No variational characterization of the cycles in the method of periodic projections

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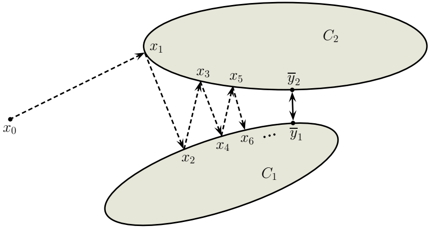
What is the problem ?

We considere 2 closed convex sets C_1 and C_2. We want to known what is the limit of the sequence $(P_{C_1} P_{C_2})^n x$ when n goes to infinite.
The origine of this problem comes from the Schwarz alternating method (1870) for the PDE.
The strong convergence was proved by Von Neuman in 1935 for the case where $C_1 \cap C_2 \neq \varnothing$. The limit in this case is the projection of x on $C_1 \cap C_2$.

What happen when $C_1 \cap C_2 = \varnothing$ and C_1 is bounded for example.

$(P_{C_1}P_{C_2})^n x \to \bar{y}_1 \in C_1$ et $P_{C_2}(P_{C_1}P_{C_2})^n x \to \bar{y}_2 \in C_2$.

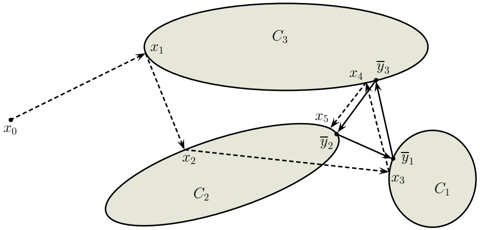


y_1 and y_2 solve the following minimization problem :

$$\min\{||y_1 - y_2||, y_1 \in C_1 \text{ et } y_2 \in C_2\}$$

We may ask the question :

What happen for 3 or more convex sets ?



Can we have a variational formulation for the limit points ?

Is there an universal Φ functions such that the cycles $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_m)$ is a solution of the following variational problem

$$\min_{y1 \in C_{\mathbf{1}}, \ldots, y_{\boldsymbol{m}} \in C_{\boldsymbol{m}}} \Phi(y_1, \ldots, y_m)$$

Open problem since ... 1965-1967

Proof Idea

Theorem

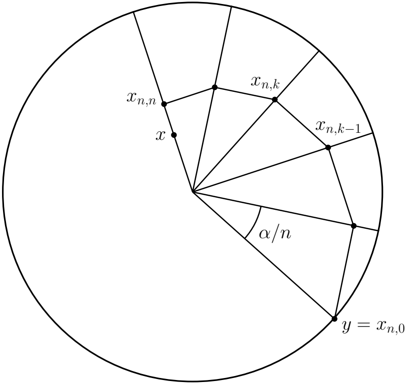
dim $H \geq 2$, let $\varphi : H \to R$ be such that its infimum on every nonempty closed convex set $C \subset H$ is attained at $P_C\ 0$. Then
(i) φ is radially increasing :

$$||x|| < ||y|| \Rightarrow \varphi(x) \leq \varphi(y)$$

(ii) Suppose that, for every nonempty closed convex set C , $P_C\ 0$ is the unique minimizer , then $\varphi(x)$ is strictly radially increasing.

$$||x|| < ||y|| \Rightarrow \varphi(x) < \varphi(y)$$

(iii) Except for at most countably many values of ρ, φ is constant on the sphere $S(0; \rho)$.


$x_{n,n}$
x
$x_{n,k}$
$x_{n,k-1}$
α/n
$y = x_{n,0}$

What should be this Φ ?

$\Phi = ||y_1 - y_2|| + ||y_2 - y_3|| + ||y_3 - y_1|| + \dots$
or
$\Phi = ||y_1 - y_2||^2 + ||y_2 - y_3||^2 + ||y_3 - y_1||^2 + \dots$
or...

Theorem (jbb,P.L. Combettes,R. Cominetti-jfa2012)

dim $H \geq 2$, $m \geq 3$, there exists no function $\Phi(y_1, \ldots, y_m)$ such that the limit cycles $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_m)$ are solutions of the variational problem :

$$\min_{y1 \in C_{\mathbf{1}}, \ldots, y_{\boldsymbol{m}} \in C_{\boldsymbol{m}}} \Phi(y_1, \ldots, y_m)$$

$m = 3$ for simplification
and we suppose the existence of Φ.
We are going to play with the convex sets C_j

First, we choose :

$$C_1 = \{z\},\ C_2 = \{0\} \text{ and } C_3 = C.$$

The limit cycle is $(\bar{x}_1 = z, \bar{x}_2 = 0, \bar{x}_3 = P_C\ 0)$.
Therefore :

$$\min_{y \in C} \Phi(z, 0, y) = \Phi(z, 0, P_C\ 0)$$

The function $y \mapsto \Phi(z, 0, y)$ depends only of the parameter $||y|| = \rho$ when z is fixed.

Next, we choose

$$C_1 = [-z, z],\ C_2 = \{0\} \text{ et } C_3 = \{\rho z\} \text{ avec } \rho > 1$$

The limit cycle is $(\bar{x}_1 = z, \bar{x}_2 = 0, \bar{x}_3 = \rho z)$.

$$\Phi(z, 0, \rho z) > \Phi(-z, 0, \rho z)$$

$$\Phi(z, 0, -\rho z) < \Phi(-z, 0, -\rho z)$$

Open Problem :

Variational formulation ? ? ? ?