Proportional Apportionment

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The Apportionment Problem

US constitution 14th Amendment:

Representatives shall be apportioned among the several States according to their respective numbers, counting the whole number of persons in each State,

- Decide the number of representatives for each state according to their populations.
- Decide the number of representatives of each party in an election according to their votes.

The Apportionment Problem

Apportionment Problem. We have positive values $p_1, \ldots, p_n$ and a number of seats H. The goal is to find non-negative integer values $x_1, \ldots, x_n$ such that $\sum_{i=1}^{n} x_i = H$.

- The values $p_1, \ldots, p_n$ can represent the population of n states. Typically, after every census, the apportionment problem is solved to decide the number of representatives for each state.
- The values $p_1, \ldots, p_n$ can represent the votes of n political parties in an election. Then, the apportionment problem is solved to find the number of elected candidates from each party.

Proportional Apportionment

Allocate H seats across the states proportionally to their populations. Each state i has a population p_i.

For example, suppose there are $H = 10$ seats to allocate and three states with populations $p_1 = 4300$, $p_2 = 3400$, and $p_3 = 2300$. The total population is 10000. Assign to each state the proportion of seats corresponding to its population:

$$p = \begin{pmatrix} 4300 \\ 3400 \\ 2300 \end{pmatrix} \xrightarrow{\lambda=0.001} q = \begin{pmatrix} 4.3 \\ 3.4 \\ 2.3 \end{pmatrix}$$

Proportional? Yes!
Feasible? No!

For a vector $(p_1, p_2, \ldots, p_n)$ and a number of seats H, the quota of i is equal to

$$q_i = \frac{p_i}{\sum_{j=1}^{n} p_j} H,$$

that is the fractional number of seats proportionally assigned to i.

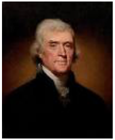
Jefferson's Approach

In 1792, Thomas Jefferson, the Secretary of State by then, proposed a solution to solve this problem. We have a divisor (multiplier) that scales the populations, and we increase or decrease its value until the following is satisfied: If we round down the scaled populations, the total summation is equal to 10.

$$\mathcal{V} = \begin{pmatrix} 4300 \\ 3400 \\ 2300 \end{pmatrix} \xrightarrow{\lambda=0.00117} \lambda\mathcal{V} = \begin{pmatrix} 5.03 \\ 3.98 \\ 2.69 \end{pmatrix} \rightarrow x = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

The value $\lambda = 0.00117$ satisfies the condition, but it is not the unique value doing the work. This way of finding the solution is known as the Jefferson/D'Hondt Method, which is widely used nowadays!

The Belgian Victor D'Hondt rediscovered this method many years later, in 1878.




Jefferson's approach

There is an alternative way of describing Jefferson's method, which is obtained by assigning the seats one by one:

Jefferson's Method. Allocate seats one by one. Next seat goes to the state maximizing the ratio $p_i/(s_i+1)$, where s_i is the current number of seats of i.

$$\frac{1}{100}p = \begin{pmatrix} \mathbf{43} \\ 34 \\ 23 \end{pmatrix} \xrightarrow{\div} \begin{pmatrix} 21.5 \\ \mathbf{34} \\ 23 \end{pmatrix} \xrightarrow{\div} \begin{pmatrix} 21.5 \\ 17 \\ \mathbf{23} \end{pmatrix} \xrightarrow{\div} \begin{pmatrix} \mathbf{21.5} \\ 17 \\ 11.5 \end{pmatrix} \xrightarrow{\div} \begin{pmatrix} 14.3 \\ \mathbf{17} \\ 11.5 \end{pmatrix} \to \dots$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \to \dots \to \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

Other Methods

Historically, several methods have been proposed by politicians and mathematicians, all with different characteristics. For example, we have the following two:

Hamilton (or Largest Remainders). Give $\lfloor q_i \rfloor$ many seats to every state i, and then allocate the remaining seats one by one according to non-increasing order of the fractional parts $q_i - \lfloor q_i \rfloor$.

Huntington-Hill. Allocate seats one by one. Next set goes the the state i maximizing the ratio $p_i / \sqrt{s_i(s_i + 1)}$, where x_i is the current number of seats of i.

US House of Representatives

- 1792-1840: Jefferson's Method ($H = 105$)
- 1852-1941: Hamilton's Method ($H = 435$)
  with some usage of Webster's method
- 1942-Present: Huntington-Hill's Method ($H = 435$)

Paradoxes

Three methods have been presented, all of them reasonable. Then, which one should we use? One way to evaluate a method is by analyzing the outcome when the population or the number of seats is changed.

Increasing the number of seats: After the 1880 census, C. W. Seaton from the Census Bureau realized that under Hamilton's method of apportionment, Alabama would receive 8 seats with a house of 299, but only 7 seats with a house of 300. That is, increasing the number of seats could reduce the allocation for some states. This is known as the Alabama Paradox.

Changing the populations: Under Hamilton's method, when the population changes, a state could lose a seat against another one, even when the population change favored the first state. This is known as the Population Paradox or the Virginia Paradox.

In 1900 Virginia's population grew faster than Main's but a seat would have been transferred from Virginia to Main.

Axiomatic Approach

One can approach the apportionment problem from an axiomatic point of view to avoid undesirable outcomes such as Alabama or Population Paradoxes. In what follows, given two vectors p and p' and two number of seats H and H', we denote by x the apportionment solution for p and H, and x' the apportionment solution for p' and H'.

Population Monotonicity Property. For every pair of states i and j,

$$\text{if } p'_i/p_i \geq p'_j/p_j, \text{ then } x'_i \geq x_i \text{ or } x'_j \leq x_j.$$

That is, if the population change favors state i over j, then i can not lose a seat against j (i.e., the method avoids the Population Paradox).

House Monotonicity Property. When $p' = p$ and $H' = H + 1$, for every state i we have $x'_i \geq x_i$.

That is, increasing the house size does not make any state lose a seat (i.e., the method avoids the Alabama Paradox).

Axiomatic Approach

A third natural property is the following:

Quota Property. For every p and every H, the apportionment method computes a solution x such that $\lfloor q_i \rfloor \leq x_i \leq \lceil q_i \rceil$ for every i.

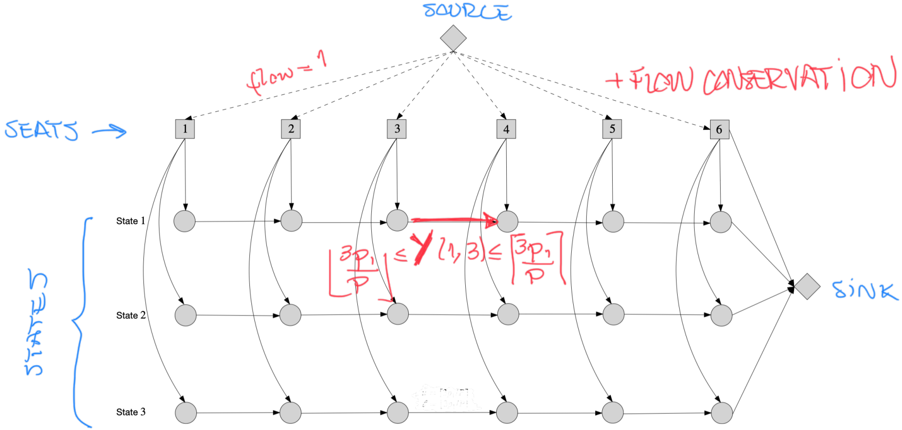
Hamilton's method violates population monotonicity (and house monotonicity) but satisfies quota. Then, a natural question is the following: Is there any method that satisfies population monotonicity and quota? The answer is **no**.

Balinski-Young Impossibility Result:
There is no apportionment method that satisfies population monotonicity and quota.

House Monotonicity and Quota

Cembrano, Correa, Schmidt-Kraepelin, Tsigonias-Dimitriadis, Verdugo (2023)

Fortunately, house monotonicity and quota are compatible. In fact we can characterize all of them. We consider an approach inspired by network flows.



House Monotonicity and Quota

Cembrano, Correa, Schmidt-Kraepelin, Tsigonias-Dimitriadis, Verdugo (2023)

Projection ($x(i,\ell)$ represents whether state i gets the ℓ-th seat)

$$\sum_{i=1}^{n} x(i,t) = 1 \quad \text{for every } t \in [L], \tag{1}$$

$$\sum_{\ell=1}^{t} x(i,\ell) \geq \lfloor tp_i/P \rfloor \quad \text{for every } i \in [n] \text{ and every } t \in [L], \tag{2}$$

$$\sum_{\ell=1}^{t} x(i,\ell) \leq \lceil tp_i/P \rceil \quad \text{for every } i \in [n] \text{ and every } t \in [L] \tag{3}$$

$$x(i,t) \geq 0 \quad \text{for every } i \in [n] \text{ and every } t \in [L]. \tag{4}$$

- LP is integral and extreme points allocates the first L seats respecting quota.
- *How large should L be to obtain every possible allocation of H seats that can be obtained by a house monotone and quota-compliant method?*.
- We derive tight bounds for this quantity.
- But in any case considering $L = \sum_{j=1}^{n} p_j$ we recover every possible apportionment method that is house monotone and quota-compliant.

Alternative characterizations by Balinski, Young (1985); Golz, Peters, Procaccia (2022).

Randomized Apportionment

Cembrano, Correa, Schmidt-Kraepelin, Tsigonias-Dimitriadis, Verdugo (2023)

> **Ex-ante Proportionality.** For every p and every H, the randomized apportionment method computes x such that $\mathbb{E}(x_i) = q_i$ for every i.

Theorem:[Golz, Peters, Procaccia (2022)] There is a method that satisfies house monotonicity, quota and ex-ante proportionality. $\longrightarrow$ We escape B-Y impossibility

Our characterization provides a simple proof:

- Set $Q(i,\ell) = p_i / \sum_{j=1}^{n} p_j$ for every ℓ, we have that

$$\sum_{\ell=1}^{H} Q(i,\ell) = \frac{p_i}{\sum_{j=1}^{n}} H = q_i,$$

- Q is feasible for the LP with $L = \sum_{j=1}^{n} p_j$.
- Q is a convex combination of extreme points (say form set S):

$$\sum_{x \in S} \lambda_x \cdot x = Q, \ \sum_{x \in S} \lambda_x = 1 \text{ and } \lambda \geq 0,$$

- Randomized method: Sample x w.p. λ_x.

Back to population monotonicity

There are three basic properties that, in general, an apportionment method M is expected to satisfy:

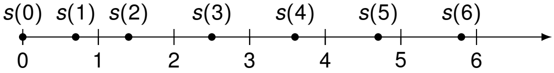
- **Homogeneity.** For every $\alpha > 0$, we have $M(p, H) = M(\alpha p, H)$.
- **Exactness.** If $H = \sum_{j=1}^{n} p_j$, then $M(p, H) = p$.
- **Symmetric.** For every permutation σ over $[n]$, $M(\sigma p, H) = \sigma M(p, H)$.

> **Theorem.** [Balinki, Young 1985] An homogeneous, exact and symmetric method is population monotone if and only if it is a divisor method.

Jefferson's method belongs to the class of divisor methods.

Divisor Methods

Jefferson's method achieves integrality in the solution by rounding down the numbers. However, why not use another rounding rule?



Divisor Method:
Given an apportionment instance with H seats and population p, and rounding rule s, the solution x with the divisor method with rounding rule $[\![\cdot]\!]_s$ is defined as follows:

- There exists $\lambda > 0$ such that $s(x_i) \leq \lambda p_i \leq s(x_i + 1)$ for each i, (i.e., $x_i = [\![\lambda p_i]\!]_s$)
- $\sum_{i=1}^{n} x_i = H$.

Divisor Methods

Jefferson/D'Hondt method is the divisor method with downwards rounding rule.
Webster/Sainte-Laguë method is the divisor method with nearest-integer rounding rule.
Adams method is the divisor method with upwards rounding rule.

The choice of the rounding rule has a crucial impact on the characteristics of the solution. Consider an instance with $H = 10$ seats and three states with populations $p_1 = 129$, $p_2 = 102$, and $p_3 = 69$. We compute the solutions using the Jefferson method and the Webster method.

$$p = \begin{pmatrix} 129 \\ 102 \\ 69 \end{pmatrix} \xrightarrow{\lambda=0.039} \lambda p = \begin{pmatrix} 5.03 \\ 3.98 \\ 2.69 \end{pmatrix} \xrightarrow{\lfloor\cdot\rfloor} x = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix},$$

$$p = \begin{pmatrix} 129 \\ 102 \\ 69 \end{pmatrix} \xrightarrow{\lambda=0.0347} \lambda p = \begin{pmatrix} 4.48 \\ 3.54 \\ 2.39 \end{pmatrix} \xrightarrow{[\cdot]} x = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}.$$

Using different rounding rules, we obtain different solutions! and one more seat in the parliament can be extremely valuable ...

Divisor Methods

Lower and upper quota

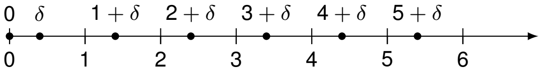
- Balinski-Young's impossibility imply that no divisor method can satisfy quota.

However:

- Jefferson is unique divisor method satisfying lower quota.
- Adams is the unique divisor method satisfying upper quota.

A method satisfies lower quota if, for every state i, it gives at least $\lfloor q_i \rfloor$, and it satisfies upper quota if it gives at most $\lceil q_i \rceil$.

Proposition:[Cembrano, Correa, Schmidt-Kraepelin, Tsigonias-Dimitriadis, Verdugo (2023)]
For every instance (p, H) there exists a value $\delta \in [0, 1]$ such that the divisor method induced by the δ threshold rounding satisfies quota.



Current efforts. Understanding randomization over divisor methods.

Beyond Single Dimension

Typically, the electoral systems have a political dimension (parties in an election) and a geographical dimension (elections per district or state).
In general, these methods can be grouped into four families:

- **Majoritarian (FPP)**, USA, Brazil, Canada
- **Proportional**, Chile, Finland, Israel, Spain
- **Mixed (MMPR)**, Germany, New Zealand, Bolivia
- **Biproportional**, Switzerland ($\approx$ Bulgaria, Italy)

	Indirect	**Majoritarian**	**Mixed**	**Proportional**
Indirect	-	12	1	5
Majoritarian	2	-	13	27
Mixed	0	1	-	8
Proportional	0	7	6	-

Table: Trends in system changes (Colomer, 2004)

Two-Dimensional Apportionment

So far, we have only considered the apportionment problem across the elements of a single dimension: Political parties (political dimension) or districts (geographic and population dimension). What if we disaggregate the votes obtained by each list across two electoral districts? For instance, suppose the aggregated votes by list

$$\mathcal{V} = \begin{pmatrix} 43 \\ 34 \\ 23 \end{pmatrix}$$

Comes from votes in two states or districts:

$$\mathcal{V} = \begin{matrix} \begin{pmatrix} 28 & 15 \\ 23 & 11 \\ 14 & 9 \end{pmatrix} & \begin{matrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{matrix} \\ \begin{matrix} d_1 & \quad d_2 \end{matrix} & \end{matrix}$$

We look for a new (fractional) matrix that is proportional (in some sense) to $\mathcal{V}$ and such that the prescribed marginals are satisfied (say $d_1 = d_2 = 5$ and $\ell_1 = 5, \ell_2 = 3, \ell_3 = 2$).

Two-Dimensional Apportionment

Solution: Find row and column multipliers (λ_i and μ_j) such that ($\mathcal{V}_{ij}\lambda_i\mu_j$) satisfies the marginals.

- Always exists
- Can be found efficiently
- Related to the problem of matrix scaling (Allen-Zhu, Li, Mendes de Oliveira, Wigderson [2017])
- Extends to multiple dimensions (not just two).

Moreover, we can show this is the unique method satisfying three natural axioms:

- Exactness: If there is a single scaling factor such that the resulting matrix satisfies the marginals then this should be the output.
- Consistency: Any part of a proportional solution must itself be proportional
- Homogeneity: Invariant to scaling of the vote matrix as long as the marginals remain the same

What about integrality?

Two-dimensional Proportionality (1989)

Biproportionality (Balinski and Demange 1989)

For each list ℓ and district d, we allocate $x_{\ell d}$ seats, such that:

- Each list ℓ receives m_ℓ seats: $\sum_{d \in D} x_{\ell d} = m_\ell$.
- Each district d is assigned s_d seats: $\sum_{\ell \in L} x_{\ell d} = s_d$.
- There exists $\lambda_\ell > 0$ for each ℓ and $\mu_d > 0$ for each d such that

$$x_{\ell d} \in \lfloor \lambda_\ell \mu_d \mathcal{V}_{\ell d} \rceil \text{ for every } (\ell, d) \in L \times D.$$

✓ Guaranteed to exist (under mild conditions)

$$\mathcal{V} = \begin{array}{c c} & \\ & \begin{pmatrix} 28 & 15 \\ 23 & 11 \\ 14 & 9 \end{pmatrix} \\ \mu & \begin{matrix} 0.5 & 1 \end{matrix} \end{array} \begin{array}{c} \lambda \\ 0.21 \\ 0.18 \\ 0.2 \\ \end{array} \to \begin{pmatrix} 2.94 & 3.15 \\ 2.07 & 1.98 \\ 1.4 & 1.8 \end{pmatrix} \to x = \begin{pmatrix} 2 & 3 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Two-dimensional Proportionality (1989) - LP Approach

Inspired by the work of Rote and Zachariasen [2007] for matrix scaling, Gaffke and Pukelsheim [2008] proposed the following integer LP to characterize biproportional apportionments.

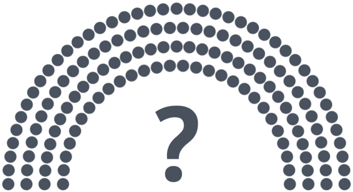
$$
\begin{aligned}
\text{minimize} \quad & \sum_{\substack{(\ell,d)\in L\times D;\\ \mathcal{V}_{\ell d}>0}} \sum_{t=1}^{H} y_{\ell d}^{t} \log\left(\frac{t}{\mathcal{V}_{\ell d}}\right) & \\
\text{subject to} \quad & \sum_{t=1}^{H} y_{\ell d}^{t} = x_{\ell d} & \text{for every } (\ell, d) \in L \times D \text{ with } \mathcal{V}_{\ell d} > 0, \\
& \sum_{d\in D:\mathcal{V}_{\ell d}>0} x_{\ell d} = m_{\ell} & \text{for every } \ell \in L, \\
& \sum_{\ell\in L:\mathcal{V}_{\ell d}>0} x_{\ell d} = s_{d} & \text{for every } d \in D, \\
& y_{\ell d}^{t} \in \{0, 1\} & \text{for every } (\ell, d) \in L \times D \text{ with } \mathcal{V}_{\ell d} > 0 \\
& & \text{and every } t \in \{1, \ldots, H\}.
\end{aligned}
$$

Specifically, they proved that given $x \in \mathbb{N}^{L\times D}$:

x is a biproportional apportionment $\iff$ There exists y such that (x, y) is an optimal solution for the linear relaxation of this problem

Existence of biproportional apportionments then follow by total unimodularity.

Beyond Two Dimensions



Historically, two dimensions have been considered when allocating seats of a house of representatives: political (lists, parties) and geographical (electoral districts).

However, representation of dimensions beyond these two is increasingly demanded: New Zealand's Parliament, Parliament of the Federation of Bosnia and Herzegovina, Chilean Constitutional Convention.

Three-dimensional Proportionality

Cembrano, Correa, and Verdugo (PNAS 2022)

What if now we want to incorporate gender balance through an additional dimension?

Three-proportional apportionment (Cembrano, Correa, Verdugo, PNAS'22)

For each list ℓ, district d and gender g, we allocate $x_{\ell dg}$ seats, such that:

- Each list ℓ receives m_ℓ seats: $\sum_{d \in D} \sum_{g \in G} x_{\ell dg} = m_\ell$.
- Each district d is assigned s_d seats: $\sum_{\ell \in L} \sum_{g \in G} x_{\ell dg} = s_d$.
- Each gender g receives half of the seats: $\sum_{\ell \in L} \sum_{d \in D} x_{\ell dg} = \frac{H}{2}$.
- There exists $\lambda_\ell > 0$ for each ℓ, $\mu_d > 0$ for each d and $\gamma_g > 0$ for each g such that

  $$\lambda_\ell \mu_d \gamma_g \mathcal{V}_{\ell dg} \leq x_{\ell dg} \leq \lambda_\ell \mu_d \gamma_g \mathcal{V}_{\ell dg} + 1 \text{ for every } (\ell, d, g) \in L \times D \times G.$$

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- There exists $\lambda_\ell > 0$ for each ℓ, $\mu_d > 0$ for each d and $\gamma_g > 0$ for each g such that

$$\lambda_\ell \mu_d \gamma_g \mathcal{V}_{\ell dg} \leq x_{\ell dg} \leq \lambda_\ell \mu_d \gamma_g \mathcal{V}_{\ell dg} + 1 \text{ for every } (\ell, d, g) \in L \times D \times G.$$

✗ Three-proportional apportionments are not guaranteed to exist. (Integer LP characterization)

✗ Deciding whether a given instance admits a three-proportional apportionment is NP-complete. (Reduction from perfect matching in 3-partite hypergraphs)

Main Result in 3D

Cembrano, Correa, and Verdugo (PNAS 2022)

Existence of approximate three-proportional apportionments

If we consider integer values u_1, u_2, u_3 such that

$$\frac{1}{u_1+2} + \frac{1}{u_2+2} + \frac{1}{u_3+2} \leq 1,$$

then (under mild conditions) there exists an apportionment x verifying:

- Each list ℓ receives $m_\ell \pm u_1$ seats.
- Each district d is assigned $s_d \pm u_2$ seats.
- Each gender g receives $H/2 \pm u_3$ seats.
- There exists $\lambda_\ell > 0$ for each ℓ, $\mu_d > 0$ for each d and $\gamma_g > 0$ for each g such that

  $\lambda_\ell \mu_d \gamma_g \mathcal{V}_{\ell dg} \leq x_{\ell dg} \leq \lambda_\ell \mu_d \gamma_g \mathcal{V}_{\ell dg} + 1$ for every $(\ell, d, g) \in L \times D \times G$.

Some possible values for (u_1, u_2, u_3): $(1,1,1)$, $(0,1,4)$, $(0,2,2)$.

Main Result in General

Cembrano, Correa, and Verdugo (PNAS 2022)

- We actually study and prove our results in a more general setting:
  - ✓ Arbitrary rounding rule instead of downwards rounding.
  - ✓ Soft bounds instead of hard marginals.
  - ✓ Arbitrary dimension. In dimension d, the sufficient condition over $u \in \mathbb{N}^d$ in the main theorem turns into

    $$\sum_{i=1}^{d} \frac{1}{u_i + 2} \leq 1.$$

    When $d = 2$ we recover the existence result of Balinski and Demange.
- Tightness: further improvements may be possible, but for instance $u_1 = u_2 = 0,\ u_3 = K$ is not possible.

Proof Idea: LP relaxation

Cembrano, Correa, and Verdugo (PNAS 2022)

$$
\begin{aligned}
\text{minimize} \quad & \sum_{\substack{(\ell,d,g)\in L\times D\times G;\\ \mathcal{V}_{\ell dg}>0}} \sum_{t=1}^{H} y_{\ell dg}^{t} \log\left(\frac{t}{\mathcal{V}_{\ell dg}}\right) \\
\text{subject to} \quad & \sum_{t=1}^{H} y_{\ell dg}^{t} = x_{\ell dg} && \text{for every } (\ell,d,g)\in L\times D\times G \text{ with } \mathcal{V}_{\ell dg}>0, \\
& \sum_{\substack{(d,g)\in D\times G:\\ \mathcal{V}_{\ell dg}>0}} x_{\ell dg} = m_{\ell} && \text{for every } \ell\in L, \\
& \sum_{\substack{(\ell,g)\in L\times G:\\ \mathcal{V}_{\ell dg}>0}} x_{\ell dg} = s_{d} && \text{for every } d\in D, \\
& \sum_{\substack{(\ell,d)\in L\times D:\\ \mathcal{V}_{\ell dg}>0}} x_{\ell dg} = \frac{H}{2} && \text{for every } g\in G, \\
& 1 \geq y_{\ell dg}^{t} \geq 0 && \text{for every } (\ell,d,g)\in L\times D\times G \text{ with } \mathcal{V}_{\ell dg}>0 \\
& && \text{and every } t\in\{1,\dots,H\}.
\end{aligned}
$$

For an optimal solution (x, y) of the LP, any $\bar{x}$ with $\bar{x}_{\ell dg} \in \{\lfloor x_{\ell dg}\rfloor, \lceil x_{\ell dg}\rceil\}$ for every (ℓ, d, g) with $\mathcal{V}_{\ell dg} > 0$ verifies the proportionality condition.

Proof Idea: Iterative Rounding Algorithm

Cembrano, Correa, and Verdugo (PNAS 2022)

1. Solve the linear relaxation. If its optimal solution x is integer, return it.

2. Otherwise, define $z \leftarrow \lfloor x \rfloor$, $y^0 \leftarrow x - \lfloor x \rfloor$ and initialize a set $\mathcal{F} \subseteq L \times D \times G$ containing the fractional entries of y^0, a set $\mathcal{F}_L \subseteq L$ containing the lists with $\geq u_1 + 2$ fractional entries of y^0, and sets $\mathcal{F}_D \subseteq D,\ \mathcal{F}_G \subseteq G$ defined analogously. Let $t \leftarrow 0$.

3. While some of the sets $\mathcal{F}_L, \mathcal{F}_D, \mathcal{F}_G$ is nonempty, consider the following LP with variables $\{y_{\ell dg}\}_{(\ell,d,g)\in\mathcal{F}}$:

$$\sum_{\substack{(d,g)\in D\times G:\\ (\ell,d,g)\in\mathcal{F}}} y_{\ell dg} = \sum_{\substack{(d,g)\in D\times G:\\ (\ell,d,g)\in\mathcal{F}}} y^t_{\ell dg} \qquad \text{for every } \ell \in \mathcal{F}_L,$$

$$\sum_{\substack{(\ell,g)\in L\times G:\\ (\ell,d,g)\in\mathcal{F}}} y_{\ell dg} = \sum_{\substack{(\ell,g)\in L\times G:\\ (\ell,d,g)\in\mathcal{F}}} y^t_{\ell dg} \qquad \text{for every } d \in \mathcal{F}_D,$$

$$\sum_{\substack{(\ell,d)\in L\times D:\\ (\ell,d,g)\in\mathcal{F}}} y_{\ell dg} = \sum_{\substack{(\ell,d)\in L\times D:\\ (\ell,d,g)\in\mathcal{F}}} y^t_{\ell dg} \qquad \text{for every } g \in \mathcal{F}_G,$$

$$y_{\ell dg} \in [0,1] \qquad \text{for every } (\ell,d,g) \in \mathcal{F}$$

Let y^{t+1} be any extreme point of the feasible region. Update $z_{\ell dg} \leftarrow z_{\ell dg} + y^{t+1}_{\ell dg}$ for each (ℓ,d,g) such that $y^{t+1}_{\ell dg}$ is integer. Update $t \leftarrow t+1$ and the sets $\mathcal{F},\ \mathcal{F}_L,\ \mathcal{F}_D,\ \mathcal{F}_G$.

4. When the cycle ends, return z.

Proof Idea: Analyzing the Algorithm

Cembrano, Correa, and Verdugo (PNAS 2022)

- The LP we solve in each iteration has more variables than equality constraints:

$$\begin{aligned}\#\text{ Eq. Constraints} = |\mathcal{F}_L| + |\mathcal{F}_D| + |\mathcal{F}_G| &\leq \sum_{i=1}^{3} \left\lfloor \frac{|\mathcal{F}|}{u_i + 2} \right\rfloor \\ &\leq \sum_{i=1}^{3} \frac{|\mathcal{F}|}{u_i + 2} \leq |\mathcal{F}| = \#\text{ Variables}\end{aligned}$$

$\Longrightarrow$ In each iteration we fix at least one fractional variable to an integer value.

- As long as a list has $u_1 + 2$ or more fractional entries, the sum over its entries remain unchanged. When the cycle ends, we have at most $u_1 + 1$ fractional entries for each list. Since the sums are integer values, after rounding we obtain a deviation of at most u_1 as claimed (analogously for each district and gender).

- Since we are rounding each entry of the optimal solution, proportionality is guaranteed.

The Chilean Constitutional Convention

2019 October 18	2019 November 15	2020 October 25	2021 May 15-16
The Social Outburst	The “Peace Agreement”	Referendum for a New Constitution	Conventional Election

Applying Our Method to the Constitutional Convention Election

Cembrano, Correa, Diaz & Verdugo (EEAMO 2021)

More than 1300 candidates!

Constitutional Convention Method (CCM): It operates one district at a time. In a first step, it assigns the seats of the district according to the D'Hondt method across the political lists. While gender balance is not achieved in the district, it replaces the least-voted candidate of the over-represented gender by the top-voted candidate of the under-represented gender and the same list.

Our methods (TPM): We look for a three-proportional apportionment such that:

- The seats (138) are apportioned across the lists through the (one-dimensional) D'Hondt method. (m_ℓ)
- Each district is assigned a number of seats predefined by law. (s_d)
- Each gender obtains 69 seats.

Variants of TPM

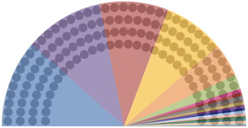
Cembrano, Correa, Diaz & Verdugo (EEAMO 2021)

We evaluate several variants of TPM and a local BPM.

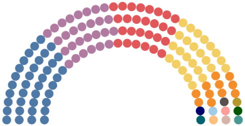
- **Biproportional method (BPM):** Creates a local apportionment of list-gender (with parity) in every district
- **Three-proportional method (TPM):** Creates a global (national-level) apportionment of list-district-gender (with parity)
- **Three-proportional with threshold (TPM3):** Equivalent to TPM, but in this method only lists with 3% of national votes or more can obtain representation.
- **Three-proportional with plurality (TPP):** Creates a global (national-level) apportionment of list-district-gender (with parity), and also includes the condition that enforces that the most voted candidate in each district gets chosen.
- **Three-proportional with plurality and threshold (TPP3):** Equivalent to TPP, but with 3% threshold for representation.

Representativeness & Proportionality

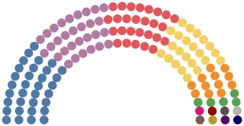
Cembrano, Correa, Diaz & Verdugo (EEAMO 2021)



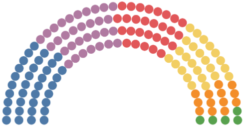
(a) Fair share



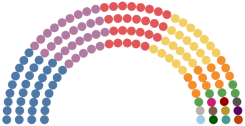
(b) CCM & BPM



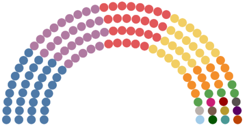
(c) TPM



(d) TPM3



(e) TPP



(f) TPP3

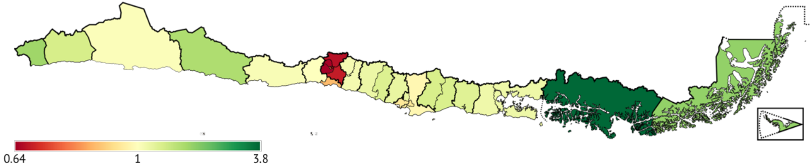
Some Relevant Metrics

Cembrano, Correa, Diaz & Verdugo (EEAMO 2021)

MALAPPORTIONMENT

	CCM	BPM	TPM	TPM3	TPP	TPP3
Av. Votes	14,126	14,103	14,048	13,848	14,406	14,198
Global Mal.	4.6	4.6	1.8	3.8	1.3	2.9
Local Mal.	13.7	13.7	18.5	19.0	18.2	18.5

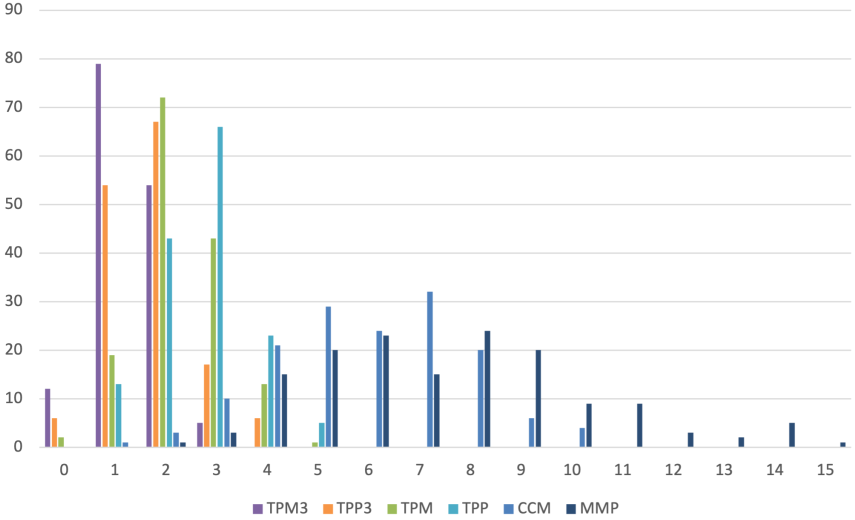
VOTING POWER



- With current districting, local methods (CCM, BPM, y MMP) distortion the voting power.
- A vote in Aysen is around six times a vote in Santiago!
- Global methods (TPM, TPP, TPM3, TPP3) do not suffer from this weakness.

Robustness: Seats Transfers

Cembrano, Correa, Diaz & Verdugo EEAMO 2021



Conclusions

- Combinatorial optimization tools can be valuable for designing electoral methods.
- These methods can be useful in order to find an answer to conciliate natural axioms desirable in electoral methods.
- Randomization can also be helpful to (ex-ante) fulfill desirable properties. Are we ready for randomized electoral systems?
- Multidimensional apportionment seems like a promising direction to deal with the complex representation demands of modern societies.
- Electoral methods induce strategic behavior in the political parties and voters and we need to be careful about this!