A New Method for Solving Pareto Eigenvalues Complementarity Problems[1]

Samir ADLY

University of Limoges, France

ADGO 2013

Playa Blanca, october 16, 2013

[1] A joint work with H. Rammal

Complementarity problems

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a map and $K \subset \mathbb{R}^n$ be a closed convex cone. The Nonlinear Complementarity Problem (NCP) is defined by

$$NCP(F,K) \begin{cases} \text{Find } z \in K \text{ such that} \\ \\ F(z) \in K^* \text{ and } \langle z, F(z) = 0. \end{cases}$$

K^* is the positive polar of K, defined by

$$K^* = \Big\{ p \in \mathbb{R}^n \ : \ \langle p, x \rangle \geq 0, \ \forall x \in K \Big\}.$$

$$K \ni z \perp F(z) \in K^*.$$

- Other formulation as a variational inequality:

$$VI(F,K) \begin{cases} \text{Find } z \in K \text{ such that} \\ \\ \langle F(z), y - z \rangle \geq 0, \ \ \forall y \in K \end{cases}$$

Linear Complementarity Problems on the positive orthant

- $K = \mathbb{R}^n_+$ and $F(z) = Mz + q$ with $M \in \mathbb{R}^{n \times n}$

$$\mathrm{LCP}(M, q): \qquad 0 \leq z \perp Mz + q \geq 0.$$

- **Existence result:**

LCP(M, q) has a unique solution for all $q \in \mathbb{R}^n$ if and only M is a P-matrix, i.e. all its principal minors are positive.

- **Numerical solvers:**
  - Lemke's algorithm
  - PATH Solver
  - Quadratic Programming with bound constraints (if M is symmetric):

$$\min_{z \geq 0} \frac{1}{2} z^T M z + q^T z.$$

PATH Solver

$$0 \leq x \perp F(x) \geq 0.$$

- Nonsmooth Normal map (S. Robinson):

$$\Phi(x) = F(x^+) + x - x^+ = 0.$$

$$0 \leq x^+ \perp F(x^+) = x^+ - x \geq 0.$$

- Nonsmooth Newton Method applied to Φ.

- Merit function: $\frac{1}{2}\|F(x^+)\|_2^2$ with Armijo line search.

- Generalization to a closed convex cone $\mathcal{K}$

$$x^+ \to P_{\mathcal{K}}.$$

Applications of Complementarity

- Mechanical engineering: unilateral contact problems with friction
- Electrical engineering: electrical circuits with diodes
- Pricing electricity markets and options
- Electricity market deregulation
- Congestion in Networks
- Structural engineering
- Economic equilibria
- Game Theory (Nash equilibria)
- transportation planning
- Crack propoagation
- Video games

Unconstrained eigenvalue problems

Let $A,\ B,\ C \in \mathbb{R}^{n\times n}$ be given.

- The standard eigenvalue problem is:

$$\left\{\begin{array}{l}\text{Find } \lambda \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \setminus \{0\} \text{ such that} \\ Ax = \lambda x\end{array}\right.$$

- The generalized eigenvalue problem is:

$$\left\{\begin{array}{l}\text{Find } \lambda \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \setminus \{0\} \text{ such that} \\ Ax = \lambda Bx\end{array}\right.$$

- The quadratic eigenvalue problem is:

$$\left\{\begin{array}{l}\text{Find } \lambda \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \setminus \{0\} \text{ such that} \\ Q(\lambda)x = 0 \text{ with } Q(\lambda) = \lambda^2 A + \lambda B + C.\end{array}\right.$$

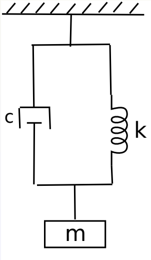
Pencil applications

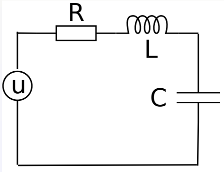
- Mechanical systems:

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f.$$

- Electrical systems:

$$L\frac{d^2 i}{dt}(t) + R\frac{di}{dt}(t) + \frac{1}{C}i(t) = u'(t).$$





$$q(t) = e^{\lambda t}x \Longrightarrow (M\lambda^2 + C\lambda + K)x = 0.$$

Constrained eigenvalue problems

Let $A,\ B,\ C \in \mathbb{R}^{n\times n}$ be given and K be a closed convex cone of $\mathbb{R}^n$. We denote by K^+ its positive polar.

- The constrained eigenvalue problem is:

$$\begin{cases} \text{Find } \lambda \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \setminus \{0\} \text{ such that} \\ K \ni x \perp (Ax - \lambda x) \in K^+ \end{cases}$$

- The constrained generalized eigenvalue problem is:

$$\begin{cases} \text{Find } \lambda \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \setminus \{0\} \text{ such that} \\ K \ni x \perp (Ax - \lambda Bx) \in K^+ \end{cases}$$

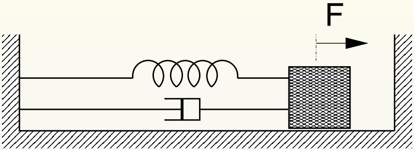
- The constrained quadratic eigenvalue problem is:

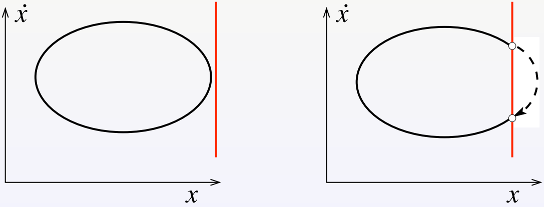
$$\begin{cases} \text{Find } \lambda \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \setminus \{0\} \text{ such that} \\ K \ni x \perp Q(\lambda)x \in K^+,\ \text{with } Q(\lambda) = \lambda^2 A + \lambda B + C. \end{cases}$$

Pencil applications

- Mechanical systems with impact and/or friction

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) \in -N_{\mathcal{K}}(q(t)).$$

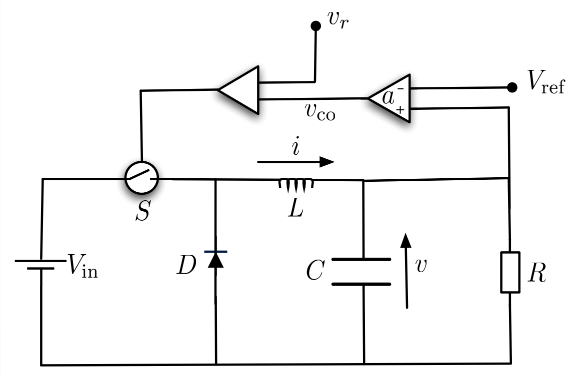




Before impact

After impact

Buck Converter as a piecewise-smooth system



$$\frac{di}{dt} = -\frac{v(t)}{L} + \begin{cases} \frac{V_{in}}{L}, & S \text{ is conducting} \\ 0, & S \text{ is blocking} \end{cases}$$

$$\frac{dv}{dt} = \frac{1}{C} i(t) - \frac{1}{RC} v(t),$$

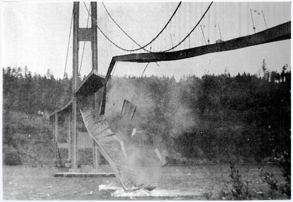
Applications of Constrained Eigenvalue Problem CEiP

A wide variety of applications require the solution of CEiP:

- Dynamic analysis of structural mechanical systems
- Vibro-acoustic systems
- Electrical circuit simulation
- Signal processing
- fluid dynamics

How can instability and unwanted resonance be avoided for a given system?

Eigenvalues corresponding to unstable modes or yielding large vibrations can be relocated or damped.



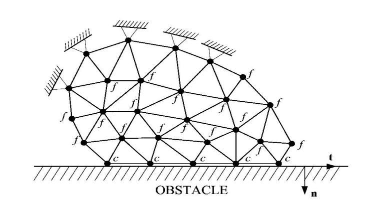
Pareto eigenvalue problems

An important particular case is given when $K = \mathbb{R}^n_+$ (pareto eigenvalue problem):

- $0 \leq x \perp (Ax - \lambda x) \geq 0.$
- $0 \leq x \perp (Ax - \lambda Bx) \geq 0.$
- $0 \leq x \perp Q(\lambda)x \geq 0,$ with $Q(\lambda) = \lambda^2 A + \lambda B + C.$

Stability Analysis of Finite Dimensional Elastic Systems with Frictional Contact.

- A. Costa, J. Martins, I. Figueiredo and J. Júdice, “The directional instability in systems with frictional contacts”, Comput. Methods Appl. Mech. Engrg., 193 (2004)357–384.



A necessary and sufficient condition for the occurrence of divergence instability along a constant admissible direction, is to find $\lambda^2 \geq 0$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $x \neq 0$ such that

$$\begin{cases} (\lambda^2 M + K)x = y, \\ y_f = 0 \\ 0 \leq x_c \perp y_c \geq 0 \end{cases} \qquad x = \begin{bmatrix} x_f \\ x_c \end{bmatrix}, \quad y = \begin{bmatrix} y_f \\ y_c \end{bmatrix}$$

Applications in mechanics

- P. Quittner (1986): Spectral analysis of variational inequalities.
- J. A. C. Martins and A. Pinto da Costa (2001): Computation of Bifurcations and Instabilities in Some Frictional Contact Problems.
- A. Pinto da Costa, J.A.C. Martins, I.N. Figueiredo and J.J. Judice (2004): The directional instability problem in systems with frictional contacts.
- J. A. C. Martins and A. Pinto da Costa (2004): Bifurcations and Instabilities in Frictional Contact Problems: Theoretical Relations, Computational Methods and Numerical Results.

Pareto Eigenvalue Complementarity Problem EiCP

Definition

Let $A \in \mathbb{M}_n(\mathbb{R})$.

$$(EiCP)\begin{cases}\text{Find } \ \lambda > 0 \text{ and } \ x \in \mathbb{R}^n \setminus \{0\}, \text{such that} \\ x \geq 0, \ \lambda x - Ax \geq 0, \ \langle x, \lambda x - Ax \rangle = 0.\end{cases}$$

$$\sigma(A) = \{\lambda > 0 : \exists \, x \in \mathbb{R}^n \setminus \{0\}, \ 0 \leq x \perp (\lambda x - Ax) \geq 0\}.$$

Let

$$\pi_n = \max_{A \in \mathbb{R}^{n \times n}} card \ [\sigma_{\mathbb{R}^n_+}(A)].$$

- We have

$$3(2^{n-1} - 1) \ \leq \ \pi_n \ \leq \ n2^{n-1} - (n-1),$$

- We have $\pi_1 = 1$, $\pi_2 = 3$ and that $\pi_3 = 9$ or 10. We note that e.g. $\pi_{20} \geq 1\,572\,861$

Definitions

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz function.

- The B-subdifferential de Φ at $z \in \mathbb{R}^n$ is defined by

  $$\partial_B \Phi(z) = \left\{ M \in \mathbb{R}^{n\times n} : \exists (z_k) \subset D_\Phi : z_k \to z, \lim_{k\to+\infty} \nabla\Phi(z_k) = M \right\}$$

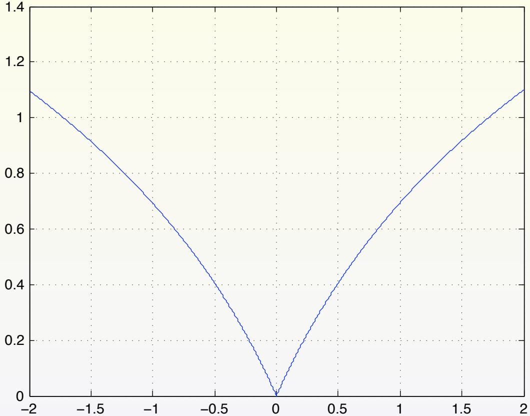
  where D_ϕ is the set of differentiability points of Φ.
- The Clarke generalized Jacobian of Φ is given by

  $$\partial\Phi(z) = \mathrm{co}\, \partial_B \Phi(z),$$

- The function Φ is said to be **semismooth** at $z \in \mathbb{R}^n$ if it is locally Lipschitz around z, directionally differentiable at z and satisfies the following condition

  $$\sup_{M\in\partial\Phi(z+h)} \|\Phi(z+h) - \Phi(z) - Mz\| = o(\|h\|).$$

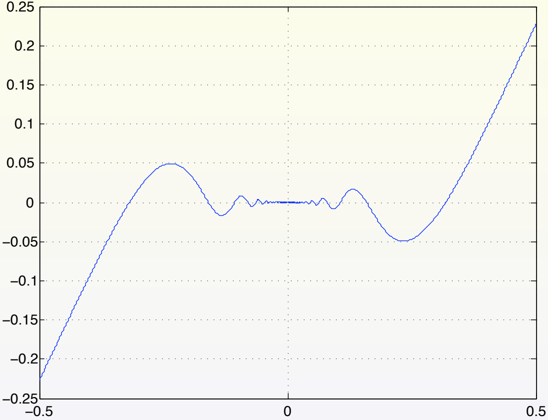
Example



$$f(x) = \ln(1 + |x|)$$

is semismooth but not differentiable at 0.

Example



$$f(x) = \begin{cases} x^2 \sin(1/x) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

is not semismooth but locally Lipschitz.

Examples of semismooth functions

- The Euclidean norm: $\|\cdot\|_2$.
- The Fischer–Burmeister function:
  $\varphi_{FB} : \mathbb{R}^2 \to \mathbb{R}, \;\; (a, b) \mapsto \varphi_{FB}(a, b) = \sqrt{a^2 + b^2} - (a + b).$
- Piecewise continuously differentiable functions:
  $(a, b) \in \mathbb{R}^2 \mapsto \min(a, b)$ or $\max(a, b)$.

SNM algorithm

1. **Initialization:** Choose an initial point z^0 and set $k = 0$.
2. **Iteration:** One has a current point z^k, If $\|\Phi(z^k)\| \leqslant 10^{-8}$, then STOP.
3. Else choose $M^k \in \partial\Phi(z^k)$ and compute h^k by solving the linear system

$$M^k h^k = -\Phi(z^k).$$

Then, set $z^{k+1} = z^k + h^k$, $k = k + 1$ and go to STEP 2.

Convergence Theorem

Theorem

Let z^* be a zero of the function Φ. Suppose the following

- *Φ is semismooth (resp. strongly semismooth) at z^* ;*
- *all matrices in $\partial\Phi(z^*)$are nonsingular.*

Then, there exists a neighborhood V of z^* such that the SNM initialized at any $z^0 \in V$ generates a sequence $(z^k)_{k\in\mathbb{N}}$ that converges superlinearly (resp. quadratically) to z^*.

Reformulation

Reformulation of EiCP[2]: by using Nonlinear Complementarity Function NCP.
$\phi : \mathbb{R}^2 \to \mathbb{R}$ is a NCP-function if and only if

$$\phi(a, b) = 0 \iff a \geq 0,\ b \geq 0,\ ab = 0.$$

We conisder

$$\phi_{\mathrm{FB}}(a, b) = a + b - \sqrt{a^2 + b^2},$$

$$\phi_{\min}(a, b) = \min(a, b).$$

[2] S. Adly and A. Seeger, *A Nonsmooth Algorithm for Cone-Constrained Eigenvalue Problems*, Springer, Computational Optimization and Applications 49, 299-318 (2011).

Resolution

Let $z = (x, y, \lambda)$ and $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^{2n+1}$ defined by

$$\Phi(z) = \Phi(x, y, \lambda) = \begin{bmatrix} U_\phi(x, y) \\ \lambda x - Ax - y \\ \langle \mathbf{1}_n, x \rangle - 1 \end{bmatrix},$$

where $U_\phi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by

$$U_\phi(x, y) = \begin{bmatrix} \phi(x_1, y_1) \\ \vdots \\ \phi(x_n, y_n) \end{bmatrix},$$

with $\phi = \phi_{\mathsf{min}}$ or $\phi = \phi_{\mathrm{FB}}$.

(x, y, λ) **is a solution of EiCP if and only if** $\Phi(x, y, \lambda) = 0_{\mathbb{R}^{2n+1}}$.

Jacobian matrix $\partial\Phi(z)$

$$\Phi(z) = \Phi(x, y, \lambda) = \begin{bmatrix} U_\phi(x, y) \\ \lambda x - Ax - y \\ \langle \mathbf{1}_n, x \rangle - 1 \end{bmatrix}.$$

Lemma

The function Φ is semismooth. Moreover, its Clarke generalized Jacobian at $z = (x, y, \lambda)$ is given by

$$\partial\Phi(z) = \left\{ \begin{bmatrix} E & F & 0 \\ \lambda \mathbf{I}_n - A & -\mathbf{I}_n & x \\ \mathbf{1}_n^T & 0 & 0 \end{bmatrix} : \ [E, F] \in \partial U_\phi(x, y) \right\}.$$

Lattice Projection Method LPM

Lemma

EiCP is equivalent to find the roots of the following nonlinear and nonsmooth function

$$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \textit{ defined by}$$

$$(x, \lambda) \mapsto f(x, \lambda) = (Ax)^+ - \lambda x.$$

Conclusion: EiCP is equivalent to solve the nonlinear eigenvalue problem (The **L**attice **P**rojection **M**ethod)

$$(\mathbf{P}_{\mathbb{R}^\mathbf{n}_+} \circ \mathbf{A})(\mathbf{x}) = \lambda \mathbf{x}.$$

Jacobian matrix of Φ_{LPM}

EiCP is equivalent to solving the nonlinear system

$$\Phi_{\mathrm{LPM}}(x, \tilde{y}, \lambda) = \begin{bmatrix} \tilde{y}^+ - \lambda x \\ Ax - \tilde{y} \\ \langle \mathbf{1}_n, x \rangle - 1 \end{bmatrix} = 0_{\mathbb{R}^{2n+1}}.$$

Lemma

The function Φ_{LPM} is semismooth. Its Clarke generalized Jacobian at $\tilde{z} = (x, \tilde{y}, \lambda)$ is given by

$$\partial \Phi_{\mathrm{LPM}}(\tilde{z}) = \left\{ \begin{bmatrix} -\lambda \mathbf{I}_n & \tilde{F} & -x \\ A & -\mathbf{I}_n & 0 \\ \mathbf{1}_n^T & 0 & 0 \end{bmatrix} : \tilde{F} \in \partial(\cdot)^+(\tilde{y}) \right\}.$$

Testing on matrices of order $3, 4$ and 5

Let the following matrices (having exactly 9, 23 **and** 57 **Pareto eigenvalues**, respectively).

$$A_1 = \begin{bmatrix} 5 & -8 & 2 \\ -4 & 9 & 1 \\ -6 & -1 & 13 \end{bmatrix}, \ A_2 = \begin{bmatrix} 132 & -106 & 18 & 81 \\ -92 & 74 & 24 & 101 \\ -2 & -44 & 195 & 7 \\ -21 & -38 & 0 & 230 \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} 788 & -780 & -256 & 156 & 191 \\ -548 & 862 & -190 & 112 & 143 \\ -456 & -548 & 1308 & 110 & 119 \\ -292 & -374 & -14 & 1402 & 28 \\ -304 & -402 & -66 & 38 & 1522 \end{bmatrix}.$$

First numerical results

1. LPM: Lattice Projection Method.
2. SNM_{FB}: SNM using ϕ_{FB} .
3. $\text{SNM}_{\mathsf{min}}$: SNM using ϕ_{min}.

Methods	A_1			A_2			A_3		
	Iter	Time	Failure	Iter	Time	Failure	Iter	Time	Failure
LPM	4	0.0003	0%	6	0.0006	0%	7	0.0004	0%
SNM_{FB}	8	0.0012	5%	10	0.0015	16%	11	0.0014	31%
$\text{SNM}_{\mathrm{min}}$	2	0.0003	47%	2	0.0008	71%	2	0.0007	95%

Background on Performance Profiles

Dolan and Moré[3] introduced the notion of a performance profile as a means to evaluate and compare the performance of the set of solvers $\mathcal{S}$ on a test set $\mathcal{P}$. The idea is to compare the performance of solver s on problem p with the best performance by any solver on this particular problem. The performance ratio is defined by

$$r(p, s) = \frac{t_{p,s}}{\min\{t_{p,s} \ : \ s \in \mathcal{S}},$$

$t_{p,s}$=computing time required to solve problem p by solver s.

[3] E. D. Dolan and J. J. Moré, *Benchmarking Optimization Software with Performance Profiles*, Math. Prog. 91, 201-213 (2002).

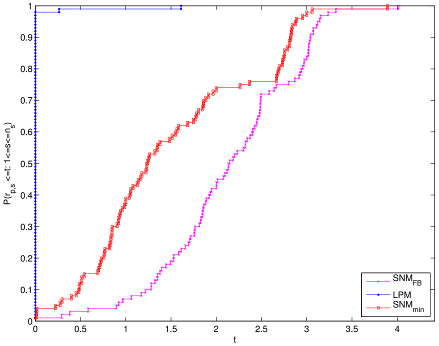
Background on Performance Profiles

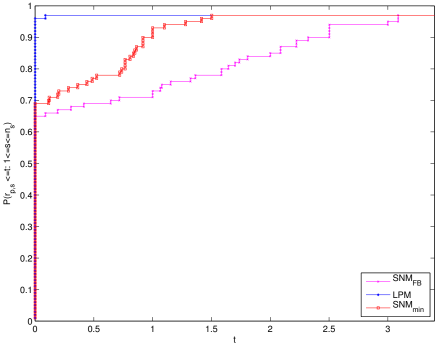
In order to obtain an overall assessment of a solver on the given model test set, we define a cumulative distribution function

$$\rho_s(\tau) = \frac{1}{n_p} \text{size}\Big\{ p \in \mathcal{P} \ : \ r(p,s) \le \tau \Big\}.$$

$\rho_s(\tau)$ is the probability that a performance ratio $r(p,s)$ is within a factor of τ of the best possible ratio.
Interpretation:
In general, $\rho_s(\tau)$ for a particular solver s gives information on the percentage of models that the solver will solve if for each model, the solver can have a maximum resource time of τ times the minimum time.
For $\tau = 1$ the probability $\rho_s(1)$ of a particular solver is the probability that the solver will win over all the others.
For large values of τ the probability function $\rho_s(\tau)$ gives information if a solver actually solves a problem.

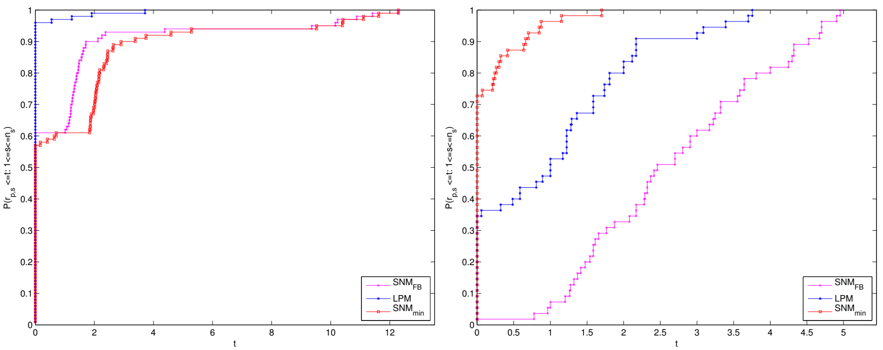
Performance profiles





- Fig1. presents the performance profiles of the three solvers corresponding to the **average computing time**.
- Fig2. **the maximum number of solution found by each solver** is the comparison criterion.

Performance profils



- In Fig1. the comparison tool is **the number of failures**.
- In Fig2. the comparison tool is **the average iterative number**.

Conclusions

- $\surd$ Reformulation of EiCP and SOCEiCP as a system of semismooth equations.
- $\surd$ Nonsingularity conditions for solving EiCP and SOCEiCP.
- $\surd$ New method LPM for solving the two problems.
- $\surd$ Numerical results and the performance of LPM.

References:

- S. Adly and A. Seeger, *A Nonsmooth Algorithm for Cone-Constrained Eigenvalue Problems*, Springer, Computational Optimization and Applications 49, 299-318 (2011).
- S. Adly and H. Rammal, *A New Method for Solving Pareto Eigenvalue Complementarity Problems*, Computational Optimization and Applications (2013).
- S. Adly and H. Rammal, *A New Method for Solving Second Order Cone Eigenvalue Complementarity Problems*, JOTA.

THANKS FOR YOUR ATTENTION!!