A nonparametric approach to model the term structure of interest rates
The case of Chile

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Abstract

Numerous studies have resorted to parametric models to infer the shape of the term structure of interest rates. Recently, however, it has been shown that nonparametric techniques may be more adequate. This is an empirical study for Chile between December 1992 and April 1998. Monte Carlo simulations, based upon a nonparametric one-factor model, suggest that Chile's downward-sloping term structure could be explained by the mean reversion process in the data. The latter could reflect medium- and long-term goals of monetary policy of the Central Bank of Chile. Some alternative explanations, such as that of the preferred habitats, might be also plausible. © 2001 Elsevier Science Inc. All rights reserved.

JEL classification: C14; E43

Keywords: Term structure of interest rates; Nonparametric estimation

1. Introduction

A topic that has been extensively studied in the finance literature is the modeling of the stochastic process describing the dynamics of the short interest rate and the term structure that
can be inferred from such a process. Indeed, several theoretical models in continuous time have been developed from which different functional forms for the term structure can be derived. Classical examples are one-factor models (see Chan, Karoli, Longstaff, & Sanders, 1992, for a description and econometric estimation of these models and an alternative parametric specification). These models assume that the instantaneous interest rate follows an Ito process, in which the drift and the variance parameters are a function of the interest rate but are independent of time. A generalization is the two-factor models, which assume that the volatility of the interest rate is also stochastic.

Unfortunately, most models in the current literature arbitrarily parameterize the stochastic process of the interest rate. Ait-Sahalia (1996b) shows that all existing parametric one-factor models can be statistically rejected when comparing the density function given by each model and that obtained from historical US data. In order to overcome this shortcoming, Ait-Sahalia (1996a, 1996b) uses semiparametric estimation to infer what parametric specification might be adequate to the data. Another article in the same line of research is by Stanton (1997), who chooses a fully nonparametric approach in the specification of a one-factor model for US data.

More recently, Jiang (1998) has compared semiparametric models and nonparametric models, also under the set-up of a one-factor model. The author concludes that empirical results for the US show that most traditional spot rate models and market prices of interest rate risk are misspecified. Boudoukh, Richardson, Stanton, and Whitelaw (1999) have extended Stanton’s (1997) work by developing a general two-factor diffusion model for interest rates using approximation methods for multifactor continuous-time Markov processes.

There is an extensive literature for the US and other industrialized countries devoted to modeling the term structure of interest rates. However, as in the case of Chile and other Latin American countries, this is almost an unexplored field. Interestingly, through the years it has been observed that the term structure in Chile is downward sloping, a phenomenon rarely found in other countries of the world.

The aim of this article is to capture the empirical properties of the term structure of interest rates in Chile by nonparametric estimation techniques. Earlier attempts by Chilean researchers have relied upon parametric models that have been questioned in recent years (e.g., Chan et al., 1992; Nelson & Siegel, 1987).

The focus of attention of this study is the term structure of nominal interest rates for the time period December 1992–April 1998. In choosing this sample period, we have excluded the troughs experienced by Chile in the early 1990s and from mid-1998 onwards. For the sake of parsimony, we estimate a nonparametric one-factor model, which seems a reasonable approximation of the dynamics of the short-term nominal rates.

Our estimation results show that the term structure of nominal interest rates is eventually downward sloping, a phenomenon also observed in the term structure of real interest rates. We believe that real and nominal interest rates are eventually decreasing with maturity because they sooner or later revert to a mean that reflects medium- and long-term goals of monetary policy of the Central Bank of Chile. Although we briefly refer to this issue in the article, we think that the existence of segmented markets (preferred habitats) might also explain this atypical shape of the term structure, particularly for long-maturity bonds.
This article is organized as follows. Section 2 presents our nonparametric model. Section 3 discusses the features of our data and shows our estimation results. Finally, Section 4 presents some concluding remarks.

2. The model

The two different approaches commonly used to model the dynamics of the short interest rate are the equilibrium and no-arbitrage models. The equilibrium models usually start from a set of assumptions about other economic variables and obtain the dynamics of the instantaneous risk-free rate as an output. The no-arbitrage models, by contrast, take the current term structure as an input to price interest rate derivatives.

In this study, we will focus on a one-factor model, a class of equilibrium models that has been extensively used in empirical work. One-factor models assume that the stochastic process that governs \( r \) involves only one source of uncertainty: the common factor, \( r \):

\[
dr_t = \mu(r_t)dt + \sigma(r_t)dZ_t,
\]

where \( \mu \), the drift parameter, and \( \sigma \), the instantaneous volatility, are a function of the short risk-free rate, \( r \), but not of time. The variable \( dZ_t \) represents the increment of a standard Brownian (Wiener) motion.

The essence of one-factor models is that the interest rates for different maturities move in the same direction for any short-time interval, but they do not change by the same amount. Therefore, the same model can give rise to a rich pattern of shapes for the term structure. Under the assumptions of a one-factor model, it is possible to obtain the term structure for any \( t \) from \( r(t) \), the instantaneous short rate, and its corresponding dynamics in a risk-neutral world (Hull, 1997). To see why, consider the price of an interest rate derivative that pays \( f_t \) at \( t = T \) [Eq. (2)]:

\[
E[e^{-\tilde{r}_T t} f_t],
\]

where \( \tilde{r} \) represents the average rate observed in the time interval \([t, T]\), and \( \tilde{E}(\cdot) \) is the expected value in a risk-neutral world. If we define \( P(t, T) \) as the price at \( t \) of a zero-coupon bond that pays $1 at \( T \), we get:

\[
P(t, T) = E[e^{-\tilde{r}_T t}],
\]

If \( R(t, T) \) is the continuously compounded interest rate at \( t \) for the period \( T - t \), the price of a zero-coupon bond can be expressed as \( P(t, T) = e^{R(t, T) (T - t)} \), such that:

\[
R(t, T) = -\frac{1}{T - t} \ln P(t, T).
\]
Substituting Eq. (3) into Eq. (4) yields:

$$R(t, T) = -\frac{1}{T-t} \ln E[e^{-\bar{\tau}(T-t)}].$$  \hspace{1cm} (5)

In most cases, one-factor models do not give rise to an analytical solution for the probability density function of the instantaneous rate. This makes the estimation of the drift and volatility parameters by maximum likelihood extremely cumbersome. Therefore, alternative estimation techniques, such as the generalized method of moments, the efficient method of moments, and nonparametric regression, have become popular in recent years. In this article, we attempt to capture the dynamics of the short rate by a nonparametric one-factor model.

To begin with, let us assume that the instantaneous risk-free rate satisfies the differential Eq. (1). Under certain regularity conditions (e.g., Oksendal, 1998), Eq. (1) can be written as follows [Eq. (6)]:

$$r_t = r_0 + \int_0^t \mu(r_\tau)d\tau + \int_0^t \sigma(r_\tau)dZ_\tau. \hspace{1cm} (6)$$

By Ito's lemma, a function of $r_t$, $f(r_t)$, has the following representation:

$$f(r_t) = f(r_0) + \int_0^t \left\{ \frac{\partial f(r_\tau)}{\partial r_\tau} \mu(r_\tau) + \frac{1}{2} \frac{\partial^2 f(r_\tau)}{\partial r_\tau^2} \sigma^2(r_\tau) \right\}d\tau + \int_0^t \frac{\partial f(r_\tau)}{\partial r_\tau} \sigma(r_\tau)dZ_\tau,$$

$$\equiv f(r_0) + \int_0^t \bar{L}f(r_\tau)d\tau + \int_0^t \bar{L}f(r_\tau)dZ_\tau, \hspace{1cm} (7)$$

where the operators $\bar{L}$ and $\tilde{L}$ are, respectively, defined by:

$$f : \bar{L}f = \mu \frac{\partial f}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial r^2} \hspace{1cm} f : \tilde{L}f = \sigma \frac{\partial f}{\partial r}.$$

Eq. (7) can be written as:\footnote{The proof is available from the author upon request}

$$f(r_t) = f(r_0) + \bar{L}f(r_0) \int_0^t dZ_\tau + it\bar{L}f(r_0) + \bar{L}^2f(r_0) \int_0^t \int_0^\tau dZ_\tau dZ_\tau + O(t^{3/2})$$

$$= f(r_0) + \bar{L}f(r_0)(Z_t - Z_0) + it\bar{L}f(r_0) + \frac{1}{2} \bar{L}^2f(r_0) \left\{ (Z_t - Z_0)^2 - t \right\} + R. \hspace{1cm} (8)$$
where \( t \tilde{L}f(r_0) \) and \( 1/2 \tilde{L}^2f(r_0)(Z_t - Z_0)^2 - t \) are of order \( t \), \( \tilde{L}f(r_0)(Z_t - Z_0) \) is of order \( t^{1/2} \) and the remaining term, \( R \), is of order \( t^{3/2} \). From Eq. (8) it follows that [Eq. (9)]:

\[
E_0[f(r_t) - f(r_0)] = t \tilde{L}f(r_0) + O(t^2),
\]

(9)

given that \( E_0[\tilde{L}f(r_0)(Z_t - Z_0)] = E_0[1/2 \tilde{L}^2f(r_0)(Z_t - Z_0)^2 - t] \to 0 \) and \( E(R) = O(t^2) \). This implies that an order one approximation of \( \tilde{L}f(r_0) \) is given by:

\[
\tilde{L}f(r_0) = \frac{1}{t} E_0[f(r_t) - f(r_0)] + O(t).
\]

(10)

Eq. (10) states that such an approximation of the function \( \tilde{L}f(r_0) \) converges to the true function \( \tilde{L}f(r_0) \) at the rate \( t \), as \( t \to 0 \). With a bit more of algebra, it can be shown that more generally:

\[
E_0[f(r_t) - f(r_0)] = \sum_{i=1}^{n} \frac{t^i}{i!} \tilde{L}^i f(r_0) + O(t^{n+1}).
\]

(11)

Eq. (11) enables us to obtain an approximation to \( f(\cdot) \) of order 1 or greater.²

In order to approximate the drift function, we let \( f(r) = r \). From the definition of \( \tilde{L}f(r) \), it follows that \( \tilde{L}f(r_0) = \mu(r_0) \). Then a first-order approximation of \( \mu \) is given by Eq. (12):

\[
\mu(r_0) = \frac{1}{t} E_0(r_t - r_0) + O(t).
\]

(12)

By the same token, the instantaneous variance, \( \sigma^2 \), can be approximated by taking the function \( f(r) = (r - r_0)^2 \). In particular, a first-order approximation of \( \sigma^2 \) is given by Eq. (13):

\[
\sigma^2(r_0) = \frac{1}{t} E_0(r_t - r_0)^2 + O(t).
\]

(13)

which implies that a first-order approximation of the instantaneous volatility is given by Eq. (14):

\[
\sigma(r_0) = \sqrt{\frac{1}{t} E_0(r_t - r_0)^2} \cdot O(t).
\]

(14)

In order to compute the term structure of interest rates from Eq. (5), we need to obtain the dynamics of the instantaneous risk-free rate in a risk-neutral world. Therefore, it is necessary to find the market price of interest rate risk. This can be obtained by noticing the following.

² This derivation is similar in spirit to Stanton's.
Suppose that the short interest rate follows Eq. (1) and we take two assets, 1 and 2, whose prices, $f_1$ and $f_2$, respectively, depend only on time and on the short risk-free rate, $r$. By Ito’s lemma, the dynamics of $f_i$, for $i = 1$ and 2, is given by Eq. (15):

$$\frac{df_i}{dt} = \left\{ \frac{\partial f_i(r)}{\partial t} + \frac{\partial f_i(r)}{\partial r} \mu(r) + \frac{1}{2} \frac{\partial^2 f_i(r)}{\partial r^2} \sigma^2(r) \right\} dt + \left\{ \frac{\partial f_i}{\partial r} \sigma(r) \right\} dZ,$$

$$\equiv \mu_i f_i dt + \sigma_i f_i dt. \quad (15)$$

where

$$\mu_i f_i \equiv \frac{\partial f_i(r)}{\partial t} + \frac{\partial f_i(r)}{\partial r} \mu(r) + \frac{1}{2} \frac{\partial^2 f_i(r)}{\partial r^2} \sigma^2(r), \quad \sigma_i f_i \equiv \frac{\partial f_i(r)}{\partial r} \sigma(r).$$

A risk-less portfolio, $\Pi$, can be formed by taking a long position on $\sigma_2 f_2$ units of asset 1, and a short position in $\sigma_1 f_1$ units of asset 2 [Eq. (16)]:

$$\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2. \quad (16)$$

It is easy to show that the dynamics of $\Pi$ is given by:

$$d\Pi = \{\mu_1 \sigma_2 f_2 f_1 - \mu_2 \sigma_1 f_1 f_2\} dt. \quad (17)$$

If each unit of asset $i$ pays a dividend (coupon) $d_i(r,t)dt$ in a time interval $dt$, it follows that the return on the portfolio equals the instantaneous risk-free rate, after taking account of the dividends (coupons). For a time interval $dt$, with $dt \to 0$,

$$d\Pi + \{(\sigma_2 f_2) d_1 - (\sigma_1 f_1) d_2\} dt = \rho \Pi dt. \quad (18)$$

where $d\Pi$ is given by Eq. (17).

Provided that $\sigma_1, \sigma_2 \neq 0$, Eq. (18) yields:

$$\frac{\mu_1 + (d_1/f_1) - \rho}{\sigma_1} = \frac{\mu_2 + (d_2/f_2) - \rho}{\sigma_2} \equiv \lambda(r), \quad (19)$$

where $\lambda(\cdot)$ is defined as the market price of interest rate risk. The two expressions on the left-hand side of Eq. (19) represent Sharpe ratios: the expected excess return on asset $i$ grows linearly at the rate $\lambda(r)\sigma_i$. Under the assumption of a one-factor model, all assets have the same risk premium per unit of volatility, because all the uncertainty comes from the common factor $r$. In addition, it can be shown that $\lambda(r)$ must be negative, that is, the expected growth rate of the instantaneous risk-free rate is greater in a risk-neutral world than in the “real” world.
When $\sigma(r) = 0$, the short risk-free rate is no longer stochastic. Therefore, the volatility of the price of an asset that depends on the short risk-free rate is also zero. The same holds for its excess return:

$$\chi(r) = 0, \quad \text{if } \sigma(r) = 0.$$  \hfill (20)

Conditions (19) and (20) suffice to prevent arbitrage opportunities and determine $\chi(r)$ (see Jiang, 1998; Stanton, 1997). We will rely upon the estimate proposed by Jiang (1998):

$$\chi(r_0) = \frac{Y_d(r_0, \tau_1, \tau_2) + \frac{1}{2} \left[ \tau_1^2 \tau_2^2 \tau_3^2 \right]^2 (r_0, \tau_1) - \tau_2^2 \tau_3^2 (r_0, \tau_1) + \tau_3 \theta(r_0, \tau_1) - \tau_1 \theta(r_0, \tau_2)}{\tau_2 \tau_3 (r_0, \tau_2) - \tau_1 \tau_3 (r_0, \tau_1)}. \hfill (21)$$

where $\tau_i = T - t_i$ is the time to maturity of the zero-coupon bond $i$. $Y_d = Y(r, \tau_1) - Y(r, \tau_2)$ is the spread between the yields of zero-coupon bonds with time to maturity $\tau_1$ and $\tau_2$, respectively, and the parameters $\theta(r, \tau_i)$ and $\gamma(r, \tau_i)$ represent the drift and the instantaneous volatility of $Y(r, \tau_i)$, respectively [Eq. (22)]:

$$dY(r, \tau_i) = \theta(r, \tau_i) dt + \gamma(r, \tau_i) dZ_i. \hfill (22)$$

Expression (21) can be computed by substituting each unknown function by the sort of (first or higher order) approximation described above.

3. Empirical analysis for Chile

Our study covers December 1992 through April 1998. In choosing this sample period, we have excluded the troughs experienced by Chile in the early 1990s and from mid-1998 onwards. Our proxy for the instantaneous risk-free rate is the average nominal rate paid on 7-day deposits by the four largest Chilean commercial banks (source: Bloomberg). This rate is indeed almost risk-less because the likelihood of default of any of the four largest commercial banks was extremely low during our sample period. Moreover, our proxy is a very good approximation of the rate paid on 7-day zero-coupon bonds issued by the Central Bank of Chile (Pagarés Descontables del Banco Central de Chile, or PDBC). However, given that these instruments are not issued in a regular basis, we work with deposits data, for which there are daily records starting December 1992.

In order to compute the market price of interest rate risk, we also use information on the interest rate paid on 30-day deposits by the four largest Chilean commercial banks between December 1992 and April 1998 (source: Bloomberg). This rate is a very good approximation.

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3 The PDBC of the Central Bank of Chile are financial instruments issued for monetary policy purposes that do not pay coupons. Their maturity ranges from 1 to 364 days.
Fig. 1 Interest rates of the Central Bank’s 30-day zero-coupon bond (PDBC) and of the commercial bank’s 30-day deposits (monthly figures. December 1992–July 1997).

of the 30-day zero-coupon bonds issued by the Central Bank of Chile (PDBC) from January 1983 through July 1997 (Fig. 1).

Given that there are monthly records for the 30-day PDBC for that time period, we test whether there are statistical differences between the 30-day deposit rates and the 30-day PDBC rates (Table 1). In order to do so, we postulate that the difference between the two monthly series is white noise. A Box–Ljung test shows that we cannot reject the null hypothesis that the partial autocorrelations of the difference between the two series for the first sixth lags are not statistically different from zero, for any significance level commonly used. The same test applied to the first twelfth lags leads to the same conclusion (Table 2).

Table 1
Statistics for 30-day PDBC and 30-day deposits for the Chilean commercial bank system (monthly figures December 1992–July 1997, annual terms)

<table>
<thead>
<tr>
<th>Annual rate</th>
<th>Observations</th>
<th>Mean</th>
<th>S.D</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>30-Day deposit</td>
<td>56</td>
<td>1.4009</td>
<td>.0447</td>
<td>0.714</td>
<td>2.713</td>
</tr>
<tr>
<td>30-Day PDBC</td>
<td>56</td>
<td>1.4004</td>
<td>.0451</td>
<td>0.780</td>
<td>2.820</td>
</tr>
<tr>
<td>Rate differential</td>
<td>56</td>
<td>5.16 x 10^-5</td>
<td>.0160</td>
<td>-.0367</td>
<td>.0750</td>
</tr>
</tbody>
</table>

Sources: Monthly bulletins of the Central Bank of Chile and Bloomberg
Table 2
White noise test for the difference between the rates of 30-day PDBC and 30-day deposits of the Chilean commercial bank system (monthly figures: December 1992–July 1997, annual terms)

<table>
<thead>
<tr>
<th>Lag</th>
<th>Box–Ljung statistic</th>
<th>Probability</th>
<th>Partial autocorrelations</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>2.06</td>
<td>.914</td>
<td>−.033 −.120 .118 −.014 −.064 0.008</td>
</tr>
<tr>
<td>12</td>
<td>4.96</td>
<td>.959</td>
<td>−.043 .014 .070 .179 .013 −.045</td>
</tr>
</tbody>
</table>

Sources: Monthly bulletins of the Central Bank of Chile and Bloomberg.

Fig. 2 shows the daily figures of nominal rates for 7- and 30-day deposits for the four main Chilean commercial banks for the time period December 1997–April 1998. In order to check the stationarity of both series, we computed correlograms and carried out augmented Dickey–Fuller tests (Table 3). As we see, the sample autocorrelation coefficients for the
Table 3
Statistics for daily figures of 7- and 30-day deposit rates (annual terms)

<table>
<thead>
<tr>
<th></th>
<th>Daily figures of 7-day deposits rates (annual terms)</th>
<th>Daily figures of 30-day deposits rates (annual terms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>.141</td>
<td>138</td>
</tr>
<tr>
<td>S.D.</td>
<td>.056</td>
<td>.047</td>
</tr>
<tr>
<td>Minimum</td>
<td>.046</td>
<td>.046</td>
</tr>
<tr>
<td>Maximum</td>
<td>.421</td>
<td>.361</td>
</tr>
<tr>
<td>( p_1 ) (monthly)</td>
<td>.287</td>
<td>.482</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>.222</td>
<td>.235</td>
</tr>
<tr>
<td>( p_3 )</td>
<td>.072</td>
<td>.103</td>
</tr>
<tr>
<td>( p_4 )</td>
<td>-.069</td>
<td>-.008</td>
</tr>
<tr>
<td>Augmented Dickey–Fuller for daily data</td>
<td>-4.204</td>
<td>-4.242</td>
</tr>
</tbody>
</table>

\( p_j \) represents the \( j \)-th autocorrelation coefficient

The augmented Dickey–Fuller statistic equals \( \tau = \gamma / \text{S.D.}(\gamma) \) in the model: \( \Delta r_t = \alpha + \beta \tau - \gamma r_{t-1} + \sum_{j=2}^{p} b_j \Delta r_{t-j} + \varepsilon_t \) with \( p = 30 \) and the standard deviation of \( \gamma \) The critical values for this test are those of MacKinnon (1994) incorporated into “TSP” 4.4.

monthly series are fairly small from the third lag onwards (they are within the 95% confidence bands). This suggests that the series of the 7- and 30-day nominal rates are both stationary. Such a conjecture is confirmed by the augmented Dickey–Fuller tests that reject the null hypothesis of a unit root in the daily series.

It is interesting to notice the high correlation between the nominal interest rates series for the maturities being considered. For the daily data, the sample correlation is 80.2%, while for the monthly data this goes up to 93.7%. We carried out a bootstrap of both series to determine the empirical distribution of the correlation coefficient (Figs. 3 and 4). For the daily data, .773

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**Bootstrap for the Correlation Coefficient**

Fig. 3. Distribution of the sample correlation coefficient of the 7- and 30-day deposit rates (monthly figures).
Fig. 4. Distribution of the sample correlation coefficient of the 7- and 30-day deposit rates (daily figures).

Fig. 5. Estimated density of the 7-day deposit rate. Note The optimal bandwidth, $h^* = 0.011$, was found by cross-validation with the statistical package “XploRe.” The density estimation was done in “S-Plus 4.5” using a normal kernel $K(x) = (2\pi)^{-1/2} \exp(-1/2 x^2)$ and taking 600 equidistant points. A 90% confidence band was obtained by Künsch (1989) algorithm for a mobile block of size 100 and 40 iterations.
accumulates 5% of the mass of the distribution, while .829 accumulates 97.5%. For the monthly data, these figures correspond to .905 and .962, respectively. These computations suggest that there is some evidence supporting the hypothesis that the 7- and 30-day deposits rates are perfectly and positively correlated, as one would expect under the set-up of a one-factor model.

Fig. 5 shows the distribution of the 7-day nominal interest rate — our proxy for the instantaneous interest rate — computed nonparametrically using a normal kernel. The raw data has a mean of 14.1% and an S.D. of 560 basis points. The first quintile of the empirical distribution is located around 10.4 annual percent, while the third quintile is around 16.6 annual percent. The minimum value is 4.6 annual percent (January 1993) while the maximum value is 42.1 annual percent (November 1993). The confidence bands for the distribution were obtained by the Künsch algorithm of moving blocks (e.g., Efron & Tibshirani, 1994).

3.1. Implementation of the nonparametric approximation

One way to estimate the nonparametric approximations of the drift and volatility functions of Section 2 is by kernels. In particular, the conditional expectations of the first-order approximations of $\mu(r)$ and $\sigma(r)$ are given, respectively, by Eqs. (23) and (24):

$$E(r_{t+\Delta} - r_t \mid r_t = r) \approx \frac{\sum_{t-1}^{T} (r_{t+\Delta} - r_t)K[(r - r_t)/h]}{\sum_{t-1}^{T} K[(r - r_t)/h]}.$$  \hspace{1cm} (23)

$$E((r_{t+\Delta} - r_t)^2 \mid r_t = r) \approx \frac{\sum_{t-1}^{T} (r_{t+\Delta} - r_t)^2K[(r - r_t)/h]}{\sum_{t-1}^{T} K[(r - r_t)/h]}.$$  \hspace{1cm} (24)

where $h$ represents the bandwidth and $\Delta$ is the time length elapsed until a new observation of the interest rate is taken. If we assume that a year has approximately 250 working days, 1 day represents $\Delta = 0.004$ years. The above approximations can be obtained from the theory of nonparametric regressions (e.g., Härdle, 1990). In particular, if $E(Y \mid X=x) = g(x)$, where $g$ is an unknown function, the kernel estimator of $g$ is given by Eq. (25):

$$g_T(x) = [Thf_T(x)]^{-1} \sum_{t=1}^{T} Y_tK\left(\frac{x - X_t}{h}\right).$$  \hspace{1cm} (25)

where $T$ is the sample size and $f_T$ is an estimator of the density of the random variable $X$ [Eq. (26)]:

$$f_T(x) = (Th)^{-1} \sum_{t=1}^{T} K\left(\frac{x - X_t}{h}\right).$$  \hspace{1cm} (26)
Fig. 6(a) shows daily changes of the 7-day rate as a function of the previous day rate. The line that crosses the cluster of points is our estimation of Eq. (23). This is depicted more clearly in Fig. 6(b), where we also include approximated 95\% confidence bands. The estimation shows that the instantaneous interest rate presents mean reversion, that is, for low interest rates the drift is positive (although close to zero), whereas for high interest rates the drift is negative. It is noticeable that the confidence bands considerably widen in the presence of unusually high interest rates.

![Graph showing daily changes of the 7-day rate as a function of the previous day rate. The line that crosses the cluster of points is our estimation of Eq. (23). This is depicted more clearly in Fig. 6(b), where we also include approximated 95\% confidence bands. The estimation shows that the instantaneous interest rate presents mean reversion, that is, for low interest rates the drift is positive (although close to zero), whereas for high interest rates the drift is negative. It is noticeable that the confidence bands considerably widen in the presence of unusually high interest rates.](image)

Fig. 6. First-order approximation of the drift function by kernel smoother. Note: Panel (b) shows the drift function (daily base) with approximated 95\% confidence bands. These were calculated as \( m_n(r) = c_n \sqrt{\hat{\sigma}^2 / \sqrt{nhf_n(r)}} \), where \( m_n(r) \) is the nonparametric estimate of \( \mu(r) \), \( c_n \) is the \((100 - \alpha)\) percentile of the normal distribution, \( \hat{\sigma}^2 = \frac{1}{\sqrt{4\pi}} \) for the kernel, \( n \) is the sample size, \( h \) is 0.09 is the bandwidth, \( f_n(r) \) is the nonparametric estimate density of the 7-day rate, \( \sigma^2_n(r) = n^{-1} \sum_{i=1}^{n} [W^\alpha_i(r)]^2 \), where \( W_t^\alpha = K_\alpha(r - R_t)Y_t \), where \( K_\alpha(r) \) is the daily change observed in the 7-day rate and \( W_t^\alpha = K_\alpha(r - R_t)Y_t \) (see Handie, 1990).
In order to obtain an alternative estimate, we fitted the drift function, \( \mu(r) \), by cubic splines. These functions behave very much like kernels but arise as the solution, \( \hat{f} \), of the minimization of the penalized sum of square residuals [Eq. (27)]:

\[
\sum_{i} (y_i - f(x_i))^2 + \eta \int (f''(t))^2 \, dt.
\]  

over all functions with continuous first and integrable second derivatives (see Härdle, 1990). The parameter \( \eta \) is similar to the bandwidth \( h \). In this case, \( y_i \) stands for the daily change in the 7-day rate and \( f \) stands for the estimate of the first-order approximation of \( \mu(r) \).

Fig. 7 shows our estimate of \( \mu(r) \) obtained by this alternative nonparametric technique. The estimated drift shows now a smoother negative trend, but the mean reversion of the instantaneous rate is still evident. The 95\% confidence bands, built with the jackknife residual, also widen, as extremely high interest rates become atypical.

Figs. 8(a and b) show a first-order approximation of the instantaneous variance obtained using a kernel smoother. As we see, this function is increasing in the instantaneous interest rate. The sharp decrease experienced by the drift as the instantaneous rate becomes larger prevents the interest rate from diverging, despite its increasing volatility. Fig. 9 shows our estimate obtained by cubic splines. This alternative method also yields an increasing variance function, but with a smoother trend than that previously observed. In this case, an eventual decline in volatility for extreme observations is not so evident.

It is important to mention that the one-factor model proposed by Chan et al. (1992) leads to a nonstationary dynamics of the short interest rate (see Jiang, 1998, for a rigorous proof). Parisi (1998) fitted Chan et al.’s model to 30-day PDBC monthly data and found that a linear drift and a concave volatility function may describe the dynamics of the data. Our

Fig. 7. First-order approximation of the drift function by a spline smoother. Note: Estimation was done with a smoothing parameter \( \eta = 0.01 \) (drift measured in daily terms). The 95\% confidence bands were obtained using the jackknife residual computed by “S-Plus 4.5.” This is defined as the ratio of (i) the difference between the weighted actual value of the dependent variable and its fitted value to (ii) the difference of 1 minus the corresponding element of the main diagonal of the smoothing matrix.
Fig 8. First-order approximation of the variance function by kernel smoother. Note: Panel (b) shows the drift function (daily base) with approximated 95% confidence bands. These were calculated as \( \hat{m}_h(r) \pm c_{\alpha} \sqrt{c_{\alpha} \hat{\sigma}^2(r)/\sqrt{n}h_n(r)} \), where \( \hat{m}_h(r) \) is the nonparametric estimate of \( \sigma^2(r) \), \( c_{\alpha} \) is the \( (100 - \alpha) \) percentile of the normal distribution, \( c_{\alpha} = 1.96 \) for the normal kernel, \( n \) is the sample size, \( h = 0.07 \) is the bandwidth, \( f_h(r) \) is the nonparametric estimate density of the 7-day rate, \( \hat{\sigma}^2(r) = \frac{1}{n} \sum_{i=1}^{n} W_i(r) |Y_i - \hat{m}_h(r)|^2 \), where \( Y_i \) is the daily change observed in the 7-day rate and \( W_i(r) = K_h(r - R_i) \), see Hardle, 1990.

Nonparametric estimation, however, suggests that the volatility is an increasing function not necessarily concave but that the drift function is nonlinear.

Fig. 10 shows our first-order approximation by kernels of the market price of interest rate risk obtained by Jiang’s (1998) formula. As we see, the market price of interest rate risk is always negative and shows a decreasing trend. In addition, the function is clearly nonlinear. We should point out that arbitrary functional forms for the market price of interest rate risk are likely to lead to misspecification errors. In particular, Chan et al.’s model estimated by Parisi for Chilean data assumes that the market price of interest rate risk is zero for all levels of the interest rate. Our estimation suggests that this is small but different from zero.
It is evident from Fig. 10 that the market price of interest rate risk decreases more rapidly as the interest rate increases. Intuitively, for high interest rates, it is more likely that the rates go down due to the mean reversion process. This leads the prices of assets to be negatively correlated with the interest rate to increase (e.g., stocks and bonds). Therefore, investors are more willing to bear risk from assets whose prices are positively correlated with the interest rate.

3.2. Monte Carlo simulations to determine the term structure of nominal rates

In this section, we illustrate how to implement our nonparametric estimation to simulate the term structure of interest rates. First, notice that the dynamics of the instantaneous interest rate in a risk-neutral world is given by:

\[ d \tilde{r}_t = (\mu(\tilde{r}_t) - \lambda(\tilde{r}_t)\sigma(\tilde{r}_t))dt + \sigma(\tilde{r}_t)d\tilde{Z}_t. \]  \hspace{1cm} (28)

where \( \tilde{r} \) denotes the short rate and \( d\tilde{Z} \) represents the increment of a standard Wiener process in a risk-neutral world. Such a transformation is valid in virtue of Girsanov’s theorem (see, for instance, Duffie, 1996). Intuitively, in a risk-neutral world the drift of the interest rate equals the risk-free rate [see Eq. (19)].

In order to obtain the term structure, we have to compute the price of a zero-coupon bond for different maturities:

\[ P(r, t, T) = E_t^F \left[ \exp\left( - \int_t^T \tilde{r}_u du \right) | \tilde{r}_t = r \right]. \]  \hspace{1cm} (29)
Fig. 10. Market price of interest risk Note: This function was obtained by computing first-order approximations of each component of Eq (28) with kernel smoothers. The market price of interest risk is measured in daily terms.

From Eq. (8) we know that:

\[ f(r_t) = f(r_0) + \frac{\mathbf{L}f(r_0)(Z_t - Z_0)}{1 + \frac{\mathbf{L}f(r_0)}{2}} (Z_t - Z_0)^2 - t + O(t^{3/2}). \]

Letting \( f(r_t) = r_t \) yields:

\[ r_t = r_0 + \sigma(r_0)(Z_t - Z_0) + \mu(r_0)t + \frac{1}{2} \sigma^2(r_0) \left\{ (Z_t - Z_0)^2 - t \right\} + O(t^{3/2}). \]

(Milshtein 1978) designed a scheme to simulate the path of \( r_t \) starting from a given point, which is based upon the approximation in Eq. (30) expressed in terms of the risk-neutral measure:

\[ \tilde{r}_t = \tilde{r}_0 + \sigma(\tilde{r}_0)(\tilde{Z}_t - \tilde{Z}_0) + \left\{ \mu(\tilde{r}_0) - \kappa_0(\tilde{r}_0)\sigma(\tilde{r}_0) \right\}t + \frac{1}{2} \sigma^2(\tilde{r}_0) \left\{ (\tilde{Z}_t - \tilde{Z}_0)^2 - t \right\} + O(t^{3/2}). \]

In order to evaluate Eq. (29), we divide the time interval \([0, T]\) into \( n \) subintervals. To simplify the notation, we let \( t = 0. \) Then Eq. (31) can be rewritten as [Eq. (32)]:

\[ \tilde{r}_{(m-1)T/n} = \tilde{r}_{mT/n} + \sigma(\tilde{r}_{mT/n})(\tilde{Z}_{(m-1)T/n} - \tilde{Z}_{mT/n}) + \left\{ \mu(\tilde{r}_{mT/n}) - \kappa_0(\tilde{r}_{mT/n})\sigma(\tilde{r}_{mT/n}) \right\} T/n + \frac{1}{2} \sigma^2(\tilde{r}_{mT/n}) \left\{ (\tilde{Z}_{(m-1)T/n} - \tilde{Z}_{mT/n})^2 - T/n \right\} + O\left( \left( \frac{T}{n} \right)^{3/2} \right). \]
Table 4

Simulations of prices of zero-coupon bonds with maturities of 0.5, 1, and 1.5 years

<table>
<thead>
<tr>
<th>Instantaneous rate (annual)</th>
<th>Price, $T=0.5$ years</th>
<th>Implicit interest rate (annual)</th>
<th>Price, $T=1$ year</th>
<th>Implicit interest rate (annual)</th>
<th>Price, $T=1.5$ years</th>
<th>Implicit interest rate (annual)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.961 (0.017)</td>
<td>7.91</td>
<td>0.899 (0.019)</td>
<td>10.56</td>
<td>0.854 (0.021)</td>
<td>10.53</td>
</tr>
<tr>
<td>10</td>
<td>0.945 (0.015)</td>
<td>11.36</td>
<td>0.879 (0.018)</td>
<td>12.88</td>
<td>0.834 (0.022)</td>
<td>12.12</td>
</tr>
<tr>
<td>15</td>
<td>0.931 (0.018)</td>
<td>14.36</td>
<td>0.864 (0.019)</td>
<td>14.64</td>
<td>0.820 (0.022)</td>
<td>13.26</td>
</tr>
<tr>
<td>25</td>
<td>0.909 (0.025)</td>
<td>18.88</td>
<td>0.836 (0.030)</td>
<td>17.94</td>
<td>0.794 (0.026)</td>
<td>15.34</td>
</tr>
</tbody>
</table>

S.D.s are between parenthesis.

with $r_0 = r, m = 0, 1, \ldots, n-1$. In order to simulate the path of $(r_t, 0 \leq t \leq T)$, we generate the families of independent random variables $(\bar{Z}_{T,n}, \bar{Z}_{T,n} - \bar{Z}_{T,n}, \ldots, \bar{Z}_{T} - \bar{Z}_{T-1/n})$, where

![Fig. 11. Term structures of nominal rates. (a) Instantaneous rate of 5%. (b) Instantaneous rate of 10%. (c) Instantaneous rate of 15%. (d) Instantaneous rate of 25%.](image)
each component is distributed as $N(0, T/n)$. For $(mT/n) \leq t \leq (m + 1)T/n$, $m = 0, \ldots, n - 1$, $r_t$ is given by Eq. (33):

$$
\tilde{r}_t = \tilde{r}_{mT/n} + \sigma(\tilde{r}_{mT/n})(\tilde{Z}_t - \tilde{Z}_{mT/n}) + \left\{ \mu(\tilde{r}_{mT/n} - \lambda_0(\tilde{r}_{mT/n})\sigma(\tilde{r}_{mT/n}) \right\}(t - T/n)
+ \frac{1}{2} \sigma^2(\tilde{r}_{mT/n}) \left\{ (\tilde{Z}_t - \tilde{Z}_{mT/n})^2 - (t - T/n) \right\} + O\left( \left( \frac{T}{n} \right)^{\frac{3}{2}} \right),
$$

(33)

Table 4 shows simulations of the price of a zero-coupon bond with maturities of 6 months, 1 year, and 1.5 years for different initial values of the instantaneous nominal rate. For relative low interest rates — between 5% and 10% — the term structure is increasing for maturities up to 1 year. However, the mean reversion process of the data begins to show for annual instantaneous rates of 15% or higher. For example, for an instantaneous annual rate of 25%,

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the term structure of nominal rates is decreasing, although it is relatively flat for maturities between 6 months and 1.5 years. Fig. 11(a-d) illustrates different term structures for maturities between 0 and 2 years, depending on the initial level of the instantaneous rate.\(^4\)

In order to test the accuracy of our estimation, we present some economic indicators for Chile for the time period 1993–1997 (Table 5). For 90–365-day nonindexed deposits, the data shows that the average rate reverts to an annual mean of approximately 15% (with an estimated S.D. of 230 basis points). Such behavior is not far from our estimation for a 1-year zero-coupon bond, at least for an instantaneous annual rate between 10% and 25%. In particular, our simulations show that, for an instantaneous rate within that range, the 1-year average nominal interest rate is 15.2% (annual terms) with an estimated S.D. of 210 basis points.

Due to indexation, most bonds issued by the Central Bank of Chile are linked to past inflation. The maturity of this debt ranges from 90 days to 20 years. The Central Bank issues nominal bonds only for maturities of 42, 90, and 360 days. Fig. 12 shows the evolution of the term structure of indexed bonds issued between February 1994 and December 1997. As we see, every month the term structure of real rates is in general inverted, although increasing in some ranges. This pattern is also observed in nominal interest rates for our sample period, as discussed above.

One interesting application of our methodology is the prediction of inflation. Table 6 shows predicted annual CPI percent variation for different time horizons using the simulated term structure of nominal interest rates when the instantaneous rate is 15% (close to the average observed for the 7-day interest rate for 1993–1997), and an average estimate of the term structure of indexed bonds (after the coupon effect has been isolated). Our estimate of the CPI percent variation over a 1-year horizon is 7.92%, which is not far off the average observed for 1993–1997: 8.38%. It is interesting to notice that expected inflation decreases with time-horizons of over one year. This probably reflects the process of declining inflation Chile has experienced over the last few years.

It still remains the issue of explaining why Chile’s term structure is eventually downward sloping. There are different theories found in the literature aimed at explaining the shape of the term structure of the interest rates (e.g., Kozicki, 1997). In particular, there is a theory, which links the interest rates spread and the business cycles, that states that the spread reflects the current stance of monetary policy. For example, a low spread (even negative) is indicative of a tight monetary policy. Such an interpretation is based upon the fact that long interest rates are a weighted average of future short rates plus a risk or term premium.

Based upon the above theory and our estimation, it is possible that real and nominal rates are eventually decreasing with maturity because they revert to a mean that reflect medium- and long-term goals of monetary policy of Chile’s Central Bank.\(^5\) In this regard, Parisi (1998) suggests that the mean reversion process of interest rates may be attenuated as

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\(^4\) We did not consider over 2-year maturities because, due to indexation, most financial instruments for maturities over 2 years are denominated in indexed ("real") rates.

\(^5\) There are no limits or direct intervention from the Central Bank in the clearing of market interest rates in Chile. The only exception is the limit established for the maximum interest rate applicable to credit operations. However, in practice, commercial banks have always operated below that level (Magendzo, 1997).
Table 5
Some economic indicators for Chile, 1993-1997

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>7-Day nominal rate paid on bank deposits</td>
<td>17.69</td>
<td>14.53</td>
<td>13.36</td>
<td>12.96</td>
<td>12.34</td>
<td>14.18</td>
</tr>
<tr>
<td>30-Day nominal rate paid on bank deposits</td>
<td>17.15</td>
<td>14.40</td>
<td>13.05</td>
<td>12.90</td>
<td>11.69</td>
<td>13.84</td>
</tr>
<tr>
<td>60-Day nominal rate paid on bank deposits</td>
<td>16.10</td>
<td>12.87</td>
<td>11.50</td>
<td>11.69</td>
<td>10.14</td>
<td>12.46</td>
</tr>
<tr>
<td>90-365-Day nominal rate paid on bank deposits</td>
<td>18.87</td>
<td>14.79</td>
<td>12.08</td>
<td>15.80</td>
<td>13.54</td>
<td>15.01</td>
</tr>
<tr>
<td>1-Year nominal rate paid on bank deposits</td>
<td>18.58</td>
<td>17.27</td>
<td>14.02</td>
<td>13.14</td>
<td>12.36</td>
<td>15.07</td>
</tr>
<tr>
<td>90-365-Day rate paid on indexed bank deposits</td>
<td>6.41</td>
<td>6.38</td>
<td>5.85</td>
<td>6.94</td>
<td>6.45</td>
<td>6.41</td>
</tr>
<tr>
<td>Annual percent variation in the CPI</td>
<td>12.2</td>
<td>8.9</td>
<td>8.2</td>
<td>6.6</td>
<td>6.0</td>
<td>8.38</td>
</tr>
<tr>
<td>Annual percent variation in the GDP</td>
<td>7.0</td>
<td>5.7</td>
<td>10.6</td>
<td>7.4</td>
<td>7.1</td>
<td>7.56</td>
</tr>
</tbody>
</table>

Source: Bloomberg and Central Bank of Chile (1998). The rates paid on deposits correspond to annual averages of observations recorded as monthly average for all commercial banks.

the Central Bank issues longer-term nonindexed instruments that allow for a better hedging of unexpected inflation.

Fig. 12 Term structure of Central Bank indexed bonds. Source: Own computations based upon data published by the Central Bank of Chile at its website www.bcentral.cl. The sample period covers February 1994 through December 1997. The interest rates were adjusted by the coupon bond effect by linear interpolations. The computations were carried out with data on indexed bonds with maturities of 90 days, 360 days, 8, 10, 12, 14, and
Table 6
Prediction of inflation based on nominal and real interest rate term structure

<table>
<thead>
<tr>
<th>Time horizon (semesters)</th>
<th>Expected CPI percent variation (annual terms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.6</td>
</tr>
<tr>
<td>2</td>
<td>7.92</td>
</tr>
<tr>
<td>3</td>
<td>7.71</td>
</tr>
<tr>
<td>4</td>
<td>6.87</td>
</tr>
</tbody>
</table>

Source: Own elaboration based on estimation reported in this article

An alternative explanation to a downward-sloping term structure might be found in the theory of preferred habitats. This states that investors prefer specific maturity ranges but that they can be induced to switch if risk premiums are high enough. In the particular case of Chile, the existence of a downward-sloping term structure may be related to the creation of the new pension system in 1981. Indeed, this gave rise to an increasing demand for long-maturity financial instruments, particularly from life insurance companies. These are enforced to hedge unexpected fluctuations in the value of their liabilities with extra capital. The existence of long-maturity instruments has made hedging simpler, and therefore mandatory increments in capital have become less frequent. One would conjecture that this might translate into a lower (or even negative) spread.

Table 7 presents the composition of the portfolio of general and life insurance companies for March 1998. As we see, a sizeable percentage of their total investment corresponds to financial instruments issued by the public sector (e.g., Central Bank and government): 29% for general insurance companies and 38% for life insurance companies.

It is important to mention that such a percentage is also high for pension funds companies (AFPs). Indeed, 31% of AFP’s portfolios is invested on indexed bonds issued by the Central Bank (source: Superintendency of Pension Fund Managing Entities, June 1999). Such a portfolio composition would support our alternative hypothesis that the demand for long-maturity instruments from these institutions would cause bond prices to increase and therefore bond yields to decrease, leading to an eventually downward-sloping term structure. But, of course, this subject deserves further research.

4. Conclusions

The term structure of interest rates term has been widely studied in the fields of theoretical and applied finance. Several authors have resorted to parametric one-factor models to

Table 7
Portfolio of general and life insurance companies, March 1998 (thousands of Chilean pesos)

<table>
<thead>
<tr>
<th>Type of company</th>
<th>Public sector bonds</th>
<th>Commercial bank deposits</th>
<th>Stocks</th>
<th>Mutual funds</th>
<th>Total investment</th>
</tr>
</thead>
<tbody>
<tr>
<td>General insurance</td>
<td>48,260,425</td>
<td>32,381,044</td>
<td>6,023,383</td>
<td>2,191,346</td>
<td>166,007,525</td>
</tr>
<tr>
<td>Life insurance</td>
<td>1,541,243,442</td>
<td>63,111,000</td>
<td>230,367,579</td>
<td>3,913,037</td>
<td>4,017,754,292</td>
</tr>
</tbody>
</table>

Source: Superintendency of Pension Fund Managing Entities
describe the dynamics of the short rate, and infer possible functional forms of the term structure. Recently, however, Aït-Sahalia (1996a, 1996b), Jiang (1998), and Stanton (1997) have shown that nonparametric techniques may be preferable.

In this article, we have presented a study on the dynamics of short nominal rates in Chile for the time period December 1992 - April 1998. Inspired by the new trend in the finance literature, we have fitted a nonparametric one-factor model obtained from stochastic Taylor series expansions. The Milshstein scheme enables us to carry out some simulations that suggest that one could explain Chile's inverted term structure by the mean reversion process observed in the data for the time period of the study. We believe that this mean reversion process may reflect medium- and long-term goals of monetary policy of the Central Bank of Chile. As the Central Bank issues longer-term nonindexed instruments that allow for a better hedging of unexpected inflation, the mean reversion of the data may be attenuated.

However, some alternative explanations, such as that of the preferred habitats, might be also plausible. Indeed, the existence of a downward-sloping term structure may be related to the creation of the new pension system in 1981, which gave rise to an increasing demand for long-maturity financial instruments.

References


