Forward–partial inverse–forward method for solving monotone inclusions: application to land use planning [1]

L. M. Briceño-Arias

Universidad Técnica Federico Santa María

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Problem

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$$\text{Find } \; x \in \mathcal{H} \quad \text{such that} \quad 0 \in Ax + Bx + N_V x\,.$$

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- $\mathcal{H}$ is a real Hilbert space,
- $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone, i.e., it is **monotone:**

$$(\forall u \in Ax)(\forall v \in Ay) \quad \langle u - v \mid x - y \rangle \geq 0$$

  and its graph is **maximal** among graphs of monotone op.
- $B\colon \mathcal{H} \to \mathcal{H}$ is monotone and χ–lipschitzian.
- V is a closed vectorial subspace of $\mathcal{H}$ ($N_V = V^{\perp}$ is the normal cone to V).

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- $B\colon \mathcal{H} \to \mathcal{H}$ is monotone and χ–lipschitzian.
- V is a closed vectorial subspace of $\mathcal{H}$ ($N_V = V^{\perp}$ is the normal cone to V).
- We suppose that the set of solutions to (P) is $Z \neq \varnothing$.

Examples

- If $A = \partial f$ and $B = \nabla g$, where f and g are convex functions, the problem (P) reduces to (+ qualification conditions)

$$\underset{x \in V}{\text{minimize}} \; f(x) + g(x).$$

Examples

- If $A = \partial f$ and $B = \nabla g$, where f and g are convex functions, the problem (P) reduces to (+ qualification conditions)

$$\underset{x \in V}{\text{minimize}} \;\; f(x) + g(x).$$

- If $\mathcal{H} = \mathsf{H}^n$, where H is a real Hilbert space, $A = \mathsf{A}_1 \times \cdots \times \mathsf{A}_n$, where $\mathsf{A}_i \colon \mathsf{H} \to 2^{\mathsf{H}}$ is maximally monotone, $B \colon (\mathsf{x}_i)_{1 \le i \le n} \mapsto (\mathsf{B}\mathsf{x}_i)_{1 \le i \le n}$, where B is single-valued, monotone, and lipschitzian, and $V = \{x \in \mathcal{H} \mid \mathsf{x}_1 = \cdots = \mathsf{x}_n\}$, the problem (P) becomes

$$\text{find} \;\; x \in \mathcal{H} \quad \text{such that} \quad 0 \in \sum_{i=1}^{n} \mathsf{A}_i \mathsf{x} + \mathsf{B}\mathsf{x}.$$

Particular case 1

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Partial inverse of A with respect to V

$$A_V : \mathcal{H} \to 2^{\mathcal{H}}$$
$$v \in A_V u \;\Leftrightarrow\; P_V v + P_{V^\perp} u \in A(P_V u + P_{V^\perp} v)$$
$$A_{\mathcal{H}} = A \text{ and } A_{\{0\}} = A^{-1}$$

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Problem of partial inverse

$$\text{find } x \in V \quad \text{and} \quad y \in V^{\perp} \quad \text{such that} \quad 0 \in A_V(x+y).$$

Particular case 1

Method: Proximal point algorithm (Martinet (1970), Rockafellar (1976)) $x_{n+1} = J_{\gamma_n A_V} x_n = (\mathrm{Id} + \gamma_n A_V)^{-1} x_n$, $\gamma_n > 0$

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Partial inverse method (Spingarn, 1983)

Let $x_0 \in V$ and $y_0 \in V^\perp$. For every $n \in \mathbb{N}$,

Step 1. Find $(p_n, q_n) \in \mathcal{H}^2$ such that $x_n + y_n = p_n + q_n$

$$\text{and } \frac{P_V q_n}{\gamma_n} + P_{V^\perp} q_n \in A\left(P_V p_n + \frac{P_{V^\perp} p_n}{\gamma_n}\right).$$

Step 2. Set $x_{n+1} = P_V p_n$ and $y_{n+1} = P_{V^\perp} q_n$. Back to step 1.

We have $x_n \rightharpoonup x$ solution to partial inverse problem.

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If $\gamma_n \equiv 1$,
$$\begin{cases} x_{n+1} = P_V J_A(x_n + y_n) \\ y_{n+1} = P_{V^\perp}(x_n + y_n - J_A(x_n + y_n)). \end{cases}$$

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Method: Forward-backward-forward (Tseng (2000)):

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \gamma_n \in \left]0, \chi^{-1}\right[\\ y_n = z_n - \gamma_n B z_n \\ p_n = J_{\gamma_n A} y_n \\ q_n = p_n - \gamma_n B p_n \\ z_{n+1} = z_n - y_n + q_n. \end{array} \right.$$

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We obtain that $(x_n)_{n \in \mathbb{N}}$ converges weakly to a solution to $(P2)$.

General case

Existing methods do not exploit the whole structure of the problem:

- Combettes (2009) and B-A & Combettes (2011) propose an algorithm that converges weakly to a solution to (P). However, it is necessary to compute $(\mathrm{Id} + B)^{-1}$.
- Combettes and Pesquet (2012) exploit the single-valued property of B. However, the algorithm does not take into advantage the normal cone structure and the use of product space techniques generates several additional auxiliary variables to be updated at each iteration.

Objectives

- To propose a new convergent method for solving problem (P) that take into advantage of all the structure of the problem.
- To generalize the previous methods: partial inverse and forward-backward-forward.
- Applications: monotone inclusions involving partial sums and land use planning and Generalized Nash equilibrium problems.

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Equivalence

x is a solution to Problem (P) if and only if

$$x \in V \quad \text{and} \quad (\exists y \in V^{\perp}) \quad \text{such that}$$

$$0 \in (\lambda A)_V \underbrace{(x + \lambda(y - P_{V^\perp} Bx))}_{z} + \lambda P_V B P_V \underbrace{(x + \lambda(y - P_{V^\perp} Bx))}_{z},$$

where $\lambda \in \,]0, +\infty[$.

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where $\lambda \in \,]0, +\infty[$.

Note that:

1. $(\lambda A)_V$ is maximally monotone.
2. $\lambda P_V B P_V$ is $\lambda\chi$–lipschitzien and monotone.
3. $Z = P_V((\lambda A)_V + \lambda P_V B P_V)^{-1}(0)$.

Main result

Let $\lambda \in \left]0, +\infty\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be une sequence in $\left]0, (\lambda\chi)^{-1}\right[$, let $x_0 \in V$ and let $y_0 \in V^{\perp}$. For every $n \in \mathbb{N}$,

Step 1. Find (p_n, q_n) such that $x_n - \gamma_n \lambda P_V B x_n + \lambda y_n = p_n + \lambda q_n$

$$\text{and } \frac{P_V q_n}{\gamma_n} + P_{V^\perp} q_n \in A\left(P_V p_n + \frac{P_{V^\perp} p_n}{\gamma_n}\right).$$

Step 2. Let $x_{n+1} = P_V p_n + \gamma_n \lambda P_V (B x_n - B P_V p_n)$

and $y_{n+1} = P_{V^\perp} q_n$. Back to Step 1.

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Then, the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are in V and $V^{\perp}$, respectively, and for a solution $\overline{x} \in Z$ and $\overline{y} \in V^{\perp} \cap (A\overline{x} + P_V B\overline{x})$ we have:

1. $x_n \rightharpoonup \overline{x}$ and $y_n \rightharpoonup \overline{y}$.
2. $x_{n+1} - x_n \to 0$ and $y_{n+1} - y_n \to 0$.
3. $P_V B x_n \to P_V B \overline{x}$.

Particular cases

- If $\lambda = 1$ and $B \equiv 0$, the proposed method becomes the partial inverse algorithm (Spingarn, 1983) that solves

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- Step 1 is not always easy to compute. If we set $\gamma_n \equiv 1$ we obtain

Let $\lambda \in]0, \beta[$, $x_0 \in V$, $y_0 \in V^{\perp}$.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} s_n = x_n - \lambda P_V B x_n + \lambda y_n \\ p_n = J_{\lambda A} s_n \\ r_n = p_n - \lambda P_V B P_V p_n \\ q_n = (s_n - p_n)/\lambda \\ x_{n+1} = P_V(x_n - s_n + r_n) \\ y_{n+1} = P_{V^{\perp}} q_n. \end{array} \right.$$

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- If $V = \mathcal{H}$ we obtain the forward–backward–forward splitting (Tseng, 2000) with a constant step size.

Composite monotone inclusions involving partial sums

Definition

Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone operators and let V be a closed vectorial subspace of $\mathcal{H}$. We define the partial sum of A and B with respect to V by

$$A \,\square_V B = (A_V + B_V)_V.$$

Note that

1. $A \,\square_{\mathcal{H}} B = A + B$
2. $A \,\square_{\{0\}} B = A \,\square\, B = (A^{-1} + B^{-1})^{-1}$ (parallel sum).

Monotone inclusion involving partial sums

Problem

Find $x \in H$ such that

$$z \in \mathcal{A}x + N_{\mathcal{U}}x + \sum_{i=1}^{m} \Big(\mathcal{L}_i^*(\mathcal{B}_i \,\square_{\mathcal{V}_i^\perp} \mathcal{D}_i)(\mathcal{L}_i x - r_i) + \mathcal{L}_i^* N_{\mathcal{V}_i}(\mathcal{L}_i x - r_i)\Big) + \mathcal{C}x$$

For every $i \in \{1, \ldots, m\}$:

1. $\mathcal{U}$ and $\mathcal{V}_i$ are closed vectorial subspaces of real Hilbert spaces H and G_i, respectively.
2. $\mathcal{A}: H \to 2^H$ and $\mathcal{B}_i : G_i \to 2^{G_i}$ are maximally monotone.
3. $\mathcal{D}_i : G_i \to 2^{G_i}$ is monotone and $(\mathcal{D}_i)_{\mathcal{V}_i^\perp}$ is ν_i-lipschitzian.
4. $\mathcal{C}: H \to H$ is monotone and μ-lipschitzian.
5. $\mathcal{L}_i : H \to G_i$ is linear and bounded.
6. $z \in H$ and $r_i \in G_i$.

Particular case

If $\mathcal{U} = H$ and, for every $i \in \{1, \ldots, m\}$, $\mathcal{V}_i = G_i$ the problem reduces to

Find $x \in H$ such that

$$z \in \mathcal{A}x + \sum_{i=1}^{m} \left(\mathcal{L}_i^* (\mathcal{B}_i \,\square\, \mathcal{D}_i)(\mathcal{L}_i x - r_i) + \mathcal{C}x, \right.$$

which is solved in Combettes-Pesquet (2012).

Equivalent formulation

KKT conditions yields

Primal-Dual inclusion

$$\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} \in \underbrace{\begin{bmatrix}\mathcal{A} & 0 & \dots & 0\\ 0 & (\mathcal{B}_1)_{\mathcal{V}_1^\perp} & \dots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \dots & (\mathcal{B}_m)_{\mathcal{V}_m^\perp}\end{bmatrix}}_{A}\begin{bmatrix}x\\v_1\\\vdots\\v_m\end{bmatrix} +$$

$$\underbrace{\begin{bmatrix}\mathcal{C} & \mathcal{L}_1^* & \dots & \mathcal{L}_m^*\\ -\mathcal{L}_1 & (\mathcal{D}_1)_{\mathcal{V}_1^\perp} & \dots & 0\\ \vdots & & \ddots & \vdots\\ -\mathcal{L}_m & 0 & \dots & (\mathcal{D}_m)_{\mathcal{V}_m^\perp}\end{bmatrix}}_{B}\begin{bmatrix}x\\v_1\\\vdots\\v_m\end{bmatrix} + \underbrace{\begin{bmatrix}N_{\mathcal{U}} & 0 & \dots & 0\\ 0 & N_{\mathcal{V}_1} & \dots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \dots & N_{\mathcal{V}_m}\end{bmatrix}}_{N_W}\begin{bmatrix}x\\v_1\\\vdots\\v_m\end{bmatrix}$$

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$$\text{Find } z \in \mathcal{H} \quad \text{s.t.} \quad 0 \in Az + Bz + N_W z,$$

where

- $\mathcal{H} = H \times G_1 \times \cdots \times G_m$.
- $A \colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone.
- $B \colon \mathcal{H} \to \mathcal{H}$ is monotone and lipschitzian.
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Our algorithm applied in this case gives a splitting convergent algorithm.

Land use planning: Notation

- C: finite set of types of households (or agents). For every $h \in C$, $H_h > 0$ is the number of agents of type h.
- N: finite set of zones. For every $i \in N$, $S_i > 0$ is the number of available houses in the zone i.
- x_{hi}: % of agents of type h which will be localized in the zone i (variable to obtain).
- Constraints: for every $i \in N$ and $h \in C$, $\sum_{h \in C} H_h x_{hi} \leq S_i$ and $\sum_{i \in N} x_{hi} = 1$.
- z_{hi}: perceived utility of agents type h with respect to the zone i. It depends on the localization of the agents of all types (externality):

$$z_{hi} = \gamma_{hi} + \sum_{g \in C} \alpha_{hg} x_{gi}.$$

Land use planning: Notation

$|C|$**-players game:** we denote $x^h = (x_{hi})_{i\in N}$ and x^{-h} as usual.

- Strategy set: $\Delta = \{x \in [0,+\infty[^{|N|} \mid \sum_{i\in N} x_i = 1\}$
- Payoff agent type h:

$$\begin{aligned} F_h(x^h, x^{-h}) &= \sum_{i\in N}(x_{hi}z_{hi} - \mu x_{hi}(\ln(x_{hi}) - 1)) \\ &= \sum_{i\in N}(x_{hi}\gamma_{hi} - \mu x_{hi}(\ln(x_{hi}) - 1) + \alpha_{hh}x_{hi}^2) \\ &\quad + \sum_{i\in N}\sum_{g\neq h}\alpha_{hg}x_{hi}x_{gi} \end{aligned}$$

- Shared constraintes: For every $i \in N$,
  $X_i = \{x = (x^h)_{h\in C} \in [0,+\infty[^{|C|+|N|} \mid \sum_{h\in C} H_h x_{hi} \leq S_i\}$.
  Denote by $X = \cap_{i\in N} X_i$.

Land use planning: Model

Generalized Nash equilibrium (GNE)

Find $\quad x = (x^h)_{h \in C} \in \Delta^{|N|} \cap X \quad$ such that

$$(\forall h \in C)(\forall y^h \in \Delta \cap X_{x^{-h}}) \quad F_h(x^h, x^{-h}) \geq F_h(y^h, x^{-h}).$$

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Particular case: If, for every g, h, $\alpha_{hg} = 0$, our game becomes the potential game:

$$\underset{x \in X \cap \Delta^{|N|}}{\text{maximize}} \sum_{h \in C} \sum_{i \in N} \left(x_{hi}\gamma_{hi} - \mu x_{hi}(\ln(x_{hi}) - 1) \right),$$

which has been proposed by Wilson (1967) (see also Roy (2004)).

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Existing methods for solving (GNE) use lagrangian multipliers or penalty methods (see Facchinei-Fischer-Piccialli, 2009; Facchinei-Kansow, 2007; Pang-Fukushima, 2005, ...)

Land use planning: Equivalent formulation

Define
$\Phi\colon \mathbb{R}^{|C|+|N|} \to \mathbb{R}^{|C|+|N|}\colon x = (x^h)_{h\in C} \mapsto (-\nabla_{x^h} F_h(x^h, x^{-h}))_{h\in C}$.

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Since, for every $h\in C$ and x^{-h}, $-F_h(\cdot,x^{-h})$ is convex, it is enough to solve:

$$\text{find} \quad x=(x^h)_{h\in C}\in X\cap\Delta^{|N|} \quad \text{such that}$$
$$(\forall y\in X\cap\Delta^{|N|}) \quad \langle \Phi(x) \mid y-x\rangle \geq 0,$$

or, equivalently, (qualification conditions hold)

Monotone inclusion

$$\text{find} \quad x\in X \quad \text{such that} \quad 0\in\Phi(x)+N_{\Delta^{|N|}}(x)+N_X(x).$$

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Monotone inclusion

$$\text{find} \quad x \in X \quad \text{such that} \quad 0 \in \Phi(x) + N_{\Delta^{|N|}}(x) + N_X(x).$$

Remark: All solution of the inclusion is NE, but not every NE is a solution of the inclusion. These special NE are called variational equilibria (see e.g. Facchinei-Kansow, 2007).

Land use planning: Equivalent formulation

Variable sustitution: $u = x - e$, where $e = (1, \dots, 1)/|N|$. Define $V = \{x \in \mathbb{R}^{|C|+|N|} \mid (\forall h \in C)\ \sum_{i \in N} x_i^h = 0\}$, which is a vectorial subspace of $\mathbb{R}^{|C|+|N|}$, and $\widetilde{X} = X - e$. Then our inclusion becomes

Modified inclusion

$$\text{find} \quad u \in \widetilde{X} \quad \text{such that} \quad 0 \in \Phi(u + e) + N_V u + N_{\widetilde{X}} u.$$

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Under suitable conditions on the constants $(\alpha_{hg})_{h \in C, g \in C}$ we can guarantee that Φ is monotone. Hence, we can apply the forward-PI-forward method for finding a solution of the land use planning problem.

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