The Role of Optimal Threats in Auction Design*

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Abstract

This paper studies revenue-maximizing auctions when buyers' outside options depend on their private information and are endogenously chosen by the seller. We show that the revenue-maximizing assignment of the object can depend crucially on the outside options that the seller can choose as threats. Depending on the shape of outside options, sometimes an optimal mechanism allocates the object in an ex-post efficient way, and, other times, buyers obtain the object more often than is efficient. Keywords: *Externalities, Optimal Auctions, Type-Dependent Outside Options: JEL D44, C7, C72.*

In this note, we consider the problem of finding an optimal selling mechanism when a buyer's payoff depends on the final allocation of the object, even if he does not win it, and the seller, as in [4], optimally chooses how to threaten a buyer in the event of non-participation. In contrast to that paper, in our setup, buyers' non-participation payoffs depend on their private information. Jehiel, Moldovanu and Stacchetti [5] are the first to allow for private information in the non-participation payoffs. They do so in a multi-dimensional setting, but, due to the complexity of the problem, they describe optimal mechanisms out of a specific class.

Without putting any restrictions on the mechanisms, we show that the revenue-maximizing assignment of the object (and not only the payments) depends heavily on what the seller can do in case a buyer does not participate (the "threats," in the language of [4]). There are cases in which the seller can strictly increase *both* revenue and efficiency by choosing outside options appropriately. Practitioners seem to know this possibility, as is suggested by the design of the U.K. spectrum auctions (see, for instance, [7]). In general, depending on the shape of outside payoffs, it is possible that the revenue-maximizing allocation involves under- or overselling compared with the socially desirable level, or it can be even ex-post efficient. Therefore, when non-participation payoffs are type-dependent and the seller can manipulate them through the choice of the mechanism, the classical trade-off between revenue maximization and efficiency is sometimes not present, and, other times, the optimal assignment involves inefficiencies of a different nature: The monopolist oversells instead of underselling. We illustrate these phenomena with purposely chosen simple examples.[1]

[1] The complete characterization with type-dependent outside options is quite challenging and beyond the scope of this note. In the working-paper version of this paper ([2]), we examine a general setup, analyze and discuss the intricacies of the problem, and identify cases in which the solution can be found explicitly. We employ that approach in Examples 1 and 2 in this paper.

In Example 1, we show that the seller, by changing the outside options that buyers face, changes not only revenue, but also the way the good is allocated. This point is made by deriving the revenue-maximizing assignment of the object for *exogenously given* outside options and by noticing that it differs across different ones. Then, we derive the optimal mechanism allowing the seller to optimally choose threats, and we show that, at the optimum, there is overselling. This example is simple because optimal threats are deterministic and can be derived independently of the allocation the seller wants to implement. However, with type-dependent outside options this is not always true: As Example 2 demonstrates, optimal threats can depend on the allocation that the seller is considering, and can be random.[2] In this example, it turns out that the optimal mechanism is ex-post efficient. This shows that, with type-dependent outside options, it is possible that the goal of a revenue-maximizing seller is completely aligned with efficiency.[3]

Our examples show that, with type-dependent outside payoffs, the seller may extract more rent by simultaneously raising the price *and* selling more often: the dream of any monopolist! Intuitively, with type-dependent outside payoffs, the seller can design outside options that hurt bad types relatively more than good types. This allows her to charge a higher price without restricting supply; good types pay due to their high valuation, and bad types pay because their outside options

[2]Other papers (see [1,11,15]) show that it is optimal for the seller to randomize between different allocations. In those papers, randomization occurs at the optimal allocation and it relaxes the incentive constraints. In [14] randomization again occurs at the optimal allocation, but there it relaxes the participation constraints. In our paper, too, randomization relaxes the participation constraints, but it occurs "off-path," at the optimal threats. This is also true in [5] for reasons we explain later.

[3]Such an alignment appears also in revenue-maximizing auctions in which the number of bidders is endogenous and participation is costly. In such cases (see, for instance, [16]), the seller finds it optimal to raise no entry fees and to set a reserve price at her valuation in order to encourage entry to intensify competition.

are relatively worse.

Technically, with type-dependent outside options, the selection of the threat and the allocation rule together determine the critical type, the type where the participation constraint binds. The critical type is, therefore, endogenous (and not always equal to the "worst" type, as in [12] or in [4] or equal to the type closest to the origin, as in [5]). With type-dependent outside options, the virtual surplus of allocating the object to a buyer (actual surplus minus information rents) is modified for all types *worse* than the critical type: For all those types the information rents are reduced (or even eliminated); how much they are reduced depends on the shape of non-participation payoffs. Hence, the "modified virtual surplus" is weakly higher than the virtual surplus of an allocation (they coincide for all types if the critical type is the worst type) and can be equal to, strictly greater than or strictly less than the actual surplus of an allocation. Depending on this comparison, the revenue-maximizing mechanism may be ex-post efficient or may induce overselling or underselling.

Myerson [12] studies revenue-maximizing mechanisms of a single unit in an independent private-value environment, where each buyer's outside payoff is some fixed constant. This seminal contribution establishes that, at a revenue-maximizing auction, the seller gives the good to the buyer with the highest virtual surplus whenever it is above the seller's valuation. Because a buyer's virtual surplus is equal to his valuation minus information rents, optimal auctions usually sell less than what is efficient.

As mentioned earlier, Jehiel, Moldovanu and Stacchetti [4] are the first to consider the case in which a buyer's payoff depends on the final allocation, even if he does not win the object, and the seller can use the design of the mechanism to affect the buyers' participation constraints. In their setup there is incomplete information about a buyer's payoff from obtaining the object himself, but there is *complete information* about his payoffs when one of his opponents obtains the object (that is, non-participation payoffs are flat with respect to his type). In contrast to our work, they show that, by choosing the appropriate outside options, the seller increases *only revenue*; the optimal allocation of the good is never affected. Also, optimal threats are independent of the allocation the seller wants to implement and are deterministic,[4] whereas both decisions can be interdependent in our case, and optimal threats can be random. The reason behind these differences is that, in their paper, non-participation payoffs are type-independent.

Even though overselling never occurs at an *optimal* mechanism when outside payoffs are type-independent, it may occur at *specific classes* of mechanisms, as shown in [3]. They examine second-price auctions with reserve prices and present an example (Example 4.3) in which, at the optimum out of this class, trade takes place more often than is ex-post efficient. The reason is that the reserve price in the second-price auction also plays the role of threats: The lower it is, the higher is the chance that trade occurs, and buyers suffer external effects if some other buyer wins, which makes buyers bid more aggressively. An optimal mechanism exploits the externalities with non-participation threats, which enable the seller to extract payments from non-winners.

We noted in the beginning that Jehiel, Moldovanu and Stacchetti [5] are the first to allow for incomplete information in the buyers' outside payoffs, examining an optimal auction problem

[4] In their setup, the optimal threat is always to give the object to the "worst" opponent. To take advantage of that, it is enough for the seller to reduce the level of the participation payoffs by a constant amount by using an instrument such as an entry fee. Then, the critical type is always the type with the lowest participation payoffs - the "worst" type - which implies that the virtual surplus of an allocation is never "modified," which, in turn, is the reason why overselling does not occur at the revenue-maximizing mechanism when there are externalities but outside payoffs are flat.

with externalities, in which each buyer's type is a multidimensional vector, and each component denotes his payoff in case another buyer obtains the good. They restrict attention to mechanisms that satisfy two properties: (i)buyers make a one-dimensional bid, and (ii)types are divided into two regions- one where trade takes place with probability one, and one where trade takes place with probability zero. Even though their model is multidimensional, whereas we look at single-dimensional cases, our examples are not nested in their setup. One way to see this is that, in their paper, the critical type is *always* the one closest to the origin, which is not true in our examples.[5]

Even if their paper bears similarities to ours in terms of qualitative results, the forces behind these similarities are often quite different. Overselling occurs at the optimal mechanism out of the class they are considering (second-price auctions with reserve prices lower than the efficient ones can be optimal). The reason behind this overselling result is that when a buyer submits a single-dimensional bid, he has to think not only about how much the object is worth to him, but also about the externalities he will suffer if another buyer obtains the good (which depend on the other components of his type). If the seller sets a high reserve price, it becomes more likely that the seller will keep the good (a zero externality alternative), which reduces the expected externalities that a buyer faces, leading to a reduction in his bid. This effect can dominate the standard revenue-enhancing effects of posting a reserve price strictly above the sellers' value. In contrast, in our paper, overselling occurs when the critical type is close to the best type, which inflates the virtual surplus. Another similarity is that the optimal allocation rule and the optimal threats can be interdependent, and, moreover, optimal threats can be random. However, in the model of [5], randomization is needed when externalities may be positive *and* negative. In our

[5] For further explanation of the non-nestedness, see footnote 6.

model, randomization can be optimal even if externalities always have the same sign, and its role is to determine the shape of non-participation payoffs that allows the seller to extract the most surplus.

In addition to the above-mentioned papers, this paper is related to the literature on mechanism design with type-dependent outside options. Lewis and Sappington [10] study an agency problem with this feature. They show that the critical type is not necessarily the "worst" one, which mitigates the inefficiencies that arise from contracting under private information. This feature also appears sometimes in our analysis, but we also show that, at times, inefficiencies are not reduced, but they change in nature (the monopolist, instead of selling too little, sells too much). Jullien [6] uses a dual approach to characterize properties of the optimal incentive scheme such as the possibility of separation, non-stochasticity, etc. In this paper, we do not rely on dual methods, and we allow for multiple agents. Krishna and Perry [9] examine efficient auctions in a setup with type-dependent non-participation payoffs, whereas our focus is revenue maximization. However, the most important difference between our paper and these earlier works is that, as in [4], we allow the mechanism designer to affect the buyers' participation constraints.

1. Example 1

Two beer manufacturers, a European one, firm 1, and a north American one, firm 2, are competing for the sponsorship of the world cup final, where one of the teams participating is Asian. Each of these firms has a dominant position in their respective markets, which are both of size α, but none has any significant presence in Asia. The sponsorship will allow the firm that gets it to penetrate the Asian market that is worth 1 at a cost of building new capacity c_i, which is private information and is uniformly and independently distributed in $C_i \equiv [0, 1]$. Hence, the payoff from winning the auction is $1 - c_i$. If nobody obtains the sponsorship, each firm's profits will be the same as before, which we normalize to zero. However, if firm i does not obtain the sponsorship but firm j does, firm i will see its dominant position in its "home turf" threatened as j becomes a global player. However, this can be prevented (and it is profitable to do so) by investing in capacity. The cost of building new capacity in the home market is proportional to its size and is given by $-\alpha c_i$. Hence firm i suffers a negative externality (which depends on its cost of building capacity c_i) if its competitor gets the sponsorship.

Let Z denote the set of all possible allocations of the sponsorship. There are three possibilities: nobody sponsors the event, z_0; firm 1 does, z_1; or firm 2 does, z_2. Let $\pi_i^{z_j}(c_i)$ denote the payoff to firm i if allocation z_j is implemented and its type is c_i. The payoffs that accrue to each firm from each of these alternatives are:

$$\begin{array}{ll} \pi_1^{z_0}(c_1) = 0 & \pi_2^{z_0}(c_2) = 0 \\ \pi_1^{z_1}(c_1) = 1 - c_1 & \pi_2^{z_1}(c_2) = -\alpha c_2 \\ \pi_1^{z_2}(c_1) = -\alpha c_1 & \pi_2^{z_2}(c_2) = 1 - c_2 \end{array}.$$ [6]

[6] The payoffs of our Example 1 can be translated in the setup of [5] by defining $v_1 = (v_1^0 = 0, v_1^1 = 1 - c_1, v_1^2 = -\alpha c_1)$. The reason why their analysis does not apply to such cases is that it requires that the distributions of types have full support (in particular, the sufficiency part of their Proposition 1; Krishna and Maenner [8] present an example where the assumption of full support fails and sufficiency fails). This rewriting fails the full-support assumption, as positive weight is assigned only along the line $v_1^2 = -\alpha + 2v_1^1$ instead of all the rectangle $[-\alpha, 0] \times [0, 1]$. Notice that in other cases, too, it may be reasonable to use a model in which the externality suffered by a buyer is proportional to his efficiency (c_i). In such cases, too, (for the same reasons as in this example) the full support assumption would be violated.

Our objective is to find the allocation mechanism that maximizes the seller's revenue. The fact that a firm is affected if the other firm gets the sponsorship, even if it is not the winner, endows the seller with an additional instrument to boost revenue: appropriate outside options or "threats" in the event that a firm refuses to participate. We use p_{-i} to denote the outside option (or non-participation assignment rule) for firm i. The seller chooses $p_{-i} \in \Delta(Z^{-i})$, where $Z^{-i} \subset Z$ is the set of allocations that are feasible without i.[7]

As usual, we can appeal to the revelation principle and search among the direct revelation mechanisms (DRM) that satisfy truth-telling and voluntary participation. A DRM here consists of an assignment rule, a payment rule and a non-participation rule (outside option), (p, x, p_{-i}). For all $c = (c_1, c_2)$, $p(c) = (p^{z_0}(c), p^{z_1}(c), p^{z_2}(c))$ specifies the probability with which each allocation prevails, and $x_i(c)$, $i \in \{1, 2\}$ specifies an expected payment for each buyer.

The allocation and payment rules (p, x) determine firms' payoffs when they participate in the mechanism. For firm 1 that has cost c_1 and reports c_1', we have

$$U_1(c_1, c_1'; p, x) = \int_0^1 \left[p^{z_0}(c_1', c_2) \cdot 0 + p^{z_1}(c_1', c_2) \cdot (1 - c_1) + p^{z_2}(c_1', c_2) \cdot (-\alpha c_1) - x_1(c_1', c_2)\right] dc_2.$$

The outside options p_{-i}, $i \in \{1, 2\}$ determine their payoffs when they do not participate in the mechanism. For firm 1, we have

$$\underline{U}_1(c_1; p_{-1}) = p_{-1}^{z_0} \cdot 0 + p_{-1}^{z_1} \cdot (1 - c_1) + p_{-1}^{z_2} \cdot (-\alpha c_1).$$

Analogous expressions hold for firm 2.

[7] In general (e.g., when values are interdependent), it may be worthwhile to condition p_{-i} on the reports of all other buyers but i (it cannot depend on $i's$ report, since he is not around!), that is $p_{-i} : C_{-i} \to \Delta(Z^{-i})$. In the examples considered in this paper, values are private, and such generality is not needed.

A DRM (p, x, p_{-i}) is feasible if it satisfies (i) resource constraints, (ii) incentive compatibility constraints and (iii) voluntary participation.

Resource constraints require that, for all c, we have that $0 \leq p^z(c) \leq 1$ and $\Sigma_{z\in Z} p^z(c) = 1$.

Letting $V_i(c_i) = \max\limits_{c_i'} \int\limits_{C_{-i}} \left(\sum\limits_{z\in Z} p^z(c_i', c_{-i})\pi_i^z(c_i) - x_i(c_i', c_{-i}) \right) f_{-i}(c_{-i})dc_{-i}$,[8] necessary and sufficient conditions for incentive compatibility are that (a) the derivative of V_i (more precisely, a selection from its subgradient, which is single-valued almost surely) evaluated at the true type, that is,

$$P_i(c_i) \equiv \int\limits_{C_{-i}} \sum_{z\in Z} p^z(c_i, c_{-i}) \frac{\partial \pi_i^z(c_i)}{\partial c_i} f_{-i}(c_{-i})dc_{-i}, \tag{1}$$

is weakly increasing, and (b)

$$V_i(c_i) = V_i(1; p, p_{-i}) - \int\limits_{c_i}^{1} P_i(s)ds \text{ for all } c_i \in C_i. \tag{2}$$

An incentive-compatible mechanism satisfies voluntary participation if $U_i(c_i, c_i; p, x) \geq \underline{U}_i(c_i; p_{-i})$ for all $i \in I$ and for all $c_i \in C_i$.

We denote by $J_z(c)$ the *virtual surplus* of allocation z.[9] For this example, the virtual surpluses are given by

[8] In the general formulas, we use f_i to denote the distribution of c_i and f_{-i} to denote the distribution of c_{-i}.

[9] In general, the virtual surplus of allocation z is defined as

$$J_z(c) \equiv \sum_{i=1}^{I} [\pi_i^z(c_i) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i^z(c_i)}{\partial c_i}].$$

Notice that we sum over all buyers because an allocation may affect all of them, and not just the ones that obtain objects. Therefore, the virtual surplus of allocation z may depend on the *whole vector of types*. In the classical case, virtual valuations are buyer-specific (instead of being allocation-specific, as they are here): for buyer i, we have $J_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$, ($v_i$ is i's valuation for the object).

$$J_{z_0}(c) = 0;\ J_{z_1}(c) = 1 - 2c_1 - 2\alpha c_2;\ J_{z_2}(c) = 1 - 2c_2 - 2\alpha c_1. \tag{3}$$

By using standard arguments, we can write the seller's problem as:

$$\max_{p,p_{-1},p_{-2}} \int_0^1 \int_0^1 [p^{z_0}(c)J_{z_0}(c) + p^{z_1}(c)J_{z_1}(c) + p^{z_2}(c)J_{z_2}(c)]dc_1dc_2 - V_1(1,p,p_{-1}) - V_2(1,p,p_{-2}) \tag{4}$$

subject to:

$P_1(c_1) \equiv -\int_0^1 [p^{z_1}(c) + 2p^{z_2}(c)]dc_2$ be increasing

$P_2(c_2) \equiv -\int_0^1 [2p^{z_1}(c) + p^{z_2}(c)]dc_1$ be increasing

$V_i(c_i) \geq \underline{U}_i(c_i; p_{-i}),\ i = 1, 2$

$0 \leq p^{z_i}(c) \leq 1,\ i = 0, 1, 2$ and $\sum_{i=0}^{2} p^{z_i}(c) = 1$

Revenue is determined by the assignment rule p, which determines the shape of participation payoffs, and by the terms $V_i(1,p,p_{-i})$. We now explain why these terms depend both on the shape of participation payoffs, which are determined by p, and on the shape of the non-participation payoffs, which are determined by p_{-i}.

The seller has the freedom to choose p_{-i} to be such that the participation payoff for all types is weakly greater than the non-participation payoffs. At an optimum, there exists a type, which we call "critical type"[10] c_i^*, where participation payoffs are exactly equal to non-participation payoffs; that is,

$$V_i(c_i^*) = \underline{U}_i(c_i^*; p_{-i}). \tag{5}$$

Then, using (5), (2) can be rewritten as

$$V_i(1; p, p_{-i}) = \underline{U}_i(c_i^*; p_{-i}) + \int_{c_i^*}^{1} P_i(s)ds. \tag{6}$$

[10] In general, there can be many critical types and any one can be chosen to stand for c_i^*.

To illustrate how $V_i(1, p, p_{-i})$ can depend on the outside options that buyers face, and how this can affect the optimal mechanism, we proceed as follows. First, we fix $p'_{-i}s$ and show how the optimal allocation rule p depends on $p'_{-i}s$. Then, we let the seller choose $p'_{-i}s$ optimally.

Optimal Allocation Rules for Fixed $p'_{-i}s$

We examine two scenarios.

Scenario 1: Flat Outside Options

In this case, if a firm does not participate, the seller withdraws the possibility of sponsorship; that is, $p_{-1} = p_{-2} = (1, 0, 0)$. Given this non-participation assignment rule, the payoff to firm i from not participating is $\pi_i^{z_0}(c_i) = 0$, which is independent of $i's$ type. Because at an incentive-compatible assignment rule V_i is decreasing in c_i, the participation constraint binds at the "worst" type $c_i^* = 1$ for all i and all p. This implies immediately that $V_1(1, p, p_{-1}) = V_2(1, p, p_{-2}) = 0$, and after substituting for the $J'_z s$ from (3), (4) becomes

$$\max_p \int_0^1 \int_0^1 [p^{z_1}(c)\,(1 - 2c_1 - 2\alpha c_2) + p^{z_2}(c)\,(1 - 2c_2 - 2\alpha c_1)] dc_1 dc_2. \tag{7}$$

The revenue-maximizing assignment[11] is obtained via pointwise maximization and is depicted in Figure 2a for the case where $\alpha = 2$.[12]

[11] It is

$$p(c) = \begin{cases} (0,1,0) & \text{if } c_2 \leq c_1 \text{ and } 1 \geq 2c_1 + 2\alpha c_2 \\ (0,0,1) & \text{if } c_1 \leq c_2 \text{ and } 1 \geq 2c_2 + 2\alpha c_1 \\ (1,0,0) & \text{if } 2c_1 + 2\alpha c_2 > 1 \text{ and } 2c_2 + 2\alpha c_1 > 1, \end{cases}$$

which, as can be routinely verified, is incentive-compatible and, hence, optimal.

[12] The optimal payment rule can be calculated from the optimal assignment rule using $x(c) = \sum_{z \in Z} p^z(c)\pi^z(c) + \int_c^1 \sum_{z \in Z} \frac{\partial \pi^z(s)}{\partial s} p^z(s) ds - V(1; p, p_{-A})$. Proposition 2 in [2] establishes why doing so is optimal and satisfies the participation constraints.

Scenario 2: Type-Dependent Outside Options

In this case, if a firm fails to participate, the seller gives the sponsorship to its competitor, the other firm - that is, $p_{-1} = (0,0,1)$ and $p_{-2} = (0,1,0)$. Here, we analyze the case in which the negative externality is very steep with respect to type c_i, in the sense that $\alpha > 1$. Then, allocation z_j gives the steepest payoff function to buyer i, for $j \neq i$ - that is,

$$\frac{d\pi_i^{z_j}(c_i)}{dc_i} \leq \frac{d\pi_i^{z}(c_i)}{dc_i} \leq 0 \text{ for all } z \in \{z_0, z_1, z_2\} \text{ and all } c_i \in C_i, \tag{8}$$

which implies that the participation constraint binds at $c_i^* = 0$ irrespective of p. This is so because $V_i(c_i)$ is a convex combination of payoff functions π^{z_0}, π^{z_1} and π^{z_2}, and has, therefore, a less negative slope than $\underline{U}_i(c_i; p_{-i}) \equiv -\alpha c_i$ when $\alpha > 1$ (see Figure 1, which depicts the case of $\alpha = 2$).

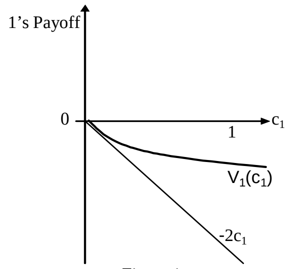

Figure 1

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Then, for $i, j = 1, 2$ (6) becomes

$$V_i(1) = \pi_i^{z_j}(0) + \int_0^1 P_i(c_i)dc_i = 0 + \int_0^1 \int_0^1 [-p^{z_i}(c) - \alpha p^{z_j}(c)]dc. \tag{9}$$

Substituting (9) and (3) in (4), it becomes

$$\max_p \int_0^1 \int_0^1 [p^{z_1}(c)\,(2 + \alpha - 2c_1 - 2\alpha c_2) + p^{z_2}(c)\,(2 + \alpha - 2c_2 - 2\alpha c_1)]dc_1 dc_2. \tag{10}$$

By comparing (7) and (10), we see how the terms $V_1(1, p, p_{-1})$ and $V_2(1, p, p_{-2})$ affect the objective function. They essentially modify virtual surpluses to:

$$\hat{J}_{z_0}(c) = 0;\ \hat{J}_{z_1}(c) = 2 + \alpha - 2c_1 - 2\alpha c_2;\ \hat{J}_{z_2}(c) = 2 + \alpha - 2c_2 - 2\alpha c_1.^{13}$$

Again, the revenue-maximizing assignment[14] of this problem is obtained via pointwise maximization, and it is depicted in Figure 2b for the case that $\alpha = 2$. Payments are derived as discussed in the previous case.

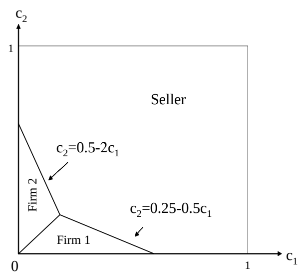

Figure 2a: Optimal Assignment in Scenario 1

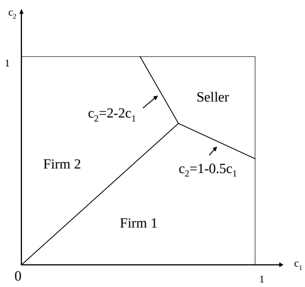

Figure 2b: Optimal Assignment in Scenario 2

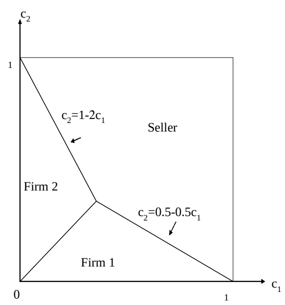

Figure 2c: Efficient Assignment

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The Optimal Mechanism

In this example, it is immediate to see that irrespective of p, the optimal way for the seller to threaten a firm is, in the event of non-participation, to give the sponsorship to the other firm. This guarantees the lowest payoff for firm i for any cost realization, and, hence, the mechanism we

[13] In this example, virtual surpluses are modified for all types, as all types are worse than the most efficient type 0.

[14] In this case, it is

$$p(c) = \begin{cases} (0,1,0) & \text{if } c_2 \leq c_1 \text{ and } 2\alpha \geq 2c_1 + 2\alpha c_2 \\ (0,0,1) & \text{if } c_1 \leq c_2 \text{ and } 2\alpha \geq 2c_2 + 2\alpha c_1 \\ (1,0,0) & \text{if } 2c_1 + 2\alpha c_2 > 2\alpha \text{ and } 2c_2 + 2\alpha c_1 > 2\alpha, \end{cases}$$

which, as can be routinely verified, is feasible and, hence, optimal.

derived in Scenario 2 is truly optimal. Notice that a seller that fails to consider the nature of the optimal threats would not only earn less money, but would also assign the sponsorship in a very different way.

Remark 1 *By comparing Figures 2a and 2b, we see that the optimal assignment rule critically depends on the outside options that each firm faces. The reason is that, in the second scenario, when firm i fails to participate, its payoff depends on its cost,* $(\pi_i^{z_j}(c_i) = -2c_i)$.

Remark 2 *It is also interesting to compare the optimal allocations in the above scenario with the ex-post efficient allocation rule*[15] *depicted in Figure 2c again for the case that* $\alpha = 2$. *Comparing Figures 2b and 2c, we see that at the revenue-maximizing assignment rule, firms 1 and 2 obtain the sponsorship for cost realizations where efficiency dictates that the seller should keep it. Thus, the seller sells the sponsorship too often compared to what would have been ex-post efficient. On the other hand, by comparing Figures 2a and 2c, we see that the seller sells less often than what efficiency dictates. Both allocations are inefficient but in different ways.*

With type-independent outside payoffs, there is always one outside option that is unambiguously best. This was also the case in this example. However, this example differs from the type-independent case because the critical type, instead of being the worst type, is the best type. This leads to a modification of the virtual surplus, which ultimately affects the way objects are allocated.

[15] The ex-post efficient allocation rule is the one that maximizes the social surplus $\pi_1^{z}(c) + \pi_2^{z}(c)$ for each (c_1, c_2). It is given by

$$p^e(c) = \begin{cases} (0,1,0) & \text{if } c_2 \leq c_1 \text{ and } c_1 + 2c_2 \leq 1 \\ (0,0,1) & \text{if } c_1 \leq c_2 \text{ and } 2c_1 + c_2 \leq 1 \\ (1,0,0) & \text{otherwise} \end{cases}.$$

This modification of the virtual surplus is the channel through which the optimal assignment takes advantage of the shape of outside options. In this example, this is achieved by putting more weight on type-sensitive options than on flat options, as in the option of giving the object to the seller, leading to overselling.

To summarize, outside options affect the optimal assignment rule only if the payoffs from non-participation are type-dependent. In the next example, we show how, in the case of type-dependent outside options, the seller can increase *both revenue and efficiency* by choosing threats optimally.

2. Example 2

Consider a small company in Silicon Valley that develops a valuable new technology. This company does not have the necessary infrastructure to reap the benefits of this new technology, so it is essentially worthless to it. There is, however, a large firm (say, firm A) that is willing to purchase the technology. The value of the new technology to firm A is given by $5 - 5c$, where c is private information and uniformly distributed on [0,1]. If A does not get the technology and no one else does either, A's payoff is zero. From [12] or [13], we know that the best the developer can do is to make a take-it-or-leave-it offer to firm A of 2.5. Then, company A will get the invention only if its cost parameter is below $\frac{1}{2}$. This maximizes ex-ante expected revenue, which is 1.25, but it is inefficient, because the developer is stuck half the time with a worthless invention, whereas company A would generate a non-negative payoff for all cost realizations.

Now, suppose that the developer can make the invention publicly available by making it open-source, in which case A's payoff is $1 - 10c$. A very efficient firm A ($c < \frac{1}{10}$) would like the new invention to become publicly available, but a less efficient one ($c > \frac{1}{10}$) would suffer in its core business because of new competitors coming in. If the developer considers threatening firm A, if it were to drop out of the sale, which threat should it use? The answer is not obvious since the developer does not know firm A's cost parameter, so it does not know which alternative "hurts more". [16]

We now show that the optimal mechanism offers the invention for sale at a price of 4.5 and threatens firm A that if it does not participate, the seller will keep the invention with probability $\frac{1}{2}$ and make it open-source with probability $\frac{1}{2}$. Faced with this lottery, company A's expected outside payoff is $0.5 - 5c$. Firm A *always(!)* agrees to buy the invention at the asking price of 4.5 since, in that case, its payoff is $5 - 5c - 4.5 = 0.5 - 5c$, which is (weakly) greater than its outside payoff. Thus, the open-source option, though never implemented, has an extraordinary effect on the revenue-maximizing allocation. It guarantees a higher expected revenue of 4.5, and makes the mechanism efficient.

Formally, there are three allocations: firm A gets the exclusive rights, z_A; the developer (seller) keeps it, z_S; or the invention becomes open-source, z_O. The payoffs that accrue to firm A in each of these allocations are

$$\pi^{z_A}(c) = 5 - 5c,\ \pi^{z_S}(c) = 0,\ \pi^{z_O}(c) = 1 - 10c.$$

Following the same procedure used in the previous example, the seller's problem can be rewritten as

$$\max_{p,p_{-A}} \int_0^1 [p^{z_A}(c)\,(5 - 10c) + p^{z_O}(c)\,(1 - 20c)]dc - V(1, p, p_{-A}) \tag{11}$$

[16] This is in contrast to Example 1, where the choice of the optimal threat was obvious, since there was an option (give the slot to the competitor) that was unambiguously worse for all types.

subject to:

$-[5p^{z_A}(c) + 10p^{z_O}(c)]$ is increasing

$V_i(c_i) \geq \underline{U}_i(c_i; p_{-i}), i = 1, 2$

$0 \leq p^z(c) \leq 1$ for all $z \in \{z_A, z_S, z_O\}$ and $\Sigma_{z \in \{z_A, z_S, z_O\}} p^z(c) = 1$.

This example is more complex than Example 1 because, as we show below, the optimal threat depends on the assignment rule p that the seller wishes to implement. We solve this example in two steps. First, we find the optimal p_{-A} for a given assignment rule p, which we denote by $p^*_{-A}(p)$, and then we solve for an optimal p.

Step 1: Find an Optimal Outside Option $p^*_{-A}(p)$

With a slight abuse of notation, let p_{-A} denote the probability that allocation z_O is chosen if A fails to participate, and let $(1 - p_{-A})$ denote the probability that allocation z_S is chosen. Then, A's payoff if it fails to participate is given by

$$\underline{U}_A(c, p_{-A}) = p_{-A} - 10p_{-A}c. \tag{12}$$

Step 1.1: In this step, we show that for any p and any optimally chosen outside option $p^*_{-A}(p)$, the critical type c_i^* is the same and equal to $\frac{1}{10}$, that is,

$$c^*(p, p^*_{-A}(p)) = \frac{1}{10} \text{ for all } p. \tag{13}$$

This result is similar to the one in [5] which shows that the critical type is always the same (it is the type closest to the origin) independently of the allocation rule used. In this example, however, the critical type is not the one closest to the origin (which corresponds to 1 in this case).[17]

[17] In this example, it just happens that the critical type is independent of the allocation rule. This is not generally true with type-dependent outside options in a single-dimensional world (see [2]).

To obtain (13), note that at an optimal mechanism $p^*_{-A}(p)$ minimizes $V(1,p,p_{-A})$ - that is, $p^*_{-A}(p) \in \arg\min_{p_{-A}} \underline{U}_A(c^*(p,p_{-A}),p_{-A}) + \int\limits_{c^*(p,p_{-A})}^{1} \frac{dV(c)}{dc}dc$, which by using (12) can be rewritten as

$$p^*_{-A}(p) \in \arg\min_{p_{-A}} p_{-A} - 10p_{-A}c^*(p,p_{-A}) + \int\limits_{c^*(p,p_{-A})}^{1} \frac{dV(c)}{dc}dc. \tag{14}$$

At a minimizer $p^*_{-A}(p)$, the total derivative of $V(1,p,p_{-A})$ with respect to p_{-A} is equal to the partial, and it is given by $\left.\frac{dV(1,p,p_{-A})}{dp_{-A}}\right|_{p_{-A}=p^*_{-A}(p)} = 1 - 10c^*(p,p_{-A}(p))$.[18] Since this derivative is strictly decreasing, positive at $p_{-A} = 0$, and negative at $p_{-A} = 1$, we conclude that the solution is interior; therefore, $\left.\frac{dV(1,p,p_{-A})}{dp_{-A}}\right|_{p_{-A}=p^*_{-A}(p)} = 1 - 10c^*(p,p_{-A}(p)) = 0$, which implies (13).

Step 1.2: Now, we move on to find $p^*_{-A}(p)$. In order to do so, we use the fact that, at the critical type c^*, the payoffs from participation and non-participation must be tangent.

Given an assignment rule $p(c) = (p^{z_A}(c), p^{z_S}(c), p^{z_O}(c))$, the payoff from participation is given by $V(c) = p^{z_A}(c)(5-5c) + p^{z_S}(c)\cdot 0 + p^{z_O}(c)(1-10c)$, and its slope is given by[19]

$$\frac{dV(c)}{dc} = -5p^{z_A}(c) - 10p^{z_O}(c), \tag{15}$$

while the slope of the outside payoff is $\frac{\partial \underline{U}_A(c,p_{-A})}{\partial c} = -10p_{-A}$. At a minimizer $p^*_{-A}(p)$, both payoffs must be tangent at the critical type $c^*(p,p_{-A}(p)) = \frac{1}{10}$. In other words,

$$\left.\frac{dV(c)}{dc}\right|_{c^*=\frac{1}{10}} = -10p_{-A}. \tag{16}$$

[18] This property is an envelope condition. We state it and prove it in Lemma A in Appendix C of [2].

[19] This is true for all c where V is differentiable, which are a set of measure 1 since V is convex. For points where V is not differentiable, the subgradient is set-valued and $p_{-A}(p)$ is not uniquely determined, but the selection does not affect the objective function since the critical type is the same for any selection.

Combining (15) and (16), we get that

$$p^*_{-A}(p) = \frac{1}{2}p^{z_A}(\frac{1}{10}) + p^{z_O}(\frac{1}{10}). \tag{17}$$

Equation (17) gives us an optimal p^*_{-A} as a function of the assignment rule p.

Step 2. Find an Optimal Allocation Rule p

Using (13) and (14), $V(1, p, p^*_{-A}(p))$ can be rewritten as

$$V(1, p, p^*_{-A}(p)) = -\int_{\frac{1}{10}}^{1} [5p^{z_A}(c) + 10p^{z_O}(c)]dc. \tag{18}$$

By substituting (18) into (11), it becomes:

$$\max_p \int_0^{\frac{1}{10}} [p^{z_A}(c)\,(5 - 10c) + p^{z_O}(c)\,(1 - 20c)]dc + \int_{\frac{1}{10}}^{1} p^{z_A}(c)\,(10 - 10c) + p^{z_O}(c)\,(11 - 20c)]dc. \tag{19}$$

Note that the objective function is modified in the region $[\frac{1}{10}, 1]$, and that the increased virtual surplus in that region increases the seller's willingness to transfer the object. A higher probability of assigning the invention (p^{z_A}), decreases the rent from the worst type, $V(1)$, by charging him a high price. He is willing to pay that high price since otherwise he would be badly hurt by the outsourcing allocation.

Pointwise maximization of (19) gives us that $p^{z_A}(c) = 1$ for all c. Hence, the optimal assignment rule is

$$p(c) = (p^{z_A}(c), p^{z_S}(c), p^{z_O}(c)) = (1, 0, 0), \tag{20}$$

which is incentive-compatible, since, irrespective of the report, firm A obtains the object with probability 1. The revenue-maximizing assignment rule is ex-post efficient since $\pi^{z_A}(c) \geq \pi^{z_S}(c)$ and $\pi^{z_A}(c) \geq \pi^{z_O}(c)$ for all $c \in [0, 1]$.

By substituting (20) into (17), we get that the optimal non-participation assignment rule is given by $p^*_{-A}(p) = \frac{1}{2}$, or, more precisely, $p_{-A}(p) = (p^{z_A}(c), p^{z_S}(c), p^{z_O}(c)) = (0, \frac{1}{2}, \frac{1}{2})$, implying that the seller randomizes between keeping the invention and the open-source option with equal probability.

Since the firm obtains the object irrespective of its type, its payment must be constant also, and it can easily be seen that 4.5 is the most that can be charged and still have participation.[20] As we mentioned earlier, when open sourcing (allocation z_O) is not an available option, the seller keeps the invention when $c \in [\frac{1}{2}, 1]$. This assignment is inefficient since, half of the time, Firm A does not obtain the invention, whereas it is always efficient that it does.

Discussion

Example 1 illustrated how, with type-dependent outside payoffs, the optimal assignment depends on the outside options that buyers face. In the same spirit, in Example 2 we saw that the introduction of the option of open sourcing, even though never implemented, increased *both* revenue (it more than tripled) and efficiency; in fact, it resulted in an ex-post efficient revenue-maximizing assignment.

Example 1 is simple because for every buyer there is a threat that is always best. Example 2 is more complex because it illustrates how, with type-dependent outside payoffs, the two steps of finding an optimal assignment p and an optimal threat (the non-participation assignment p_{-i}) are interrelated: For a given assignment rule p, there is a specific threat that best exploits the shape of participation payoffs determined by p (Step 1). Once we have an optimal $p^*_{-i}(p)$, we then

[20] As explained in Example 1, the payment is obtained using $x(c) = \sum_{z \in Z} p^z(c)\pi^z(c) + \int_c^1 \sum_{z \in Z} \frac{\partial \pi^z(s)}{\partial s} p^z(s) ds - V(1; p, p_{-A})$.

optimize with respect to p (Step 2). This interrelationship is what makes a general characterization of revenue-maximizing mechanisms with type-dependent outside options quite challenging without additional assumptions. For a list of some tractable cases, see [2].

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