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Brøndsted-Rockafellar property of subdifferentials of prox-bounded functions

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SUBDIFFERENTIAL OF CONVEX FUNCTIONS

Everywhere X is a Banach space. A set-valued operator $T : X \rightrightarrows X^*$, or graph $T \subset X \times X^*$, is *monotone* provided

$$\langle y^* - x^*, y - x \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in T,$$

and *maximal monotone* provided it is monotone and not properly contained in another monotone operator.

The *subdifferential* $\partial f : X \rightrightarrows X^*$ of a convex $f : X \to \left]-\infty, +\infty\right]$ is

$$\partial f(x) := \left\{ x^* \in X^* : \langle x^*, y - x \rangle + f(x) \leq f(y), \, \forall y \in X \right\},$$

and the *duality operator* $J : X \rightrightarrows X^*$ is

$$J(x) := \left\{ x^* \in X^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2 \right\}.$$

It is easily verified that $J(x) = \partial j(x)$ where $j(x) = (1/2)\|x\|^2$.

> Theorem (Rockafellar, 1970) Let X be a Banach space. The subdifferential ∂f of a proper convex lower semicontinuous function $f : X \to \left]-\infty, +\infty\right]$ is maximal monotone.

PROOF WHEN $X = H$ IS HILBERT (taken from Brezis, 1973)

By Hahn-Banach, $f \geq \ell + \alpha$ for some $\ell \in X^*$ and $\alpha \in \mathbb{R}$, and $j + \ell$ is coercive ($j(x) + \ell(x) = (1/2)\|x\|^2 + \ell(x) \to +\infty$ as $\|x\| \to +\infty$), so

$$f + j \text{ is coercive.}$$

Hence $f + j$ attains its minimum at some $\bar{x} \in H$, so $0 \in \partial(f + j)(\bar{x})$.

Since $\partial j = \nabla j = I$ (identity on H), we readily get $0 \in (\partial f + I)(\bar{x})$, so $0 \in R(\partial f + I)$. We conclude that

$$X^* = R(\partial f + I).$$

This is easily seen to imply that ∂f is maximal monotone.

(This is the elementary part in Minty's characterization of maximal monotonicity (1962).)

PROOF IN THE GENERAL BANACH CASE: *STEP 1*

Claim:

$$0 \in \overline{R(\partial f + J)}. \tag{1}$$

First, $f \geq \ell + \alpha$ for some $\ell \in X^*$ and $\alpha \in \mathbb{R}$, and $j + \ell$ bounded below, so $f + j$ is bounded below.

Next, let $\varepsilon > 0$ arbitrary and let $y_\varepsilon \in \operatorname{dom} f$ such that

$$(f + j)(y_\varepsilon) \leq (f + j)(y) + \varepsilon^2, \ \forall y \in X.$$

By Brøndsted-Rockafellar approximation theorem (1965),
$\exists x^*_\varepsilon \in X^*$ with $\|x^*_\varepsilon\| \leq \varepsilon$ and $z_\varepsilon \in X$ such that $x^*_\varepsilon \in \partial(f + j)(z_\varepsilon)$.
By Rockafellar's sum rule (1966), $x^*_\varepsilon \in \partial f(z_\varepsilon) + J(z_\varepsilon)$.

Conclusion: $\exists x^*_\varepsilon \in R(\partial f + J)$ with $\|x^*_\varepsilon\| \leq \varepsilon$, proving the claim.

PROOF: STEP 2

Let $(x, x^*) \in X \times X^*$ such that

$$\langle y^* - x^*, y - x \rangle \geq 0, \ \forall (y, y^*) \in \partial f. \tag{2}$$

Applying (1) to $f(x + .) - x^*$, we get

$$x^* \in \overline{R(\partial f(x + .) + J)}.$$

Thus, there are $(x_n^*) \subset X^*$ with $x_n^* \to x^*$ and $(h_n) \subset X$ such that $x_n^* \in \partial f(x + h_n) + J(h_n)$. Let $(y_n^*) \subset X^*$ such that

$$y_n^* \in \partial f(x + h_n) \quad \text{and} \quad x_n^* - y_n^* \in J(h_n).$$

By definition of J, we have

$$\langle x_n^* - y_n^*, h_n \rangle = \|x_n^* - y_n^*\|^2 = \|h_n\|^2. \tag{3}$$

From (2) and $y_n^* \in \partial f(x + h_n)$, we get $\langle x^* - y_n^*, x + h_n - x \rangle \leq 0$, so

$$\|h_n\|^2 = \langle x_n^* - x^*, h_n \rangle + \langle x^* - y_n^*, x + h_n - x \rangle \leq \langle x_n^* - x^*, h_n \rangle \leq \|x_n^* - x^*\| \|h_n\|.$$

Hence, $h_n \to 0$, so, by (3), $\|x_n^* - y_n^*\| \to 0$, therefore $y_n^* \to x^*$. Since ∂f has closed graph and $y_n^* \in \partial f(x + h_n)$, we conclude that $x^* \in \partial f(x)$.

OTHER PROOFS OF MAXIMALITY OF ∂f FOR CONVEX f

1/ *f everywhere finite and continuous*:
• Minty (1964), Phelps (1989), using mean value theorem and link between subderivative and subdifferential

2/ *f lsc, X Hilbert*:
• Moreau (1965), via prox functions and duality theory,
• Brezis (1973), showing directly that $\partial f + I$ is onto

3/ *f lsc, X Banach*: all proofs use a variational principle and another tool
• Rockafellar (1970): continuity of $(f+j)^*$ in X^* and link between $(\partial f)^{-1}$ and ∂f^* in $X^{**} \times X^*$,
• Taylor (1973) and Borwein (1982): subderivative mean value inequality and link between subderivative and subdifferential,
• Zagrodny (1988?), Simons (1991), Luc (1993), etc: subdifferential mean value inequality,
• Thibault (1999): limiting convex subdifferential calculus,
• Marques Alves-Svaiter (2008), Simons (2009): conjugate functions and Fenchel duality formula or subdifferential sum rule.

BEYOND THE CONVEX CASE: MAIN TOOLS

Let be given a 'subdifferential' ∂ that associates a subset $\partial f(x) \subset X^*$ to each $x \in X$ and each function f on X so that $\partial f(x)$ coincides with the convex subdifferential when f is convex.

The two main tools in the convex situation were:

- *Brøndsted-Rockafellar's approximation theorem (1965)*
- *Rockafellar's subdifferential sum rule (1966)*.

They will be respectively replaced by:

> Ekeland Variational Principle (1974). For any lsc function f on X, $\bar{x} \in \operatorname{dom} f$ and $\varepsilon > 0$ such that $f(\bar{x}) \leq \inf f(X) + \varepsilon$, and for any $\lambda > 0$, there is $x_\lambda \in X$ s.t. $\|x_\lambda - \bar{x}\| \leq \lambda$, $f(x_\lambda) \leq f(\bar{x})$, and
> $$x \mapsto f(x) + (\varepsilon/\lambda)\|x - x_\lambda\| \text{ has a minimum at } x_\lambda.$$

> Subdifferential Separation Principle. For any lsc functions f, φ on X with φ convex Lipschitz near $\bar{x} \in \operatorname{dom} f \cap \operatorname{dom} \varphi$,
> $$f + \varphi \text{ has a local minimum at } \bar{x} \Longrightarrow 0 \in \partial f(\bar{x}) + \partial\varphi(\bar{x}).$$

SUBDIFFERENTIALS SATISFYING THE SEPARATION PRINCIPLE

The main examples of pairs (X, ∂) for which the Subdifferential Separation Principle holds are:

- the Clarke subdifferential ∂_C in arbitrary Banach spaces,
- the limiting Fréchet subdifferential $\widehat{\partial}_F$ in Asplund spaces,
- the limiting Hadamard subdifferential $\widehat{\partial}_H$ in separable spaces,
- the limiting proximal subdifferential $\widehat{\partial}_P$ in Hilbert spaces.

For more details, see, e.g., Jules-Lassonde (2013, 2013b).

COMBINING THE TOOLS

Set $\operatorname{dom} f^* = \{x^* \in X^* : \inf(f - x^*)(X) > -\infty\}$.

Proposition Let X Banach, $f : X \to]-\infty, +\infty]$ proper lsc, $\varphi : X \to \mathbb{R}$ convex loc. Lispchitz. Then, $\operatorname{dom}(f + \varphi)^* \subset \operatorname{cl}(R(\partial f + \partial\varphi))$.

Proof. Let $x^* \in \operatorname{dom}(f + \varphi)^*$ and let $\varepsilon > 0$. There is $\bar{x} \in X$ s.t.

$$(f + \varphi - x^*)(\bar{x}) \leq \inf(f + \varphi - x^*)(X) + \varepsilon^2,$$

so, by Ekeland's variational principle, there is $x_\varepsilon \in X$ such that $x \mapsto f(x) + \varphi(x) + \langle -x^*, x\rangle + \varepsilon\|x - x_\varepsilon\|$ attains its minimum at x_ε. Now, applying the Separation Principle with the convex locally Lipschitz $\psi : x \mapsto \varphi(x) + \langle -x^*, x\rangle + \varepsilon\|x - x_\varepsilon\|$ we obtain $x^*_\varepsilon \in \partial f(x_\varepsilon)$ such that $-x^*_\varepsilon \in \partial\psi(x_\varepsilon) = \partial\varphi(x_\varepsilon) - x^* + \varepsilon B_{X^*}$. So, there is $y^*_\varepsilon \in \partial\varphi(x_\varepsilon)$ such that $\|x^* - y^*_\varepsilon - x^*_\varepsilon\| \leq \varepsilon$. Thus, for every $\varepsilon > 0$ the ball $B(x^*, \varepsilon)$ contains $x^*_\varepsilon + y^*_\varepsilon \in \partial f(x_\varepsilon) + \partial\varphi(x_\varepsilon) \subset R(\partial f + \partial\varphi)$. This means that $x^* \in \operatorname{cl}(R(\partial f + \partial\varphi))$. ∎

The case $\varphi = 0$ and $f = \delta_C$ with C nonempty closed convex set says that the set $R(\partial\delta_C)$ of functionals in X^* that attain their supremum on C is dense in the set $\operatorname{dom}\delta^*_C$ of all those functionals which are bounded above on C (Bishop-Phelps).

PROX-BOUNDED FUNCTIONS

A function f is called *prox-bounded* if there exists $\lambda > 0$ such that the function $f + \lambda j$ is bounded below; the infimum λ_f of the set of all such λ is called the *threshold* of prox-boundedness for f:

$$\lambda_f := \inf\{\lambda > 0 : \inf(f + \lambda j) > -\infty\}.$$

Any convex lsc function g is prox-bounded with threshold $\lambda_g = 0$, the sum $f + g$ of a prox-bounded f and of a convex lsc g is prox-bounded with $\lambda_{f+g} \leq \lambda_f$, for every $x^* \in X^*$, $\lambda_{f+x^*} = \lambda_f$, and for every $x \in X$, $f(x + .) + \lambda j$ is bounded below for any $\lambda > \lambda_f$ (see Rockafellar-Wets book (1998)).

Consequence: if f is prox-bounded, then for every $\lambda > \lambda_f$,

$$\forall x \in X,\ \operatorname{dom}(f(x + .) + \lambda j)^* = X^*.$$

From this and the previous result we get:

Proposition Let X Banach and let $f : X \to \,]-\infty, +\infty]$ be lsc and prox-bounded with threshold λ_f. Then, for every $\lambda > \lambda_f$,

$$\forall x \in X,\ \operatorname{cl}(R(\partial f(x + .) + \lambda J)) = X^*.$$

GOING FURTHER: MONOTONE ABSORPTION

Given $T : X \rightrightarrows X^*$, or $T \subset X \times X^*$, and $\varepsilon \geq 0$, we let

$$T^\varepsilon := \{ (x, x^*) \in X \times X^* : \langle y^* - x^*, y - x \rangle \geq -\varepsilon, \ \forall (y, y^*) \in T \}$$

be the set of pairs (x, x^*) *ε-monotonically related* to T.

An operator T is monotone provided $T \subset T^0$ and monotone maximal provided $T = T^0$.

A non necessarily monotone operator T is declared to be *monotone absorbing* provided $T^0 \subset \overline{T}$ (norm-closure).

A non necessarily monotone operator T is declared to be *widely monotone absorbing* with threshold $\lambda_T \geq 0$ provided for every $\lambda > \lambda_T$ one has

$$\forall \varepsilon \geq 0, \ T^\varepsilon \subset \overline{\left(T + \sqrt{\lambda^{-1}\varepsilon}\, B_X \times \sqrt{\lambda\varepsilon}\, B_{X^*} \right)}.$$

Equivalently: $\forall \varepsilon \geq 0$, $(x, x^*) \in T^\varepsilon \Rightarrow$
$\exists (x_n, x_n^*) \subset T : \lim_n \|x - x_n\| \leq \sqrt{\lambda^{-1}\varepsilon}$ and $\lim_n \|x^* - x_n^*\| \leq \sqrt{\lambda\varepsilon}$.

SUFFICIENT CONDITION FOR WIDE MONOTONE ABSORPTION

Proposition Let $T : X \rightrightarrows X^*$ and $\lambda > 0$. Assume that

$$\forall x \in X,\ \mathrm{cl}\,(R(T(x+.)+\lambda J) = X^*. \tag{4}$$

Then:

$$\forall \varepsilon \geq 0, \quad T^\varepsilon \subset \mathrm{cl}\left(T + \sqrt{\lambda^{-1}\varepsilon} B_X \times \sqrt{\lambda\varepsilon} B_{X^*}\right). \tag{5}$$

Proof. Let $\varepsilon \geq 0$ and let $(x, x^*) \in T^\varepsilon$. Since $T(x+.)+\lambda J$ has a dense range, we can find $(x_n^*) \subset X^*$ with $x_n^* \to x^*$ and $(y_n) \subset X$ such that $x_n^* \in T(x+y_n)+\lambda J y_n$. Let $(y_n^*) \subset X^*$ such that

$$y_n^* \in T(x+y_n) \quad \text{and} \quad x_n^* - y_n^* \in \lambda J y_n.$$

By definition of J, we have

$$\lambda^{-1}\langle x_n^* - y_n^*, y_n\rangle = \|\lambda^{-1}(x_n^* - y_n^*)\|^2 = \|y_n\|^2. \tag{6}$$

But $x^* \in T^\varepsilon x$ and $y_n^* \in T(x+y_n)$, so $\langle x^* - y_n^*, y_n\rangle \leq \varepsilon$, hence

$$\lambda\|y_n\|^2 = \langle x_n^* - x^*, y_n\rangle + \langle x^* - y_n^*, y_n\rangle \leq \langle x_n^* - x^*, y_n\rangle + \varepsilon \leq \|x_n^* - x^*\|\|y_n\| + \varepsilon.$$

Therefore, $\lambda\|y_n\|^2 - \|x_n^* - x^*\|\|y_n\| - \varepsilon \le 0$, so we must have

$$\|y_n\| \le (\|x_n^* - x^*\| + \sqrt{\|x_n^* - x^*\|^2 + 4\varepsilon\lambda})/2\lambda. \tag{7}$$

From (7) we derive that $\limsup_n \|y_n\| \le \sqrt{\lambda^{-1}\varepsilon}$, so, by (6),

$$\limsup_n \|x_n^* - y_n^*\| = \limsup_n \lambda\|y_n\| \le \sqrt{\lambda\varepsilon}.$$

In conclusion we have $(x + y_n, y_n^*) \in T$ with

$$\limsup_n \|x - (x + y_n)\| \le \sqrt{\lambda^{-1}\varepsilon}, \quad \limsup_n \|x^* - y_n^*\| \le \sqrt{\lambda\varepsilon},$$

and without loss of generality we can replace $\limsup_n$ by $\lim_n$. ∎

Open problem: We don't know whether the converse (5) $\Rightarrow$ (4) is true.

WIDE MONOTONE ABSORPTION PROPERTY OF SUBDIFFERENTIALS OF PROX-BOUNDED FUNCTIONS

Combining the last two propositions gives:

Theorem Let X Banach and $f : X \to]-\infty, +\infty]$ lsc, prox-bounded with threshold $\lambda_f \geq 0$. Then:

$$\forall \lambda > \lambda_f, \forall \varepsilon \geq 0, \quad (\partial f)^\varepsilon \subset \mathrm{cl}\left(\partial f + \sqrt{\lambda^{-1}\varepsilon} B_X \times \sqrt{\lambda\varepsilon} B_{X^*}\right).$$

Equivalently: for all $\lambda > \lambda_f$ and $\varepsilon \geq 0$, $(x^*, x) \in (\partial f)^\varepsilon \Rightarrow$

$$\exists ((x_n^*, x_n))_n \subset \partial f : \lim_n \|x - x_n\| \leq \sqrt{\lambda^{-1}\varepsilon} \ \& \ \lim_n \|x^* - x_n^*\| \leq \sqrt{\lambda\varepsilon}.$$

In case $\lambda_f = 0$ (in particular for a convex f), the wide monotone absorption property is equivalent to the so-called *maximal monotonicity of Brøndsted-Rockafellar type* studied in Simons (1999, 2008) and others, hence the above theorem extends known results for convex functions to the class of prox-bounded non necessarily convex functions, with a more direct proof.

REFERENCES

J. M. Borwein, A note on ε-subgradients and maximal monotonicity, *Pacific J. Math.* **103** (1982), 307–314

H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Co., Amsterdam, 1973; North-Holland Mathematics Studies, No. 5. Notas de Matemática (50)

A. Brøndsted and R. T. Rockafellar, On the subdifferentiability of convex functions, *Proc. Amer. Math. Soc.* **16** (1965), 605–611

I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47** (1974), 324–353

F. Jules and M. Lassonde, Subdifferential estimate of the directional derivative, optimality criterion and separation principles, *Optimization* **62** (2013), 1267–1288

F. Jules and M. Lassonde, Subdifferential test for optimality, *J. Global Optim.*, in press, doi:10.1007/s10898-013-0078-6

M. Lassonde, Brøndsted-Rockafellar property of subdifferentials of prox-bounded functions, arXiv:1306.5466

D. T. Luc, On the maximal monotonicity of subdifferentials, *Acta Math. Vietnam.* **18** (1993), 99–106

M. Marques Alves and B. F. Svaiter, A new proof for maximal monotonicity of subdifferential operators, *J. Convex Anal.* **15** (2008), 345–348

G. J. Minty, Monotone (nonlinear) operators in Hilbert space, *Duke Math. J.* **29** (1962), 341–346

G. J. Minty, On the monotonicity of the gradient of a convex function, *Pacific J. Math.* **14** (1964), 243–247

J.-J. Moreau, Proximité et dualité dans un espace hilbertien, *Bull. Soc. Math. France* **935** (1965), 273–299

R. R. Phelps, *Convex functions, monotone operators and differentiability*, volume 1364 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, second edition, 1993.

R. T. Rockafellar, Extension of Fenchel's duality theorem for convex functions, *Duke Math. J.* **33** (1966), 81–89

R. T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific J. Math.* **33** (1970), 209–216

R. T. Rockafellar and R. J.-B. Wets, *Variational analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin, 1998, xiv+733 pp.

S. Simons, The least slope of a convex function and the maximal monotonicity of its subdifferential, *J. Optim. Theory Appl.* **71** (1991), 127–136

S. Simons, Maximal monotone multifunctions of Brøndsted-Rockafellar type, *Set-Valued Anal.* **7** (1999), 255–294

S. Simons, *From Hahn-Banach to monotonicity*, volume 1693 of *Lecture Notes in Mathematics*, Springer, New York, second edition, 2008

S. Simons, A new proof of the maximal monotonicity of subdifferentials, *J. Convex Anal.* **16** (2009), 165–168

P. D. Taylor, Subgradients of a convex function obtained from a directional derivative, *Pacific J. Math.* **44** (1973), 739–747

L. Thibault, Limiting convex subdifferential calculus with applications to integration and maximal monotonicity of subdifferential, *Constructive, experimental, and nonlinear analysis (Limoges, 1999)*, CMS Conf. Proc. **27**, pp. 279–289, Amer. Math. Soc., Providence, RI, 2000