Technical Appendix 1: Gibbs Sampler for structural demand model

In this Appendix we describe the procedure to sample each of the parameters of the model from their full-conditional posterior distribution according to the assumptions in Section 4. We note that in the case of the full-conditional posterior simulation of θ_i, ξ_t and ν_t we use a MH step with candidate vectors generated from a normal distribution with mean equal to the current value (random walk) and a variance matrix proportional to the current value of D, Σ^d and Σ^c, respectively (see C.3, C.4 and C.5 for details).

a) Sampling Choices (y_{it}):

1. In each iteration (k) randomly select $N/2$ pairs of consumers without replacement and enumerate these pairs. Let (i_{1p}, i_{2p}) be the indexes of consumers in pair p and $(z^{(k)}_{i_{1p}t}, z^{(k)}_{i_{2p}t})$ their choices in period t in the current iteration k.

2. For each period t and starting from the first pair, successively and jointly draw the choices of each pair of consumers $(z^{(k+1)}_{i_{1p}t}, z^{(k+1)}_{i_{2p}t})$ from their full-conditional posterior distribution. The pair (p) subscript is dropped for notational convenience. In addition, define $\omega_i = 1$ if $\psi_i > 0$ and, otherwise $\omega_i = 0$. We proceed by assigning $(z^{(k+1)}_{i_1t}, z^{(k+1)}_{i_2t}) = (z^{(k)}_{i_2t}, z^{(k)}_{i_1t})$ according to the following probability:

$$
f\left((z^{(k+1)}_{i_1t}, z^{(k+1)}_{i_2t}) = (z^{(k)}_{i_2t}, z^{(k)}_{i_1t}) \mid *\right)
= \begin{cases}
\dfrac{\mathrm{I}_{\left\{\omega_{i_1} c_{i_1j} z^{(k)}_{i_1j} + \omega_{i_2} c_{i_2j} z^{(k)}_{i_2j} = \omega_{i_1} c_{i_1j} z^{(k)}_{i_2j} + \omega_{i_2} c_{i_2j} z^{(k)}_{i_1j}\right\}}{1 + \dfrac{\prod_{j=1}^{J} p_{i_1jt}^{z^{(k)}_{i_2jt}} p_{i_2jt}^{z^{(k)}_{i_1jt}} h\left(c_{i_1jt+1} \middle| c_{i_1jt}, z^{(k)}_{i_2jt}\right) h\left(c_{i_2jt+1} \middle| c_{i_2jt}, z^{(k)}_{i_1jt}\right)}{\prod_{j=1}^{J} p_{i_1jt}^{z^{(k)}_{i_1jt}} p_{i_2jt}^{z^{(k)}_{i_2jt}} h\left(c_{i_1jt+1} \middle| c_{i_1jt}, z^{(k)}_{i_1jt}\right) h\left(c_{i_2jt+1} \middle| c_{i_2jt}, z^{(k)}_{i_2jt}\right)}}, & 1 \le t \le T-1; \\[3em]
\dfrac{\mathrm{I}_{\left\{\omega_{i_1} c_{i_1j} z^{(k)}_{i_1j} + \omega_{i_2} c_{i_2j} z^{(k)}_{i_2j} = \omega_{i_1} c_{i_1j} z^{(k)}_{i_2j} + \omega_{i_2} c_{i_2j} z^{(k)}_{i_1j}\right\}}{1 + \dfrac{\prod_{j=1}^{J} p_{i_1jt}^{z^{(k)}_{i_2jt}} p_{i_2jt}^{z^{(k)}_{i_1jt}}}{\prod_{j=1}^{J} p_{i_1jt}^{z^{(k)}_{i_1jt}} p_{i_2jt}^{z^{(k)}_{i_2jt}}}}, & \text{t=T,}
\end{cases}
$$

where $h(\cdot|\cdot,\cdot)$ is the likelihood contribution of next-period coupons based on current coupons and choices. This function is defined as follows:

$$(26)\ \ h(c_{ijt+1}|\ c_{ijt}, z_{ijt}) = \left(r_{ijt+1}(c_{ijt}, z_{ijt})\right)^{c_{ijt+1}} \left(1 - r_{ijt+1}(c_{ijt}, z_{ijt})\right)^{1-c_{ijt+1}},\ \ t = 1, .., T-1.$$

Finally, with the complement of the probability defined in equation (26), the pair of choices remain at their current values by assigning: $(z_{i_1}^{(k+1)}, z_{i_2}^{(k+1)}) = (z_{i_1}^{(k)}, z_{i_2}^{(k)})$.

b) Sampling Coupons (c_{it}):

In every iteration k, for each period t and for every consumer i, successively draw $c_{it}^{(k+1)}$ as follows:

1. Let b_i the brand chosen by consumer i in period t (i.e., $z_{ib_it} = 1$).

2. Let c_{it}^* be such that:

   (a) If $\omega_i = 1$, $c_{ib_it}^* = c_{ib_it}^{(k)}$ (this condition is required in order to satisfy condition (14)), and define the set $\mathcal{Q}_i = \{1, ..., J\}\backslash\{b_i\}$, otherwise, if $\omega_i = 0$ define $\mathcal{Q}_i = \{1, ..., J\}$.

   (b) If $\delta_{b't} = 1$, generate $c_{ib't}^*$ from a Bernoulli distribution with probability 0.5, for all $b \in \mathcal{Q}_i$; otherwise, set $c_{ib't}^* = 0$.

3. Accept c_{it}^*, according to the following MH probability:

$$P\left(c_{it}^{(k+1)} = c_{it}^*\right) = \begin{cases} \dfrac{\prod_{j=1}^{J} p_{ijt}(c_{it}^*)^{z_{ijt}}\ h(c_{ijt+1}|c_{ijt}^*, z_{ijt})\ r_{ij1}^{c_{ijt}^*}(1-r_{ij1})^{(1-c_{ijt}^*)}}{\prod_{j=1}^{J} p_{ijt}(c_{it}^{(k)})^{z_{ijt}}\ h(c_{ijt+1}|c_{ijt}^{(k)}, z_{ijt})\ r_{ij1}^{c_{ijt}^{(k)}}(1-r_{ij1})^{(1-c_{ijt}^{(k)})}}, & t = 1; \\[2em] \dfrac{\prod_{j=1}^{J} p_{ijt}(c_{it}^*)^{z_{ijt}}\ h(c_{ijt+1}|c_{ijt}^*, z_{ijt})\ h(c_{ijt}^*|c_{ijt-1}, z_{ijt-1})}{\prod_{j=1}^{J} p_{ijt}(c_{it}^{(k)})^{z_{ijt}}\ h(c_{ijt+1}|c_{ijt}^{(k)}, z_{ijt})\ h(c_{ijt}^{(k)}|c_{ijt-1}, z_{ijt-1})}, & 2 \le t \le T-1; \\[2em] \dfrac{\prod_{j=1}^{J} p_{ijt}(c_{it}^*)^{z_{ijt}}\ h(c_{ijt}^*|c_{ijt-1}, z_{ijt-1})}{\prod_{j=1}^{J} p_{ijt}(c_{it}^{(k)})^{z_{ijt}}\ h(c_{ijt}^{(k)}|c_{ijt-1}, z_{ijt-1})}, & t = T, \end{cases}$$

otherwise, assign $c_{it}^{(k+1)} = c_{it}^{(k)}$.

Also note that the acceptance rate of candidate vectors of coupons can be increased if some elements of $\mathcal{Q}_i$ are removed at random in each iteration (i.e., if the value of c_{ijt} is left constant for some components in a given iteration). For example, for the data set used in the empirical application, if four elements are removed at random from $\mathcal{Q}_i$, then the acceptance rate exhibits values closer to 25% instead of values below 10%.

c) Sampling θ_i:

Define $c^r_{ijt} = c_{ijt} z_{ijt} \mathrm{I}_{\{\psi_i > 0\}}$. In every iteration k, for each consumer i, successively draw θ_i using a Metropolis-Hastings step specified as follows:

1. Let $l_i = 0$, if $\max_{j,t} c^r_{ijt} = 1$; otherwise, set $l_i = -\infty$.

2. Let $u_i = 0$, if $\max_{j,t} c^r_{ijt} = 0$ and $\max_{j,t} z_{ijt} c_{ijt} = 1$; otherwise, set $u_i = +\infty$.

3. Generate a candidate value ψ_i^* from a normal distribution with mean equal to $\psi_i^{(k)}$ (the value of ψ_i in the current iteration), variance equal to $a_\psi \cdot D_\psi^{(k)}$ and truncated in the interval (l_i, u_i). We note that D_ψ is the diagonal element of D that corresponds to the coupon redemption utility coefficient (ψ_i), while a_ψ is a scalar tuning parameter for the jumping kernel used in the MH step.

4. Generate a candidate value ϕ_i^* from a normal distribution centered on the current value (ϕ_i^k) and with variance matrix equal to $a_\phi D_\phi^{(k)}$. We note that $D_\phi^{(k)}$ is the current value of the variance-covariance matrix of ϕ_i and a_ϕ is a scalar tuning parameter for this jumping kernel. In our simulation experiment we used $a_\phi = 0.28$ and $a_\psi = 6 \cdot 0.28$ in order to obtain a MH acceptance rate for θ_i between 20% and 30%.

5. Define $\theta_i^* = (\phi_i^{*\prime}, \psi_i^*)'$.

6. Accept θ_i^* with the following MH probability:

$$\alpha_{MH,\theta_i} = \min \left\{ \frac{\frac{\phi(\theta_i^*;\bar{\theta},D) \prod_{j=1}^{J} \prod_{t=1}^{T} p_{ijt}(\theta_i^*)^{z_{ijt}}}{\phi(\psi_i^*;\psi^{(k)},a_\psi D_\psi,l_i,u_i)}}{\frac{\phi(\theta_i^{(k)};\bar{\theta},D) \prod_{j=1}^{J} \prod_{t=1}^{T} p_{ijt}(\theta_i^{(k)})^{z_{ijt}}}{\phi(\psi^{(k)};\psi_i^*,a_\psi D_\psi,l_i,u_i)}}, 1 \right\},$$

otherwise assign $\theta_i^{(k+1)} = \theta_i^{(k)}$, where $\phi(\,\cdot\,; \psi_i, a_\psi D_\psi, l_i, u_i)$ denotes the density of a normal distribution with mean ψ_i and variance $a_\psi D_\psi$ truncated between (l_i, u_i); and $p_{ijt}(\theta_i)$ denotes the probability that consumer i chooses brand j in period t when her vector of preference coefficients is equal to θ_i.

d) Sampling $\tilde{\xi}_t$ and ξ_t:

1. Let $\overline{\xi}_t = \Sigma^d_{\xi,\eta}(\Sigma^d_{\eta,\eta})^{-1}\eta_t$ and let $\Sigma^d_{\xi|\eta} = \Sigma^d_{\xi,\xi} - \Sigma^d_{\xi,\eta}(\Sigma^d_{\eta,\eta})^{-1}\Sigma^d_{\eta,\xi}$, where $\eta_t = \text{price}_{jt} - w'_{jt}\upsilon_j$.
2. Generate $\tilde{\xi}^*_t$ from a normal distribution with mean $\tilde{\xi}^{(k)}_t$ and variance matrix $a_\xi \Sigma^d_{\xi|\eta}$, where a_ξ is a unidimensional MH tuning parameter (in our numerical experiment we used $a_\xi = 0.55$).
3. Let $\xi^*_{j1} = \tilde{\xi}^*_{j1}/\sqrt{1-\gamma^2_{dj}}$ and $\xi^*_{jt} = \gamma_{dj}\xi^*_{jt-1} + \tilde{\xi}^*_{jt}$.
4. Accept ξ^*_t and $\tilde{\xi}^*_t$ with the following MH probability:

$$\alpha_{MH,\xi_t} = \min\left\{ \frac{\phi(\tilde{\xi}^*_t; \overline{\xi}_t, \Sigma^d_{\xi|\eta}) \prod_{i=1}^{N}\prod_{j=1}^{J} p_{ijt}(\xi^*_t)^{z_{ijt}}}{\phi(\tilde{\xi}^{(k)}_t; \overline{\xi}_t, \Sigma^d_{\xi|\eta}) \prod_{i=1}^{N}\prod_{j=1}^{J} p_{ijt}(\xi^{(k)}_t)^{z_{ijt}}}, 1 \right\},$$

otherwise assign $\xi^{(k+1)}_t = \xi^{(k)}_t$ and $\tilde{\xi}^{(k+1)}_t = \tilde{\xi}^{(k)}_t$, where $p_{ijt}(\xi_t)$ denotes the probability that consumer i chooses brand j in period t when the vector of common demand shocks is equal to ξ_t.

e) Sampling ν_t:

1. Generate ν^*_t from a normal distribution centered on $\nu^{(k)}_t$ and with variance matrix $a_\nu \Sigma^c$, where Σ^c is the current value of the covariance matrix of ν_t (we used $a_\nu = 0.4$ in the numerical experiment).
2. Accept ν^*_t with the following MH probability:

$$\alpha_{MH,\nu_t} = \min\left\{ \frac{\phi(\nu^*_t; 0, \Sigma^c) \prod_{i=1}^{N}\prod_{j=1}^{J} r_{ijt}(\nu^*_t)^{c_{ijt}}(1 - r_{ijt}(\nu^*_t))^{1-c_{ijt}}}{\phi(\nu^{(k)}_t; 0, \Sigma^c) \prod_{i=1}^{N}\prod_{j=1}^{J} r_{ijt}(\nu^{(k)}_t)^{c_{ijt}}(1 - r_{ijt}(\nu^{(k)}_t))^{1-c_{ijt}}}, 1 \right\},$$

otherwise assign $\nu_t^{(k+1)} = \nu_t^{(k)}$, where $r_{jt}(\cdot)$ denotes the probability that consumer i has a coupon available for brand j in period t as a function of the value of ν_t.

f) Sampling δ_t:

1. If in iteration k there is a positive number of coupons assigned to brand j in period t (i.e., if $\sum_{i=1}^{N} c_{ijt} > 0$), then set $\delta_{jt}^{(k+1)} = 1$ (see condition (11)).

2. If no coupons are assigned, set $\delta_{jt}^{(k+1)} = 1$ with the following probability:

$$P(\delta_{jt}^{(k+1)} = 1 \mid *) = \frac{\left(\frac{1}{1+e^{\alpha_j+\nu_{jt}+\rho m_{jt}}}\right)^N q_j}{\left(\frac{1}{1+e^{\alpha_j+\nu_{jt}+\rho m_{jt}}}\right)^N q_j + (1-q_j)},$$

   otherwise, set $\delta_{jt}^{(k+1)} = 0$.

g) Sampling $\overline{\theta}$:

1. Define $A_{\overline{\theta}}$ and $B_{\overline{\theta}}$ as follows:

$$A_{\overline{\theta}} = B_{\overline{\theta}}(D^{-1}\sum_{i=1}^{N} \theta_i)$$

$$B_{\overline{\theta}} = \left(V_0^{-1} + ND^{-1}\right)^{-1}$$

   where $V_{0,\overline{\theta}}^{-1}$ is the inverse of the prior variance for $\overline{\theta}$ ($V_{0,\overline{\theta}} = 10^5 \mathrm{I}_5$ in the simulation experiment).

2. Generate $\overline{\theta}$ from a normal distribution with mean $A_{\overline{\theta}}$ and variance $B_{\overline{\theta}}$.

h) Sampling D:

1. Let K be equal to the number of columns of D.

2. Generate D from an $\mathrm{IW}(K+2+N, (K+2)I_K + \sum_{i=1}^{N}(\theta_i - \overline{\theta})(\theta_i - \overline{\theta})')$.

i) Sampling $\alpha_1, ..., \alpha_J$ and ρ:

1. Let $\widetilde{\alpha} = (\alpha_1, ..., \alpha_J, \rho)'$.

2. Let $X_{c,t}$ be a matrix such that $X_{c,t} = [\mathrm{I}_J \; m_t]$, where I_J denotes an identity matrix with J rows and columns, while m_t is a matrix that contains the values in period t of marketing variables assumed to be coordinated with coupon promotion.

3. Let $E_t = x_{c,t}\widetilde{\alpha} + \nu_t$.

4. Let $A_{\widetilde{\alpha}} = B_{\widetilde{\alpha}} \left(\sum_{t=1}^{T} X'_{c,t}(\Sigma^c)^{-1} E_t \right)$ and $B_{\widetilde{\alpha}} = \left(V_{0,\widetilde{\alpha}}^{-1} + \sum_{t=1}^{T} X'_{c,t}(\Sigma^c)^{-1} X_{c,t} \right)^{-1}$, where $V_{0,\widetilde{\alpha}}$ is the prior variance of $\widetilde{\alpha}$ ($V_{0,\widetilde{\alpha}} = 10^4 \mathrm{I}_4$ in the simulation experiment).

5. Generate $\widetilde{\alpha}^{(k+1)}$ from a normal distribution with mean $A_{\widetilde{\alpha}}$ and variance $B_{\widetilde{\alpha}}$ and set $\nu_t = E_t - x_{c,t}\widetilde{\alpha}^{(k+1)}$.

j) Sampling α_{J+1}:

1. Generate α^*_{J+1} from a normal distribution with mean $\alpha^{(k)}_{J+1}$ and variance $\sigma^2_{\alpha_{J+1}}$.

2. Accept α^*_{J+1} with the following MH probability:

$$\alpha_{MH,\alpha_{J+1}} = \min \left\{ \frac{\phi(\alpha^*_{J+1}; 0, V_{0,\alpha_{J+1}}) \prod_{i=1}^{N} \prod_{j=1}^{J} \prod_{t=2}^{T} r_{ijt}(\alpha^*_{J+1})^{c_{ijt}} (1 - r_{ijt}(\alpha^*_{J+1}))^{1-c_{ijt}}}{\phi(\alpha^{(k)}_{J+1}; 0, V_{0,\alpha_{J+1}}) \prod_{i=1}^{N} \prod_{j=1}^{J} \prod_{t=2}^{T} r_{ijt}(\alpha^{(k)}_{J+1})^{c_{ijt}} (1 - r_{ijt}(\alpha^{(k)}_{J+1}))^{1-c_{ijt}}}, 1 \right\},$$

otherwise, let $\alpha^{(k+1)}_{J+1} = \alpha^{(k)}_{J+1}$ (in the numerical experiment, we set $V_{0,\alpha_{J+1}} = 10$).

k) Sampling q:

1. Generate q_j from the following beta distribution: $\mathrm{Beta}\left(1 + \sum_{t=1}^{T} \delta_{jt}, 1 + T - \sum_{t=1}^{T} \delta_{jt}\right)$.

l) Sampling υ:

1. Let $\upsilon = (\upsilon'_1, .., \upsilon'_J)'$.

2. Let $zp_{j1} = \sqrt{1 - \gamma_{pj}}\,(\mathrm{price}_{j1} - \Sigma^d_{\mathrm{price},\xi}(\Sigma^d_{\xi,\xi})^{-1}\xi_1)$ and let $zp_t = \mathrm{price}_t - \gamma_p \mathrm{price}_{t-1} - \Sigma^d_{\mathrm{price},\xi}(\Sigma^d_{\xi,\xi})^{-1}\xi_t$ for $t \geq 2$.

3. Let $\Sigma^d_{\text{price,price}|\xi} = \Sigma^d_{\text{price,price}} - \Sigma^d_{\text{price},\xi}(\Sigma^d_{\xi,\xi})^{-1}\Sigma^d_{\xi,\text{price}}$.

4. Let $\tilde{w}_{j1} = \sqrt{1-\gamma_{pj}}\ w_{j1}$ and $\tilde{w}_t = w_{jt} - \gamma_{pj} w_{jt-1}$ for $t \geq 2$.

5. Let W_t a block-diagonal matrix, where the j^{th} block corresponds to $\tilde{w}'_{jt}$, for $j = 1, .., J$.

6. Let $A_\upsilon = B_\upsilon \left(\sum_{t=1}^{T} W'_t (\Sigma^d_{\text{price,price}|\xi})^{-1} zp_t \right)$ and $B_\upsilon = \left(V^{-1}_{0,\upsilon} + \sum_{t=1}^{T} W'_t (\Sigma^d_{\text{price,price}|\xi})^{-1} W_t \right)^{-1}$, where $V_{0,\upsilon}$ is the prior variance of υ ($V_{0,\upsilon} = 100\text{I}_4$ in the simulation experiment).

7. Generate $\upsilon^{(k+1)}$ from a normal distribution with mean A_υ and variance B_υ.

m) Sampling Σ^d:

1. Let $\tilde{\zeta}_t = (\tilde{\eta}'_t, \tilde{\xi}'_t)'$.

2. Generate Σ^d from an $\text{IW}(J+2+T, (J+2)\ 0.01\ \text{I}_{2J} + \sum_{t=1}^{T} \tilde{\zeta}_t \tilde{\zeta}'_t)$.

n) Sampling Σ^c:

1. Generate Σ^c from an $\text{IW}(J+2+T-1, (J+2)\ \text{I}_J + \sum_{t=2}^{T} \nu_t \nu'_t)$.

o) Sampling γ:

1. Let $\zeta_t = (\eta'_t, \xi'_t)'$.

2. Let $\tilde{\zeta}_t = (\tilde{\eta}'_t, \tilde{\xi}'_t)'$.

3. Let D_t a diagonal matrix with diagonal elements equal to the components of ζ_t.

4. Let $A_\gamma = B_\gamma \left(\sum_{t=1}^{T-1} D'_t (\Sigma^d)^{-1} \zeta_{t+1} \right)$ and $B_\upsilon = \left(V^{-1}_{0,\gamma} + \sum_{t=1}^{T} D'_t (\Sigma^d)^{-1} D_t \right)^{-1}$, where $V_{0,\gamma}$ is the prior variance of γ ($V_{0,\gamma}$ equals the identity matrix in the simulation experiment).

5. Generate γ^* from a normal distribution with mean A_γ and variance B_γ.

6. Let $\tilde{\eta}^*_{j1} = \sqrt{1-(\gamma^*_{pj})^2}\ \eta_{j1}$, $\tilde{\xi}^*_{j1} = \sqrt{1-(\gamma^*_{dj})^2}\ \xi_{j1}$. Let $\tilde{\zeta}^*_1 = (\tilde{\eta}^{*\prime}_1, \tilde{\xi}^{*\prime}_1)'$.

7. Accept γ^* according to the following MH probability:

$$\alpha_{MH,\gamma} = \min\left\{ \frac{\phi(\tilde{\zeta}^*_1; 0, \Sigma^d)}{\phi(\tilde{\zeta}^{(k)}_1; 0, \Sigma^d)}, 1 \right\},$$

otherwise, set $\gamma^{(k+1)} = \gamma^{(k)}$.

Technical Appendix 2: Estimation of the marginal likelihood

In what follows we derive an estimator of the marginal likelihood by generalizing the harmonic mean method proposed by Newton and Raftery (1994). This generalization is needed for the aggregate estimation procedures presented in this paper that are based on augmenting the aggregate data (A) with unobserved sequences of choices (Z) and coupons (C).

As before, define $\omega_i = 1$ if $\psi_i > 0$ and, otherwise $\omega_i = 0$. Let ω define a vector with the values of ω_i for all consumers. In addition, Let $\Omega_{\mathcal{M}}$ denote the set of all values of (Z, C, ω) consistent with the aggregate data (A) under model $\mathcal{M}$ and let φ denote the collection of parameters that determine the likelihood of the augmented choices and coupons (i.e., $\varphi = \{\theta, r\}$). We are interested in computing $p(A|\mathcal{M})$, the marginal likelihood of the aggregate data A under model $\mathcal{M}$. For notational convenience, we drop the model subscript ($\mathcal{M}$) and we refer to $p(A|\mathcal{M})$ and $\Omega_{\mathcal{M}}$ simply as $p(A)$ and Ω, respectively. By noting that $\int p(\varphi)d\varphi = 1$, it is straightforward to verify that the marginal likelihood $p(A)$ satisfies the following equation:

$$\frac{1}{p(A)} = \frac{1}{|\Omega|} \sum_{(Z,C,\omega)\in\Omega} \int \frac{p(\varphi)}{p(A)} d\varphi. \tag{27}$$

Using Bayes Law and noting that $p(A|Z, C, \omega, \varphi) = 1$ for any pair $(Z, C, \omega) \in \Omega$, the following identity can be easily derived:

$$\frac{1}{p(A)} = \frac{p(Z, C, \omega, \varphi|A)}{p(Z, C, \omega, \varphi)}, \qquad \forall (Z, C, \omega) \in \Omega. \tag{28}$$

Using this identity in equation (27) we obtain:

$$\frac{1}{p(A)} = \frac{1}{|\Omega|}\sum_{(Z,C,\omega)\in\Omega}\int \frac{p(\varphi)}{p(Z,C,\omega,\varphi)}p(Z,C,\omega,\varphi|A)d\varphi$$
$$= \frac{1}{|\Omega|}\sum_{(Z,C,\omega)\in\Omega}\int \frac{1}{p(Z,C,\omega|\varphi)}p(Z,C,\omega,\varphi|A)d\varphi$$
$$= \frac{1}{|\Omega|}\mathrm{E}\left[\frac{1}{p(Z,C,\omega|\varphi)}\Big|A\right]. \tag{29}$$

Consequently, using equation (29) we can estimate $p(A)$ as follows:

$$\widehat{p}(A) = \frac{|\Omega|}{\frac{1}{m}\sum_{l=1}^{m}\frac{1}{p(Z^{(l)},C^{(l)},\omega^{(l)}|\varphi^{(l)})}}, \tag{30}$$

where each quadruplet $\left(Z^{(l)},C^{(l)},\omega^{(l)},\varphi^{(l)}\right)$ is drawn from the posterior distribution $p(Z,C,\omega,\varphi|A)$. Therefore, this estimator corresponds to the harmonic mean of the likelihood of the augmented choices and coupons amplified by $|\Omega|$, where the values for $\left(Z^{(l)},C^{(l)},\omega^{(l)},\varphi^{(l)}\right)$ can be obtained from the MCMC output.

Finally, we note that if two models $\mathcal{M}_1$ and $\mathcal{M}_2$ share the same set of feasible combinations of choices and coupons (i.e., $\Omega_{\mathcal{M}_1} = \Omega_{\mathcal{M}_2} = \Omega$), then for the purposes of model selection, it is not necessary to compute $|\Omega|$, which is constant for these two models and, thus, it is not needed to compute the corresponding Bayes factors.

References

[1] Newton, Michael A. and Adrian E. Raftery (1994), "Approximating Bayesian Inference with the Weighted Likelihood Bootstrap," *Journal of the Royal Statistical Society (B)*, 56: 3-48.