB1, the objective function is composed of two parts, an approximation of the transportation costs in period t and backorder penalty costs. Both parts are functions of the decision variables r_i. Constraint (12) ensures that we do not exceed the total vehicle capacity, and constraints Eq. (13) enforce that we deliver the exact amount of the outstanding demand only to customers included in the delivery in period t. Constraint Eq. (14) defines the vector of delivery amounts used in approximating the transportation cost function.

The main outcome from solving SUB1 is the backorder decisions evaluated as $B_{it} = \delta_{i,t}$ for every customer i CD that has $z_i = 0$, and accordingly $w_{it} = 0$, in the solution of SUB1. The delivery amounts, w_{it}, for customers in set CD that have $z_i = 1$ in the solution of SUB1 are not decided yet as future demand requirements may be added. These decisions are investigated through subproblem SUB2. For every other customer j CD, $w_{jt} = 0$, $B_{jt} = 0$ and $I_{jt} = I_{jt-1} - d_{jt}$.

Let FD be the set of customers that have $z_i = 1$ in the solution of SUB1. Consider the integer variable $u_{i\tau}$ to decide whether to deliver customer i's demand for period τ in the current period t, where $\tau > t$. Let Q^r denote the total remaining vehicle capacity, i.e. $Q_r = \sum_{v=1}^{V} q_v - \sum_{i \in CD} r_i, andletT_i^{\max}$ be the latest period where customer i's demand can be considered without violating its storage capacity constraint, i.e. $T_i^{\max} = \min\left\{\arg\max_L \left(\sum_{\tau=t+1}^{L} d_{i\tau} \leqslant C_i\right), T\right\}$. We also define $T^{\max} = \max_i(T_i^{\max})$.

Let $w_{i\tau}^{PL}$ be the planned delivery amount for customer i in a future period $\tau > t$. The values of $w_{i\tau}^{PL}$ are initially calculated using the PLNDLV$(t+1)$ procedure as described in Subsection 4.1 with a small modification to make sure that for every customer $j \in FD$, initial values of $w_{j\tau}^{PL} = d_{jt}$. If it is not possible to achieve this condition in a future period τ for customer $j \in FD$, $T_j^{\max}$ is set to $\tau - 1$. The $w_{i\tau}^{PL}$ values for customers that do not belong to set FD are fixed; however, the values of $w_{i\tau}^{PL}$ for customers in set FD change with the change of the $u_{i\tau}$ decision variables. The following problem decides whether to include future demand for any customer in the current delivery by minimizing the total transportation and inventory costs and satisfying capacity limits. This part is formulated as follows:

[SUB2] – Inventory decisions subproblem

$$\text{Min } \sum_{\tau=t+1}^{T^{\max}} ATC(\Omega_\tau) \sum_{i \in FD} \sum_{\tau=t+1}^{T_i^{\max}} [(\tau - t) h_i d_{i,\tau}] u_{i\tau}$$

Subject to :

$$\sum_{i \in FD} \sum_{\tau=t+1}^{T_i^{\max}} d_{i\tau} u_{i\tau} \leqslant Q^r \tag{16}$$

$$U_{i\tau-1} \geq u_{i\tau} \qquad \tau = t+1, \ldots, T_i^{\max} \quad \forall i \in FD \tag{17}$$

$$W_{i\tau}^{PL} = d_{i\tau}(1 - u_{i\tau}) \qquad \tau = t+1, \ldots, T_i^{\max} \quad \forall i \in FD \tag{18}$$

$$\Omega_\tau(\omega_{i\tau} : \omega_{i\tau} = w_{i\tau}^{PL}, i = 1, \ldots, N) \qquad \tau = t+1, \ldots, T^{max} \tag{19}$$

$$U_{i\tau} = 0 \text{ or } 1 \cdot \qquad \tau = t+1, \ldots, T_i^{\max} \quad \forall i \in FD \tag{20}$$

Constraint Eq. (16) represents the available vehicle capacity limit. For simplification, the customers' storage limits are represented by the time index ($T_i^{\max}$), which is computed in advance as described earlier. The precedence constraints Eq. (17) are added to represent the fact that future demand in a certain period is to be considered only if the customer's preceding period demand is fulfilled. Constraints Eq. (18) define the relationship between the future planned delivery amounts for customers in set FD and the decision variables $u_{i\tau}$. When the delivery amount in period t changes, there may be changes in the transportation cost in that period. The formulation of SUB2 neglects such changes.By solving SUB2, the delivery amounts for customers in set FD can be calculated as $w_{it} = r_i + \sum_{\tau=t+1}^{T_i^{\max}} d_{i\tau} u_{i\tau}$. Accordingly, the inventory and backorder decision variables in period t can be easily calculated. Finally, delivery routes in period t are decided by solving a VRP using the resulting delivery amounts. The flow chart in Fig. 1 summarizes the major steps of the proposed heuristic. The following subsection provides the algorithmic solutions for both subproblems and their related analyses.

4.3. Solving subproblems

The two subproblems are resource allocation problems in which the scarce resource is the associated available vehicle capacity and the main decision variables, z_i and u_{it}, are binary variables. Accordingly, both of them can be solved optimally using dynamic programming (DP) as described in Taha (1992). However, with the increase of the problem size, mainly due to the number of customers and the planning horizon, the DP implementations suffer from the curse of dimensionality. In this section, we present efficient heuristics that can be used instead. First, we present the following result that characterizes optimal solutions to subproblem SUB1.

Proposition 2. *There is an optimal solution to SUB1 that makes deliveries to customer i only if the quantity delivered satisfies* $r_i > ETC_i(\Omega_t)/\pi_i$ *Also, every optimal solution to SUB1 only makes deliveries if* $r_i > ETC_i(\Omega_t)/\pi_i$.

Proof. Assume that in the optimal solution to SUB1, some customer i is delivered ri that satisfies $r_i > ETC_i(\Omega_t)/\pi_i$ or equivalently $\pi_i(\delta_{i,t} - r_i) + ATR_t(\Omega_t) \geq \pi_i\delta_{i,t} + ATR_t(\Omega_t^{(i)})$. If we consider the modified solution obtained by setting $z_i = r_i = 0$, then the previous inequality shows that the modified solution, which is feasible, is at least as good as the optimal solution. In the case when $r_i > ETC_i(\Omega_t)/\pi_i$ then the modified solution is strictly better. Thus, the original solution cannot be optimal. □

Proposition 2 gives a necessary condition for the optimality of the delivery decision made for a specific customer; however, satisfying this condition for all customers that have planned deliveries does not guarantee optimality for the solution of SUB1. Yet, since backorder decisions are generally not preferable, we will consider solutions that have this characteristic sufficiently good. We design the following algorithm that utilizes this rule.

Let $DL_k = \{dl: dl \subseteq CD \text{ and } |dl| = |CD| - k\}$, where $|.|$ denotes the size of a set. We define $f^{SUB1}(dl)$ as the objective function value of subproblem SUB1 when $z_i = 1$ for every customer $i \in dl$ and $z_j = 0$ for every customer $j \in CD - dl$, where $dl \in DL_k$ for some k. If the vehicle capacity constraint of SUB1 associated with setting $z_i = 1$ for all customers in a set dl is not satisfied, we define $f^{SUB1}(dl) = \infty$ .The following list describes the steps of a breadth-first-based heuristic approach that searches for efficient solutions to SUB1.

Procedure SUBALG1

1. Let k = 0 and $dl^{\min} = CD$;
2. If $f^{SUB1}(dl^{\min}) \neq \infty$ and $r_i \geq ETC_i(\Omega_t)/\pi_i \quad \forall i \in dl^{\min}$ then go to 9;
3. For every $dl \in DL_k evaluate f^{SUB1}(dl)$;

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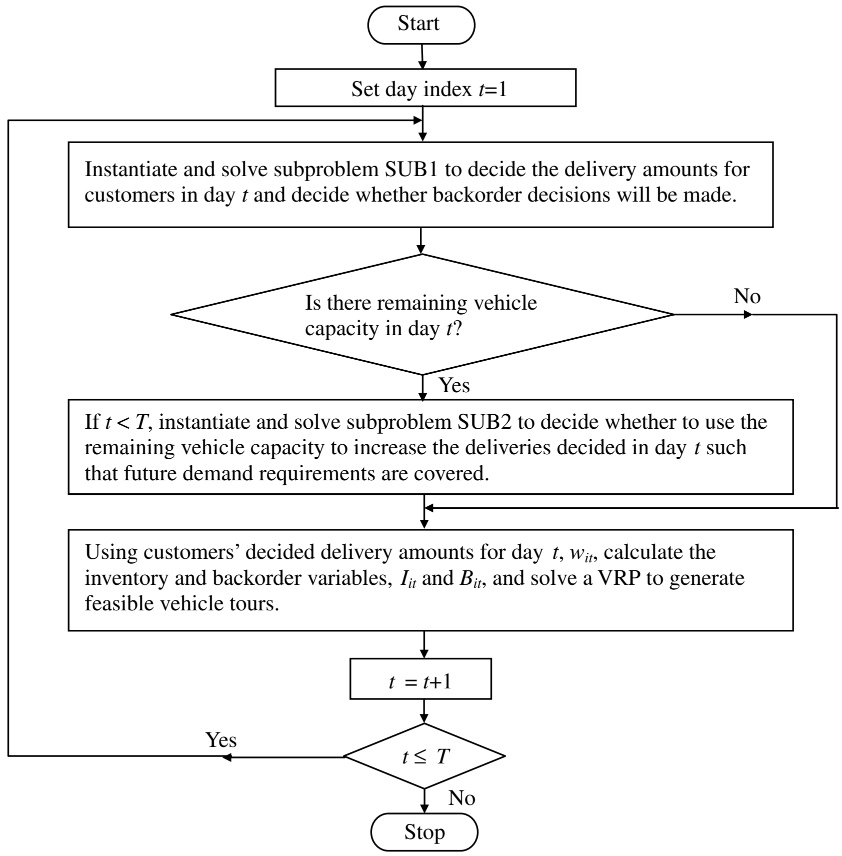

Fig. 1. An outline of ETCH.

4. Find dl^s from set DL_k that has the minimum $f^{SUB1}(dl)$ selected from the members of DL_k that satisfy the following conditions (tie-breaking is arbitrary):
   a. $f^{SUB1}(dl) \neq \infty$ and $r_i \geq ETC_i(\Omega_t)/\pi_i \quad \forall i \in dl$;
5. If $dl^s \neq \emptyset$ then let $dl^{min} = dl^s$ go to 9;
6. Find dl^* from set DL_k that has the minimum $f^{SUB1}(dl)$; tie-breaking is arbitrary
7. If $f^{SUB1}(dl*) < f^{SUB1}(dl^{min})$ then let $dl^{min} = dl*$;
8. If $k < |CD|$ then let $k = k + 1$, go to 3;
9. Generate a solution for SUB1 in which deliveries are only made to customers in set dl^{min};

SUBALG1 evaluates the $f^{SUB1}(dl)$ value for every set $dl \in DL_k$ at values of $k = 0,\ldots,|CD|$. If at some level of k, the condition that $r_i > ETC_i(\Omega_t)/\pi_i$ is satisfied for all $i \in dl$, we find an approximate solution and the algorithm terminates. However, if steps 2, 4 and 5 are removed, the algorithm guarantees that an optimal solution for SUB1 has been identified.

Subproblem SUB2 can be illustrated graphically. Consider the sample case for SUB2 illustrated in Fig. 2. The decision variables $u_{i\tau}$ are represented by directed arcs, where the cost saving associated with each arc $S_{i\tau-t} = ETC_i(\Omega_\tau) - (\tau - t)h_i\, d_{i\tau}$. A solid vertical line is drawn to represent the time limit T_i^{max} for customer i. Starting from node 0, arcs are to be selected using the order given by their directions, such that the total cost saving is maximized and the vehicle capacity constraint is satisfied. We note here that if one or more arcs in a given period are selected, the saving values $S_{i\tau-t}$ of the unselected arcs in the same period will be changed due to changes in the transportation cost estimates and therefore have to be recalculated.

Inspired by this graphical representation, subproblem SUB2 can be dealt with as precedence constrained knapsack problem (PCKP) in which the coefficients of the objective function, $S_{i\tau-t}$, are dependent on the decision variables. The PCKP is known to be NP-hard (Garey & Johnson, 1979); however, Johnson and Niemi (1983) provide a dynamic programming algorithm for the PCKP that can solve the problem in a pseudo-polynomial time, given that the underlying precedence graph is a tree, which is fortunately a property of SUB2 as can be seen in Fig. 2.

We present here a simpler algorithm based on a greedy search that selects the next possible arc (see Fig. 2) that has the maximum positive saving. This algorithm does not guarantee optimality to the solution of SUB2; however, it can produce relatively good solutions in polynomial time. The following steps describe the algorithm.

Procedure SUBALG2

1. Let $D^{max} = Q^r$ and $TD = FD$;
2. For every customer i in set TD, Let $\Delta t_i = 1$;
3. Find customer j in set TD that has the largest positive value of $(ETC_j\Omega_{t+\Delta t_j}) - \Delta t_j\, h_j\, d_{j,t+\Delta t_j})$; If none found then terminate;
4. If $D^{max} \geq d_{j,t+\Delta t_j}$ then
   Let $D^{max} = D^{max} - d_{j,t+\Delta t_j}$;
   Add $d_{j,t+\Delta t_j}$ to customer j's delivery amount and updatetransportation cost estimates in period t+Δt_j;
   Let $\Delta t_j = \Delta t_j + 1$;
   If $\Delta_j > T_j^{max}$ then remove customer j from set TD;
   End-If
   Else remove customer j from set TD;
5. If $TD = \emptyset$ then terminate; Else go to step 3.

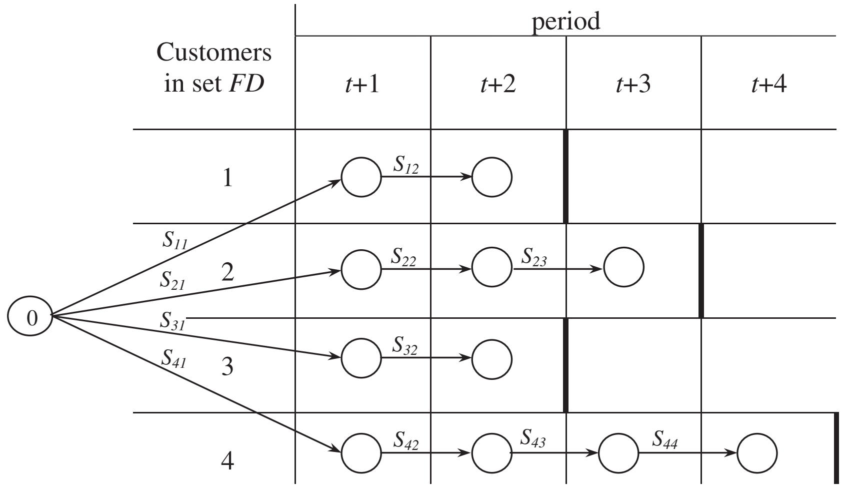

Fig. 2. Graphical illustration of subproblem SUB2 for a sample case.

5. Improvement heuristic

There are two apparent limitations of ETCH. The first is due to the myopic nature of the decisions conducted. This myopic nature stems from the strategy used at each period for solving the two subproblems, as it aims to optimize inventory allocation decisions at the studied period without considering the impact of such decisions on the optimality of the overall solution. The second limitation is concerned with not allowing for partial fulfillment of demand, that is exact demand requirements in the current and future periods must be considered in the delivery schedule. This may prevent ETCH from achieving further savings, especially in transportation and backordering costs. To overcome these limitations, we introduce in this section an improvement heuristic. In this heuristic, transitions from a given solution to its neighborhood are conducted using the idea of exchanging customers' delivery amounts between periods. These delivery exchanges are conducted through guiding rules that tend to reduce the total cost.

5.1. Neighborhood search structure

First, we define in this section a neighborhood search structure to be used in the developed improvement heuristic. The act of reducing a customer's delivery in a given period t and adding the reduced amount to another period is referred to as *delivery exchange*. If a delivery exchange is made to period $\tau < t$, it is referred to as *backward delivery exchange*. In this case, there may be an increase in the inventory carrying cost for the customer or a reduction in backordering cost when a backorder exists in a preceding period to which the transferred amount is added. A *forward delivery exchange* occurs when a delivery exchange is made to period $\tau > t$. In this case, either a reduction in the inventory carrying cost will be gained or a shortage cost will be incurred depending on the amount exchanged. A forward delivery exchange may be needed to create more capacity in period t, which could be more profitable for other customers' backward delivery exchanges.

Let $\hat{w}_{i,t\to\tau}$ denote the amount of delivery exchange made from period t to period τ for customer i. Let $\Delta IC(\hat{w}_{i,t\to\tau})$ represent the overall change in inventory carrying and shortage costs (positive if increased) associated with the delivery exchange $\hat{w}_{i,t\to\tau}$. From Proposition 1, we know that the reduction (increase) of the delivery amount made to a specific customer is associated with either reduction (increase) in the transportation costs or the transportation costs remain unchanged. Let $TCR_t(\hat{w}_{i,t\to\tau})$ and $TCI_\tau(\hat{w}_{i,t\to\tau})$ denote, respectively, the amounts of transportation cost reduction in period t and the transportation cost increase in period τ that will result from a delivery exchange $\hat{w}_{i,t} \to \tau$. By using backward and forward delivery exchanges, a transition from an incumbent solution to its neighborhood can be achieved.

A single delivery exchange may not be profitable, yet a combination of delivery exchanges, applied in a specific sequence, can lead to a reduction in the total cost. Generally, a better solution can be obtained by searching for an ordered set of exchanges, DX, that maximize the resultant cost saving $\sum_{\hat{w}_{i,t\to\tau}\varepsilon DX} TCR_t(\hat{w}_{i,t\to\tau}) - TCI_\tau(\hat{w}_{i,t\to\tau}) - \Delta IC_\tau(\hat{w}_{i,t\to\tau})$ while maintaining the vehicle and customer capacity constraints. The following subsection discusses some of the guiding rules that can use for this purpose.

5.2. Guidelines for delivery exchanges

For a given solution, the first step in constructing useful delivery exchanges is to look for reductions to the delivery amounts at a selected period so that savings in transportation costs in that period can be achieved, and additions of delivery amounts to customers that have backorders at the end of that period such that their associated shortage costs is reduced.

In Proposition 1, it is shown that the reduction in transportation costs as a result of reducing the delivery amount to a customer occurs at discrete values of the amount reduced. The reason for such discrete changes is due to changes in the vehicle tours which are directly related to the usage of vehicle capacities. Therefore, reductions to delivery amounts that will result in reducing transportation cost can be determined by studying the relationship between the total delivery amount and the vehicle capacities. In the case when there is a backorder decision for a customer in a given period, reduction to shortage costs can be achieved by increasing the delivery made to the customer. The amount of increase is bounded by the total amount of backorder. In this case backward delivery exchanges from future periods are needed.

After deciding the suitable amounts of delivery reduction and addition, the next step is to select the mechanism by which the reduced or added amount can be exchanged to or from another period (the ordered set of delivery exchanges) such that reduction in the total cost can be achieved. Abdelmaguid and Dessouky (2006) list a set of different delivery exchange rules that can be used to effectively guide a neighborhood search algorithm. These rules are adopted here for the developed improvement heuristic.

5.3. The improvement heuristic

The developed improvement heuristic can be considered as a complementary phase to ETCH in which partial fulfillments of demand and their associated cost reductions are investigated. Contrary to ETCH, the improvement heuristic conducts its iterations starting from period T backward to period 1. The reasons for such backward movement is to provide a remedy for the myopic decisions of ETCH. In its search for the best ordered set of delivery exchanges, the improvement heuristic employs the previously described delivery exchange rules repeatedly at a given period in a systematic fashion. We refer to the improvement heuristic as the Backward Delivery Exchanges Heuristic (BDXH). The following list describes its main steps.

Procedure BDXH

1. Let s^* be the initial solution obtained by ETCH;
2. Let $t = T$;
3. Let set $S1 = \{s^*\}$ and set $S2 = \emptyset$;
4. For every solution in set S1 do
   - 4.1. Let R represent the set of customers that either have scheduled deliveries or backorders in period t.
   - 4.2. For each customer i in set R find all possible reductions/additions (Δw_{it}) to the delivery amount of customer i in which either a transportation cost saving or a reduction in the shortage cost can be achieved.
   - 4.3. For every possible Δw_{it} found in step 4.2, generate suitable delivery exchanges for that amount in period t using the delivery exchange rules described earlier
   - 4.4. Generate all the resulting neighborhood solutions for the delivery exchanges found in step 4.3, and add them to set $S2$ such that solutions are stored in an increasing order of their costs and solutions are not repeated. If the allowed maximum size of set $S2$ is exceeded, solutions with the worst costs are eliminated
5. If the cost of the best solution in set $S2$ is less than the cost of s^*, then let s^* be that solution
6. Let $S1 = S2$, repeat step 4 until there is no further delivery exchanges in period t that can be made
7. Let $t = t - 1$, if $t > 0$ then go to step 3, otherwise STOP. The best solution found is s^*.

The most time consuming part of the BDXH is the generation of the neighborhood solutions in step 4.4, as a vehicle routing problem has to be solved for every period in the planning horizon in which a change in the delivery schedule occurs. The loop conducted in step 4 has a polynomial time complexity that is a function of the maximum size allowed for set $S2$. In the conducted experiments this maximum size is selected to be twice the number of customers.

6. Experimentation and results

Two versions of ETCH have been implemented. In the first one, optimal solutions for the two subproblems are generated using a complete breadth-first search for SUB1 and a dynamic programming algorithm for SUB2. We refer to this implementation as ETCH-O. The second version uses the provided breadth-first heuristic for SUB1 and the greedy-search algorithm for SUB2, and is referred to as ETCH-H. The improvement heuristic is then applied using the initial solutions generated by each version of ETCH. Accordingly, we refer to the results obtained by the improvement heuristic as BDXH-O and BDXH-H depending on the initial solution used. These heuristics are programmed and compiled using Borland C++ Builder version 3 and benchmarked against the lower and upper bounds obtained by AMPL-CPLEX 8.1 running under an Intel Pentium 4 processor running with a clock speed of 2.40 GHz with 1GB RAM.

6.1. Experimental design

First, we consider two different scenarios to examine the effectiveness of the developed heuristics under different circumstances. These scenarios simulate the integrated inventory-distribution decisions faced by manufacturing companies that deal with small number of customers, each located in a different major city. An example for similar cases in the literature can be found in Fumero and Vercellis (1999).

The first scenario is designed to test the quality of the inventory holding decisions of ETCH; while, in the second one, some parameters are tuned to provide conditions in which backorder decisions are economical, so that the backorder decisions of the ETCH are assessed. The main factors that are controlled to produce such cases are the ratio of the available vehicle capacity to the average daily demand by customers, the average unit shortage cost and the transportation cost per unit distance.

In both scenarios, customers are allocated in a square of 20×20 distance units and their coordinates are generated using a uniform distribution within these limits. The depot is located in the middle of the square. Customers' unit holding costs are generated using a normal distribution with a mean of 0.1 and a standard deviation of 0.02, and each customer has a storage capacity of 120 items. A constant value of 10 for the vehicle usage cost (f_{vt}) is used.

In the first scenario, the transportation cost per unit distance is set to 1, the customers' unit shortage costs are generated using a normal distribution with a mean of 5 and a standard deviation of 0.5, and the customers' demands are generated using a uniform distribution from 25 to 50 items per day. In the second scenario, we set the parameter values so it is optimal to carry backorders. In this scenario, the transportation cost per unit distance is set to 2, the customers' unit shortage costs are generated using a normal distribution with a mean of 3 and a standard deviation of 0.5, and the customers' demands are generated using a uniform distribution from 5 to 50 items per day.

For each scenario, sixty problems have been generated by varying the number of customers (N), the number of planning periods (T) and the number of homogenous vehicles (V). We generate three levels of N (5), (10), and (15), two levels of T (5) and (7), and two levels of V (1) and (2). For each problem setting defined by a combination of N, T, and V, we randomly generate five problems. The total vehicle capacity in the first scenario is selected to be fixed at 500, 1000, and 1500 for each level of N, respectively. In the second scenario, the selected total vehicle capacities are 150, 300, and 450.

The naming convention used for the test problems starts with a number that refers to the scenario. After a hyphen, two digits are assigned for the number of customers, followed by a digit representing the length of the planning horizon. The next digit represents the number of vehicles. Finally, the replicate number is given at the last digit after a hyphen. Thus, the problem 1-0551-1 represents the first replicate of the first scenario with 5 customers, a planning horizon of 5 periods and 1 vehicle.

6.2. Results and discussion

The detailed experimental results are listed in Tables 3 and 4 in the Appendix A. These tables list the cost components of solutions obtained by each version of the constructive and improvement heuristics along with the CPLEX lower and upper bounds. An * next to the lower bound in the tables indicates that CPLEX was able to find the optimal solution within the one hour time limit.

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Table 1
Average computational times (in minutes) for the developed heuristics

N	T	V	# Binary variables	First scenario				Second scenario			
				ETCH-O	BDXH-O	ETCH-H	BDXH-H	ETCH-O	BDXH-O	ETCH-H	BDXH-H
5	5	1	150	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	7	1	210	0.00	0.01	0.00	0.01	0.00	0.00	0.00	0.00
5	5	2	300	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	7	2	420	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	5	1	550	0.00	0.15	0.00	0.18	0.01	0.00	0.00	0.00
10	7	1	770	0.00	0.25	0.00	0.22	0.01	0.01	0.00	0.01
10	5	2	1100	0.00	0.05	0.00	0.07	0.00	0.00	0.00	0.00
10	7	2	1540	0.05	0.35	0.00	0.64	0.18	0.02	0.00	0.02
15	5	1	1200	0.26	1.85	0.00	2.38	0.20	0.04	0.00	0.05
15	7	1	1680	0.65	5.48	0.00	7.35	0.14	0.16	0.00	0.12
15	5	2	2400	0.22	0.75	0.00	0.94	0.10	0.08	0.00	0.05
15	7	2	3360	0.56	1.95	0.00	2.51	0.21	0.09	0.00	0.07

Table 2
Average results for the third scenario problems

N	T	V	# Binary variables	CPLEX UB LB diff%	ETCH-H		BDXH-H	
					LB diff%	Time (min)	LB diff%	Time (min)
20	7	2	5880	75.91	34.16	0.00	28.87	0.54
25	7	2	9100	126.39	37.10	0.00	31.26	1.31
30	7	2	13020	200.84	39.83	0.01	34.47	3.33

The percentage differences between the total cost obtained by each heuristic and the lower bound are used as performance indicators. The percentage difference, also referred to as optimality gap, is calculated by taking the ratio of the difference between the heuristic's total cost and the lower bound to the lower bound. A comparison against the lower bound provides a measure of deviation from optimality. The CPLEX upper bound in a maximum of one-hour running time is used as an alternate heuristic and its percentage difference against the lower bound is similarly calculated.

For each problem setting, the average of the percentage differences of the 5 replicates is calculated and plotted against the number of binary variables of that setting as shown in Fig. 3 and 6. Table 1 summarizes the average computational time of the developed heuristics for each problem set in both scenarios.

As shown in Figs. 3 and 4, the combined constructive and improvement heuristics outperform the CPLEX upper bound for instances with ten customers and more in the first scenario and 15 customers in the second scenario. While the growth of the CPLEX optimality gap seems to be exponential with the increase of the number of binary variables, the optimality gap for the developed heuristics is below 30% on average and remains almost level with the increase of the number of binary variables.

In the first scenario, the ETCH-O version of the constructive heuristic is on average 2% closer to the lower bound than ETCH-H. However, after applying the improvement heuristic on both versions this difference reduces to only 0.4%. In the second scenario, this difference is 1% and slightly increases with the application of the improvement heuristic to 1.25%.

Reductions to the total cost as a result of applying the improvement heuristic are evident. In the first scenario the improvement heuristic provides reductions in the optimality gap of 6.1% and 7.7% on average over solutions generated by ETCH-O and ETCH-H, respectively. While in the second scenario, these figures are 4% and 3.8%, respectively.

In the case of small problem instances of 150 and 210 binary variables, for which CPLEX was able to find optimal solutions within the one-hour time limit, we can see that the improvement heuristic can reach a relatively good optimality gap of less than 1% in the case of 150 binary variables and less than 5% for the case of 210 binary variables for the first scenario problems. These figures are higher in the case of the second scenario problems. However for larger problem instances, it is hard to judge the quality of the lower bounds obtained by CPLEX, and so the optimality gaps obtained can not give a clear cut measure of how far the results obtained by the developed heuristics are from the optimal solutions.

The computational time for ETCH-H is found to be less than one second in all the cases tested. For ETCH-O, due to the dynamic programming part of the algorithm, the computational time increases with the increase of the problem size; however on average, it has not reached the 90 seconds limit in all problem sets. The increase of computational time of ETCH-O is mostly attributed to the increase in both N and T; while, the number of vehicles, V, does not seem to have a significant effect on computational time. The computational time of the improvement heuristic seems to increase at a higher rate with the increase of the problem size in the first scenario as compared to the second one. We attribute this to the increase of the ratio of the total vehicle capacity to the average daily customers' demand, which increases the number of possible delivery exchanges and neighborhood solutions generated at each iteration of BDXH.

From the previous results we conclude the following. The ETCH-O version of the constructive heuristic is capable of generating slightly better solutions compared to ETCH-H with up to 2% difference on average in the optimality gap. However, with the increase of the problem size, especially the number of customers and the number of planning periods, the computational time of ETCH-O will be significantly higher than the computational time of ETCH-H. The consideration of partial fulfillment of demand and the mechanism of delivery exchanges implemented by the improvement heuristic seem to offer improvements to the optimality gap that can reach more than 3.8% on average. However, the computational time of BDXH will be considerably higher with the increase of the ratio between vehicle capacity and the average customers demand per period.

6.3. Experimental results for larger problem instances

To investigate the performance of the developed heuristics with larger problem sizes, we construct an additional experimental set based on a third scenario. In this scenario, medium vehicle capacity to average daily demand ratio is used such that a situation in the middle of the first two extreme scenarios is addressed. This scenario considers similar parameters as in the second one with some modifications to reduce the frequency in which backorder decisions are needed. The main difference between the parameters

Table 3
Detailed costs for the first scenario problems

Problem	CPLEX bounds		ETCH-O				BDXH starting with ETCH-O				ETCH-H				BDXH starting with ETCH-H			
	UB	LB	Hold	Short	Transp	Total	Hold	Short	Transp	Total	Hold.	Short.	Transp	Total	Hold	Short	Transp	Total
1-0551-1	205.84	205.84*	71.84	0	134	205.84	71.84	0	134	205.84	64.6	0	146	210.6	79.54	0	128	207.54
1-0551-2	150.74	150.74*	45.74	0	105	150.74	45.74	0	105	150.74	54.94	0	98	152.94	54.94	0	98	152.94
1-0551-3	186.6	186.6*	47.64	0	154	201.64	48.6	0	138	186.6	54.36	0	151	205.36	48.6	0	138	186.6
1-0551-4	200.8	200.8*	59.18	0	146	205.18	58.3	0	146	204.3	59.34	0	165	224.34	58.3	0	146	204.3
1-0551-5	184.8	184.8*	53.85	0	136	189.85	49.35	0	136	185.35	58.89	0	132	190.89	49.35	0	136	185.35
1-0571-1	278.96	278.96*	64.59	0	238	302.59	70.81	0	211	281.81	83.55	0	220	303.55	79.14	0	203	282.14
1-0571-2	268.68	268.68*	84.04	0	219	303.04	70.98	0	202	272.98	103.1	0	215	318.1	101.1	0	181	282.1
1-0571-3	273.07	273.07*	85.8	0	225	310.8	76.4	0	198	274.4	93.29	0	220	313.29	75.07	0	198	273.07
1-0571-4	312.25	312.25*	80.49	0	269	349.49	80.49	0	269	349.49	80.49	0	269	349.49	80.49	0	269	349.49
1-0571-5	310.98	310.98*	77.76	0	266	343.76	86.04	0	228	314.04	111.85	0	237	348.85	96.4	0	222	318.4
1-0552-1	212.41	205.11	50.99	0	194	244.99	43.69	0	178	221.69	50.99	0	194	244.99	43.69	0	178	221.69
1-0552-2	254.28	254.28*	56.94	0	231	287.94	69.28	0	185	254.28	63.98	0	205	268.98	69.28	0	185	254.28
1-0552-3	220.86	220.86*	87.56	0	178	265.56	59.98	0	164	223.98	64.83	0	189	253.83	50.26	0	178	228.26
1-0552-4	250.35	250.35*	72.83	0	182	254.83	72.83	0	182	254.83	72.83	0	182	254.83	72.83	0	182	254.83
1-0552-5	235.09	233.33	59.92	0	186	245.92	59.92	0	186	245.92	59.92	0	186	245.92	59.92	0	186	245.92
1-0572-1	319.22	302.88	88.47	0	282	370.47	102.24	0	246	348.24	106.61	0	273	379.61	87.38	0	249	336.38
1-0572-2	289.15	274.02	84.54	0	234	318.54	85.33	0	205	290.33	111.25	0	213	324.25	91.67	0	218	309.67
1-0572-3	270.66	253.78	77.79	0	258	335.79	59.71	0	212	271.71	70.85	0	229	299.85	70.85	0	229	299.85
1-0572-4	278.68	258.79	84.27	0	212	296.27	76.57	0	214	290.57	79.25	0	220	299.25	72.79	0	214	286.79
1-0572-5	292.03	271.68	81.6	0	227	308.6	81.6	0	227	308.6	80.44	0	234	314.44	68.91	0	239	307.91
1-1051-1	327.09	306.82	111.09	0	233	344.09	108.97	0	218	326.97	111.09	0	233	344.09	108.97	0	218	326.97
1-1051-2	286.17	251.17	82.76	0	203	285.76	93.41	0	183	276.41	69	0	223	292	81.6	0	196	277.6
1-1051-3	300.69	295.9	91.18	0	210	301.18	90.69	0	210	300.69	27.94	0	322	349.94	90.69	0	210	300.69
1-1051-4	291.13	260.2	80.72	0	222	302.72	90.09	0	192	282.09	93.32	0	208	301.32	88.13	0	192	280.13
1-1051-5	269.47	218.9	76.31	0	180	256.31	69.63	0	180	249.63	19.73	0	272	291.73	69.63	0	180	249.63
1-1071-1	451.45	413.73	149.91	0	350	499.91	142.84	0	309	451.84	169.67	0	318	487.67	139.26	0	316	455.26
1-1071-2	454.86	374.32	128.26	0	327	455.26	132.2	0	288	420.2	151.68	0	296	447.68	124.21	0	312	436.21
1-1071-3	495.2	410.98	159.74	0	360	519.74	130.65	0	337	467.65	168.99	0	345	513.99	169.01	0	299	468.01
1-1071-4	489.67	428.21	134.76	0	352	486.76	142.4	0	319	461.4	123.09	0	383	506.09	149.44	0	313	462.44
1-1071-5	399.07	370.35	142.91	0	258	400.91	142.91	0	258	400.91	128.7	0	282	410.7	130.96	0	267	397.96
1-1052-1	325.57	268.63	67.56	0	255	322.56	67.56	0	255	322.56	72.96	0	251	323.96	67.56	0	255	322.56
1-1052-2	376.66	296.12	94.05	0	241	335.05	94.05	0	241	335.05	88.35	0	269	357.35	90.95	0	250	340.95
1-1052-3	326.41	268.99	83.89	0	255	338.89	105.27	0	205	310.27	49.51	0	335	384.51	56.93	0	259	315.93
1-1052-4	367.04	295.17	85.23	0	289	374.23	84.05	0	262	346.05	63.42	0	290	353.42	73.25	0	273	346.25
1-1052-5	342.21	264.07	104.29	0	206	310.29	125.73	0	183	308.73	72.61	0	253	325.61	110.97	0	199	309.97
1-1072-1	637.37	401.1	154.82	0	312	466.82	135.28	0	328	463.28	149.97	0	351	500.97	152.77	0	319	471.77
1-1072-2	690.6	466.94	159.73	0	413	572.73	178.22	0	351	529.22	126.52	0	436	562.52	137.73	0	404	541.73
1-1072-3	508.91	367.88	145.7	0	325	470.7	91.5	0	340	431.5	114.39	0	342	456.39	137.14	0	294	431.14
1-1072-4	551.38	413.52	130.69	0	400	530.69	125.02	0	366	491.02	74.34	0	453	527.34	104.77	0	396	500.77
1-1072-5	531.64	392.88	117.72	0	358	475.72	137.84	0	317	454.84	103.68	0	441	544.68	128.6	0	327	455.6
1-1551-1	458.73	337.92	140.23	0	264	404.23	133.05	0	271	404.05	99.2	0	324	423.2	153.54	0	249	402.54
1-1551-2	414.62	294.14	115.54	0	237	352.54	131.76	0	218	349.76	57.97	0	317	374.97	113.54	0	237	350.54
1-1551-3	430.06	319.32	94.82	0	296	390.82	130.4	0	254	384.4	77.23	0	321	398.23	130.4	0	254	384.4
1-1551-4	420.33	314.08	111.04	0	258	369.04	109.88	0	258	367.88	86.94	0	300	386.94	109.88	0	258	367.88
1-1551-5	425.91	315.85	96.84	0	280	376.84	121.22	0	248	369.22	58.79	0	340	398.79	140.36	0	234	374.36
1-1571-1	733.36	452.81	159.71	0	414	573.71	180.57	0	343	523.57	43.7	0	551	594.7	180.57	0	343	523.57
1-1571-2	654.56	454.07	139.81	0	417	556.81	194.99	0	332	526.99	125.1	0	453	578.1	194.99	0	332	526.99
1-1571-3	553.61	405.71	179.02	0	304	483.02	179.02	0	304	483.02	18.59	0	511	529.59	179.02	0	304	483.02
1-1571-4	649.16	469.47	200.88	0	365	565.88	196.69	0	345	541.69	158.27	0	425	583.27	203.57	0	345	548.57
1-1571-5	666.62	440.91	182.4	0	378	560.4	176.48	0	336	512.48	116.19	0	440	556.19	176.48	0	336	512.48
1-1552-1	667.69	357.51	174.33	0	297	471.33	130.58	0	309	439.58	91.42	0	344	435.42	112	0	316	428

1-1552-2	797.32	369.08	144.09	0	345	489.09	138.07	0	305	443.07	109.32	0	340	449.32	124.15	0	320	444.15
1-1552-3	435.36	374.53	121.12	0	323	444.12	164.27	0	273	437.27	78.64	0	363	441.64	48.83	0	373	421.83
1-1552-4	492.81	358.67	128.58	0	317	445.58	42.07	0	389	431.07	61.6	0	398	459.6	53.84	0	369	422.84
1-1552-5	474.11	343.76	126.95	0	295	421.95	61.9	0	340	401.9	16.4	0	402	418.4	66.7	0	335	401.7
1-1572-1	751.64	488.79	173.18	0	394	567.18	148.95	0	407	555.95	173.18	0	394	567.18	148.95	0	407	555.95
1-1572-2	1038.4	497.06	199.83	0	455	654.83	145.26	0	459	604.26	100.67	0	567	667.67	124.67	0	477	601.67
1-1572-3	933.26	520.94	220.53	0	430	650.53	193.4	0	426	619.4	127.51	0	512	639.51	150.34	0	476	626.34
1-1572-4	869.07	472.66	207.12	0	402	609.12	182.74	0	419	601.74	32.99	0	587	619.99	61.5	0	519	580.5
1-1572-5	969.49	469.56	178.05	0	481	659.05	155.3	0	446	601.3	128.11	0	503	631.11	153.33	0	449	602.33

* Optimal solution found.

used in the third scenario as compared to the second one is that the travel cost per unit distance is set to 1 and customers daily demand is generated using a uniform distribution between 0 and 25. We consider three different levels of the number of customers, N: 20, 25, and 30, and a total vehicle capacity of 300, 350, and 400 at each level of N, respectively. We only consider one level for both T and V at 7 and 2, respectively. Five random replicates are generated at each level of N. We use the previously defined naming convention for the third scenario problems.

The detailed cost results for the third scenario problems are shown in Table 5 in the Appendix A. CPLEX lower and upper bounds are obtained after a running time of three hours. Due to the inability of the optimization routines to find solutions for the two subproblems for these large problem instances, we only ran the heuristic versions, ETCH-H and BDXH-H. The average cost and time results for the ETCH-H version and the improvement heuristic BDXH-H are shown in Table 2.

We can see that the rate of increase of the heuristics optimality gaps is almost constant with the increase of the number of customers. When we compare this with the exponential rate of increase for the CPLEX upper bound percentage difference, we can see the potential benefit of the developed constructive and improvement heuristics for larger problem sizes. In terms of computational time, the ETCH-H version of the constructive heuristic remains below one second for larger problems with up to 30 customers; while, the improvement heuristic has an increasing computational time.

7. Conclusion and future work

This article addressed the inventory routing problem with backlogging in which multiperiod vehicle routing and inventory holding and backlogging decisions for a set of customers are to be made. We considered an environment in which the demand at each customer is relatively small compared to the vehicle capacity, and the customers are closely located such that a consolidated shipping strategy is appropriate. We presented a constructive heuristic based on the idea of allocating single transportation cost estimates for each customer. Two subproblems, comparing inventory holding and backlogging decisions with these transportation cost estimates, are formulated and their solution methods are incorporated in the developed heuristic. The main idea behind the constructive heuristic as seen in the formulation of the two subproblems is to consider only delivery plans in which fulfillment of part of the current or the future demand requirements in a currently studied period is not allowed. An improvement heuristic is developed to overcome some of the limitations of the constructive heuristic. This improvement heuristic is based on the idea of exchanging delivery amounts between periods to allow for partial fulfillments of demands and exploit associated reductions in costs. A mixed integer programming formulation is provided and used to obtain lower and upper bounds using AMPL-CPLEX to assess the performance of the developed heuristics.

For small sized problems with up to 15 customers, the experimental results show that the developed constructive heuristic can achieve solutions that are on average not farther than 30% from the optimal in a few minutes. This figure can be reduced to 25% by applying the improvement heuristic which shows the significance of allowing partial fulfillment of demand. With the increase of problem size, the optimality gap of the developed heuristics increases with almost a constant rate and results can be obtained in a few minutes. This shows the potential benefit of the developed heuristics for larger problem sizes.

The studied problem and the developed heuristic approaches can give insights for solving other problems in the manufacturing

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Table 4
Detailed costs for the second scenario problems

Problem	CPLEX bounds		ETCH-O				BDXH starting with ETCH-O				ETCH-H				BDXH starting with ETCH-H			
	UB	LB	Hold	Short	Transp	Total	Hold	Short	Transp	Total	Hold.	Short.	Transp	Total	Hold	Short	Transp	Total
2-0551-1	649.8	649.8*	4.44	430.44	387	821.88	12.54	310.7	387	710.24	8.34	350.82	387	746.16	12.66	300.62	387	700.28
2-0551-2	468	468*	5.46	54.81	477	537.27	10.86	0	489	499.86	5.46	54.81	477	537.27	10.86	0	489	499.86
2-0551-3	400	400*	7.56	79.3	377	463.86	4.74	42.7	388	435.44	7.56	79.3	377	463.86	4.74	42.7	388	435.44
2-0551-4	475.29	475.29*	5.11	19.84	451	475.95	5.11	19.84	451	475.95	5.11	19.84	451	475.95	5.11	19.84	451	475.95
2-0551-5	426.01	426.01*	6.75	132.06	394	532.81	8.35	33.66	408	450.01	8.43	132.06	394	534.49	8.55	33.66	408	450.21
2-0571-1	522.97	522.97*	19.65	0	621	640.65	20.55	0	615	635.55	26.2	22.8	609	658	26.2	22.8	609	658
2-0571-2	557.89	557.89*	7.91	198.69	439	645.6	8.63	169.19	442	619.82	7.91	198.69	439	645.6	8.63	169.19	442	619.82
2-0571-3	434.86	434.86*	12.86	43.6	458	514.46	17.32	37.06	444	498.38	12.86	43.6	458	514.46	17.32	37.06	444	498.38
2-0571-4	536.42	536.42*	22.7	67.15	567	656.85	24	41.25	567	632.25	26.32	48.75	571	646.07	30.74	48.75	562	641.49
2-0571-5	498.08	498.08*	12.62	33.6	536	582.22	12.62	33.6	536	582.22	13.72	33.6	536	583.32	12.62	33.6	536	582.22
2-0552-1	522.82	509	11.53	0	553	564.53	11.53	0	553	564.53	11.53	0	553	564.53	11.53	0	553	564.53
2-0552-2	940.47	933.76	0	545.04	495	1040.04	0.2	533.94	496	1030.14	1.2	569.23	482	1052.43	1.2	564.93	485	1051.13
2-0552-3	512.44	497.98	20.54	54.36	550	624.9	21.53	14.52	574	610.05	20.54	54.36	550	624.9	21.53	14.52	574	610.05
2-0552-4	537.37	519.91	9.99	137.86	545	692.85	7.02	135.4	501	643.42	9.99	137.86	545	692.85	7.02	135.4	501	643.42
2-0552-5	553.2	536.52	5.7	0	614	619.7	5.7	0	614	619.7	5.7	0	614	619.7	5.7	0	614	619.7
2-0572-1	828.6	789.04	5.81	50.58	914	970.39	6.2	0	902	908.2	5.81	50.58	914	970.39	6.2	0	902	908.2
2-0572-2	988.31	943.43	11.4	108.64	939	1059.04	7.88	69.84	949	1026.72	11.4	108.64	939	1059.04	7.88	69.84	949	1026.72
2-0572-3	864.23	793.38	6.63	52.5	872	931.13	6.51	7.5	883	897.01	6.63	52.5	872	931.13	6.51	7.5	883	897.01
2-0572-4	786.53	738.55	12.92	119	845	976.92	11.96	119	806	936.96	12.92	119	845	976.92	11.96	119	806	936.96
2-0572-5	771.35	728.76	12.06	163.6	812	987.66	11.09	61.8	844	916.89	31.15	91.08	837	959.23	21.51	61.8	848	931.31
2-1051-1	528.69	509.59	24.93	0	549	573.93	24.11	0	546	570.11	24.67	0	554	578.67	26.51	0	541	567.51
2-1051-2	487.7	423.78	47.26	0	448	495.26	47.26	0	448	495.26	47.26	0	448	495.26	47.26	0	448	495.26
2-1051-3	724.13	660.23	2.8	238.36	548	789.16	3.55	196.5	550	750.05	2.25	240.18	550	792.43	3.55	215.19	550	768.74
2-1051-4	456	445.86	21.22	0	442	463.22	21.22	0	442	463.22	19.12	0	452	471.12	19.12	0	452	471.12
2-1051-5	591.03	546.62	26.37	42.42	560	628.79	27.72	39.39	544	611.11	19.26	42.42	576	637.68	19.26	39.39	579	637.65
2-1071-1	784.36	728.48	32.39	64.25	729	825.64	32.39	64.25	729	825.64	28.49	64.25	738	830.74	27.29	64.25	739	830.54
2-1071-2	842.4	730.1	35.37	37.68	752	825.05	35.37	37.68	752	825.05	37.67	37.68	744	819.35	37.67	37.68	744	819.35
2-1071-3	748.65	668.8	24.75	0	778	802.75	33.39	0	763	796.39	46.51	0	699	745.51	52.39	0	691	743.39
2-1071-4	897.24	799.72	37.7	45.23	889	971.93	44	45.23	874	963.23	43.42	45.23	877	965.65	49.52	45.23	858	952.75
2-1071-5	763.69	712.32	28.72	139.52	730	898.24	29.57	72.67	754	856.24	37.45	139.52	721	897.97	30.61	139.52	726	896.13
2-1052-1	829.24	758.39	8.42	117.65	748	874.07	6.62	117.65	730	854.27	11.97	157.45	745	914.42	7.92	157.45	736	901.37
2-1052-2	676.22	566.94	35.34	0	662	697.34	33.14	0	641	674.14	36.24	0	704	740.24	34.56	0	640	674.56
2-1052-3	759.4	659.22	19.84	60.68	727	807.52	21.14	47.6	727	795.74	21.14	60.68	715	796.82	21.14	47.6	727	795.74
2-1052-4	630.89	509.46	26.17	0	669	695.17	23.28	0	658	681.28	26.8	0	667	693.8	23.28	0	658	681.28
2-1052-5	799.24	718.07	3	140.88	727	870.88	4.56	81.55	741	827.11	3	140.88	727	870.88	4.56	81.55	741	827.11
2-1072-1	955.63	808.46	37.19	47.79	934	1018.98	35.94	37.24	893	966.18	43.3	105.63	862	1010.93	33.42	53.26	902	988.68
2-1072-2	1266.9	1029.21	31.53	137.62	1135	1304.15	31.74	131.92	1107	1270.66	31.53	137.62	1135	1304.15	31.74	131.92	1107	1270.66
2-1072-3	1037.6	857.06	45.14	48.06	1015	1108.2	51.35	48.06	948	1047.41	54.36	35.49	1028	1117.85	53.49	24.03	1001	1078.52
2-1072-4	1135.9	896.78	39.79	76.06	1062	1177.85	36.1	63.9	1056	1156	50.18	100.64	1012	1162.82	42.62	97.92	962	1102.54
2-1072-5	938.11	750.97	27.41	65.36	877	969.77	30.66	24.08	864	918.74	20.89	73.6	922	1016.49	25.13	12.88	923	961.01
2-1551-1	823.1	736.42	22.01	66.91	716	804.92	63.86	22.1	716	801.96	22.29	66.91	738	827.2	63.86	19.98	738	821.84
2-1551-2	781.1	725.42	6.8	0	779	785.8	6.8	0	779	785.8	12.67	0	767	779.67	12.02	0	763	775.02
2-1551-3	800.63	666.94	20.19	115.65	623	758.84	20.74	95.09	628	743.83	21.34	115.65	675	811.99	18.67	110.51	673	802.18
2-1551-4	739.67	608.49	39.51	0	672	711.51	39.51	0	672	711.51	38.72	0	690	728.72	34.32	0	693	727.32
2-1551-5	1012.9	971.66	3.8	273.47	809	1086.27	6.52	223.09	810	1039.61	2.24	338.2	822	1162.44	2.24	338.2	822	1162.44
2-1571-1	1095.1	747.3	114.14	0	756	870.14	114.14	0	756	870.14	100.39	0	882	982.39	87.69	0	879	966.69
2-1571-2	1097.7	660.8	38.31	0	857	895.31	51.67	0	824	875.67	81.34	0	819	900.34	91.32	0	776	867.32
2-1571-3	1217.2	800.45	94.91	0	929	1023.91	99.49	0	908	1007.49	112.1	26.59	919	1057.69	105.27	6.88	923	1035.15
2-1571-4	1095.3	803.99	81.7	0	930	1011.7	80.01	0	928	1008.01	77.02	0	971	1048.02	74.07	0	950	1024.07
2-1571-5	1383.8	1130.8	27.74	267.13	990	1284.87	27.92	260.69	990	1278.61	30.25	281.01	1014	1325.26	26.23	263.72	1024	1313.95
2-1552-1	924.1	620.96	39.04	14.6	804	857.64	41.15	14.6	747	802.75	56.44	14.6	751	822.04	53.81	14.6	739	807.41

2-1552-2	818.36	595.9	39.02	0	712	751.02	41.57	0	669	710.57	43.04	0	746	789.04	43.99	0	670	713.99
2-1552-3	1103.7		55.26	22.47	887	964.73	50.76	0	824	874.76	48.89	17.8	875	941.69	50.99	0	817	867.99
2-1552-4	1086.4	923.82	14.05	118.06	922	1054.11	14.05	104.7	931	1049.75	14.05	126.45	936	1076.5	14.41	126.45	931	1071.86
2-1552-5	1125.5	729.65	30.91	0	903	933.91	29.47	0	880	909.47	31.11	0	909	940.11	25.51	0	891	916.51
2-1572-1	1375.1	881.63	52.65	0	1132	1184.65	52.83	0	1074	1126.83	51.92	0	1194	1245.92	47.36	0	1117	1164.36
2-1572-2	1415.2	972.09	15.98	88.97	1140	1244.95	12.02	28	1135	1175.02	25.56	72.41	1137	1234.97	21.84	46.65	1126	1194.49
2-1572-3	1768.9	1042.43	79.5	40.5	1212	1332	69.89	40.5	1189	1299.39	65.88	40.5	1244	1350.38	65.91	40.5	1155	1261.41
2-1572-4	1328.7	920.04	82.52	0	1137	1219.52	92.32	0	1053	1145.32	99.43	15.65	1080	1195.08	106.14	0	1009	1115.14
2-1572-5	1575.2	1117.42	18.58	137.43	1176	1332.01	19.12	130.17	1138	1287.29	20.89	188.07	1180	1388.96	20.96	160.37	1188	1369.33

* Optimal solution found.

industry that have wider scope. The integration of manufacturing and logistic decisions at the operational planning level, as found in Chandra and Fisher (1994), Fumero and Vercellis (1999) and Lei, Liu, Ruszczynski, and Park (2006), is a good example of one such problem.

Appendix A. Appendix

A.1. An illustration of the subtour elimination mechanism in the developed MILP model

To illustrate the role of constraints Eq. (4) in eliminating subtours in the proposed MILP model, let us start with an MILP model for the IRPB that does not contain constraints Eq. (4) and call it IRPB$^{-(4)}$. Consider a case in which there is only one vehicle and three customers. Then, constraint Eq. (5) can be rewritten as follows:

$$\sum_{\substack{l=0\\ l\neq i}}^{3} y_{lit}^{1} - \sum_{\substack{k=0\\ k\neq i}}^{3} y_{ikt}^{1} = d_{it} - I_{it-1} + B_{it-1} + I_{it} - B_{it} \qquad i = 1,\ldots,3 \quad \text{and} \quad t = 1,\ldots,T \tag{21}$$

The right-hand-side of the above equation represents the amount that will be delivered to customer i in period t. Let us denote this quantity by ς_{it}. Notice that ς_{it} is unrestricted in sign since constraints Eq. (4) are excluded. Let us consider one time period and let us drop the indexes for both the time period and the vehicle for brevity. Then, the above equation is reduced to:

$$\sum_{\substack{l=0\\ 1\neq i}}^{3} y_{li} - \sum_{\substack{k=0\\ k\neq i}}^{3} y_{ik} = \varsigma_i \qquad i = 1,\ldots,3 \tag{22}$$

The above equation is quite familiar in network flow models as it is equivalent to saying that the difference between the amount of inflow and the amount of outflow to and out of node i equals the quantity delivered to that node. Now, let us consider a simple numerical example in which $\varsigma_1 = 2$, $\varsigma_2 = -5$ and $\varsigma_3 = 3$. Fig 5(a) illustrates one feasible vehicle tour for this case that satisfies all the vehicle routing constraints of IRPB$^{-(4)}$, yet it contains the subtour 1-2-3-1.

Notice that constraints Eq. (4) are non-negativity constraints for the delivery quantities ς_i which when added to the MILP model we would not obtain a negative value for ς_2. The two feasible vehicle tours illustrated in Fig. 5(b) and (c) represent two different feasible solutions when the value of ς_2 equals zero and greater than zero, respectively. Based on the required delivery quantities ς_1, ς_2, and ς_3, the values for the continuous variables y_{ij} will be determined as required by constraints Eq. (5), which in turn will force the binary decision variables x_{ij} to take the value of one as necessitated by constraints Eq. (3). Accordingly, new arcs will be added to the vehicle tour, which in turn must satisfy constraints Eq. (1) and (2). It can be easily shown that the subtour 1-2-3-1 in both cases shown in Fig. 5(b) and (c) can not occur, for otherwise constraints Eq. (1) and (2) will be violated. This logic can be easily extended for the case of more than one vehicle.

Furthermore, for the cases in which nodes (o_1, o_2, ..., o_N) have zero delivery quantities, subtours that come in the form o_1–o_x–...–o_1 would not be efficient since an additional unnecessary transportation cost associated with the their arcs will be added.

From the above analysis, It is evident that constraints Eq. (4) which mandates that the quantity delivered to any node by a given vehicle should be greater than or equal to zero is necessary for eliminating subtours in the developed MILP model.

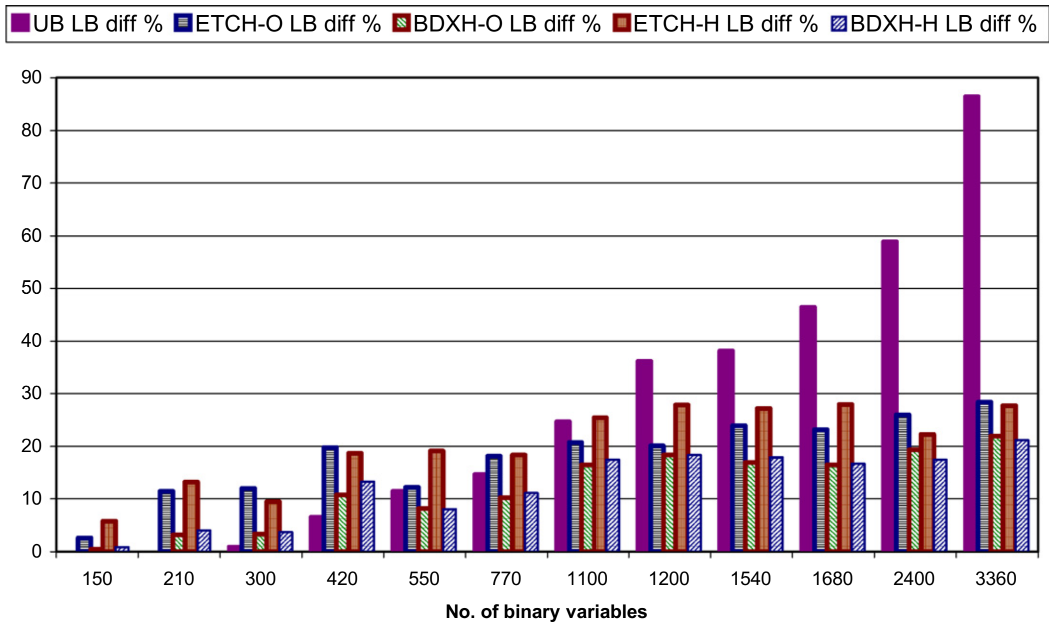


Fig. 3. Average percentage differences against lower bounds for the first scenario problems.

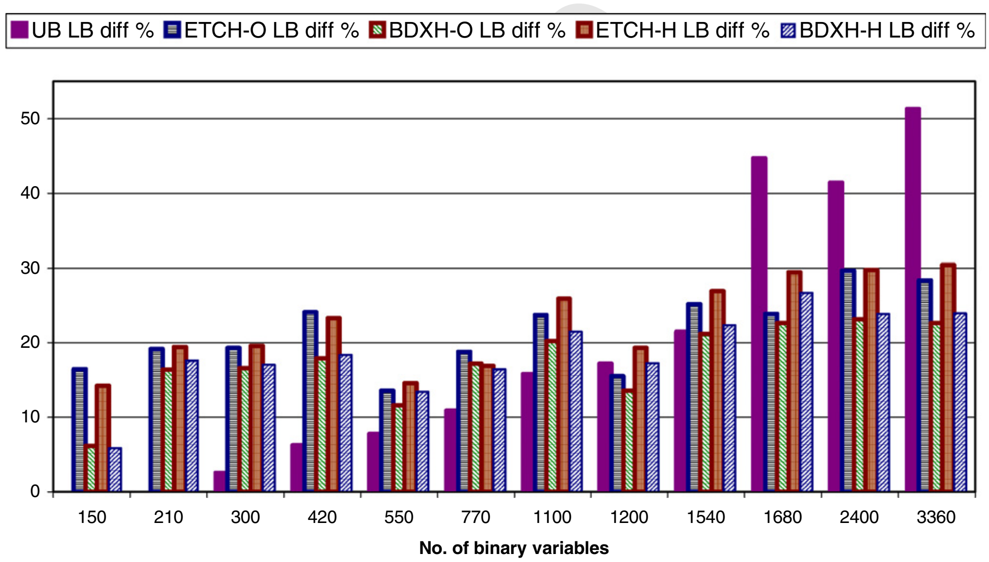
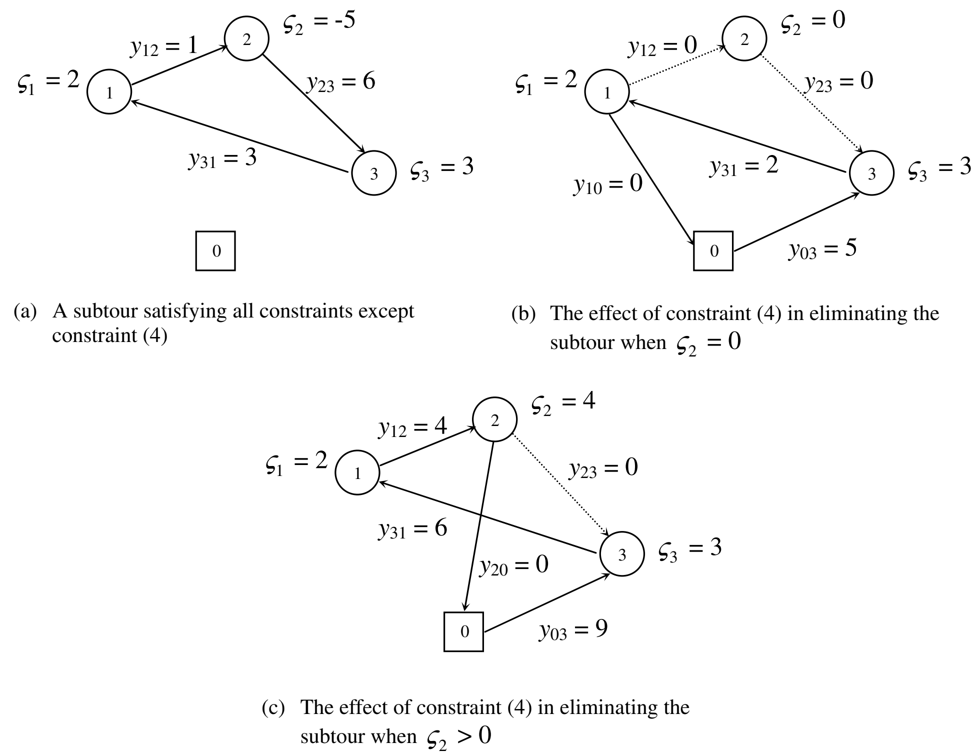


Fig. 4. Average percentage differences against lower bounds for the second scenario problems.

Table 5
Detailed costs for the third scenario problems

Problem	CPLEX bounds		ETCH-H				BDXH starting with ETCH-H			
	UB	LB	Hold	Short	Transp	Total	Hold	Short	Transp	Total
3-2072-1	892.42	510.34	64.76	26.88	605	696.64	61.52	5.34	605	671.86
3-2072-2	811.23	467.85	73.5	6.1	580	659.6	66.73	6.1	552	624.83
3-2072-3	802.6	495.58	75	10.63	597	682.63	55.62	0	593	648.62
3-2072-4	890.79	473.97	80.73	3.08	556	639.81	78.41	3.08	531	612.49
3-2072-5	1175.38	647.95	14.3	2.95	764	781.25	13.51	2.95	755	771.46
3-2572-1	1265.5	570.43	58.96	6.26	684	749.22	51.28	0	668	719.28
3-2572-2	1295.92	613.47	72.77	17.14	743	832.91	77.42	17.14	701	795.56
3-2572-3	1347.05	608.41	52.53	8.78	760	821.31	52.97	6.24	738	797.21
3-2572-4	1411.5	566.68	69.38	17.46	701	787.84	69.19	7.68	695	771.87
3-2572-5	1280.35	560.61	71.9	3.47	734	809.37	73.33	0	674	747.33
3-3072-1	1823	570.69	51.05	16.25	781	848.3	48.52	16.25	743	807.77
3-3072-2	1739.72	596.14	55.4	6.18	766	827.58	53.77	6.18	729	788.95
3-3072-3	1981.65	653.8	51.16	0	883	934.16	42.61	0	851	893.61
3-3072-4	1794.65	653.37	50.61	10.47	827	888.08	43.48	0	814	857.48
3-3072-5	2138.36	678.46	31.59	0	870	901.59	29.75	0	856	885.75



(a) A subtour satisfying all constraints except constraint (4)

(b) The effect of constraint (4) in eliminating the subtour when $\varsigma_2 = 0$

(c) The effect of constraint (4) in eliminating the subtour when $\varsigma_2 > 0$

Fig. 5. Illustration of subtour elimination by constraints Eq. (4).

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