Optimal Allocation Mechanisms with Single-Dimensional Private Information[*]

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Abstract

We study revenue-maximizing allocation mechanisms for multiple heterogeneous objects when buyers care about the entire allocation, and not just about the ones they obtain. Buyers' payoff depends on their cost parameter and, possibly, on their competitors' costs. Costs are independently distributed across buyers, and both the buyers and the seller are risk-neutral. The formulation allows for complements, substitutes and externalities. We identify a number of novel characteristics of revenue-maximizing mechanisms: First, we find that revenue-maximizing reserve prices depend on the bids of other buyers. Second, we find that when non-participation payoffs are type-dependent, revenue-maximizing auctions may sell too often, or they may even be ex-post efficient. Keywords: *Multi-Unit Auctions, Type-Dependent Outside Options, Externalities, Mechanism Design, Interdependent Values: JEL D44, C7, C72.*

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1. Introduction

In many important allocation problems, all market participants, and not only the winners of items, are affected by the ultimate allocation of the object(s). In this paper, we study revenue-maximizing mechanisms when this is the case. One recent example of this is the acquisition of Wachovia:[1] The two potential buyers, Citibank and Wells Fargo, entered into fierce negotiations to determine who would gain control of Wachovia's assets. The reason was that, in addition to Wachovia's assets, Citibank's position in retail banking in the Eastern U.S. was at stake.[2] Another example of a buyer caring about the entire allocation of the objects is the allocation of "sponsored-link" positions on a search engine: Each advertiser cares not only about which slot he gets, but also about the identity and characteristics of the other advertisers that obtain a position for the same search entry. Millions of such sponsored links are auctioned off, and this is an important source of revenue for search engines and other internet portals.[3]

We analyze revenue-maximizing auctions in a multi-object allocation problem where buyers' payoffs depend on the *entire* allocation of the objects, not merely on the ones they obtain. Therefore, the auction outcome may affect buyers regardless of whether or not they win any objects, and regardless of whether or not they participate in the auction. Non-participation payoffs may then very well depend on their cost (type). In our model, described in Section 2, objects can be heterogeneous, and they can simultaneously be complements for some buyers and substitutes for others. Buyers are risk-neutral and their payoffs depend on their single-dimensional costs, which are private information, and on their competitors' costs (interdependent values) and can be non-linear.

We identify a number of novel characteristics of revenue-maximizing mechanisms: First, we find that revenue-maximizing reserve prices depend on the bids of other buyers. This happens not only in the case of interdependent values, but, perhaps more surprisingly, in private-value setups where buyers care about the entire allocation of the objects. Hence, we see that, in general, simple reserve prices and/or entry fees will not maximize revenue, in contrast to the classical private-values case (see Myerson (1981) or Riley and Samuelson (1981)) and to the findings of Jehiel, Moldovanu and Stacchetti (1996), where auctions with flat reserve prices and entry fees are revenue-maximizing. Second, we find that revenue-maximizing auctions may sell too often, or they may even be ex-post efficient. This happens if non-participation payoffs are type-dependent, in which case a revenue-maximizing assignment of the objects can depend crucially on the outside options that buyers face. Therefore, outside options can affect the *degree of efficiency* of revenue-maximizing auctions. This is, again, in contrast to the classical case, and to Jehiel, Moldovanu and Stacchetti (1996), where with the use of the reserve prices, the seller, much like the classical monopolist[4], restricts supply to boost revenue. Another difference is that, in contrast to our paper, outside options in Jehiel, Moldovanu

[1] See "Citigroup and Wells Fargo Said to Be Bidding for Wachovia," *New York Times*, September 28, 2008 and "Citi Concedes Wachovia to Wells Fargo," *New York Times,* October 9, 2008.

[2] Wells Fargo has a prevalent position in the western U.S. and virtually no presence in the east, while Citibank is present in both territories. Wachovia's prevalent position in the east would, therefore, allow Wells Fargo to become a much tougher competitor for Citibank.

[3] Works that have looked at other aspects of sponsored-links markets are Athey and Ellison (2008), Edelman, Ostrovsky, and M. Schwarz (2007) and Varian (2007).

[4] See Bulow and Roberts (1989) for an insightful discussion relating the theory of revenue-maximizing auctions to the classical monopoly theory.

and Stacchetti (1996) affect only the transfers between the buyers and the seller and not the way the object is allocated.[5]

On a broader level, this paper provides an elegant formulation that allows one to analyze in a unified framework numerous scenarios that have been addressed in the literature, as well as many more.[6] The main technical innovation of our paper is to show how the presence of type-dependent non-participation payoffs modifies the virtual surpluses of allocations and how this modification affects the efficiency properties of revenue-maximizing mechanisms. In particular, we show that the optimal assignment rule can depend crucially on the outside options (for buyers) that the seller can choose as threats; that overselling or even ex-post efficiency may occur; and that the optimal threat-allocation rule can be random. Figueroa and Skreta (2009a) consists of two examples illustrating how the efficiency properties of revenue-maximizing mechanisms change when non-participation payoffs are type-dependent, whereas this paper contains the general theory for solving such problems. Moreover, this paper sheds light on the reasons for the phenomena in Figueroa and Skreta (2009a) by identifying the general forces behind them.

The work on revenue-maximizing auctions has had a huge impact on various aspects of economics. One reason is that the solution method is elegant and simple: In principle, one has to solve a complicated optimal control problem, but because of the linearity of the problem and the structure of the feasible set, one can solve it using simple pointwise optimization. Our problem is generally not solvable with such Myerson-like techniques for at least two reasons: 1) Buyers' participation payoffs can depend non-linearly on their types; and 2) non-participation payoffs can depend on buyers' types.

When buyers' payoffs depend non-linearly on their types, it is possible that, despite strictly monotonic virtual surpluses, the solution derived via pointwise optimization is not incentive-compatible. This is illustrated in a simple example in Figueroa and Skreta (2009b). Thus, one has to explicitly account for the incentive-compatibility constraints. In this paper, we focus on identifying the classes of problems where the incentive-compatibility conditions do not bind.

The second reason why Myerson-like techniques fail is the fact that non-participation payoffs can depend on buyers' own private information. In such cases, the shape of participation and non-participation payoffs (which both depend on the mechanism chosen by the seller) together determine the type where the participation constraints bind-the critical type. This set of types determines the modification of the virtual surplus of an allocation: For types between the critical and the best type, the only distortion comes from the incentive-compatibility constraints that reduce the surplus of an allocation by the information rents. For types between the critical and the worst type, on top of this distortion, there is another distortion-introduced by the participation constraints-which goes in the other direction. Ultimately, the degree of efficiency of the revenue-maximizing assignment depends on how these two distortions balance out. Thus, the way the goods

[5] For more on this point, as well on further comments on related literature, see Figueroa and Skreta (2009a).

[6] An incomplete list of environments included in our formulation are the ones in Myerson (1981), Gale (1990), Dana and Spier (1994), Milgrom (1996), Branco (1996), Jehiel, Moldovanu and Stacchetti (1996), Levin (1997), Brocas (2007) and Aseff and Chade (2008).

are allocated depends on the vector of critical types, and the reverse, making the problem fundamentally non-linear.[7] For all these cases, we can describe some qualitative features of the revenue-maximizing mechanisms; however, a general analytical expression of a solution is not possible, just as in other non-linear revenue-maximizing auction problems. See, for example, Maskin and Riley (1984), which analyzes the problem with risk-averse buyers.

Despite the complications of the allocation problems that we examine, we are able to identify interesting classes of problems that are solvable via Myerson-like techniques, but that, at the same time, are among the classes of problems where the novel features of overselling or even efficiency of the revenue-maximizing allocation appear. This is done in Section 4 of this paper. In these cases, the vector of critical types does not depend on the allocation that the seller chooses, because (roughly) buyers' outside payoffs have extreme slopes. The analytical solutions of these cases show the possibilities of efficiency and "overselling." We choose to state these observations as possibilities rather than to describe the complete list of cases where they would be true because this would be a very long and tedious task. Whether efficiency, "overselling" or "underselling" occurs depends on the vector of critical types. These features will also be present when revenue depends non-linearly in the assignment rule.

In some sense, our analysis highlights how far one can push Myerson-like techniques within the framework of allocation problems with risk-neutral buyers and single-dimensional private information. Indeed, because of the generality of our framework, we hit a number of boundaries of these techniques. Given the large number of applications that these techniques have had across many subfields of economics, demonstrating their reach can further extend the number of applications significantly.

Our multi-unit model is very versatile, but has the drawback that private information is single-dimensional. For a discussion of why this assumption can sometimes be satisfactory, see Levin (1997). Other papers that study revenue-maximizing multi-unit auctions when private information is single-dimensional are Maskin and Riley (1989), who analyze the case of unit demands and continuously divisible goods; Gale (1990), the case of discrete goods and superadditive valuations; and, finally, Levin (1997) the case of complements. A number of papers on revenue-maximizing multi-unit auctions model types as being multi-dimensional. With multi-dimensional types, the characterization of the optimum is extremely difficult. Significant progress has been made, but no analytical solution or general algorithm is known. Important contributions include Jehiel,Moldovanu and Stacchetti (1999),[8] Armstrong (2000), Avery and Hendershott (2000) and Jehiel and Moldovanu (2001). This paper is less general in the dimensionality of the types, but much more general in all other dimensions.

This paper is also related to the literature on mechanism design with type-dependent outside options

[7] It is very important to stress that the virtual surplus is modified only when outside-payoffs are type-dependent. Thus, overselling *cannot* occur when there are externalities (positive or negative), but the outside options are flat, as is the case in Jehiel, Moldovanu and Stacchetti (1996). Also, the presence of externalities is just one instance where outside options may be type-dependent, but there can be many more. Consider, for instance, a procurement setting where bidders have to give up the possibility of undertaking other projects in order to participate in the current auction.

[8] Jehiel-Moldovanu and Stacchetti (1999) consider the design of optimal auctions of a single unit in the presence of type-dependent externalities and *multi-dimensional types.* A buyer's type is a vector, where each component indicates his/her utility as a function of who gets the object. The multi-dimensionality of types makes the complete characterization intractable.

and, most notably, to the paper by Krishna and Perry (2000), who examine efficient mechanisms, whereas our focus is revenue maximization. Lewis and Sappington (1989) study an agency problem where the outside option of the agent is type-dependent. Among other things, the fact that the critical type is not necessarily the "worst" one mitigates the inefficiencies that arise from contracting under private information. This feature also appears at times in our analysis, but we also show that inefficiencies sometimes are not reduced, but they change in nature, and the monopolist, instead of selling too little, sells too much. Jullien (2000) uses a dual approach to characterize properties of the optimal incentive scheme, such as the possibility of separation, non-stochasticity, etc. In this paper, we do not rely on dual methods. Other differences from Jullien (2000) are that we allow for multiple agents and for the principle to choose the outside options that agents face.

2. The Environment and main Definitions

A risk-neutral seller owns N indivisible, possibly heterogeneous, objects that are of 0 value to her and faces I risk-neutral buyers. Both N and I are finite natural numbers. The seller (indexed by zero) can bundle these N objects in any way she sees fit. An allocation z is an assignment of objects to the buyers and to the seller. It is a vector with N components, where each component stands for an object and specifies who gets it; therefore, the set of possible allocations is finite and given by $Z \subseteq [I \cup \{0\}]^N$. Buyer i's valuation from allocation z is denoted by $\pi_i^z(c)$, where $c = (c_i, c_{-i})$ stands for the buyers' cost parameters. Values can, therefore, be *interdependent*. Buyer i's cost parameter c_i is private information and is distributed on $C_i = [\underline{c_i}, \overline{c_i}]$, with $0 \leq \underline{c_i} \leq \overline{c_i} < \infty$, according to a distribution F_i that has a strictly positive and continuous density f_i. Costs are *independently* distributed across buyers. The joint probability density function is $f(c) = \times_{i \in I} f_i(c_i)$, where $c \in C = \times_{i \in I} C_i$; we also use $f_{-i}(c_{-i}) = \times_{\substack{j \in I \\ j \neq i}} f_j(c_j)$.

We assume that, for all $i \in I$, $\pi_i^z(\cdot, c_{-i})$ is *decreasing*, *convex* and *differentiable* for all z and c_{-i}. We impose no restrictions on how π_i depends on z or c_{-i}. This formulation allows buyers to demand many objects that may be complements or substitutes and for externalities, that can be type- and identity-dependent.

A crucial feature of the model is that a buyer may care about the entire allocation of the objects and not only about the objects he obtains. Thus, it is quite possible that $\pi_i^z(c_i, c_{-i}) \neq 0$ even if allocation z does not include any objects for i and even if i is not taking part in the auction. In such a case, non-participation payoffs may depend on $i's$ type. This dependence introduces a number of technical difficulties and is the reason why revenue-maximizing assignments may be ex-post efficient and/or involve overselling.

The seller wants to design a revenue-maximizing mechanism, and the buyers aim to maximize expected surplus. By the revelation principle, it is without loss of generality to restrict attention to truth-telling equilibria of direct revelation games where *all* buyers participate. To see this, note that the set of possible allocations is $Z = \{I \cup \{0\}\}^N$, which is larger the more buyers that participate. The seller can then replicate an equilibrium outcome of some auction, where a subset of the buyers (for some realizations of their private information) do not participate, with a mechanism where all buyers participate that induces the original allocation for participating and non-participating buyers.

A *direct revelation mechanism,(DRM)*, $M = (p, x, \left(p^{-i}\right)_{i\in I})$ consists of an *assignment rule* $p : C \longrightarrow \Delta(Z)$, a *payment rule* $x : C \longrightarrow \mathbb{R}^I$ and a *non-participation assignment rule* p^{-i} out of $\mathcal{P}^{-i} = \{p^{-i} : C_{-i} \rightarrow \Delta(Z^{-i})\}$, where $Z^{-i} \subset Z$ is the set of allocations that are feasible without i.[9]

The assignment rule specifies the probability of each allocation for a given vector of reports. We denote by $p^z(c)$ the probability that allocation z is implemented when the vector of reports is c. Observe that the assignment rule has as many components as the number of possible allocations. The payment rule x specifies, for each vector of reports c, a vector of payments, one for each buyer. Finally, the non-participation assignment rule specifies the allocation that prevails if i refuses to participate. If i does not participate, he neither submits a message nor makes or receives any payments.

We assume that the seller chooses the non-participation assignment rule so as to maximize ex-ante expected revenue. If the seller does not have such commitment power, then $\mathcal{P}^{-i}$ would contain all the assignment rules that are feasible *and* revenue-maximizing when i is not around (therefore, $\mathcal{P}^{-i}_{NC}$ is a subset of $\{p^{-i} : C_{-i} \rightarrow \Delta(Z^{-i})\}$). It is worth stressing that the crucial qualitative features of our results depend on the fact that outside payoffs are type-dependent, and not on the exact elements of $\mathcal{P}^{-i}$. Of course, to find the revenue-maximizing p given $\mathcal{P}^{-i}$ is a different problem than finding the revenue-maximizing p given some other set $\hat{\mathcal{P}}^{-i}$, so the exact solution may differ.

We now proceed to describe the seller's and the buyers' payoffs. The interim expected utility of a buyer of type c_i when he *participates* and declares c_i' is $U_i(c_i, c_i'; (p, x)) = E_{c_{-i}}\left[\sum_{z\in Z}(p^z(c_i', c_{-i})\pi_i^z(c_i, c_{-i})) - x_i(c_i', c_{-i})\right]$, whereas his maximized payoff is given by $V_i(c_i) \equiv U_i(c_i, c_i; (p, x))$. The payoff that accrues to buyer i from *non-participation* depends on his type c_i and on what allocations will prevail in that case, which are determined by p^{-i}:

$$\underline{U}_i(c_i, p^{-i}) = E_{c_{-i}}\left[\sum_{z\in Z^{-i}}(p^{-i})^z(c_{-i})\pi_i^z(c_i, c_{-i})\right],$$

where $(p^{-i})^z$ denotes the probability assigned to allocation z by p^{-i}.

The timing is as follows:

Stage 0: The seller chooses a mechanism $(p, x, \left(p^{-i}\right)_{i\in I})$.

Stage 1: Buyers decide whether or not to participate, and which report to make. If all make a report, the mechanism determines the assignment of objects and the payments. If buyer i decides not to participate, the objects are assigned according to $\{p^{-i}\}$. If two or more buyers do not participate, then an arbitrary allocation that is feasible in that case-for example, the status quo- is implemented.

[9]Note that this formulation is flexible enough to accommodate a number of alternative scenarios. All one needs to do is define Z^{-i} appropriately. For example, if players have veto rights, Z^{-i} can be specified as containing just the status quo allocation. Also, the special case in which one can block oneself from paying anything, but has no rights at all over outcomes, can be accommodated by specifying $Z^{-i} = Z$. In an auction setup, it seems rather natural to assume that the set of feasible allocations when buyer i is not around contains all allocations that do not involve buyer i receiving any goods (that is, we cannot force objects to a non-participating buyer).

In order for a mechanism to be feasible, all buyers must choose to participate and to report their true type. We are capturing a one-shot scenario. Given that others participate and tell the truth about their types, is it a best response for buyer i to participate and tell the truth about his type? In such a one-shot scenario, buyers are not making inferences about the type of a non-participating buyer.

We now provide a formal definition of what it entails for a direct revelation mechanism to be feasible.

Definition 1. (Feasible Mechanisms) A mechanism $(p, x, (p^{-i})_{i \in I})$ is *feasible* iff it satisfies

(IC) **"incentive constraints,"** a buyer's strategy is such that
$U_i(c_i, c_i; (p, x)) \geq U_i(c_i, c_i'; (p, x))$ for all $c_i, c_i' \in C_i$, and $i \in I$
(PC) **"voluntary participation constraints,"**
$U_i(c_i, c_i; (p, x)) \geq \underline{U}_i(c_i, p^{-i})$ for all $c_i \in C_i$, and $i \in I$
(RES) **"resource constraints"** $\sum_{z \in Z} p^z(c) = 1$, $p^z(c) \geq 0$ for all $c \in C$

To summarize, feasibility requires that (1) buyers prefer to tell the truth about their cost parameter; (2) buyers choose voluntarily to participate; and (3) p is a probability distribution over Z.[10]

Observations:

1. We assume that all buyers participate in the mechanism. This is without loss of generality since any equilibrium in which a subset of buyers do not participate can be replicated with one in which all participate as follows: All buyers participate in a mechanism that assigns everyone the allocation of the original equilibrium. Moreover, the outside options, the p^{-i}'s, are the same as in the original equilibrium; that is, the participating buyers of the original equilibrium face the same p^{-i}, and the non-participating ones are assigned their original allocation from non-participation.

2. Then, given observation 1, the allocation that prevails when two or more buyers fail to participate is irrelevant. This is because when we check for feasibility of a mechanism, we look for Bayes-Nash Equilibria (BNE); hence, we look only for individual deviations (as opposed to coalitional deviations). In short, we check whether it is in each buyer's best interest to participate in the mechanism and to report truthfully, given that everybody else does so.

With the help of the revelation principle, the seller's problem can be written as

$$\max \int_C \sum_{i=1}^{I} x_i(c) f(c) dc \tag{1}$$

subject to (p, x) being "feasible."

This completes the description of our model and the seller's problem, and we now proceed with our analysis. Proofs of the results not presented in the main text can be found in Appendix A.

[10] Notice that Z contains the allocation where the seller keeps all the objects; thus, $\sum_{z \in Z} p^z(c) = 1$.

3. Analysis of the Problem

The seller's objective is to maximize expected revenue subject to incentive, participation and resource constraints. We start by studying the implications of these constraints.

Implications of Incentive Compatibility Given a *DRM* (p,x), buyer $i's$ maximized payoff,

$$V_i(c_i) = \max_{c_i'} \int\limits_{C_{-i}} \left(\sum_{z \in Z} p^z(c_i', c_{-i}) \pi_i^z(c_i, c_{-i}) - x_i(c_i', c_{-i}) \right) f_{-i}(c_{-i}) dc_{-i}, \tag{2}$$

is convex, since it is a maximum of convex functions. In the next Lemma, we show that the incentive constraints translate into the requirement that the derivative of V_i,

$$P_i(c_i) \equiv \int\limits_{C_{-i}} \sum_{z \in Z} p^z(c_i, c_{-i}) \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i}, \tag{3}$$

(more precisely, a selection from its subgradient, which is single-valued almost surely), evaluated at the true type is weakly increasing.[11]

Lemma 1 *A mechanism (p,x) is incentive-compatible iff*

$$P_i(c_i') \geq P_i(c_i) \qquad \textit{for all } c_i' > c_i \tag{4}$$

$$V_i(c_i) = V_i(\overline{c}_i) - \int\limits_{c_i}^{\overline{c_i}} P_i(s)ds \quad \textit{for all } c_i \in C_i. \tag{5}$$

Let

$$J_z(c) \equiv \sum_{i=1}^{I} [\pi_i^z(c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i}] \tag{6}$$

denote the *virtual surplus of allocation* z. Notice that we are summing over all buyers because an allocation may affect all of them, and not just the ones that obtain objects. Therefore, the virtual surplus of allocation z may depend on the whole vector of types:[12] This may be true not only when values are interdependent, but also in a private values setup when buyers care about the entire allocation of the item(s). To see this, suppose that the profits to firm i from winning a firm take-over is $1 - c_i$, whereas its competitors' payoffs are $-c_j$. Then, when i wins, (6) becomes $1 - c_i - \Sigma_{j \neq i} \left[c_j - \frac{F_j(c_j)}{f_j(c_j)} \right]$.

With the help of Lemma 1 and using standard arguments, we can write buyer i's expected payment as a function of the assignment rule p, and the payoff that accrues to his worst type,[13] $V_i(\overline{c}_i)$ as

$$\sum_{i=1}^{I} \int_C x_i(c) f(c) dc = \int\limits_{C} \sum_{z \in Z} p^z(c) J_z(c) f(c) dc - \sum_{i=1}^{I} V_i(\overline{c}_i). \tag{7}$$

Now we turn to examine the implications of the participation constraints.

[11] In the classical case, where there is only one object and $i's$ payoff from obtaining the object is v_i, (see Myerson (1981)), the analog of P_i is $P_i(v_i) = \int\limits_{V_{-i}} p(v_i, v_{-i}) f_{-i}(v_{-i}) dv_{-i}$.

[12] In Myerson (1981), virtual valuations are buyer-specific. For buyer i, we have $J_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$.

[13] For more details, see Appendix A.

Implications of Participation Constraints Since the seller's revenue is decreasing in $V_i(\overline{c}_i)$, at a solution, this term must be as small as possible subject to the participation constraint $V_i(c_i) \geq \underline{U}_i(c_i, p^{-i})$ for all $c_i \in C_i$. At an optimum, there exists a type, which we call "critical type"[14] c_i^*, where participation payoffs are exactly equal to non-participation payoffs; that is,

$$V_i(c_i^*) = \underline{U}_i(c_i^*, p^{-i}). \tag{8}$$

This type can be any type in C_i. See Figure 1:

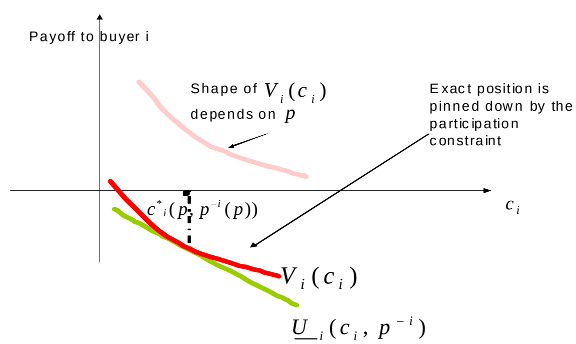

Figure 1: PC can bind anywhere

From (5) and (8), we see that $V_i(\overline{c}_i)$ depends on p through two channels: P_i and $c_i^*(p, p^{-i})$ as follows:

$$V_i(\overline{c}_i) = \underline{U}_i(c_i^*(p, p^{-i}), p^{-i}) + \int_{c_i^*(p,p^{-i})}^{\overline{c_i}} P_i(s)ds. \tag{9}$$

Recall that we assume that the seller can commit to choose p^{-i} in order to maximize revenue ex-ante. We now show how this can achieved:

Revenue-maximizing non-participation assignments for fixed p For a given p, a revenue-maximizing p^{-i} must be chosen in order to minimize $V_i(\overline{c}_i)$, subject to the voluntary participation constraints-namely,

$$p^{-i}(p) \in \arg \min_{\rho^{-i} \in \mathcal{P}^{-i}} \underline{U}_i(c_i^*(p, \rho^{-i}), \rho^{-i}) + \int_{c_i^*(p,\rho^{-i})}^{\overline{c_i}} P_i(s)ds, \tag{10}$$

subject to $c_i^*(p, \rho^{-i})$ satisfying:

$$c_i^*(p, \rho^{-i}) \in \arg \min_{c_i} \left[-\int_{c_i}^{\overline{c}_i} P_i(s)ds - \underline{U}_i(c_i, \rho^{-i}) \right]. \tag{11}$$

Hence, for each assignment of the objects, p, there is a potentially different revenue-maximizing "threat" $p^{-i}(p)$, which can be random.[15] Additionally, the dependence of c_i^* on ρ^{-i} and on p adds an additional level of complication, as for any candidate solution of (10) there is possibly a different c_i^* satisfying (11).

[14] In general, there can be many critical types, and any one can be chosen to stand for c_i^*.

[15] Example 2 in Figueroa and Skreta (2008) has this feature.

By substituting a solution of the program described in (10) into (9), we have that if ρ^{-i} is chosen optimally, we have that

$$V_i(\overline{c}_i; p, p^{-i}(p)) = \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}) + \int\limits_{c_i^*(p,p^{-i}(p))}^{\overline{c_i}} P_i(s)ds. \tag{12}$$

Modified Virtual Surpluses We now demonstrate how the presence of type-dependent outside options modifies the virtual surpluses of allocations. By substituting (12) into (7), the objective function of the seller's problem can be rewritten as

$$\int\limits_{C} \sum_{z \in Z} p^z(c) J_z(c) f(c) dc - \sum_{i=1}^{I} \left[\underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}) + \int\limits_{c_i^*(p,p^{-i}(p))}^{\overline{c_i}} P_i(s)ds \right]. \tag{13}$$

Recalling that $P_i(c_i) = \int\limits_{C_{-i}} \sum\limits_{z \in Z} p^z(c) \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i}$, and by rearranging the terms in (13), we can rewrite it as

$$\int\limits_{C} \sum_{z \in Z} p^z(c) \left[J_z(c) - \sum_{i=1}^{I} 1_{c_i \geq c_i^*(p,p^{-i}(p))} \frac{\partial \pi_i^z(c)}{\partial c_i} \frac{1}{f_i(c_i)} \right] f(c) dc - \sum_{i=1}^{I} \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}).$$

We define the *"modified virtual surplus"* of allocation z as

$$\hat{J}_z(c) \equiv J_z(c) - \sum_{i=1}^{I} 1_{c_i \geq c_i^*(p,p^{-i}(p))} \frac{\partial \pi_i^z(c)}{\partial c_i} \frac{1}{f_i(c_i)}. \tag{14}$$

Observe that the modified virtual surplus depends on p and on p^{-i} through $c_i^*(p, p^{-i}(p))$, which depends, in turn, on the shape of the participation payoffs, which are determined by p, and on the shape of non-participation payoffs, which are determined by $\{p^{-i}\}_{i \in I}$.

It is useful to compare the modified virtual surplus of an allocation z, $\hat{J}_z$, with the virtual surplus of that allocation, J_z, and with the actual surplus of that allocation S_z, which is given by $S_z(c) = \sum\limits_{i=1}^{I} \pi_i^z(c_i, c_{-i})$. This is interesting because the degree of efficiency of a revenue-maximizing mechanism depends on these comparisons.

If $c_i^* = \bar{c}_i$ for all i, the modified virtual surplus coincides with the virtual surplus; hence,[16]

$$\hat{J}_z(c) = J_z(c). \tag{15}$$

This is because the virtual surplus is modified only for $c_i \geq c_i^*$.

If, on the other hand, $c_i^* = \underline{c}_i$ for all i, then[17]

$$\hat{J}_z(c) = J_z(c) - \sum_{i=1}^{I} \frac{\partial \pi_i^z(c)}{\partial c_i} \frac{1}{f_i(c_i)}, \tag{16}$$

[16] With constant with respect to own type outside options, the critical type is always the worst type. See, for instance, Myerson (1981) or Jehiel, Moldovanu and Stacchetti (1996). This implies that the modified virtual surplus is equal to the virtual surplus and independent of the assignment rule p. In this case, a revenue-maximizing p is independent of the outside options that buyers face, and it has a simple characterization, because revenue is always linear in p. This is true even if, as in this paper and in Jehiel, Moldovanu and Stacchetti (1996), the seller can choose p^{-i}. The reason is that, when outside options give a type-independent payoff, they are essentially just a number. All the seller needs to do is to choose the option that guarantees the lowest number for i. In that case, optimal threats p^{-i} are independent of p and deterministic. In contrast, with type-dependent outside options, p_{-i} can depend on p, can be random and cannot be chosen by simple inspection.

[17] Example 1 in Figueroa and Skreta (2009a) has this feature.

which can be rewritten as

$$\hat{J}_z(c) = \sum_{i=1}^{I} \left[\pi_i^z(c) + \frac{F_i(c_i) - 1}{f_i(c_i)} \frac{\partial \pi_i^z(c)}{\partial c_i} \right]. \tag{17}$$

In this case, $\hat{J}_z(c) > J_z(c)$ because $\sum_{i=1}^{I} \frac{\partial \pi_i^z(c)}{\partial c_i} \frac{1}{f_i(c_i)}$ is negative, which follows from the fact that π_i^z is decreasing in c_i. Moreover, since the amount $\left(\frac{F_i(c_i)-1}{f_i(c_i)} \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i} \right)$ is positive, we also have that the "modified virtual surplus" of allocation z is actually larger than the actual surplus of allocation z-that is, $\hat{J}_z(c) \geq S_z(c)$.

Finally, if c_i^* is interior[18] for all i-namely, $c_i^* \in (\underline{c}_i, \bar{c}_i)$-then $\hat{J}_z(c)$ depends on how a vector c compares to the vector c^*. Consider, for instance, $(\tilde{c}_i, \tilde{c}_{-i})$, where for all i we have that $\tilde{c}_i < c_i^*$; then, it holds that $\hat{J}_z(\tilde{c}) = J_z(\tilde{c})$, as in (15), and at that $\tilde{c}$ the modified virtual surplus is less than $S_z(\hat{c})$. Now, take a $(\hat{c}_i, \hat{c}_{-i})$, where for all i we have that $\hat{c}_i \geq c_i^*$; then, it holds that $\hat{J}_z(\hat{c}) = J_z(\hat{c}) - \sum_{i=1}^{I} \frac{\partial \pi_i^z(\hat{c})}{\partial c_i} \frac{1}{f_i(\hat{c}_i)}$, as in (16), and at $\hat{c}$ we have that $\hat{J}_z(\hat{c}) > J_z(\hat{c})$ and $\hat{J}_z(\hat{c}) \geq S_z(\hat{c})$. For a vector (c_i, c_{-i}) where $c_i > c_i^*$ for some i, and $c_j \leq c_j^*$ for some j, we can see from (14), that there is no modification to J_z for j, but there is for i. Then, we can still conclude that $\hat{J}_z(c) \geq J_z(c)$, but depending on the exact comparison of (c_i, c_{-i}) with (c_i^*, c_{-i}^*), both $\hat{J}_z(c) \geq S_z(c)$ and $\hat{J}_z(c) < S_z(c)$ are possible.

How the modified virtual surplus of an allocation, (the $\hat{J}_z$), compares with the actual virtual surplus of that allocation, (the S_z), is important because it affects the degree of efficiency of the revenue-maximizing mechanisms.

Revenue-maximizing Mechanisms Here, we put together all the implications we derived in the previous section and describe the conditions that revenue-maximizing mechanisms satisfy.

Using (14), the seller's objective function given by (13) can be rewritten as

$$\int_C \sum_{z \in Z} p^z(c) \hat{J}_z(c) f(c) dc - \sum_{i=1}^{I} \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}). \tag{18}$$

The following Proposition characterizes necessary conditions of revenue-maximizing mechanisms.

Proposition 2 *If, in a mechanism, the allocation and non-participation rules $(p, \{p^{-i}\}_{i \in I})$ satisfy that (i) the assignment function p maximizes (18) subject to resource constraints and (4); (ii) $p^{-i} = p^{-i}(p)$ according to (10); and (iii) the payment function x for all i is given by:*

$$x_i(c) = \sum_{z \in Z} p^z(c) \pi_i^z(c) + \int_{c_i}^{\overline{c_i}} \sum_{z \in Z} p^z(s, c_{-i}) \frac{\partial \pi_i^z(s, c_{-i})}{\partial s} ds - V_i(\overline{c}_i; p, p^{-i}(p)), \tag{19}$$

with $V_i(\overline{c}_i; p, p^{-i}(p))$ given by (12), then it is revenue-maximizing.

Proof. We have already argued that in a revenue-maximizing mechanism, there must exist at least one type for each buyer where the participation constraint binds- that is, a type where (8) is satisfied. This type is denoted by $c_i^*(p, p^{-i})$, and it satisfies (11). These are the implications of the participation constraints on the solutions.

[18] For an illustration of such a case, see Example 2 in Figueroa and Skreta (2009a).

The implications of the incentive constraints are that revenue can be expressed as in (7). Combining these implications, we showed how we can express revenue by (18). Since the amount of revenue the seller can extract depends also on the shape and location of non-participation payoffs, then in a revenue-maximizing mechanism p^{-i} has to satisfy (10).

Now, in order for a mechanism to be a valid solution, it must have an allocation rule p that satisfies (4) and resource constraints.

Finally, if, in a mechanism, the payment rule is given by (19), then for all $i \in I$, $i's$ payoff is the lowest it can be, while ensuring voluntary participation since c_i^* is indifferent between participating or not participating. To see this, note that by substituting (12) into (19), and by taking expectations with respect to c_{-i}, we obtain that

$$\begin{aligned}\int\limits_{C_{-i}} x_i(c) f_{-i}(c_{-i}) dc_{-i} \quad &= \quad \int\limits_{C_{-i}} \left[\sum_{z \in Z} p^z(c) \pi_i^z(c) + \int\limits_{c_i}^{\overline{c_i}} \sum_{z \in Z} p^z(s, c_{-i}) \frac{\partial \pi_i^z(s, c_{-i})}{\partial s} ds \right] f_{-i}(c_{-i}) dc_{-i} \\ &\quad -\underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}) - \int\limits_{c_i^*(p, p^{-i}(p))}^{\overline{c_i}} P_i(s) ds. \end{aligned} \tag{20}$$

By recalling (3), (20) implies that

$$V_i(c_i) = \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}) - \int\limits_{c_i}^{c_i^*(p, p^{-i}(p))} P_i(s) ds,$$

from which we immediately get that

$$V_i(c_i^*) = \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}).$$

From these considerations, it follows that a mechanism $(p, x, \left(p^{-i}\right)_{i \in I})$ that satisfies all these conditions is revenue-maximizing. ■

Proposition 2 is in the same spirit as Lemma 3 in Myerson (1981). As in that paper, we have revenue equivalence. Any two mechanisms that allocate the objects in the same way and give the same expected payoff to the worst type generate the same revenue. There are, however, important differences. The most important one is that in our problem, the objective function can depend non-linearly on p. To see this, notice that c_i^* may depend on the whole shape of $p(.)$ non-linearly (both directly and indirectly through $p^{-i}(p)$). Moreover, revenue depends on c_i^* through $V_i(\bar{c}_i)$, and this effect, as we have seen, can be decomposed between an effect on $\hat{J}_z$ and another on the term $\sum_{i=1}^{I} \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i})$. Because the seller's objective function can depend non-linearly on p[19], and because of the interdependence of p, p^{-i} and c_i^* it is, in general, impossible to find an analytical expression for the revenue-maximizing assignment. However, the problem has enough structure to allow the use of variational methods once one has the specifics of the problem in hand (the F_i's and the π's). In particular, if the functions $\pi_i^z(\cdot, c_{-i})$ are smooth enough, then $c_i^*(p, p^{-i}(p))$ is a differentiable function of p, thus guaranteeing that the objective function is differentiable and, hence, continuous. It is not

[19]For an example, see Appendix B.

hard to show that the feasible set is sequentially compact. A continuous function over a sequentially compact set has a maximum. The solution will depend on the particular shapes of π_i^z and of the distributions F_i.

Despite the fact that, in general, we cannot get an explicit expression for the revenue-maximizing assignment, we can say the following: First, when virtual surpluses of allocations depend on more than one cost parameter, simple auctions with flat entry fees and reserve prices are likely to be outperformed by ones where reserve prices depend on other buyers' bids. Second, because modified virtual surpluses are (weakly) greater than virtual surpluses, and can be even greater than actual surpluses, sometimes revenue-maximizing auctions may oversell or may be even ex-post efficient. Finally, it is possible that both the revenue-maximizing assignment and the non-participation assignment are random. In fact, the revenue-maximizing assignment can be random even if revenue is linear in the assignment rule. An example with this feature is analyzed in Figueroa and Skreta (2009b). For an example where the revenue-maximizing non-participation assignment rule is random, see Figueroa and Skreta (2009a).

Additionally, there are interesting cases where the problem becomes linear and, hence, analytical solutions can be obtained through a procedure similar to the one used in Myerson (1981). Their analytical tractability allows one to clearly see the role of the shape of outside options for the efficiency properties and other characteristics of revenue-maximizing mechanisms vis-a-vis the case of type-independent outside options studied in Myerson (1981), and in Jehiel, Moldovanu and Stacchetti (1996) for the case of externalities. These cases are described in the following section.

4. Revenue-maximizing Mechanisms with Critical Types Independent of p

In many cases with interesting economic insights, critical types are independent from p when p^{-i} is optimally chosen. Whether or not this happens, depends on how sensitive the outside payoff of a buyer is with respect to his own type, relative to the one of participation payoffs. This can occur in many cases, including: (i) the case where outside options can depend on p and on the type of competitors, but not on the buyer's type; (ii) the somewhat opposite case, where the outside option is steep in the buyer's type; and (iii) an intermediate case where both options are present: The buyer can be threatened with an allocation that yields him a type-independent payoff, and with an allocation where the payoff is very steep with respect to type. When there is a critical type that is independent of p when p^{-i} is optimally chosen, then revenue is linear in the allocation rule.

Below, we describe conditions on the shape of $\pi_i^z(\cdot, c_{-i})$ and on the sets $Z^{-i\prime}s$ under which each of these cases prevails. The analysis of these cases illustrates the main economic insights of the influence of outside options on the shape of revenue-maximizing mechanisms.

4.1 Environments with Critical Types Independent of p

We now present the three previously-described environments. A more detailed description can be found in Appendix C. In what follows, we use the notation:

$$\bar{\pi}_i^z(c_i) \equiv \int_{C_{-i}} \pi_i^z(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}.$$

Case 1: Flat Payoff from Worst Allocation for i

Suppose that there is an allocation in $z_i^F \in Z^{-i}$, that gives i a type-independent payoff- that is, $\pi_i^{z_i^F}(c) = \pi_i^{z_i^F}(c_{-i})$ for all c_i- and it satisfies the following two conditions:

$$\begin{aligned} 0 &= \frac{d\bar{\pi}_i^{z_i^F}(c_i)}{dc_i} \geq \frac{d\bar{\pi}_i^z(c_i)}{dc_i} \text{ for all } z \in Z \\ \pi_i^{z_i^F}(c_{-i}) &\leq \pi_i^z(c_i, c_{-i}) \text{ for all } z \in Z^{-i} \text{ and } c_i \in C_i. \end{aligned}$$

Then, a revenue-maximizing outside option from the seller's perspective is $(p^{-i})^{z_i^F} = 1$ since it solves, for all p,

$$p^{-i}(p) \in \arg \min_{\rho^{-i} \in \mathcal{P}^{-i}} \underline{U}_i(c_i^*(p, \rho^{-i}), \rho^{-i}) + \int_{c_i^*(p,\rho^{-i})}^{\overline{c_i}} P_i(s) ds, \tag{21}$$

implying that

$$\underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}(p)) = \bar{\pi}_i^{z_i^F}(\bar{c}_i).$$

When outside options are type-independent, then

$$c_i^*(p, p^{-i}(p)) = \bar{c}_i. \tag{22}$$

This is because $V_i(c_i)$ is decreasing in c_i : If outside options are type-independent, then it is immediate that the participation constraint binds at the highest cost type, namely $c_i^* = \overline{c}_i$, irrespectively of the exact shape of V_i.

Environments that fall in this category are those in Myerson (1981) and in Jehiel, Moldovanu and Stacchetti (1996). In terms of applications, this assumption is satisfied whenever the outside options are independent from the parameter that affects the payoffs of participation in the auction. For example, it could be satisfied in a procurement setting for some specialized project. The firms' private information affects their cost of production of the project, but not their profits if they stay out of the competition.

Case 2: Very Steep Payoff from Worst Allocation for i

Another case is the polar opposite of the previous one. Here, the worst allocation for buyer i is type-dependent, and very sharply so. More precisely, there exists an allocation $z_i^S \in Z^{-i}$, at which $i's$ payoff is very sensitive to type, and guarantees the lowest payoff at $\underline{c}_i$:[20]

[20] Such a case is illustrated in Example 1 in Figueroa and Skreta (2009a).

$$\begin{aligned}\frac{d\bar{\pi}_i^{z_i^S}(c_i)}{dc_i} &\leq \frac{d\bar{\pi}_i^{z}(c_i)}{dc_i} \text{ for all } z \in Z, \\ \bar{\pi}_i^{z_i^S}(\underline{c}_i) &\leq \bar{\pi}_i^{z}(\underline{c}_i) \text{ for all } z \in Z.\end{aligned}$$

It is easy to see (the details are in Appendix C) that the revenue-maximizing outside option from the seller's perspective is $(p^{-i})^{z_i^S} = 1$ for all p since, for all p, it solves

$$p^{-i}(p) \in \arg\min_{\rho^{-i}\in\mathcal{P}^{-i}} \underline{U}_i(c_i^*(p,\rho^{-i}),\rho^{-i}) + \int_{c_i^*(p,\rho^{-i})}^{\overline{c_i}} P_i(s)ds. \tag{23}$$

In that case, we have

$$\begin{aligned} c_i^*(p,p^{-i}(p)) &= \underline{c}_i \text{ and} \\ \underline{U}_i(c_i^*(p,p^{-i}(p)),p^{-i}(p)) &= \bar{\pi}_i^{z_i^S}(\underline{c}_i);\end{aligned} \tag{24}$$

see Figure 2:

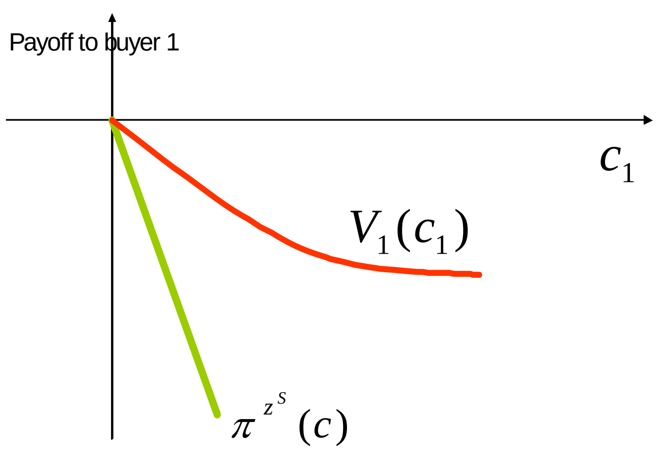

Figure 2: PC binds at BEST type

To understand why the critical type is the lowest cost (best type) in this case, recall that the critical type is the one where gains from trade are minimal. Now, in this case, as the cost gets higher, the outside options worsen faster than any feasible participation payoff. This implies that the gains from trade are minimal for the lowest cost. For the same reason, the participation constraint binds at the best type (the highest valuation) for the seller in the classical bilateral trading problem in Myerson and Satterthwaite (1983).

Case 3: Coexistence of Flat and Very Steep Worst Allocations for i

Another interesting case is the one where options like z_i^S and z_i^F coexist, and it is not obvious which one the seller should use because

$$\begin{aligned}\frac{d\bar{\pi}_i^{z_i^S}(c_i)}{dc_i} &\leq \frac{d\bar{\pi}_i^{z}(c_i)}{dc_i} \leq \frac{d\bar{\pi}_i^{z_i^F}(c_i)}{dc_i} \text{ for all } z \in Z, c_i \in C_i \\ \bar{\pi}_i^{z_i^S}(\underline{c}_i) &\geq \bar{\pi}_i^{z_i^F}(\underline{c}_i).\end{aligned}$$

As one can see from Figure 3, for some types, z_i^F hurts more, and for others z_i^S.

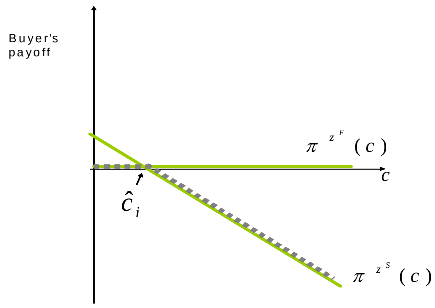

Figure 3: PC binds at an interior type.

In this case[21], the solution to

$$p^{-i}(p) \in \arg\min_{\rho^{-i}\in\mathcal{P}^{-i}} \rho^{-i}\bar{\pi}_i^{z_i^F}(c_i^*) + (1-\rho^{-i})\bar{\pi}_i^{z_i^S}(c_i^*) + \int_{c_i^*}^{\overline{c_i}} P_i(s)ds, \tag{25}$$

is such that

$$\begin{aligned} c_i^*(p, p^{-i}(p)) &= \hat{c}_i \text{ and} \\ \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}(p)) &= \bar{\pi}_i^{z_i^F}(\hat{c}_i) = \bar{\pi}_i^{z_i^S}(\hat{c}_i) \text{ for all } p \text{ and } p^{-i}(p) \end{aligned} \tag{26}$$

where $\hat{c}_i$ is the type where the payoffs cross-that is,

$$\bar{\pi}_i^{z_i^F}(\hat{c}_i) = \bar{\pi}_i^{z_i^S}(\hat{c}_i). \tag{27}$$

The critical type is independent of p, despite the fact that the revenue-maximizing p^{-i} can depend on p. This is because (8) implies that if c_i^* is interior, V_i and $\underline{U}_i$ must be tangent at c_i^*; namely, it must be the case that

$$\frac{\partial \underline{U}_i(c_i^*, p^{-i})}{\partial c_i} \in \partial V_i(c_i^*). \tag{28}$$

Then, for every possible assignment rule p, when the seller chooses $p^{-i} \in \mathcal{P}^{-i}$ optimally-that is, according to (10)- the following are true:

$$\begin{aligned} c_i^*(p, p^{-i}(p)) &\equiv c_i^* \text{ and} \\ \underline{U}_i(c_i^*, p^{-i}(p)) &\equiv \underline{U}_i(c_i^*). \end{aligned} \tag{29}$$

In contrast to the previous two cases the critical type can be interior in this case.

Summing up, in all these cases[22], neither $c_i^*(p, p^{-i}(p))$, nor the level of $\underline{U}_i(., p^{-i}(p))$, evaluated at the critical type c_i^*, depend on p.

[21] Such a case is illustrated in Example 2 in Figueroa and Skreta (2009a).

[22] These are not the only cases where revenue will be linear in p, but they are suggestive of the classes of environments that are likely to exhibit this property.

Proposition 3 *If (29) is satisfied, the seller's expected revenue can be expressed as a linear function of the assignment rule,*

$$\int_C \sum_{z \in Z} p^z(c) \hat{J}_z(c) f(c) dc - \sum_{i=1}^{I} \underline{U}_i(c_i^*), \tag{30}$$

where $\hat{J}_z$ is the modified virtual surplus of allocation z defined in (14).

We now discuss the solution of such problems.

4.2 Revenue-maximizing Mechanisms

When revenue can be expressed by (30), we can break down the characterization of revenue-maximizing mechanisms into two steps: First, find a revenue-maximizing non-participation assignment rule $\{p^{-i}(p)\}_{i \in I}$, as we have done in (21), (23), or (25), and then find a revenue-maximizing assignment rule p that solves:

$$\max_{p \in \Delta(Z)} \int_C \sum_{z \in Z} p^z(c) \hat{J}_z(c) f(c) dc \tag{31}$$
$$s.t.\ P_i \text{ increasing.}$$

This problem has a structure similar to the classical one in Myerson (1981), but with modified virtual surpluses, and can be solved using relatively conventional methods. Despite this, the qualitative features of the solution will often exhibit stark differences from the classical one.

The solution is straightforward if the assignment rule that solves the relaxed program,

$$\max_{p \in \Delta(Z)} \int_C \sum_{z \in Z} p^z(c) \hat{J}_z(c) f(c) dc,$$

also satisfies the requirement of P_i being increasing since, in that case, the relaxed program can be solved by pointwise maximization. Following Myerson (1981), we will refer to this as the *regular* case. On the other hand, in the *general* case, pointwise optimization will lead to a mechanism that may not be feasible.

In the classical problem, a sufficient condition for the problem to be regular is that the virtual surpluses are increasing. A mild condition on the distribution function F_i (MHR) guarantees that. Unfortunately, in our more general environment, the problem fails to be regular even if virtual surpluses (or modified virtual surpluses) are monotonic, so Myerson's technique of obtaining 'ironed' virtual valuations will not work. Dealing with these complications is beyond the theme of this paper, the primary focus of which is the effect of type-dependent outside options.[23]

We now state a condition that guarantees that pointwise optimization will lead to a feasible solution. Before stating the Assumption, let us provide some explanation. Recall that IC requires P_i to be increasing in c_i. Pointwise optimization assigns probability one to the allocation with the highest virtual surplus at each vector of types. Along a region where there is no switch, one allocation, say z_1, is selected throughout,

[23] In Figueroa and Skreta (2009b), we illustrate this phenomenon and show a way to solve the general case, which does not impose additional assumptions, such as differentiability, on the mechanism. There, we argue that in the general case an optimal mechanism will involve randomizations between allocations. Such lotteries are quite surprising given that buyers are risk-neutral and types are single-dimensional.

and $P_i(c_i) = \int\limits_{C_{-i}} \frac{\partial \pi_i^{z_1}(c_i,c_{-i})}{\partial c_i} f_{-i}(c_{-i})dc_{-i}$, which is increasing by the convexity of π_i. Incentive compatibility can be violated, though, when the seller wishes to switch, say, from allocation z_1 to z_2. At such a point c, we have that $\hat{J}_{z_2}(c) \geq \hat{J}_{z_1}(c)$ and IC requires that P_i does not decrease-namely, $\int\limits_{C_{-i}} \frac{\partial \pi_i^{z_2}(c_i,c_{-i})}{\partial c_i} f_{-i}(c_{-i})dc_{-i} \geq \int\limits_{C_{-i}} \frac{\partial \pi_i^{z_1}(c_i,c_{-i})}{\partial c_i} f_{-i}(c_{-i})dc_{-i}$. Our condition guarantees precisely this.

Assumption 4 [24] *Let z_1,$z_2 \in Z$ be any two allocations. For a given cost realization (c_i, c_{-i}), if $z_1 \in arg \max_{z\in Z} \hat{J}_z(c_i^-, c_{-i})$ and $z_2 \in arg \max_{z\in Z} \hat{J}_z(c_i^+, c_{-i})$, then $\frac{\partial \bar{\pi}_i^{z_2}(c_i)}{\partial c_i} \geq \frac{\partial \bar{\pi}_i^{z_1}(c_i)}{\partial c_i}$.*[25]

We now state another condition that is more stringent, but often easier to verify than Assumption 4. Note that this assumption requires knowledge of $\hat{J}_z$, which depends on c^*, but c^* is independent of p in the cases we consider here.

Assumption 5 *For all i and for all c_{-i}, when $\frac{\partial \hat{J}_{z_2}(c_i,c_{-i})}{\partial c_i} \geq \frac{\partial \hat{J}_{z_1}(c_i,c_{-i})}{\partial c_i}$, then $\frac{\partial \bar{\pi}_i^{z_2}(c_i)}{\partial c_i} \geq \frac{\partial \bar{\pi}_i^{z_1}(c_i)}{\partial c_i}$.*

Lemma 6 *Assumption 5 is sufficient for Assumption 4.*

For the special class where payoffs are linear in their own type, there is an even simpler condition that is sufficient for Assumption 4−namely, the well known monotone hazard rate condition. Hence, Assumption 4 generalizes the standard regularity condition.

Lemma 7 *If the expected payoff functions are of the linear form $\bar{\pi}_i^z(c_i) \equiv A_i^z + B_i^z c_i$, $\frac{F_i(c_i)}{f_i(c_i)}$ and $\frac{F_i(c_i)-1}{f_i(c_i)}$ are increasing in c_i for all i, then Assumption 4 is satisfied.*

With the help of Assumption 4, it is straightforward to find a revenue-maximizing assignment rule, which is described in the following result.

Proposition 8 *Suppose that (29) holds. If Assumption 4 is satisfied, then a revenue-maximizing allocation p is given by:* [26]

$$p^{z^*}(c) = \begin{cases} 1 & \text{if } z^* \in arg \max\limits_z \hat{J}_z(c) \\ 0 & \text{otherwise} \end{cases} .$$

The qualitative features of the solution depend on whether the conditions in (29) are satisfied for $c_i^* = \overline{c}_i$, $c_i^* = \underline{c}_i$, or $c_i^* \in (\underline{c}_i, \overline{c}_i)$. If $c_i^* = \bar{c}_i$, then $\hat{J}_z(c) < S_z(c)$ and the seller sells less often than is efficient. When the conditions in (29) are satisfied for $c_i^* = \underline{c}_i$, $\hat{J}_z(c) \geq S_z(c)$ and overselling occurs, as stated in the next corollary:

Corollary 9 *Suppose that $c_i^*(p, p^{-i}(p)) = \underline{c}_i$ for all i. Suppose, also, that when the seller keeps all objects, every buyer gets a payoff independent of his type- zero, for example. Then, at a revenue-maximizing assignment rule, the seller keeps all the objects* ***less*** *often than is ex-post efficient.*

[24] This condition has similar flavor to condition 5.1 in the environment of Jehiel and Moldovanu (2001b). We are grateful to Benny Moldovanu for bringing this connection to our attention.

[25] The notation c_i^- means limit from the left to c_i and c_i^+ means limit from the right to c_i.

[26] Ties can be broken arbitrarily. If for fixed c_{-i} there is an interval, subset of C_i, with a tie between two allocations, Assumption 4 implies that the partial derivatives are equal, so the selection does not affect incentive compatibility.

As noted in the introduction, "overselling" is in contrast with a standard intuition from monopoly theory, where the monopolist restricts supply in order to generate higher revenue. The intuition behind overselling in our context is as follows: With type-dependent outside payoffs, the seller may be able to design outside options that hurt bad types relatively more than good types. This allows her to charge a higher price without restricting supply; good types pay due to their high valuation, and bad types pay because their outside options are relatively worse.

The reason that the seller can design outside options that hurt bad types relatively more than good types is as follows. When outside options are type-dependent, the critical type is endogenous, and the virtual surplus of allocating the object to a buyer (actual surplus minus information rents) is increased for all types worse than the critical type (the information rents are reduced). Hence, the "modified virtual surplus" can be weakly higher than the actual surplus of an allocation. Depending on this comparison, the revenue-maximizing mechanism may be ex-post efficient or may even induce overselling.

When $c_i^* \in (\underline{c}_i, \overline{c}_i)$, then $\hat{J}_z(c) < S_z(c)$ for some type profiles, and $\hat{J}_z(c) \geq S_z(c)$ for others. Here, underselling and overselling can occur simultaneously (the seller keeps the objects in some cases where she should sell and sells them in cases where she should keep them), or even ex-post efficiency can occur.

As already discussed, revenue will be non-linear in p when $c_i^*(p, p^{-i}(p))$ depends on p. In such cases, the analysis can proceed on a case-by-case basis. However, the main qualitative features of the revenue-maximizing assignments discussed above for the linear case remain: The degree of efficiency of the revenue-maximizing assignments depends on the relation between the modified virtual surpluses and the real surpluses. In turn, this last relationship depends on the actual values of $c_i^*(p, p^{-i}(p))$, for $i \in I$, at a revenue-maximizing p.

Another difference from the standard case is that the revenue-maximizing reserve price that a buyer faces will often depend on the other buyers' reports. This is because when values are interdependent, or when buyers care about the entire allocation of the objects, the virtual surplus of an allocation (6) and, hence, the modified virtual surplus of an allocation can depend on the entire vector of reports.

5. Concluding Remarks

This paper shows that key intuitions from earlier work on revenue-maximizing auctions, such as flat reserve prices and underselling, fail to generalize. In our analysis, it turns out that revenue-maximizing reserve prices should often depend on other buyers' bids. We also show that type-dependent non-participation payoffs change the nature of the distortions that arise from the presence of asymmetric information. The designer, by creating the "appropriate" outside options, can increase both revenue and the overall efficiency of the mechanism.

More broadly, this work presents a very general allocation problem, formalized in an elegant way, that encompasses virtually all works with quasi-linear payoffs, single-dimensional private information and risk-neutral buyers. Potential applications of our model, other than the aforementioned ones, include the allocation of rights to a new technology, of positions in teams, of students to schools, and many more. Our model is so versatile that it encompasses various papers in the previous literature. But we do not only reformulate previous work: Our model has features, such as type-dependent non-participation payoffs and the possibility of non-linear payoffs, that lead to new phenomena and complications previously unaddressed in the literature. We identify those difficulties and show how far one can go using Myerson-like techniques. In some complementary work (Figueroa and Skreta (2009b)), we show an instance where those techniques are insufficient even with virtual surpluses strictly monotonic in type.

6. Appendix A

Proof of Lemma 1[27]

By the convexity of $\pi_i^z(\cdot, c_{-i})$, we have that V_i is a maximum of convex functions, so it is convex and, therefore, differentiable a.e. It is also easy to check that the following are equivalent:

(a) (p, x) is incentive-compatible

(b) $P_i(c_i) \in \partial V_i(c_i)$

(c) $U_i(c_i, c_i; (p, x)) = V_i(c_i)$

We now use these equivalent statements to prove necessity and sufficiency in our Lemma.

($\Longrightarrow$) Here, we use the fact that incentive compatibility implies (b). A result in Krishna and Maenner (2001) then implies (5). By the convexity of V_i, we know that ∂V_i is monotone, so:

$$(P_i(c_i) - P_i(c_i'))(c_i - c_i') \geq 0.$$

This immediately implies (4).

($\Longleftarrow$) To prove that (4) implies incentive compatibility, it's enough to show that $P_i(c_i) \in \partial V_i(c_i)$. By (4) and (5),

$$\begin{aligned} V_i(c_i') - V_i(c_i) &= \int_{c_i}^{c_i'} P_i(s) ds \\ &\geq P_i(c_i)(c_i' - c_i), \end{aligned}$$

which shows $P_i(c_i) \in \partial V_i(c_i)$. ■

Expected Payment in an Incentive-Compatible Mechanism[28]

Recall that

$$V_i(c_i) = \int_{C_{-i}} \left[\sum_{z \in Z} p^z(c) \pi_i^z(c) - x_i(c) \right] f_{-i}(c_{-i}) dc_{-i}. \tag{32}$$

By integrating (32) with respect to c_i, and by rearranging, we get that

$$\int_C x_i(c) f(c) dc = \int_C \sum_{z \in Z} p^z(c) \pi_i^z(c) f(c) dc - \int_{C_i} V_i(c_i) f_i(c_i) dc_i. \tag{33}$$

[27] This proof is relatively standard (see, for instance, Jehiel, Moldovanu and Stacchetti (1999)) and is included for completeness.

[28] This proof is standard and is included for completeness.

Integrating the second condition in (5) over C_{-i} and by changing the order of integration, we get:

$$\begin{aligned}
\int\limits_{C_i} V_i(c_i)dc_i &= \int\limits_{C_i} [V_i(\overline{c}_i) - \int\limits_{c_i}^{\overline{c_i}} P_i(s_i)ds_i] f_i(c_i)dc_i \\
&= V_i(\overline{c}_i) - \int\limits_{C_i} P_i(s_i) \int\limits_{\underline{c_i}}^{s_i} f_i(c_i)dc_i ds_i \\
&= V_i(\overline{c}_i) - \int\limits_{C_i} P_i(c_i)F_i(c_i)dc_i \\
&= V_i(\overline{c}_i) - \int\limits_{C_i}\int\limits_{C_{-i}} \sum_{z\in Z} p^z(c_i,c_{-i}) \frac{\partial \pi_i^z(c_i,c_{-i})}{\partial c_i} f_{-i}(c_{-i})dc_{-i}F_i(c_i)dc_i \\
&= V_i(\overline{c}_i) - \int\limits_{C} \sum_{z\in Z} p^z(c_i,c_{-i}) \frac{\partial \pi_i^z(c_i,c_{-i})}{\partial c_i} \frac{F_i(c_i)}{f_i(c_i)} f(c)dc.
\end{aligned}$$

Combining (33) with the last expression, the result follows. ■

Proof of Lemma 6

If there exists a point (c_i, c_{-i}) such that $z_1 \in arg \max\limits_{z\in Z} \hat{J}_z(c_i^-, c_{-i})$ and $z_2 \in arg \max\limits_{z\in Z} \hat{J}_z(c_i^+, c_{-i})$, then it must be the case that $\frac{\partial J_{z_2}(c_i,c_{-i})}{\partial c_i} \geq \frac{\partial J_{z_1}(c_i,c_{-i})}{\partial c_i}$. If Assumption 5 is satisfied, then we have that $\frac{d\bar{\pi}_i^{z_2}(c_i)}{dc_i} \geq \frac{d\bar{\pi}_i^{z_1}(c_i)}{dc_i}$, which implies that Assumption 4 is also satisfied. ■

Proof of Lemma 7

We just need to prove that Assumption 5 is satisfied. Suppose that $c_i < c_i^*$ and that $\frac{\partial J_{z_1}(c_i,c_{-i})}{\partial c_i} \geq \frac{\partial J_{z_2}(c_i,c_{-i})}{\partial c_i}$. By the linearity assumption, we have that $B_i^{z_1}\left[1 + \left(\frac{F_i(c_i)}{f_i(c_i)}\right)'\right] \geq B_i^{z_2}\left[1 + \left(\frac{F_i(c_i)}{f_i(c_i)}\right)'\right]$. Then, since $\left(\frac{F_i(c_i)}{f_i(c_i)}\right)' \geq 0$ by assumption, we get $B_i^{z_1} \geq B_i^{z_2}$, which is equivalent to $\frac{d\bar{\pi}_i^{z_1}(c_i)}{dc_i} \geq \frac{d\bar{\pi}_i^{z_2}(c_i)}{dc_i}$ under the linearity assumption. The proof is analogous for $c_i > c_i^*$. ■

Proof of Proposition 8

The solution proposed corresponds to pointwise maximization, so the only possibility that is not revenue-maximizing is that it is not feasible. To check feasibility, remember that $P_i(c_i) = \int\limits_{C_{-i}} \sum\limits_{z\in Z} p^z(c_i,c_{-i})\frac{\partial\pi_i^z(c_i,c_{-i})}{\partial c_i} f_{-i}(c_{-i})dc_{-i}$ and consider a fixed c_{-i}. In a region of cost realizations where $\overline{z} \in arg \max\limits_{z\in Z} \hat{J}_z(c)$, the allocation rule $p(c)$ does not change since, along this region, $p^{\overline{z}}(c) = 1$. Then, $P_i(c_i)$ is increasing by the convexity of $\pi_i^{\overline{z}}(\cdot, c_{-i})$. For a c_i^* where $z_1 \in arg \max\limits_{z\in Z} \hat{J}_z(c_i^{*-}, c_{-i})$ and $z_2 \in arg \max\limits_{z\in Z} \hat{J}_z(c_i^{*+}, c_{-i})$, $p^{z_1}(c_i^{*-}, c_{-i}) = 1$ and $p^{z_2}(c_i^{*+}, c_{-i}) = 1$, $P_i(c_i)$ is increasing by Assumption 4. ■

Proof of Corollary 9

Let's denote by z_0 the allocation where the seller keeps all the objects and consider a fixed realization of types c. Since $\pi_i^{z_0}(c)$ is constant for all i, its derivative vanishes, and we have that $J_{z_0}(c) = \sum\limits_{i=1}^{N} \pi_i^{z_0}(c) = S_{z_0}(c)$. On the other hand, for every allocation z, its virtual surplus is given by

$$J_z(c) = \sum_{i=1}^{N}\left[\pi_i^z(c) + \frac{\partial\pi_i^z(c)}{\partial c_i}\frac{F_i(c_i)-1}{f_i(c_i)}\right] > S_z(c) \equiv \sum_{i=1}^{N}\pi_i^z(c).$$

Then, it is easy to see that the set where the seller keeps the objects, $\left\{c | z_0 \in arg \max\limits_{z} S_z(c)\right\}$, is a subset of the set where it would be efficient to keep them, $\left\{c|z_0 \in arg\max_z J_z(c)\right\}$. ■

7. Appendix B: An Example where Revenue Depends Non-Linearly on p.

Suppose that there is one buyer and three possible allocations z_1, z_2, z_3 and that c is uniformly distributed on $[0,1]$. The payoffs of the allocations are $\pi^{z_1}(c) = 10 - 10c$, $\pi^{z_2}(c) = 0$ and $\pi^{z_3}(c) = -5c$, where $c \in [\underline{c}, \overline{c}]$. Then, it is easy to see that, irrespective of p, a revenue-maximizing non-participation assignment rule is $(p^{-1})^{z_3} = 1$, so the non-participation assignment rule assigns probability one to allocation z_3. An assignment rule $p(c) = (p^{z_1}(c), p^{z_2}(c), p^{z_3}(c))$ induces a surplus

$$V(c) = V(\overline{c}; p, p^{-1}) - \int_{c}^{\overline{c}} P(s)ds,$$

which, at the points where it is differentiable, satisfies $\frac{dV(c)}{dc} = P(c) = -10p^{z_1}(c) - 5p^{z_3}(c)$. The type where the participation constraint binds depends on how $P(c)$, which is the slope of the payoff from participating in the mechanism, compares to the slope of the payoff from not participating, which is given by -5. The critical type c^* depends non-linearly on p, and it is given by

$$c^*(p, p^{-1}) = \begin{cases} \underline{c} & \text{if } -5 \leq -10p^{z_1}(0) - 5p^{z_3}(0) \\ \overline{c} & \text{if } -5 \geq -10p^{z_1}(1) - 5p^{z_3}(1) \\ c^* & \text{otherwise} \end{cases},$$

where c^* satisfies that $-10p^{z_1}(c^{*^-}) - 5p^{z_3}(c^{*^-}) \leq -5 \leq -10p^{z_1}(c^{*^+}) - 5p^{z_3}(c^{*^+})$. Since

$$V(\overline{c}, p, p^{-1}) = -5c^*(p, p^{-1}) + \int_{c^*(p,p^{-1})}^{\overline{c}} [-10p^{z_1}(c) - 5p^{z_3}(c)]dc,$$

we have that the objective function is non-linear in the assignment rule p.

8. Appendix C: Two Specific Environments where Critical Types are Independent of p.

I: Steep Outside Options: Participation Constraints bind at the best type $c_i^* = \underline{c}_i$.

We now provide the precise conditions for the case of "very responsive" outside options and argue that, under those conditions, (29) is satisfied at $c_i^* = \underline{c}_i$.

Recall that we use $\overline{\pi}_i^z(c_i) = \int_{C_{-i}} \pi_i^z(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}$ to denote the expected payoff to agent i if allocation z is implemented.

Assumption 10 *Suppose that outside options are steep, in the sense that for all $i \in I$, there exists an allocation $z_i^S \in Z^{-i}$ such that*

$$\frac{d\overline{\pi}_i^{z_i^S}(c_i)}{dc_i} \leq \frac{d\overline{\pi}_i^z(c_i)}{dc_i} \textit{ for all } z \in Z \tag{34}$$

and

$$\overline{\pi}_i^{z_i^S}(\underline{c}_i) \leq \overline{\pi}_i^z(\underline{c}_i) \textit{ for all } z \in Z. \tag{35}$$

Proposition 11 *Under Assumption 10, it follows that for all p (a) $(\hat{p}^{-i})^z \equiv \begin{cases} 1 & \text{if } z = z_i^S \\ 0 & \text{if not} \end{cases}$, for all i is a revenue-maximizing non-participation assignment rule, (b) $c_i^* = \underline{c}_i$, for all i, and (c) $\underline{U}_i(c_i^*) \equiv \bar{\pi}_i^{z_i^S}(\underline{c}_i)$.*

Proof. (a)The optimality of $\hat{p}^{-i}$ follows immediately from (34) and (35).

(b) Now we show that $c_i^* = \underline{c}_i$, by establishing that if the participation constraint is satisfied at $c_i = \underline{c}_i$, then it is satisfied for all $c_i \in C_i$. This follows from three observations.

(i) $P_i(c_i) \in \partial V_i(c_i)$,

(ii)

$$\begin{aligned} P_i(c_i) &= \int\limits_{C_{-i}} \sum_{z\in Z} p^z(c) \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i} \\ &\geq \int\limits_{C_{-i}} \sum_{z\in Z} p^z(c) \frac{\partial \pi_i^{z_i^S}(c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i} = \frac{d\bar{\pi}_i^{z_i^S}(c_i)}{dc_i} \end{aligned}$$

(iii) $V_i(\underline{c}_i) \geq \bar{\pi}_i^{z_i^S}(\underline{c}_i)$.

Observations (i) and (ii) imply that the derivative of V_i is always greater than the derivative of $\bar{\pi}_i^{z_i^S}$. These two, together with (iii), imply that $V(c_i) \geq \bar{\pi}_i^{z_i^S}(c_i)$ for all $c_i \in C_i$.

(c) Finally, it follows immediately that $\underline{U}_i(c_i^*) \equiv \bar{\pi}_i^{z_i^S}(\underline{c}_i)$. ■

II: Coexistence of Steep and Flat Outside Options: Participation Constraints bind at interior types $c_i^* \in (\underline{c}_i, \bar{c}_i)$.

Suppose that there are two extreme allocations for each buyer, one that gives the flattest payoff z_i^S, and one that gives the steepest, z_i^F. If the flattest option were to be used, then $c_i^* = \bar{c}_i$, and if the steepest option were to be used, then $c_i^* = \underline{c}_i$. When neither of these two options is clearly worse, it turns out that a revenue-maximizing $p^{-i}(p)$ randomizes between the two options, and the participation constraint always binds at the type who is indifferent between z_i^S and z_i^F. We now describe the precise conditions and establish the claim.

Assumption 12 *Suppose that $Z^{-i} = \{z_i^S, z_i^F\}$ and that $\frac{d\bar{\pi}_i^{z_i^S}(c_i)}{dc_i} \leq \frac{d\bar{\pi}_i^z(c_i)}{dc_i} \leq \frac{d\bar{\pi}_i^{z_i^F}(c_i)}{dc_i}$ for all $z \in Z$ and $c_i \in C_i$ and $\bar{\pi}_i^{z_i^S}(\underline{c}_i) \geq \bar{\pi}_i^{z_i^F}(\underline{c}_i)$. Suppose, also, that either (i) values are private or (ii) the seller can only use non-participation assignment rules that do not depend on the types of other players (that is $p^{-i} \in \mathcal{P}^{-i} \Longrightarrow p^{-i}(c_{-i}) \equiv p^{-i}$).*

Proposition 13 *Under Assumption 12, it follows that (a) for all p, the critical type is $c_i^* = \hat{c}_i$ where $\hat{c}_i$ satisfies*

$$\bar{\pi}_i^{z_i^S}(\hat{c}_i) = \bar{\pi}_i^{z_i^F}(\hat{c}_i); \tag{36}$$

(b) a revenue-maximizing p^{-i} given p is determined by the condition $(p^{-i}(p))^{z_i^S} \frac{d\bar{\pi}_i^{z_i^S}(\hat{c}_i)}{dc_i} + (1-(p^{-i}(p))^{z_i^F}) \frac{d\bar{\pi}_i^{z_i^F}(\hat{c}_i)}{dc_i} \in \partial V_i(\hat{c}_i)$; and (c) for all p, we have $\underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}(p)) = \bar{\pi}_i^{z_i^F}(\hat{c}_i) = \bar{\pi}_i^{z_i^S}(\hat{c}_i)$.

Proof: To prove this Proposition, we first prove the following Lemma:

Lemma A.

$$\left.\frac{dV_i(\bar{c}_i)}{d(\rho^{-i})^z}\right|_{\rho^{-i}=p^{-i}(p)} = \left.\frac{\partial V_i(\bar{c}_i)}{\partial(\rho^{-i})^z}\right|_{\rho^{-i}=p^{-i}(p)} = \bar{\pi}_i^z(c_i^*(p,p^{-i}(p))),\ \text{for all } z \in Z^{-i}. \tag{37}$$

Proof. We suppose for simplicity that the derivative $\frac{\partial c_i^*(p,\rho^{-i})}{\partial \rho^{-i}}$ is well defined, (otherwise, we can do all the analysis with subgradients). Then, differentiating $V_i(\bar{c}_i) = \underline{U}_i(c_i^*(p,\rho^{-i}),\rho^{-i}) + \int\limits_{c_i^*(p,\rho^{-i})}^{\bar{c}_i} P_i(s)ds$ with respect to $(\rho^{-i})^z$ we obtain that

$$\frac{dV_i(\bar{c}_i)}{d(\rho^{-i})^z} = \frac{\partial \underline{U}_i(c_i^*(p,\rho^{-i}),\rho^{-i})}{\partial(\rho^{-i})^z} + \left[\frac{\partial \underline{U}_i(c_i^*(p,\rho^{-i}),\rho^{-i})}{\partial c_i} - P_i(c_i^*(p,\rho^{-i}))\right]\frac{\partial c_i^*(p,\rho^{-i})}{\partial(\rho^{-i})^z}. \tag{38}$$

Given an assignment rule p and a non-participation assignment rule ρ^{-i}, we know that in a revenue-maximizing mechanism, $c_i^*(p,\rho^{-i})$ satisfies $c_i^*(p,\rho^{-i}) \in \arg\min_{c_i}\left[-\int\limits_{c_i}^{\bar{c}_i} P_i(s)ds - \underline{U}_i(c_i,\rho^{-i})\right]$. Depending on whether $c_i^*(p,\rho^{-i}) \in (\underline{c_i},\bar{c}_i)$, or $c_i^*(p,\rho^{-i}) = \underline{c_i}$ or $c_i^*(p,\rho^{-i}) = \bar{c}_i$, there are three cases to consider.

Case 1: $c_i^*(p,\rho^{-i}) \in (\underline{c}_i,\bar{c}_i)$

Since $c_i^*(p,\rho^{-i}) \in \arg\min_{c_i}\left[-\int\limits_{c_i}^{\bar{c}_i} P_i(s)ds - \underline{U}_i(c_i,\rho^{-i})\right]$, and is an interior solution, it must satisfy

$$\left.\frac{dV_i(c_i)}{dc_i}\right|_{c_i=c_i^*(p,\rho^{-i})} = \left.\frac{\partial \underline{U}_i(c_i(p,\rho^{-i}),\rho^{-i})}{\partial c_i}\right|_{c_i=c_i^*(p,\rho^{-i})}. \tag{39}$$

Then, recall that $V_i(c_i) = V_i(\bar{c}_i) - \int\limits_{c_i}^{\overline{c_i}} P_i(s)ds$, which implies that

$$\left.\frac{dV_i(c_i)}{dc_i}\right|_{c_i=c_i^*(p,\rho^{-i})} = P_i(c_i^*(p,\rho^{-i})). \tag{40}$$

Then, substituting (39) and (40) into (38), we obtain that

$$\left.\frac{dV_i(\bar{c}_i)}{d(\rho^{-i})^z}\right|_{\rho^{-i}=p^{-i}(p)} = \frac{\partial \underline{U}_i(c_i^*(p,p^{-i}(p)),p^{-i}(p))}{\partial(\rho^{-i})^z} = \bar{\pi}_i^z(c_i^*(p,p^{-i}(p))),\ \text{for all } z \in Z^{-i},$$

which is what we wanted to show.

Case 2: $c_i^*(p,\rho^{-i}) = \underline{c_i}$

If p and ρ^{-i}such that $c_i^*(p,\rho^{-i}) = \underline{c_i}$, and we change the z^{th} component of the non-participation assignment rule p^{-i}, then two things can happen. One possibility is that

$$\frac{\partial c_i^*(p,\rho^{-i})}{\partial(\rho^{-i})^z} = 0;$$

in that case, (38) reduces to (37). Another possibility is that we move to a c_i^* in the interior, in which case we are back to Case 1.[29]

Case 3: $c_i^*(p,\rho^{-i}) = \bar{c}_i$

This case is identical to the previous one. ■

[29] Note that since both V_i and $\underline{U}_i$ are smooth a.s in c_i, changing $(p^{-i})^z$ slightly cannot result in c_i^* moving from $\underline{c}_i$ to $\bar{c}_i$.

Now, we prove the Proposition.

(a) Because there are only z_i^S and z_i^F in Z^{-i}, we can write

$$V_i(\bar{c}_i) = \rho^{-i}\bar{\pi}_i^{z_i^S}(c_i^*(p,\rho^{-i})) + (1-\rho^{-i})\bar{\pi}_i^{z_i^F}(c_i^*(p,\rho^{-i})) + \int_{c_i^*(p,\rho^{-i})}^{\bar{c}_i} P_i(s)ds.$$

Also (37), implies

$$\begin{aligned} \left.\frac{dV_i(\bar{c}_i)}{d\rho^{-i}}\right|_{\rho^{-i}=p^{-i}(p)} &= \left.\frac{\partial V_i(\bar{c}_i)}{\partial\rho^{-i}}\right|_{\rho^{-i}=p^{-i}(p)} \\ &= \bar{\pi}_i^{z_i^S}(c_i^*(p,\rho^{-i})) - \bar{\pi}_i^{z_i^F}(c_i^*(p,\rho^{-i})). \end{aligned} \tag{41}$$

When ρ^{-i} is in a neighborhood of 0, then the outside option is flat and $c_i^* = \bar{c}_i$. When ρ^{-i} is in a neighborhood of 1, then the outside option is very steep and $c_i^* = \underline{c}_i$. This means that $\left.\frac{\partial c_i^*(p,\rho^{-i})}{\partial\rho^{-i}}\right|_{\rho^{-i}=0} = \left.\frac{\partial c_i^*(p,\rho^{-i})}{\partial\rho^{-i}}\right|_{\rho^{-i}=1} = 0$, and also we get that

$$\begin{aligned} \left.\frac{dV_i(\bar{c}_i)}{d\rho^{-i}}\right|_{\rho^{-i}=0} &= \bar{\pi}_i^{z_i^S}(c_i^*(p,0)) - \bar{\pi}_i^{z_i^F}(c_i^*(p,0)) \\ &= \bar{\pi}_i^{z_i^S}(\bar{c}_i) - \bar{\pi}_i^{z_i^F}(\bar{c}_i) < 0 \\ \left.\frac{dV_i(\bar{c}_i)}{d\rho^{-i}}\right|_{\rho^{-i}=1} &= \bar{\pi}_i^{z_i^S}(c_i^*(p,1)) - \bar{\pi}_i^{z_i^F}(c_i^*(p,1)) \\ &= \bar{\pi}_i^{z_i^S}(\underline{c}_i) - \bar{\pi}_i^{z_i^F}(\underline{c}_i) > 0. \end{aligned}$$

These two inequalities imply that the optimally chosen ρ^{-i}-that is, $p^{-i}(p)$- is interior, so it satisfies the FONC $\left.\frac{dV_i(\bar{c}_i)}{d\rho^{-i}}\right|_{\rho^{-i}=p^{-i}(p)} = 0$. This implies, from (41), that $\bar{\pi}_i^{z_i^S}(c_i^*(p,\rho^{-i})) = \bar{\pi}_i^{z_i^F}(c_i^*(p,\rho^{-i}))$, from which we get that irrespective of p, we have that $c_i^* = \hat{c}_i$, where $\hat{c}_i$ satisfies (36). Moreover, because of the assumptions, the functions $\bar{\pi}_i^{z_i^S}$ and $\bar{\pi}_i^{z_i^F}$ cross at most once, so c_i^* is uniquely determined.

(b) By (28), it follows immediately that a revenue-maximizing p^{-i} given p must satisfy that $p^{-i}(p)\frac{d\bar{\pi}_i^{z_i^S}(\hat{c}_i)}{dc_i} + (1-p^{-i}(p))\frac{d\bar{\pi}_i^{z_i^F}(\hat{c}_i)}{dc_i} \in \partial V_i(\hat{c}_i)$.

(c) It is immediate. ■

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