A Note on the Comparative Statics of Optimal Procurement Auctions*

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Abstract

We consider a buyer who must procure a service from one of n potential sellers, whose production costs are private information. We find a necessary and sufficient condition such that a distributional upgrade on a seller's cost distribution implies a lower expected procurement cost for the buyer. We also show that even under the strongest assumption about this upgrade made in the literature so far, the seller can be worse off, even if this upgrade is costless. Keywords: *Procurement Auctions, Mechanism Design, Distributional Upgrade. JEL D44, C7, C72.*

1. Introduction

In many procurement circumstances, it is possible for one of the sellers to improve his cost distribution before the procurement mechanism actually takes place. This change could be exogenous, for example due to a change in its input prices, or endogenous, due to a costly investment that improves the production technology. It is very important to understand the change in the buyer's expected cost and the sellers expected profit when such a distributional improvement occurs. If the seller benefits very little from it, then he will not invest much in researching a new technology, and if he is worse-off after a change, he will likely block opportunities for an exogenous improvement. In the same spirit, if the buyer benefits the most, then it is likely she will subsidize the investment. A situation where a seller can improve his cost distribution therefore sheds light on the the motivation of a downstream firm to buy one of his upstream providers. If such a provider does not have the incentives to improve his technology and participate in the subsequent procurement auction, but the buyer would benefit greatly by him doing so, then the buyer will likely try to buy the provider and do the investment himself. To start an analysis of these important economic questions, it is crucial to study the changes in the sellers cost and the buyers profit when such an improvement occurs. We do this in a simple model of static procurement.

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We consider a buyer who must procure a service from one of n potential sellers, whose production costs are private information. A simple reformulation of the model in [6] allows to characterize the optimal mechanism used by the buyer, the expected procurement cost and the expected profits of a seller.

For the buyer, facing a better seller is good since there is a higher probability of her having low costs but, on the other hand, it may be bad since having a better distribution can imply that the informational rent she can extract is also higher.

For the seller, having a better distribution is good since, *ceteris paribus*, it increases her probabilities of winning the auction and the informational rent she can extract. However, since this better distribution is observed by the buyer and the mechanism is changed against the better seller, there is a negative effect associated to it.

For the reasons outlined above, the comparative statics for both the buyer and the seller with respect to the cost distribution of a seller are not obvious. We are interested in the circumstances under which it is desirable for the buyer to face one "better" seller (with a better cost distribution) and for the seller to have such a cost distribution. In other words, we are interested in the comparative statics of the buyer's expected cost and a seller's expected profit with respect to a distributional upgrade on a seller.

Up to now, this problem has not been solved. That is, the right notion of distributional improvement that guarantees that the buyer is better off has not been characterized. We provide a natural and weak necessary condition on the distributional upgrade under which the buyer is better off. On the other hand, with a very simple counterexample, we show that for even for the strongest concept of distributional improvement used in the literature, the seller can be worse off when her cost distribution improves, even if this improvement is costless. The buyer biases the mechanism against the better seller so much, that the potential gains of a better cost distribution and a higher informational rent are offset by a decreased chance of winning due to this adjusted mechanism.

This last result strengthens the results of underinvestment in Dasgupta [3] in a symmetric environment, and also the result in Arozamena and Cantillon [1], where underinvestment is the result of a strong reaction by other bidders in a first price auction. Here, the underinvestment result is extreme, since the socially efficient investment for a costless technology is infinity and the seller would choose to invest 0. Moreover, the seller would be strictly worse off in case of an exogenous improvement in her distribution.

2. Model

Consider a buyer who wants to procure a good or service and faces n potential suppliers indexed by $i = 1, ..., n$. If the buyer decides to carry out the task by himself, it would cost him an amount of money $c_0 \geq \underline{c}$. Suppliers' costs to perform the task (which are private information) are distributed independently across firms. Firm i obtains her cost from a differentiable distribution $F_i(\cdot)$, $i \geq 2$, with support $C \equiv [\underline{c}, \bar{c}]$. However, competitor 1 (from now on the *upgrader*) draws his cost from a differentiable distribution $F(\cdot, I)$ with the same support as before. I is a parameter that indexes the supplier's efficiency, and we assume that, as $I \geq 0$ increases, the distribution *improves.* For notational convenience we use $f_i(\cdot) \equiv F_i'(\cdot)$ if $i \geq 2$, and $\frac{\partial F}{\partial c}(c, I)$ for the upgrader.

We make the standard *regularity* assumption (first stated in [6]), which guarantees that the optimal mechanism can be found using pointwise maximization.

Assumption 1 *For every $i \geq 2$ and $I \geq 0$, the functions $J_i(c) = c + \frac{F_i(c)}{f_i(c)}$ and $J_I(c) = c + \frac{F(c,I)}{\frac{\partial F}{\partial c}(c,I)}$ are increasing.*

For technical reasons, we also need:

Assumption 2 *For every $c \in C$, $I \mapsto J_I^{-1}(c)$ is differentiable.*

There are several "distributional improvements" that may apply to the context presented here. We introduce two widely-used notions, the first one being the most commonly used in statistics and economics:

Definition 3 ***(First Order Stochastic Dominance):*** *We will say that $\{F(\cdot, I)\}_{I \in \mathbb{R}_+}$ is a family of distributional improvements in the sense of first order stochastic dominance (FOSD) if, for every fixed $c \in C$, $F(c, \cdot)$ is increasing. In other words, the probability of obtaining a cost below $c \in C$ is increasing in I.*

The next one has been of great use in the auction literature (see for example [4]):

Definition 4 ***(Monotone Likelihood Ratio Property):*** *We will say that $\{F(\cdot, I)\}_{I \in \mathbb{R}_+}$ is a family of distributional improvements in the sense of the monotone likelihood ratio property (MLRP) if, for every $I' < I \in \mathbb{R}_+$ and $c' < c \in C$,*

$$\frac{\frac{\partial F}{\partial c}(c', I')}{\frac{\partial F}{\partial c}(c, I')} < \frac{\frac{\partial F}{\partial c}(c', I)}{\frac{\partial F}{\partial c}(c, I)} \tag{1}$$

That is, as I increases, it is more likely to obtain lower costs relative *to higher ones. This condition is equivalent to $(c, I) \mapsto \frac{\partial F}{\partial c}(c, I)$ being log-submodular.*

The following well-known result relates both definitions and shows that MLRP is stronger than FOSD:

Lemma 5 *If $\{F(\cdot, I)\}_{I \in \mathbb{R}_+}$ is a family of distributional improvements in the sense of MLRP, then, it is a family of distributional improvements in the sense of FOSD.*

Proof. See, for example, [4]. ■

Observation: Another well-known result shows that MLRP also implies that $\frac{F(c,I)}{\frac{\partial F}{\partial c}(c,I)}$ is increasing in I for all $c \in C$. This term corresponds to the informational rent of a seller of type c, and the fact that it increases with I makes the effect of a better seller unclear for the buyer: a better seller has lower costs but also extracts a higher informational rent.

Finally, define $C^n = \{c^n = (c_1, ..., c_n) | \ c_i \in C \ \forall i = 1, ..., n\}$ and assume that for $i \geq 2$, $f_i(\cdot) > 0$ and $\forall I \geq 0$, $\frac{\partial F}{\partial c}(\cdot, I) > 0$, a.e. in C.

3. Comparative Statics

We now consider an upgrader with cost distribution $F(\cdot, I)$, and perform comparative statics over the procurement cost with respect to the parameter I. The buyer's problem is to choose transfer functions $t_i : C^n \to I\!R$ (payments to the sellers) and winning probability functions $q_i : C^n \to [0,1]$ (probabilities of buying), $i = 1, ..., n$. Under the regularity assumptions, it is direct that the expected optimal mechanism corresponds to (see [6])

$$q_1^*(c_1, ..., c_n) = \begin{cases} 1 & J_I(c_1) \leq \min\{c_0, J_i(c_i) | \; i \geq 2\} \\ 0 & \sim \end{cases} \tag{2}$$

$$q_i^*(c_1, ..., c_n) = \begin{cases} 1 & J_i(c_i) < \min\{c_0, J_I(c_1), J_l(c_l) | \; l \neq i, \; l \geq 2\} \\ 0 & \sim \end{cases} \tag{3}$$

$i = 2, ..., n.$

which yields a procurement cost of:

$$\mathcal{C}(I) \quad = \quad \int\limits_{C^n} \left[J_I(c_1) q_1^*(c^n) + \sum_{l \geq 2} J_l(c_l) q_l^*(c^n) + c_0 \left(1 - \sum_{i \geq 1} q_i^*(c^n) \right) \right] \frac{\partial F}{\partial c_1}(c_1, I) \left(\prod_{j \geq 2} f_j(c_j) \right) dc^n \tag{4}$$

On the one hand, an increase in I implies that the distribution $F(\cdot, I)$ puts more weight on low-cost realizations, therefore reducing the total cost $\mathcal{C}(I)$. On the other hand, due to informational asymmetries, the buyer pays (in expected terms) an amount $J_I(c)$ to the upgrader. This term can be increasing in I for a sufficiently strong distributional upgrade (for example one that satisfies MLRP), therefore increasing the total cost $\mathcal{C}(I)$. Then, for a fixed mechanism, an increase in I could affect $\mathcal{C}(I)$ in both ways.

However, under these new circumstances, the buyer adapts the mechanism (through a change in the optimal rules q_i^*) and can give a disadvantage to a "better" seller who has a bigger virtual cost, tilting the balance and decreasing the total cost when I increases.

Our main purpose is to establish which conditions on the family $\{F(\cdot, I)\}_{I \geq 0}$ imply that the expected procurement cost is reduced. The main proposition, stated below, shows that a natural and weak condition like FOSD suffices to imply the result.

Proposition 6 *Suppose that for every $c \in C$ the function $F(c, \cdot)$ is differentiable. A sufficient pointwise conditions on the the family $\{F(\cdot, I)\}_{I \geq 0}$ under which the expected procurement cost decreases is:*

$$\forall I \geq 0, \; \forall c \in [\underline{c}, J_I^{-1}(c_0)], \; \frac{\partial F}{\partial I}(c, I) \geq 0 \tag{5}$$

As a consequence, if the mentioned family satisfies FOSD, the expected procurement cost decreases when facing a better competitor.

Proof. See Appendix. ■

Observation: Condition (5) can indeed be much weaker than FOSD, since it must only be satisfied in the interval $[\underline{c}, J_I^{-1}(c_0)]$, which corresponds to the set where the upgrader has a "chance" of winning, since $J_I(c) < c_0$ in this region. This interval is small if c_0, the reservation cost, is small, since in that case the purchase does not occur often because of the seller has an attractive opportunity of doing the project himself.

If a property like MLRP is satisfied, then $J_I(c)$ is increasing in I for all c. This implies that the interval $[\underline{c}, J_I^{-1}(c_0)]$ shrinks with I, making the condition weaker as I increases. This is true since as I increases, the seller with a better cost distribution faces more disadvantageous mechanisms, and therefore the region where he has a chance of being assigned the project is reduced.

As we can see, the tradeoff mentioned in the introduction (a buyer likes a better seller since he has in average lower costs, but the other hand dislikes one, since he can extract higher informational rents) always works in the buyer's favor. The total surplus generated is clearly bigger with a better seller, and the possibly higher informational rents he is able to obtain are not enough to offset this fact. This is true since the buyer modifies the mechanism, giving an ex-ante disadvantage to the better seller, and therefore controlling the rent he is able to extract.

We conclude by pointing out that the strength of this last effect, where the buyer biases the mechanism against the better seller to extract more rent, can be enough to make the seller worse off, even if the distributional upgrade is for free. This last fact is particularly important since it sheds light on the make-or-buy problem. The buyer would like the seller to invest in new technologies that improve his cost distribution, but he would have to give incentives (either monetary or through the commitment to a less disadvantageous mechanism) for this to happen. Moreover, if investment is non-observable or commitment not possible, the buyer will have incentives to buy the provider, invest and produce the good or service himself.

Example: Suppose $n = 2$, $C = [0, 1]$ and $c_0 = +\infty$. Consider $F_2(c) = c$ and $F(c, I) = c^{\frac{1}{1+I}}$, $I \geq 0$. This family of distributions corresponds to a seller which upgrades the reliability of his technology, that is the maximum of $I + 1$ independent draws follows a uniform distribution in [0,1]. Moreover, this family satisfies MLRP and, as a consequence, FOSD. The upgrader's expected utility when his distribution is $F(\cdot, I)$ corresponds to[1]

$$\Pi(I) = \int\limits_C \Pi(c, c) \frac{\partial F}{\partial c}(c, I) dc = \int\limits_C Q^*(c) F(c, I) dc$$

with $Q^*(c) = \int\limits_C q^*(c, s) f(s) ds$. Using that $q^*(c, s) = 1 \Leftrightarrow J_I(c) \leq J_2(s)$ (from (2) and (3)), $J_I(c) = c(2+I)$ and $J_2(c) = 2c$, thus $J_2^{-1}(J_I(c)) = \frac{c(2+I)}{2}$ and we have

$$\Pi(I) = \int\limits_{\underline{c}}^{\frac{2}{2+I}} \left[1 - \frac{c(2+I)}{2}\right] c^{\frac{1}{2+I}} dc = \frac{(1+I)^2}{(2+I)(3+2I)} \left(\frac{2}{2+I}\right)^{\frac{2+I}{1+I}}$$

[1] The second equality comes from incentive compatibility, integration by parts and the fact that $\pi(\overline{c}, \overline{c}) = 0$ in an optimal mechanism.

The next result establishes that no positive investment level is profitable for the seller, even when it is costless. Though investment increases profits by reducing the expected cost in the case of winning the competition, this effect is out-weighted by more disadvantageous rules imposed by the buyer, which allow him to extract a larger fraction of the seller's informational rent.

Proposition 7 *For all* $I \geq 0$, $\frac{d}{dI}(\Pi(I)) < 0$.

Proof. See Appendix ■

□

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4. Appendix: Proofs

We first rewrite the procurement cost in the next lemma:

Lemma 8 *The expected procurement cost can be written as*

$$\begin{aligned}
\mathcal{C}(I) &= \sum_{i\geq 2} \int_{\underline{c}}^{J_i^{-1}(c_0)} f_i(c)\left(\prod_{l\neq i}[1-F_l(J_l^{-1}(J_i(c)))]\right) J_I^{-1}(J_i(c))F(J_I^{-1}(J_i(c)),I)dc \\
&+ \left(\prod_{l\geq 2}[1-F_l(J_l^{-1}(c_0))]\right) J_I^{-1}(c_0)F(J_I^{-1}(c_0),I) \\
&+ \sum_{i\geq 2} \int_{\underline{c}}^{J_i^{-1}(c_0)} f_i(c)\left(\prod_{l\neq i}[1-F_l(J_l^{-1}(J_i(c)))]\right) J_i(c)[1-F(J_I^{-1}(J_i(c)),I)]dc \\
&+ c_0\left(\prod_{l\geq 2}[1-F_l(J_l^{-1}(c_0))]\right)[1-F(J_I^{-1}(c_0),I)] \qquad (6)
\end{aligned}$$

Proof of Lemma 8: Recall that $c^n = (c_1,...,c_n)$ and define $H(c^n,I)$ as

$$H(c^n,I) \equiv \left[J_I(c_1)q_1^*(c^n) + \sum_{l\geq 2} J_l(c_l)q_l^*(c^n) + c_0\left(1-\sum_{i\geq 1} q_i^*(c^n)\right)\right]\frac{\partial F}{\partial c_1}(c_1,I)\prod_{i\geq 2} f_i(c_i)$$

Considering

$$A = \{c^n \in C^n|\ J_I(c_1)\leq c_0,\ J_I(c_1)\leq J_i(c_i),\ \forall i\geq 2\}$$

the set of cost-vectors in which the *upgrader* wins the procurement auction we can write

$$\mathcal{C}(I) = \int_A H(c^n,I)dc^n + \int_{C^n\setminus A} H(c^n,I)dc^n$$

Set A can be written as $A = A_0 \cup \left(\bigcup_{i\geq 2} A_i\right)$ with

$$\begin{aligned}
A_0 &= \{c^n \in C^n|\ J_I(c_1)\leq c_0\ \wedge\ c_0 < J_i(c_i),\ \forall i\geq 2\} \\
&= \{c^n \in C^n|\ c_1 < J_I^{-1}(c_0)\ \wedge\ J_i^{-1}(c_0)\leq c_i,\ \forall i\geq 2\}
\end{aligned}$$

$$\begin{aligned}
A_i &= \{c^n \in C^n|\ J_I(c_1)\leq J_i(c_i)\ \wedge\ J_i(c_i)\leq c_0\ \wedge\ (J_i(c_i)\leq J_l(c_l),\ l\geq i)\ \wedge\ (J_i(c_i) < J_l(c_l),\ i>l)\} \\
&= \{c^n \in C^n|\ c_1\leq J_I^{-1}(J_i(c_i))\ \wedge c_i\leq J_i^{-1}(c_0)\ \wedge\ (J_l^{-1}(J_i(c_i))\leq c_l,\ l\geq i)\ \wedge\ (J_l^{-1}(J_i(c_i)) < c_l,\ i>l)\}
\end{aligned}$$

and it is quite easy to see that $A_j \cap A_i = \emptyset$ if $i\neq j$ $i,j\in\{0,2,3,...,n\}$. Note that in A_i the *upgrader* wins the procurement auction and seller i reports de lowest virtual cost among all the upgrader's rivals. On the other hand, in A_0 the same agent wins the competition but no other firm submits a bid below the reserve cost c_0. Implicitly in our above definitions, among the lowest virtual costs, the upgrader wins the procurement auction, which certainly doesn't increase expected expenditures for the buyer. As a direct consequence,

$$\int_A H(c^n,I)dc^n = \sum_{i=0,i\geq 2}\int_{A_i} H(c^n,I)dc^n$$

Now, define $t_l(\cdot) \equiv J_l^{-1}(J_i(\cdot))$ for $l \geq 2,\ l \neq i$ and $t_I(\cdot) \equiv J_I^{-1}(J_i(\cdot))$ Integrating over A_i yields

$$\int\limits_{A_i} H(c^n, I)dc^n = \int\limits_{\underline{c}}^{J_i^{-1}(c_0)} \int\limits_{t_2(c_i)}^{\bar{c}} ... \int\limits_{t_{i-1}(c_i)}^{\bar{c}} \int\limits_{t_{i+1}(c_i)}^{\bar{c}} ... \int\limits_{t_n(c_i)}^{\bar{c}} \int\limits_{\underline{c}}^{t_I(c_i)} J_I(c_1) \frac{\partial F}{\partial c_1}(c_1, I) \left(\prod_{l \geq 2} f_l(c_l) \right) dc^n$$

and observing that $J_I(c_1)\frac{\partial F}{\partial c_1}(c_1, I) = \left[c_1 + \frac{F(c_1,I)}{\frac{\partial F}{\partial c_1}(c_1,I)} \right] \frac{\partial F}{\partial c_1}(c_1, I) = \frac{d}{dc_1}(c_1 F(c_1, I))$ we obtain

$$\int\limits_{A_i} H(c^n, I)dc^n = \int\limits_{\underline{c}}^{J_i^{-1}(c_0)} f_i(c) \left(\prod_{l \neq i} [1 - F_l(J_l^{-1}(J_i(c)))] \right) J_I^{-1}(J_i(c)) F(J_I^{-1}(J_i(c)), I) dc$$

Analogously,

$$\begin{aligned} \int\limits_{A_0} H(c^n, I)dc^n &= \int\limits_{J_2^{-1}(c_0)}^{\bar{c}} ... \int\limits_{J_n^{-1}(c_0)}^{\bar{c}} \int\limits_{\underline{c}}^{J_I^{-1}(c_0)} J_I(c_1) \frac{\partial F}{\partial c_1}(c_1, I) \left(\prod_{l \geq 2} f_l(c_l) \right) dc^n \\ &= \left(\prod_{l \geq 2} [1 - F_l(J_l^{-1}(c_0))] \right) J_I^{-1}(c_0) F(J_I^{-1}(c_0), I) \qquad (7) \end{aligned}$$

Thus,

$$\begin{aligned} \int\limits_{A} H(c^n, I)dc^n &= \sum_{i \geq 2} \int\limits_{\underline{c}}^{J_i^{-1}(c_0)} f_i(c) \left(\prod_{l \neq i} [1 - F_l(J_l^{-1}(J_i(c)))] \right) J_I^{-1}(J_i(c)) F(J_I^{-1}(J_i(c)), I) dc \\ &+ \left(\prod_{l \geq 2} [1 - F_l(J_l^{-1}(c_0))] \right) J_I^{-1}(c_0) F(J_I^{-1}(c_0), I) \qquad (8) \end{aligned}$$

On the other hand,

$$C^n \setminus A^n = \{c^n \in C^n | \ (\exists l \geq 2,\ J_l(c_l) < J_I(c_1)\ \wedge\ J_l(c_l) \leq c_0)\ \vee (c_0 < J_I(c_1),\ c_0 < J_i(c_i),\ \forall i \geq 2)\}$$

is the set over which the upgrader loses the procurement auction. As before, this set can be partitioned as $C^n \setminus A = B_0 \cup \left(\bigcup_{j \geq 2} B_j \right)$ with

$$B_0 = \{c^n \in C^n | \ J_I^{-1}(c_0) < c_1\ \wedge\ J_i^{-1}(c_0) < c_i,\ \forall i \geq 2\} \qquad (9)$$

$$B_i = \{c^n \in C^n | \ c_i \leq J_i^{-1}(c_0)\ \wedge\ (J_l^{-1}(J_i(c_i)) \leq c_l,\ i \leq l)\ \wedge (J_l^{-1}(J_i(c_i)) < c_l,\ i < l)\ \wedge\ J_I^{-1}(J_i(c_i)) < c_1\}$$

Set B_0 represents the zone in which the project is not assigned and B_i corresponds to the region where firm $i \geq 2$ wins the competition. Implicitly in the definition of these sets we assume that, in case of equal lowest-virtual-costs, the task is assigned to the lowest-index competitor, which certainly doesn't increase expected procurement expenditures. Then we can write

$$\int_{C^n \setminus A} H(c^n, I) dc^n = \sum_{i=0, i\geq 2} \int_{B_i} H(c^n, I) dc^n$$

It is direct that

$$\begin{aligned}
\int_{B_i} H(c^n, I) dc^n &= \int_{\underline{c}}^{J_i^{-1}(c_0)} \int_{t_2(c_i)}^{\bar{c}} \ldots \int_{t_{i-1}(c_i)}^{\bar{c}} \int_{t_{i+1}(c_i)}^{\bar{c}} \ldots \int_{t_n(c_i)}^{\bar{c}} \int_{t_I(c_i)}^{\bar{c}} J_i(c_i) \frac{\partial F}{\partial c_1}(c_1, I) \left(\prod_{l\geq 2} f(c_l) \right) dc^n \\
&= \int_{\underline{c}}^{J_i^{-1}(c_0)} \left(\prod_{l\neq i} [1 - F_l(J_l^{-1}(J_i(c_i)))] \right) [c_i f_i(c_i) + F_i(c_i)][1 - F(J_I^{-1}(J_i(c_i)), I)] dc_i \\
&= \int_{\underline{c}}^{J_i^{-1}(c_0)} f_i(c) \left(\prod_{l\neq i} [1 - F_l(J_l^{-1}(J_i(c)))] \right) J_i(c)[1 - F(J_I^{-1}(J_i(c)), I)] dc \qquad (10)
\end{aligned}$$

Also,

$$\begin{aligned}
\int_{B_0} H(c^n, I) dc^n &= \int_{J_2^{-1}(c_0)}^{\bar{c}} \ldots \int_{J_n^{-1}(c_0)}^{\bar{c}} \int_{J_I^{-1}(c_0)}^{\bar{c}} c_0 \frac{\partial F}{\partial c_1}(c_1, I) \left(\prod_{l\geq 2} f_l(c_l) \right) dc^n \\
&= c_0 \left(\prod_{l\geq 2} [1 - F_l(J_l^{-1}(c_0))] \right) [1 - F(J_I^{-1}(c_0), I)] \qquad (11)
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\int_{C^n \setminus A} H(c^n, I) dc^n &= \sum_{i\geq 2} \int_{\underline{c}}^{J_i^{-1}(c_0)} f_i(c) \left(\prod_{l\neq i} [1 - F_l(J_l^{-1}(J_i(c)))] \right) J_i(c)[1 - F(J_I^{-1}(J_i(c)), I)] dc \\
&\quad + c_0 \left(\prod_{l\geq 2} [1 - F_l(J_l^{-1}(c_0))] \right) [1 - F(J_I^{-1}(c_0), I)] \qquad (12)
\end{aligned}$$

which concludes the proof.

□

Proof of Proposition 6: Define

$$\alpha_i(c) \equiv f_i(c) \left(\prod_{l\neq i} [1 - F_l(J_l^{-1}(J_i(c)))] \right)$$

thus, using lemma 8

$$\begin{aligned}
\mathcal{C}(I) &= \sum_{i\geq 2} \int_{\underline{c}}^{J_i^{-1}(c_0)} \alpha_i(c)\{J_I^{-1}(J_i(c))F(J_I^{-1}(J_i(c)),I) + [1-F(J_I^{-1}(J_i(c)),I)]J_i(c)\}dc \qquad (13)\\
&+ \left(\prod_{l\geq 2}[1-F_l(J_l^{-1}(c_0))]\right)\{J_I^{-1}(c_0)F(J_I^{-1}(c_0),I) + [1-F(J_I^{-1}(c_0),I)]c_0\} \qquad (14)
\end{aligned}$$

Therefore, under suitable integrability conditions [2]

$$\begin{aligned}
\mathcal{C}'(I) &= \sum_{i\geq 2} \int_{\underline{c}}^{J_i^{-1}(c_0)} \alpha_i(c)\frac{\partial}{\partial I}\{F(J_I^{-1}(J_i(c)),I)[J_I^{-1}(J_i(c)) - J_i(c)]\}dc \qquad (15)\\
&+ \left(\prod_{l\geq 2}[1-F_l(J_l^{-1}(c_0))]\right)\frac{\partial}{\partial I}\{F(J_I^{-1}(c_0),I)[J_I^{-1}(c_0)-c_0]\} \qquad (16)
\end{aligned}$$

Define $L(c,I) \equiv F(J_I^{-1}(c),I)[J_I^{-1}(c)-c]$. Thus,

$$\begin{aligned}
\frac{\partial L}{\partial I}(c,I) &= \left[\frac{\partial F}{\partial t}(J_I^{-1}(c),I)\frac{\partial}{\partial I}(J_I^{-1}(c)) + \frac{\partial F}{\partial I}(J_I^{-1}(c),I)\right][J_I^{-1}(c)-c]\\
&\quad + F(J_I^{-1}(c),I)\frac{\partial}{\partial I}(J_I^{-1}(c))\\
&= \frac{\partial}{\partial I}(J_I^{-1}(c))\left[\frac{\partial F}{\partial t}(J_I^{-1}(c),I)[J_I^{-1}(c)-c] + F(J_I^{-1}(c),I)\right]\\
&\quad + \frac{\partial F}{\partial I}(J_I^{-1}(c),I)[J_I^{-1}(c)-c] \qquad (17)
\end{aligned}$$

Recall that $J_I(t) = t + \frac{F(t,I)}{\frac{\partial F}{\partial t}(t,I)}$, so, evaluating at $t = J_I^{-1}(c)$ we obtain

$$J_I^{-1}(c) - c = -\frac{F(J_I^{-1}(c),I)}{\frac{\partial F}{\partial t}(J_I^{-1}(c),I)}$$

Then,

$$\frac{\partial L}{\partial I}(c,I) = \frac{\partial F}{\partial I}(J_I^{-1}(c),I)[J_I^{-1}(c)-c] \qquad (18)$$

Therefore,

$$\begin{aligned}
\mathcal{C}'(I) &= \sum_{i\geq 2} \int_{\underline{c}}^{J_i^{-1}(c_0)} \alpha_i(c)\frac{\partial F}{\partial I}(J_I^{-1}(J_i(c)),I)[J_I^{-1}(J_i(c)) - J_i(c)]dc \qquad (19)\\
&+ \left(\prod_{l\geq 2}[1-F_l(J_l^{-1}(c_0))]\right)\frac{\partial F}{\partial I}(J_I^{-1}(c_0),I)[J_I^{-1}(c_0)-c_0] \qquad (20)
\end{aligned}$$

[2] For example, if we define $L(c,I) \equiv J_I^{-1}(J_i(c))F(J_I^{-1}(J_i(c)),I) + [1-F(J_I^{-1}(J_i(c)),I)]J_i(c)$, we need $L(\cdot,I)$ to be measurable for all I and $\left|\frac{\partial L}{\partial I}(c,I)\right| \leq K(c)$ where $K \in \mathcal{L}^1(C)$.

Since $\alpha_i(c) \geq 0$ and $J_I^{-1}(c) - c \leq 0$, $\forall c \in C$, a sufficient condition to obtain $\mathcal{C}'(I) \leq 0$ is

$$\forall i \geq 2,\ \forall c \in [\underline{c}, J_i^{-1}(c_0)],\ \frac{\partial F}{\partial I}(J_I^{-1}(J_i(c)), I) \geq 0$$

and

$$\frac{\partial F}{\partial I}(J_I^{-1}(c_0), I) \geq 0$$

which are equivalent to

$$\forall c \in [\underline{c}, J_I^{-1}(c_0)],\ \frac{\partial F}{\partial I}(c, I) \geq 0$$

since $J_I(\underline{c}) = J_i(\underline{c}) = \underline{c}$ and $J_I(\cdot)$ and $J_i(\cdot)$, $i \geq 2$, are increasing functions.

□

Proof of Proposition 7: In order to show that $\Pi(I)$ is strictly decreasing for all $I \geq 0$, we will prove that $\frac{d}{dI}(log(\Pi(I))) < 0$, which is obviously equivalent. Recall that

$$\Pi(I) = \int_{\underline{c}}^{\frac{2}{2+I}} \left[1 - \frac{c(2+I)}{2}\right] c^{\frac{1}{2+I}} dc = \frac{(1+I)^2}{(2+I)(3+2I)} \left(\frac{2}{2+I}\right)^{\frac{2+I}{1+I}}$$

so, we have that $log(\Pi(I))$ satisfies

$$\frac{d}{dI}(log(\Pi(I))) = \frac{1}{1+I} - \frac{1}{2+I} - \frac{2}{3+2I} + \frac{1}{(1+I)^2}\left[log\left(\frac{2+I}{2}\right)\right]$$

Also, since $log(x) \leq x - 1$, it is direct that $log\left(\frac{2+I}{2}\right) \leq \frac{I}{2}$. Then,

$$\begin{aligned} \frac{d}{dI}(log(\Pi(I))) &< \frac{1}{1+I} - \frac{1}{2+I} - \frac{2}{3+2I} + \frac{1}{(1+I)^2}\frac{I}{2} \\ &= \frac{2(1+I)(2+I)(3+2I) - 2(3+2I)(1+I)^2 - 4(2+I)(1+I)^2 + I(2+I)(3+2I)}{2(2+I)(3+2I)(1+I)^2} \end{aligned}$$

As a consequence it suffices to show that the numerator in the above expression is negative for any possible I (the denominator is always strictly positive). After straightforward algebra we obtain that the numerator is equal to $-2I^3 - 5I^2 - 4I - 2$, which is strictly less that zero, concluding the proof.

□