Loyalty inducing programs and competition with homogeneous goods

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Abstract

We analyze a market where two firms producing a homogenous good compete by means of two mechanisms: prices and a loyalty bonus. We assume that firms act simultaneously when posting their loyalty bonus and prices. Consumers who purchase from a firm in the first period must return the bonus in case they switch providers in the second period. They fully anticipate the effects on future prices of accepting the bonus and maximize their total surplus over both periods. We first show that there is no equilibrium with prices and bonuses equal to zero. We then show the existence of a SPNE where firms are able to obtain half the monopoly profits using large bonuses in the first period and high prices in the second period. We completely characterize all the symmetric equilibria of the game and show that, in general, firms obtain positive profits even when they compete in prices, the good is homogenous, and consumers are forward-looking. Finally we show that if firms are allowed to discriminate between old and new customers, the standard zero price equilibria reappear.

JEL Classification: L13.
KEYWORDS: Bertrand competition, switching costs, loyalty programs.

1 Introduction

Introductory offers that require minimum staying periods for clients are commonplace in the real world. The Book of the Month Club offers five books for US$1, but there is a minimum staying period of two years during which the client must buy at least three additional books. Similarly, cell phone companies often provide expensive phones at a nominal price in exchange for the commitment of staying with the company for a given period, and consumers are charged penalties if they decide to switch before the period is over. These commercial strategies, which we denote as *introductory offer programs* act similarly to *loyalty programs*, in which repeated buying from a provider leads to price reductions or gifts. The standard example of loyalty programs are the frequent flyer mile programs, another example being points in supermarkets. The common feature of these programs is that they raise the cost for consumers of switching providers once they have started buying from one of them, either because clients are required to compensate the company, or because there is a reduction in the expected benefits from continuing to buy from the original provider.

The literature (see below for a review) has shown that switching costs can lead to reductions in competition, even when goods or services are homogenous, because firms can raise the price differential up to the level of the switching cost before consumers decide to change providers, generating rents for the firms. Thus, in the presence of switching costs, even firms that produce homogenous products do not compete head on. However, in most of the literature of which we are aware, switching costs are exogenous; or alternatively, either goods, firms or consumers are non-homogenous. A significant exception is an unpublished paper by Banerjee and Summers [1987], who analyze frequent flyer commercial strategies and show that they can be designed to facilitate collusion, even with homogeneous goods. The loyalty program generates endogenous switching costs, and the paper shows that they allow the providers to achieve the collusive outcome. Their model uses the ad-hoc procedure of having one firm be chosen as a price leader in each period, which simplifies the strategic analysis.

In this paper we use a model of introductory offer programs with endogenous switching costs similar to that of Banerjee and Summers [1987], with the important difference that we consider firms acting simultaneously, which we believe is the more natural approach to the problem of two identical firms selling homogenous products. This is a complex problem, because homogeneity leads to a lack of continuity in the payoff function, requiring the use of mixed strategies in the solution. We use a two period model, in which firms offer bonuses to agents who sign contracts to stay with the firm for the two subsequent periods. In each of the two periods, firms set prices and compete for the market. If an agent decides to switch companies in the second period (because of the rival's lower prices), she must pay back the bonus she received before signing up.

We use the results of a previous paper (Infante, Figueroa, and Fischer [2007]) that characterizes the mixed strategy equilibria to the second period game, for any possible pair of bonuses. Using that continuation game we derive the first period equilibrium prices for any possible pair of bonuses. Finally, we find the optimal choice of an introductory offer program, thus solving the complete game.

In our first result, we show that there is no equilibrium with zero bonuses and zero prices, which helps explain why bonuses (or, equivalently, loyalty programs) are a standard commercial strategy. Next, we show that there is a subgame perfect symmetric equilibrium that maximizes joint profits, which is remarkable given that there are no exogenous switching costs, and firms and consumers are homogenous. In this equilibrium, firms offer large bonuses, and split the market in both periods. In the second period they "milk" their consumers, knowing that it is costly for them to switch providers. Even though consumers are forward looking when choosing a firm, in the equilibrium they cannot avoid being milked by the companies. There are other symmetric equilibria with lower or zero profits, but they are dominated, in terms of profits, by the equilibrium described above. We therefore consider it unlikely that firms will coordinate in these equilibria. We also analyze the structure of the symmetric equilibria of the game and show that there are no equilibria with small bonuses.

Finally, we show that this equilibrium is not robust to the possibility of firms discriminating between habitual and new consumers. In such a case, the equilibrium implies zero profits as in a standard Bertrand competition. Note that, contrary to the usual results, in this case price discrimination eliminates rents.

Our analysis highlights the potential problems presented by an apparently competitive situation, when firms are allowed to use more complex strategies than just posting prices. It is possible for firms to obtain positive profits in equilibrium by endogenously choosing switching costs and subsequently charging prices above marginal cost. This result is due to two complementary effects. In the second period, firms do not let prices fall to zero, since customers are locked-in by the bonuses they would have to return in case they switch providers. In the first period, once the bonuses are chosen, firms do not want to compete fiercely in order to capture the whole market by lowering prices. They anticipate that facing a competitor with no inherited customers in the second period is worse than facing a competitor with half of the market. A competitor with no inherited customer base will price very aggressively to capture clients in the second period, unlike a competitor with positive market share, who has incentives to charge high prices to its locked-in customers. If consumers could band together and all (or a sufficiently large majority) chose to buy from one firm, its rival would behave aggressively and the high price equilibrium would collapse. By choosing randomly to buy from one of the two firms when offered equal conditions, individual consumers impose a negative externality on the remaining consumers, leading to the high price equilibrium.

Finally, note that these arguments imply that if we allow price discrimination between consumers, this last effect will disappear and we recover the standard zero profit equilibrium. Hence, if bonuses are used as a competitive strategy by firms, the policy prescription is to allow price discrimination to enhance consumer welfare.

Existing Literature: There is an extensive literature on the subject of switching costs and their effect on competitive outcomes. Here we focus on models where firms compete over a finite number of periods.[1] For exogenously given switching costs, the simplest case is when these are large, so there is no possibility for firms to attract their rivals' customers (as reviewed, for example, in Klemperer [1995]). Clearly, in these cases firms can act as monopolies on their locked-in customers. Shilony [1977] considers the case of exogenous switching costs which are relatively small and equal for all firms. In the resulting equilibrium firms play mixed strategies where firms can either price aggressively to poach the rivals' market share or charge high prices to extract more rent from their captured market.

More complex models endogenize the market share of each firm using two periods. In the first, consumers choose a firm, and this choice induces a second period cost of switching to another firm. Basu and Bell [1991] and Padilla [1992] consider models where there are two types of consumers in the second period: those who face large switching costs, and those who are new to the market and therefore free to choose

[1] For the infinite period case see, for example, Farrell and Shapiro [1988].

between firms. Competition for the new entering consumers is tempered by the fact that it reduces profits derived from the locked-in consumers, who cannot switch firms because the cost is too large. Another approach considers consumers with (exogenous) Hotelling preferences over firms in the first period. Klemperer [1995], for example, considers this case, combined with the condition that there are large switching costs in the second period. Since firms know that they can extract the full monopoly rent from their locked-in consumers in the second period, they lower their first period price in order to increase market share. Klemperer [1987] extends this environment to allow for a fraction of consumers that change their preferences, and for the entry of new consumers. As before, consumers who maintain their preferences have large switching costs, so they are effectively locked-in. Competition for consumers whose tastes have changed as well as for new consumers, is mitigated by the existence of locked-in consumers, as in Basu and Bell [1991] and Padilla [1992]. In these models, even with price competition for market share, firms are better off with the existence of switching costs.

In Chen [1997], consumers do not differ in their initial proximity to a firm, as in the previous papers. Instead, they differ in their intrinsic loyalty to the firm from whom they will purchase in the first period. Firms compete in the first period to attract consumers, and in the second period, they compete to attract the less loyal among their opponent's customer base. In this model, the possibility of price discrimination induces more competition, decreasing firms' profits. [2]

In all previous cases, switching costs are either exogenous (Klemperer [1995], Shilony [1977]), due to consumer preferences (Chen [1997]) or to firm characteristics (Klemperer [1987], Fudenberg and Tirole [2000]). In our case the switching costs are endogenously generated by a strategic decision of firms at the beginning of the first period: the size of a gift to consumers, which must be returned if a consumer switches firms[3]. The paper closest in spirit to ours is Banerjee and Summers [1987]. In their two period model they consider firms that offer a discount in the first period to attract consumers. The discount is awarded only to consumers that repeat their purchase in the second period (as in a frequent mileage program). Consumers do not have any preferences over firms and the price reduction offered by firms is a strategic variable chosen at the beginning of the game. Next, firms select their first period price and consumers choose between firms. In the second and final period, firms set the price, and those consumers who do not switch receive the discount. The discount becomes an endogenous switching cost, as in our model. Discounts do not dissipate all rents, and firms obtain a positive payoff. The main difference with our model is that the authors consider that one firm sets the price first, thus simplifying the strategic analysis of the game. In the present paper, firms choose their strategic variables simultaneously, which seems to us a more realistic assumption. Interestingly enough, even though the second period equilibrium is much more complex and involves mixed strategies, the main result of Banerjee and Summers [1987] is preserved: firms can use gifts or discounts to extract strictly positive profits.[4]

In section 2 we present the model. In section 3, we characterize the equilibria for the second period, given market structure and bonuses. In section 4 we characterize the equilibria of the entire game. In section 5 we study the robustness of the previous result to the possibility of price discrimination among

[2] Price discrimination between locked-in and other consumers is analyzed in several of the previous models. In all cases price discrimination increases competition compared to imposing one single price for all consumers, but in general firms are still better off when consumers face switching costs.

[3] See also Caminal and Matutes [1990]

[4] Moreover, the uniform price assumption is essential. We can show that allowing price discrimination leads to a zero rent equilibrium in both cases.

firms. Finally, in section 6, we conclude.

2 The Model

Two identical firms, i and j, engage in price competition over two periods. The first period is divided into two subperiods, 0 and 1, and we denote the three stages of the game by 0, 1, and 2. In period zero the firms compete in gifts B_i and B_j to attract consumers. In period one, they choose the price they charge in that stage (denoted by p_{1i} and p_{1j}).[5] Once the gifts and first period prices are known, forward looking consumers (who have rational expectations about future prices) choose the firm that minimizes their total expenditure. Finally, in period two, firms choose prices (denoted p_{2i} and p_{2j}), taking into consideration that the gift offered in period zero is now a switching cost, since consumers must return it if they want to switch firms. The time line of the model is shown in Figure 1. We assume that the demand for the good is completely inelastic up to a reserve price, which we normalize to one. Therefore second period prices lie between zero and one. We also assume that bonuses offered by firms are restricted to this interval, but we allow firms to offer negative prices in the first period.[6]

There is a continuum of consumers of mass one. In period 1, each consumer chooses the firm that minimizes her expected total expenditure, including the bonus from the firm . This implies that given gifts B_i, B_j, prices p_{1i}, p_{1j}, and their expectations about future prices p_{2i}, p_{2j}.[7] there are three possible ways in which the mass of consumers can be divided. Either they all purchase from firm i or firm j, or the market is split in two equal halves (if their expected payments are equal). Therefore μ_i, the inherited market share of firm i in period 2 may only take three values: $\mu_i \in \{0, \frac{1}{2}, 1\}$. The total expected expenditure of a consumer if she chooses to purchase from firm i after being informed of prices in period 1 is given by

$$p_{1i} - B_i + \mathbb{E}\left(\min\{p_{2i}, p_{2j} + B_i\}\right),$$

where $\mathbb{E}$ is the expectations operator. If a consumer decides to purchase from firm i, she will receive a bonus B_i as an incentive, will pay p_{1i} in the first period, and then in the second period she will pay the minimum between p_{2i} (in case she doesn't switch providers) and $p_{2j} + B_i$ (if she is better off by returning firm i's bonus and purchasing from j). Therefore, consumers will be indifferent between firms (and therefore $\mu_i = \frac{1}{2}$) if the strategies of the firms satisfy

$$p_{1i} - B_i + \mathbb{E}\Big(\min\{p_{2i}, p_{2j} + B_i\}\Big) = p_{1j} - B_j + \mathbb{E}\Big(\min\{p_{2j}, p_{2i} + B_j\}\Big) \tag{2.1}$$

The challenge is to calculate the expected second period prices in expression (2.1). These expectations depend on the equilibria that arise in the second period, which we characterize in subsection 4.1. Since we simplify notation by assuming that firms do not have production costs, their payoffs are

$$\pi_i(B_i, B_j, p_{1i}, p_{1j}, p_{2i}, p_{2j}) = \mu_i(B_i, B_j, p_{1i}, p_{1j}, p_{2i}, p_{2j})(p_{1i} - B_i) + \pi_{2i}(p_{2i}, p_{2j}, B_i, B_j, \mu_i)$$

[5] The value of the gift is distinguished from the first period price because the size of the gift determines the behavior in period two.
[6] This is a feasible strategy to try to capture more market share for the second period.
[7] Here we put emphasis on the fact that in many cases the equilibria in the second period are in mixed strategies.

where $\pi_{2i}(p_{2i}, p_{2j}, B_i, B_j, \mu_i)$ is firm i's payoff in the second period.[8]

The second period equilibria of the game have been characterized in a previous paper Infante et al. [2007], where we describe the equilibria of a single period game with price competition in homogenous goods and asymmetric switching costs. We summarize the results of this paper in section 3 and we refer the reader to Infante et al. [2007] for a full discussion of these equilibria.

3 Characterization of the Second Period Equilibria

In this section we summarize the results we obtained in Infante et al. [2007], which characterize the equilibria that arise in the second period for any combination of gifts B_i and B_j, and any market structure μ_i. This is essential to evaluate expected second period prices, as rationally forecasted by consumers in (2.1), i.e. $\mathbb{E}\left(\min\{p_{2i}, p_{2j} + B_i\}\right)$.

The simplest second period equilibria arises when one firm inherits the entire market. We have the following intuitive result:

Proposition 3.1. *If $\mu_i = 1$, then in equilibrium the firms charge $p_{2i} = B_i$ and $p_{2j} = 0$, giving them a second period payoff of $\pi_{2i} = B_i$ and $\pi_{2j} = 0$.*

If firm i charges a price above B_i, it can be profitably undercut by firm j. On the other hand, firm j gains nothing from pricing above zero given that firm i sets a price equal to its bonus, since to switch to firm j, customers would have to pay back the bonus B_i to firm i, and in addition pay the positive price p_j, so they would be worse off.

For the rest of this subsection we characterize the equilibria when the market is divided among firms, i.e., when $\mu_i = \frac{1}{2}$. These equilibria depend strongly on the gifts chosen in period zero. The simplest of these characterizations occurs when switching costs are relatively high and trying to compete for the whole market is unprofitable.

Proposition 3.2. *If $B_i, B_j \geq \frac{1}{2}$ then in equilibrium both firms charge the monopoly price ($p_{2i} = p_{2j} = 1$), giving them a second period payoff of $\pi_{2i} = \frac{1}{2}$ and $\pi_{2j} = \frac{1}{2}$.*

When switching costs are lower, the equilibria that arise in the second period are in mixed strategies. They conform to two different types. The first type, which we denote *single sided poaching* equilibria, is one where only one firm prices aggressively and therefore captures the rival's market share with positive probability. The other type is denoted by *double sided poaching* equilibria, where both firms capture their rival's market share with positive probability. For these equilibria we use auxiliary variables $\underline{p}_i$ and $\bar{p}_i$ that represent firm i's minimum and maximum price in the support of the price distribution of the firm. First, we present single sided poaching equilibria, which by Infante et al. [2007] are characterized by

[8] Observe that neither the payments of consumers nor the profits of firms are affected by the bonus being paid either at the beginning (an initial discount) or at the end (a loyalty program) of the game. Hence, our model is equivalent to the one in Banerjee and Summers [1987].

Proposition 3.3. *If $B_i < \frac{1}{2}$ and $B_i + B_j > \frac{1}{2}$ then firm j's pricing strategy has the following cumulative distribution,*

$$F_j(p) = \begin{cases} 0 & p < \underline{p}_j \\ \left(1 - \frac{2V_i - B_i}{p}\right) & \underline{p}_j \le p < \bar{p}_i - B_i \\ \left(1 - \frac{2V_i - B_i}{\bar{p}_i - B_i}\right) & \bar{p}_i - B_i \le p < \bar{p}_j \\ 1 & p \ge \bar{p}_j \end{cases}$$

and firm i's pricing strategy has the following cumulative distribution

$$F_i(p) = \begin{cases} 0 & p < \underline{p}_i \\ 2\left(1 - \frac{V_j}{p - B_i}\right) & \underline{p}_j + B_i \le p < \bar{p}_i \\ 1 & p \ge \bar{p}_i \end{cases}$$

where $\underline{p}_i = B_i + \frac{1}{2}$, $\underline{p}_j = \frac{1}{2}$, $\bar{p}_i = \bar{p}_j = 1$ and the firms' expected payoffs are,

$$V_i = \frac{B_i}{2} + \frac{1}{4}, \quad V_j = \frac{1}{2}$$

When firms compete in a double sided poaching equilibrium their strategies may take on two forms, which depend on the difference between the maximum and minimum price that firms impose in this period. From Infante et al. [2007], we have the following two auxiliary results, which will be useful in proving the propositions below.

Lemma 3.4. *If the equilibrium strategies are such that there is double sided poaching and that*

I) $\bar{p}_i - \underline{p}_i < B_i + B_j$

II) $\bar{p}_j - \underline{p}_j \le B_i + B_j$

then firm j's price has the following cumulative distribution,

$$F_j(p) = \begin{cases} 0 & p < \underline{p}_j \\ \left(1 - \frac{2V_i - B_i}{p}\right) & \underline{p}_j \le p < \bar{p}_i - B_i \\ \left(1 - \frac{2V_i - B_i}{\bar{p}_i - B_i}\right) & \bar{p}_i - B_i \le p < \underline{p}_i + B_j \\ 2\left(1 - \frac{V_i}{p - B_j}\right) & \underline{p}_i + B_j \le p < \bar{p}_j \\ 1 & p \ge \bar{p}_j \end{cases} \tag{3.1}$$

with $\underline{p}_i = 2V_j - B_j$, $\bar{p}_i = 1$ and V_i, V_j represent the firms expected payoff.

Lemma 3.5. *If the equilibrium strategies are such that there is double sided poaching and that*

I) $\bar{p}_i - \underline{p}_i = B_i + B_j$

II) $\bar{p}_j - \underline{p}_j \le B_i + B_j$

then firm j's price has the following cumulative distribution,

$$F_j(p) = \begin{cases} 0 & p < \underline{p}_j \\ \left(1 - \frac{2V_i - B_i}{p}\right) & \underline{p}_j \leq p < \bar{p}_i - B_i \\ 2\left(1 - \frac{V_i}{p - B_j}\right) & \underline{p}_i + B_j \leq p < \bar{p}_j \\ 1 & p \geq \bar{p}_j \end{cases} \tag{3.2}$$

with $\underline{p}_i = 2V_j - B_j$, $\bar{p}_i = 2V_j + B_i$ and V_i, V_j represent the firms expected payoff.

Depending on the values of B_i and B_j each firm will use one of the aforementioned strategies. Note however, that the lemma is incomplete, in the sense that the V_i, V_j must be consistent with the choice of bonuses (conditions I), II)) and that they correspond to the payoffs associated to using these strategies. Using these two lemmas, there are potentially three types of double sided poaching equilibria,

Proposition 3.6. *The optimal strategies characterized in Lemmas 3.4 and 3.5 allow us to characterize the equilibrium as follows:*

***Case** I: If $1 - (2V_j - B_j) < B_i + B_j$ and $1 - (2V_i - B_i) < B_i + B_j$, where*

$$V_i = \frac{1}{4}\left(\frac{3B_i + B_j - (B_i + B_j)^2 + \xi(B_i, B_j)}{2 - B_i - B_j}\right),$$

$$V_j = \frac{1}{4}\left(\frac{3B_j + B_i - (B_i + B_j)^2 + \xi(B_i, B_j)}{2 - B_i - B_j}\right),$$

and $\xi(x, y)$ takes the following expression, [9]

$$\xi(x, y) = (-26xy - 11x^2 - 11y^2 + 22xy^2 + 22x^2y + 2x^3 + 2y^3 - 4xy^3 - 10x^2y^2 - 4x^3y + x^4 + y^4 + 8x + 8y)^{\frac{1}{2}}$$

then both firms use the strategy characterized by equation 3.1 and their expected payoff are V_i and V_j respectively.

***Case** II: If $1 - (2V_j - B_j) < B_i + B_j$ and $2V_i + B_j < 1$, where*

$$V_i = \frac{1}{4}\left(3B_i + B_j - 1 + \alpha(B_i, B_j)\right),$$
$$V_j = \frac{1}{4}\left(1 + B_j - B_i + (2B_i - 1)\alpha(B_i, B_j)\right)^{\frac{1}{2}},$$

and $\alpha(x, y)$ takes the following expression,

$$\alpha(x, y) = (y^2 - 2xy - 3x^2 + 2x + 2y + 1)^{\frac{1}{2}}$$

then firm i uses the strategy characterized by equation 3.2 and firm j uses the strategy characterized

[9] Note that $\xi(x, y) = \xi(y, x)$.

by equation 3.1 and their expected payoff are V_i and V_j respectively.[10]

Case *III: If $2V_j + B_i < 1$ and $2V_i + B_j < 1$, where*

$$V_i = \frac{(1+\sqrt{5})B_i + 2B_j}{4}$$

and

$$V_j = \frac{(1+\sqrt{5})B_j + 2B_i}{4}.$$

then both firms use the strategy characterized by equation 3.2 and their expected payoff are V_i and V_j respectively.

The subsets of the bonus space $(B_i, B_j) \in [0,1]^2$ that lead to each of these second period equilibria are described graphically in Figure 2, 3, and 4. In particular, note that equilibria of case I correspond to relatively high bonuses, and that equilibria of case II exclude symmetric bonuses. Case III equilibria occur when the bonuses are relatively small.[11]The proof of these results for each of the equilibria just described may be found in Infante et al. [2007].

Even if the exact expression of the mixed strategies may seem complicated, their structure is quite intuitive. Consider, for example, the case of the single-sided poaching equilibria described in Proposition 3.3. If firm j is the poaching firm, it will use prices $p \in [\frac{1}{2}, 1-B_i]$ with positive probability, and will also have an atom at the monopoly price $p = 1$ (see Figure 5). Firm i, being the poached firm, will use prices in $[B_i + \frac{1}{2}, 1]$, with an atom at $p = 1$. For both firms, offering a price $p = 1$ is an attempt to extract surplus from their own consumers. For firm i this strategy is risky, since for any price in $[\frac{1}{2}, 1-B_i]$ by firm j, it will lose all of its customer base. For firm j, however, there is no such risk. Even if firm i charges the lowest possible price $p_i = B_i + \frac{1}{2}$, consumers do not switch, since $B_i + \frac{1}{2} + B_j \geq 1$ in the area of B_i, B_j space where single-sided poaching equilibrium are defined (see Figure 3). Note, finally, that the interval $[B_i + \frac{1}{2}, 1]$ where the poached firm j puts positive probability is a translation by B_i of the support of firm j's strategy, $[\frac{1}{2}, 1-B_i]$. For any price used by the poached firm there is a corresponding price used by the poaching firm that guarantees that all consumers will select the poaching firm as provider. A similar analysis can be applied to double sided poaching equilibria, for instance in the case of Type III price distributions shown in Figure 6 for the case of asymmetric bonuses.

4 Characterization of Equilibria of the Complete Game

In this section we will use the results obtained in section 3 to characterize the equilibria for the complete game. As usual, we proceed by backward induction. In 4.1, we characterize consumers' behavior for given initial bonus offerings and first period prices. Then, in 4.2, we find the optimal prices posted by firms in the first period for any pair of initial bonuses. In 4.3, we characterize the firms optimal bonus policy, and prove our two main results. First, that there is no equilibrium with zero prices and zero bonuses, and second,

[10]We can interchange the roles of i and j in this case.

[11]We use later the fact that along the diagonal, bonuses satisfy $B_i \leq 1/4$ in Case III.

that there is an equilibrium in which firms obtain zero rents in the first period but obtain monopoly profits in the second period. Finally, in 4.4, we prove the existence of other symmetric equilibria (in bonuses) in which both firms obtain lower profits than in the one characterized before.

4.1 Expected Payments

In this subsection we use the results of the previous section to determine the expected second period payments by consumers in the different classes of equilibria.

From our previous results we observe that a firm that captures the whole market will charge a price equal to the period 0 bonus, and that its rival charges zero (see Proposition 3.1). Therefore, assuming that firm i has the entire market, expected second period payments are

$$\mathbb{E}\left(\min\{p_{2i}, p_{2j}+B_i\}\right) = \mathbb{E}\left(\min\{B_i, 0+B_i\}\right) = B_i$$

$$\mathbb{E}\left(\min\{p_{2j}, p_{2i}+B_j\}\right) = \mathbb{E}\left(\min\{0, B_i+B_j\}\right) = 0$$

On the other hand, in the case in which both firms inherit market share from the previous period ($\mu_i = \frac{1}{2}$) there are five possible equilibria, which depend on the configuration of bonuses. The simplest case occurs in the case of large bonuses, which lead to a pure strategy equilibrium. If $B_i, B_j \geq \frac{1}{2}$, then both firms charge the consumers' reserve price and therefore expected second period payments are

$$\mathbb{E}\left(\min\{p_{2i}, p_{2j}+B_i\}\right) = \mathbb{E}\left(\min\{1, 1+B_i\}\right) = 1$$

$$\mathbb{E}\left(\min\{p_{2j}, p_{2i}+B_j\}\right) = \mathbb{E}\left(\min\{1, 1+B_j\}\right) = 1$$

The other equilibria involve mixed strategies and the expressions for the expected payment in the second period are more complex. We are left with four possible outcomes, one where only one firm poaches consumers from its rival, and three in which both firms can poach clients from their rival. We have the following proposition,

Proposition 4.1. *In the case of a mixed strategy equilibrium in the second period, we have the following expressions for* $\mathbb{E}\left(\min\{p_{2i}, p_{2j}+B_i\}\right)$*:* [12]

I) Single Sided Poaching: One firm can poach from its rival (see Proposition 3.3; we consider the case in which firm j can poach from firm i),

$$\mathbb{E}\left(min\{p_{2i}, p_{2j}+B_i\}\right) = \frac{3}{2} + B_i - \frac{1}{2(1-B_i)} - \frac{1}{2}\left[\ln(1-B_i) - \ln(\frac{1}{2})\right]$$
$$\mathbb{E}\left(min\{p_{2j}, p_{2i}+B_j\}\right) = \frac{1}{2} + \frac{B_i}{2(1-B_i)} + \frac{1}{2}\left[\ln(1-B_i) - \ln(\frac{1}{2})\right]$$

II) Double Sided Poaching: Both firms can poach from their rival (see Proposition 3.6)

[12] To simplify the exposition we write "can poach" for the more precise statement "poaches with positive probability".

i) Case I

$$\mathbb{E}\left(\min\{p_{2i}, p_{2j}+B_i\}\right) = 2V_j - (2V_i - B_i)\left[\frac{2V_j}{1-B_i} + \ln(1-B_i) - \ln(2V_i - B_i)\right]$$
$$+ (2V_j - B_j)\left[\frac{2V_i}{1-B_j} + \ln(1-B_j) - \ln(2V_j - B_j)\right]$$
$$\mathbb{E}\left(\min\{p_{2j}, p_{2i}+B_j\}\right) = 2V_i - (2V_j - B_j)\left[\frac{2V_i}{1-B_j} + \ln(1-B_j) - \ln(2V_j - B_j)\right]$$
$$+ (2V_i - B_i)\left[\frac{2V_j}{1-B_i} + \ln(1-B_i) - \ln(2V_i - B_i)\right]$$

ii) Case II (we consider the case in which $B_j < B_i$),

$$\mathbb{E}\left(\min\{p_{2i}, p_{2j}+B_i\}\right) = 4V_j - B_j - (2V_i - B_i)\left[\frac{2V_j}{1-B_i} + \ln(1-B_i) - \ln(2V_i - B_i)\right]$$
$$+ (2V_j - B_j)\left[\ln(2V_i) - \ln(2V_j - B_j)\right]$$
$$\mathbb{E}\left(\min\{p_{2j}, p_{2i}+B_j\}\right) = 2V_i - 2V_j + B_j - (2V_j - B_j)\left[\ln(2V_i) - \ln(2V_j - B_j)\right]$$
$$+ (2V_i - B_i)\left[\frac{2V_j}{1-B_i} + \ln(1-B_i) - \ln(2V_i - B_i)\right]$$

iii) Case III

$$\mathbb{E}\left(\min\{p_{2i}, p_{2j}+B_i\}\right) = B_i - B_j + 4V_j - 2V_i - (2V_i - B_i)\left[\ln(2V_j) - \ln(2V_i - B_i)\right]$$
$$+ (2V_j - B_j)\left[\ln(2V_i) - \ln(2V_j - B_j)\right]$$
$$\mathbb{E}\left(min\{p_{2j}, p_{2i}+B_j\}\right) = B_j - B_i + 4V_i - 2V_j - (2V_j - B_j)\left[\ln(2V_i) - \ln(2V_j - B_j)\right]$$
$$+ (2V_i - B_i)\left[\ln(2V_j) - \ln(2V_i - B_i)\right]$$

Idea of Proof:
To calculate these expected values we first characterize the distribution of the minimum of the two possible payments that consumers face in the second period, as a function of the price distributions used by each firm. Then, using the distributions characterized in section 3 we can calculate the expected value of second period payments. See 7.1 in the Appendix for details.

The previous results provide a complete characterization of the expected payments that consumers face in the second period. Their choice of firm for the second period depends on the bonuses and on the first period prices chosen by firms, and on the class of equilibria that can occur in the second period. If firms choose strategies so that equation 2.1 is satisfied, there will be a divided market in the first period, and firms will inherit one half of the market in the second period. Any deviation from equation 2.1 will result in one firm having all of the consumers, excluding its rival from the market.

4.2 First Period Price Decision

In this subsection we examine firms' period 1 pricing strategies. Since we have characterized the decisions that consumers face when choosing a firm at the end of the first period, we can analyze the firms' optimal price strategies in period one. We focus on equilibria in which firms share the market, i.e. for given bonuses B_i, B_j, the prices satisfy equation 2.1. Rewriting the expression in terms of p_{1i}, p_{1j} we get

$$p_{1i} - p_{1j} = \underbrace{B_i - B_j - \Big[\mathbb{E}\Big(\min\{p_{2i}, p_{2j} + B_i\}\Big) - \mathbb{E}\Big(\min\{p_{2j}, p_{2i} + B_j\}\Big)\Big]}_{=d} \tag{4.1}$$

If prices satisfying this equation are to be part of an equilibrium, we must ensure that firms do not want to deviate from those prices. Therefore we must find conditions that ensure that firms would prefer to share, rather than capture the whole market. If firm i has the entire market in the second period, it will charge a price equal to the bonus offered in the first period, which gives it the following payoff,

$$\pi_{1i}(\text{entire market}) = (p_{1i} - B_i) + B_i = p_{1i}.$$

The payoff when it splits the market is

$$\pi_{1i}(\text{half the market}) = \frac{1}{2}(p_{1i} - B_i) + V_i$$

where V_i is i's expected payoff in the second period, which depends on (B_i, B_j). Therefore, the condition in which firm i prefers to share the market with j is

$$\begin{aligned} \pi_{1i}(\text{entire market}) &\leq \pi_{1i}(\text{half the market}) \\ p_{1i} &\leq 2V_i - B_i := c_i \end{aligned} \tag{4.2}$$

analogously for j we have

$$p_{1j} \leq 2V_j - B_j := c_j \tag{4.3}$$

It is immediate that

$$p^*_{1i} = \min\{c_i, c_j + d\} \tag{4.4}$$

$$p^*_{1j} = \min\{c_j, c_i - d\} \tag{4.5}$$

satisfy 4.1 - 4.3.
Given that these prices satisfy equation 4.2 and 4.3, no firm would lower its price in order to capture the entire market. In fact, they would not gain by having the entire market, even at current prices.

For these prices to constitute an equilibrium, the expected payoff of the firms must be nonnegative.[13] In the next section we verify this fact for each pair of p^*_{1i}, p^*_{1j}, so as to derive our equilibrium.

4.3 Choice of Bonus in Period Zero

In this subsection we examine the problem of choosing the optimal bonus in period zero. We derive our two main results in this section. The first one is that price competition does not lead to an equilibrium with marginal cost pricing and zero rents. Therefore, even with completely rational and forward-looking customers and an homogenous good, price competition does not necessarily eliminate profits. The second

[13] If not, a firm could always raise its price, thus losing its customers and getting zero profit.

result is that there is a subgame perfect equilibrium with maximal rent extraction in the second period. The equilibrium is sustained by large bonuses ($B_i = B_j = \frac{1}{2}$)[14] that allow each firm to milk its customer base in the second period.

We begin with the following result that shows that the intuitive result that competition between firms will drive bonuses, prices and rents to zero is incorrect:

Theorem 4.2. *The following strategies do not constitute an equilibrium:*

$$B_i = B_j = 0$$
$$p^*_{1i} = p^*_{1j} = 0$$
$$p^*_{2i} = p^*_{2j} = 0$$

Proof:
We consider bonus deviations from this equilibrium. Let us suppose that firm j decides to raise its bonus by a small $\epsilon > 0$. If the continuation equilibrium is such that the firms divide the market, the second period equilibrium is the one characterized by case III of Proposition 3.6 (See figure 4). Therefore the firms' expected payoffs in the second period take the following form,

$$V_i = \frac{\epsilon}{2}$$

$$V_j = \frac{(1+\sqrt{5})\epsilon}{4}$$

If the optimal first period prices are such that both firms have positive market shares, these prices must satisfy equations 4.4 and 4.5. In section 7.2 of the appendix[15] we show that $c_j < c_i - d$, giving us the following first period prices,

$$p^*_{i1} = c_j + d = (2-\sqrt{5})\epsilon + (3-\sqrt{5})\ln\left(\frac{1+\sqrt{5}}{2}\right)\epsilon$$

$$p^*_{1j} = c_j = \frac{\sqrt{5}-1}{2}$$

giving each firm the following expected payoff for the entire game,

$$\begin{aligned}
\pi_i(0,\epsilon) &= \frac{1}{2}(p^*_{1i} - 0) + V_i \\
&= \frac{\epsilon}{2}\left((2-\sqrt{5}) + (3-\sqrt{5})\ln\left(\frac{1+\sqrt{5}}{2}\right)\right) + \frac{\epsilon}{2} > 0 \\
\pi_j(\epsilon,0) &= \frac{1}{2}(p^*_{1j} - \epsilon) + V_j \\
&= \frac{\epsilon}{2}(\sqrt{5}-1) > 0
\end{aligned}$$

Note that p^*_{1i} and p^*_{1j} are in fact optimal because they correspond to the prices characterized by equations 4.4 and 4.5, and payoffs are positive under these prices, so neither firms wants to raise its price and receive

[14]This is the smallest bonus level at which firm choose to charge the monopoly price.
[15]See Claim 1 in Appendix for details.

zero profits. Therefore, these prices constitute a continuation equilibrium for bonuses $(B_i, B_j) = (0, \epsilon)$ and, since firm j's payoff is positive we conclude that $(B_i, B_j) = (0, 0)$ is not a Nash equilibrium. ■

Next we state our main positive result: there exists a subgame perfect equilibrium of this game where both firms obtain positive profits. Note that there is no poaching along the equilibrium path.

Theorem 4.3. *The following strategies form an equilibrium path of the game,*

$$B_i = B_j = \frac{1}{2}$$
$$p^*_{1i} = p_{1j}* = \frac{1}{2}$$
$$p^*_{2i} = p_{2j}* = 1,$$

and the firms' payoffs are

$$\pi_{1i} = \pi_{1j} = \frac{1}{2}.$$

Proof:

We use equations (4.1), (4.4), (4.5) to derive the first period prices of the continuation equilibrium corresponding to bonuses $B_i = B_j = 1/2$. Next we need to show that that the selection of prices in period one gives a non negative payoff to each firm. First, observe that the payoffs for both firms when they choose $(B_i, B_j) = (1/2, 1/2)$ correspond to those of proposition 3.2. Therefore, we have a pure strategy continuation equilibrium in the second period, with both firms charging the monopoly price 1,

$$p^*_{2i} = p^*_{2i} = 1$$
$$\pi^*_{2i} = \pi^*_{2i} = \frac{1}{2}$$

It is easy to see from (4.1) that $d = 0$ and therefore according to equations 4.4 and 4.5, the first period prices are

$$p^*_{1i} = \min\{c_i, c_j\} = \min\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$$
$$p^*_{1j} = \min\{c_j, c_i\} = \min\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$$

with a total payoff

$$\pi_{1i} = \frac{1}{2}(p^*_{1i} - B_i) + V_i = \frac{1}{2}(\frac{1}{2} - \frac{1}{2}) + \frac{1}{2} = \frac{1}{2} > 0$$
$$\pi_{1i} = \frac{1}{2}(p^*_{1j} - B_j) + V_j = \frac{1}{2}(\frac{1}{2} - \frac{1}{2}) + \frac{1}{2} = \frac{1}{2} > 0$$

Now we must prove that for any feasible deviation by one firm from bonuses $B_i = B_j = 1/2$, leads to a reduction in that firm's payoff.

Raising the bonus: if firm j raises its bonus by any feasible amount $\epsilon \in [0, 1/2]$, the second period price strategies will still be $p^*_{2i} = p^*_{2j} = 1$ (by proposition 3.2) and consumer's preferences will be determined by,

$$\begin{aligned} d &= \frac{1}{2} - (\frac{1}{2} + \epsilon) - (\mathbb{E}\Big(\min\{1, 1 + \frac{1}{2}\}\Big) - \mathbb{E}\Big(\min\{1, 1 + \frac{1}{2} + \epsilon\}\Big) \\ &= -\epsilon \end{aligned}$$

Therefore first period prices are,

$$\begin{aligned} p^*_{1i} &= \min\{c_i, c_j + d\} = \min\{\frac{2}{2} - \frac{1}{2}, \frac{2}{2} - \frac{1}{2} - \epsilon - \epsilon\} = \frac{1}{2} - 2\epsilon \\ p^*_{1j} &= \min\{c_j, c_i - d\} = \min\{\frac{2}{2} - \frac{1}{2} - \epsilon, \frac{2}{2} - \frac{1}{2} + \epsilon\} = \frac{1}{2} - \epsilon \end{aligned}$$

giving firms the following payoffs,

$$\begin{aligned} \pi_{1i} &= \frac{1}{2}(p^*_{1i} - B_i) + V_i = \frac{1}{2}(\frac{1}{2} - 2\epsilon - \frac{1}{2}) + \frac{1}{2} \\ &= \frac{1}{2} - \epsilon \geq 0 \\ \pi_{1j} &= \frac{1}{2}(p^*_{1j} - B_j) + V_j = \frac{1}{2}(\frac{1}{2} - \epsilon - \frac{1}{2} - \epsilon) + \frac{1}{2} \\ &= \frac{1}{2} - \epsilon \geq 0 \end{aligned}$$

Since the payoffs are non negative and satisfy all the requirements of a continuation equilibria, we have that p^*_{1i}, p^*_{1j} are the first period equilibrium prices corresponding to the bonuses $(B_i, B_j) = (1/2, 1/2 + \epsilon)$. Since j's payoff is lower under the deviation, it has no incentive to raise its gift from $B_j = 1/2$.

Reducing the bonus: to study the firms' behavior when one of them reduces its gift, we must remember that there are two possible equilibria that can arise in the second period for gift levels $(B_i, B_j) = (1/2, 1/2 - \epsilon)$. One of those is a single sided poaching equilibrium in which only one firm poaches from its rival with positive probability, the other is a double sided poaching equilibria in which both firms poach their rival's consumers with positive probability. Let us first study the case of single sided poaching equilibrium.

Single Sided Poaching Equilibrium:(see Infante et al. [2007] for details of this type of equilibrium) Suppose that firm j decides to reduce the gift it offers by $\epsilon \in (0, 1/2]$ and that the firms find themselves in an single sided poaching equilibrium in which only firm i can poach consumers from j.[16] Then the second period payoffs are $V_i = 1/2$, $V_j = B_j/2 + 1/4$ (by proposition 3.3), and according to proposition 4.1 we have

$$\begin{aligned} d = B_i - B_j - \Big(\Big[\frac{1}{2} + \frac{B_j}{2(1 - B_j)} + \frac{1}{2}\Big(\ln(1 - B_j) - \ln(\frac{1}{2})\Big) \Big] \\ + \Big[\frac{3}{2} + B_j - \frac{1}{2(1 - B_j)} - \frac{1}{2}\Big(\ln(1 - B_j) - \ln(\frac{1}{2})\Big) \Big] \Big) \end{aligned}$$

[16]Given the nature of this type of equilibrium in the second period, it is not feasible for j to reduce its gift and also be the firm that can potentially poach its rival's customers.

We have $c_j > c_i - d$ [17], and therefore

$$
\begin{aligned}
p^*_{1i} &= \min\{c_i, c_j + d\}\\
&= c_i = 2V_i - B_i = \frac{1}{2}\\
p^*_{1j} &= \min\{c_j, c_i - d\}\\
&= c_i - d = 2V_i - B_i - d
\end{aligned}
$$

Replacing the values for B_i, B_j, V_i and V_j we get the following expression,

$$
\begin{aligned}
p^*_{1j} &= 2V_i - B_i - d\\
&= 2V_i - B_i - \Big(B_i - B_j - \Big[\frac{1}{2} + \frac{B_j}{2(1-B_j)} + \frac{1}{2}(\ln(1-B_j) - \ln(\frac{1}{2}))\Big]\\
&\qquad + \Big[\frac{3}{2} + B_j - \frac{1}{2(1-B_j)} - \frac{1}{2}(\ln(1-B_j) - \ln(\frac{1}{2}))\Big]\Big)\\
&= -1 + \ln(1+2\epsilon) + \frac{\frac{3}{2} - \epsilon}{2(\frac{1}{2}+\epsilon)}\\
&= \frac{-2 - 4\epsilon + 3 - 2\epsilon}{2(1+2\epsilon)} + \ln(1+2\epsilon)\\
&= \frac{1-6\epsilon}{2(1+2\epsilon)} + \ln(1+2\epsilon)
\end{aligned}
$$

We then compute the total payoff for each firm,

$$
\begin{aligned}
\pi_{1i} &= \frac{1}{2}(\frac{1}{2} - \frac{1}{2}) + \frac{1}{2} = \frac{1}{2}\\
\pi_{1j} &= \frac{1}{2}\Big(\frac{1-6\epsilon}{2(1+2\epsilon)} + \ln(1+2\epsilon) - \frac{1}{2} + \epsilon\Big) + \frac{\frac{1}{2} - \epsilon}{2} + \frac{1}{4}\\
&= \frac{1}{2}\Big(\frac{1-6\epsilon}{2(1+2\epsilon)} + \ln(1+2\epsilon)\Big) + \frac{1}{4}
\end{aligned}
$$

To ensure that p^*_{1i} and p^*_{1j} constitute equilibrium responses, the expected payoffs of each firm must be nonnegative, which is obviously true for firm i. For firm j we have

$$
\begin{aligned}
\pi_{1j} &= \frac{1}{2}\Big(\frac{1-6\epsilon}{2(1+2\epsilon)} + \underbrace{\ln(1+2\epsilon)}_{>0 \text{ for } \epsilon>0}\Big) + \frac{1}{4}\\
&> \frac{1}{2}\Big(\frac{1-6\epsilon}{2(1+2\epsilon)}\Big) + \frac{1}{4}\\
&\geq 0, \qquad \forall \epsilon \in (0, \frac{1}{2}]
\end{aligned}
$$

Now it only remains to prove that firm j's payoff is smaller than its payoff before the deviation, which means we must prove $\pi_{1j}(B_j = \frac{1}{2} - \epsilon, B_i = \frac{1}{2}) \leq \pi_{1j}(B_j = \frac{1}{2}, B_i = \frac{1}{2})$. This reduces to proving that the following

[17] See Claim 2 in Appendix for details.

expression is negative:

$$\frac{1-6\epsilon}{4(1+2\epsilon)}+\frac{1}{2}\ln(1+2\epsilon)+\frac{1}{4}-\frac{1}{2}\leq\frac{1}{4}\Big[\frac{1-6\epsilon}{1+2\epsilon}+2(1+2\epsilon-1)-1\Big]$$
$$=\frac{1}{4}\Big[\frac{-4\epsilon+8\epsilon^2}{1+2\epsilon}\Big],$$

true for all $\epsilon\in(0,\frac{1}{2}]$.

Double Sided Poaching Equilibrium: (see Infante et al. [2007] for details) Suppose that j decides to reduce the gift it offers by $\epsilon\in(0,1/2]$ and that firms find themselves in a double sided poaching equilibrium. Note that there are two possible double sided poaching equilibria that can arise from this deviation, namely, Case I and Case II equilibria.[18] First, we study the case of "small" deviations by firm j, that result in equilibria characterized by Case I. Since $B_i=1/2$ and that $B_j=1/2-\epsilon$ we have the following expressions for the firms expected payoff in the second period,[19]

$$V_i(B_i=\frac{1}{2},B_j=\frac{1}{2}-\epsilon)=\frac{1}{2}-\frac{\epsilon^2}{2(1+\epsilon)}$$

$$V_j(B_j=\frac{1}{2}-\epsilon,B_i=\frac{1}{2})=\frac{1}{2}-\frac{\epsilon}{2}$$

According to Proposition 4.1 in this case we have the following expression for d,

$$d=B_i-B_j-\Big(2V_j-2V_i-2(2V_i-B_i)\Big[\frac{2Vj}{1-B_i}+\ln(1-B_i)-\ln(2V_i-B_i)\Big]$$
$$+2(2V_j-B_j)\Big[\frac{2V_i}{1-B_j}+\ln(1-B_j)-\ln(2V_j-B_j)\Big]\Big)$$

We have that $c_i<c_j+d$ [20], and therefore,

$$p^*_{1i}=\min\{c_i,c_j+d\}$$
$$=c_i=2V_i-B_i$$
$$p^*_{1j}=\min\{c_j,c_i-d\}$$
$$=c_i-d=2V_i-B_i-d$$

With these first period prices, firms have the following payoffs,

$$\pi_{1i}=\frac{1}{2}(p^*_{1i}-B_i)+V_i=2V_i-B_i$$
$$\pi_{1j}=\frac{1}{2}(p^*_{1j}-B_j)+V_j=\frac{1}{2}(2V_i-B_i-d-B_j)+V_j$$

[18]See Figure 4
[19]See Claim 3 in Appendix for details.
[20]See Claim 4 in Appendix for details

By replacing the values of B_i, B_j, V_i, and V_j, we can see that firm i's payoff is positive,

$$\begin{aligned}\pi_{1i} &= 2V_i - B_i \\ &= 2\Big(\frac{1}{2} - \frac{\epsilon^2}{2(1+\epsilon)}\Big) - \frac{1}{2} \\ &= \frac{1}{2} - \frac{\epsilon^2}{1+\epsilon} > 0\end{aligned}$$

We also prove that the same is true for firm j [21]. Therefore p_{1i}^*, p_{1j}^* constitute an equilibrium. Now we verify that firm j's payoff is smaller than before the deviation, i.e., $\pi_{1j}(B_j = 1/2 - \epsilon, B_i = 1/2) \leq \pi_{1j}(B_j = 1/2, B_i = 1/2)$,

$$\begin{aligned}\pi_{1j}(B_j = \frac{1}{2} - \epsilon, B_i = \frac{1}{2}) &= \frac{1}{2}(2V_i - B_i - d - B_j) + V_j \\ &< \frac{1}{2}(\underbrace{(2V_j - B_j)}_{=c_j} - B_j) + V_j \\ &= 2V_j - B_j \\ &= \frac{1}{2}\end{aligned}$$

Therefore j's payoff decreases when the bonus is reduced to $B_j = 1/2 - \epsilon$, and consequently it does not have incentives to do so. [22]

Now we analyze the case in which the deviation chosen by firm j falls within the equilibria characterized by Case II. With $B_i = 1/2$ and $B_j = 1/2 - \epsilon$ we have the following expression for the firm's expected payoff in the second period,[23]

$$V_i(B_i = \frac{1}{2}, B_j = \frac{1}{2} - \epsilon) = \frac{1}{4}(1 - \epsilon + (\epsilon^2 - 2\epsilon + 2)^{\frac{1}{2}})$$
$$V_j(B_j = \frac{1}{2} - \epsilon, B_i = \frac{1}{2}) = \frac{1-\epsilon}{2}$$

According to Proposition 4.1, in this case we have the following expression for d,

$$\begin{aligned}d = B_i - B_j - \Big(&6V_j - 2V_i - 2B_j - \frac{4V_j(2V_i - B_i)}{1 - B_i} - 2(2V_i - B_i)[\ln(1 - B_i) - \ln(2V_i - B_i)] \\ &+ 2(2V_j - B_j)[\ln(2V_i) - \ln(2V_j - B_j)]\Big)\end{aligned}$$

Since this type of equilibrium occurs when j's deviation from $1/2$ is relatively large, in the second period we must have that $\bar{p}_j \leq 1$ and given the expression for $\bar{p}_j$ this implies that for an equilibrium to exist, the

[21] See Claim 5 in the Appendix for details

[22] It might be counterintuitive that when firm j lowers its bond, firm i must lower its first period price to be able retain half of the market. This makes sense when we realize that firm j's best first period response to offering a lower bonus in period zero is to lower its first period price by more than the reduction in the bond, forcing i to lower its price to maintain competitiveness.

[23] See Claim 6 in the Appendix for details

deviation is bounded below: $\epsilon \geq \frac{\sqrt{17}-1}{8}$.[24] For this case we have $c_i < c_j - d$ [25], and therefore,

$$\begin{aligned}
p^*_{1i} &= \min\{c_i, c_j + d\} \\
&= c_i = 2V_i - B_i \\
p^*_{1j} &= \min\{c_j, c_i - d\} \\
&= c_i - d = 2V_i - B_i - d
\end{aligned}$$

With these first period prices, the firms will have the following payoffs,

$$\pi_{1i} = \frac{1}{2}(p^*_{1i} - B_i) + V_i = 2V_i - B_i$$

$$\pi_{1j} = \frac{1}{2}(p^*_{1j} - B_j) + V_j = \frac{1}{2}(2V_i - B_i - d - B_j) + V_j$$

By replacing the values of B_i, B_j, V_i, and V_j, we can see that firm i's payoff is positive,

$$\begin{aligned}
\pi_{1i} &= 2V_i - B_i \\
&= \frac{1}{2}(1 - \epsilon + (\epsilon^2 - 2\epsilon + 2)^{\frac{1}{2}}) - \frac{1}{2} \\
&= \frac{1}{2} - \frac{\epsilon}{2} + \frac{(\epsilon^2 - 2\epsilon + 2)^{\frac{1}{2}}}{2} > 0
\end{aligned}$$

We must also show that π_{1j} is positive[26]. Therefore p^*_{1i} and p^*_{1j} constitute an equilibrium. Finally we must show that firm j's payoff is smaller than before changing the bonus, i.e., $\pi_{1j}(B_j = \frac{1}{2} - \epsilon, B_i = \frac{1}{2}) \leq \pi_{1j}(B_j = \frac{1}{2}, B_i = \frac{1}{2})$,

$$\begin{aligned}
\pi_{1j}(B_j = \frac{1}{2} - \epsilon, B_i = \frac{1}{2}) &= \frac{1}{2}(2V_i - B_i - d - B_j) + V_j \\
&< \frac{1}{2}(\underbrace{(2V_j - B_j)}_{=c_j} - B_j) + V_j \\
&= 2V_j - B_j \\
&= 1 - \epsilon - \frac{1}{2} + \epsilon \\
&= \frac{1}{2} \\
&= \pi(B_j = \frac{1}{2}, B_i = \frac{1}{2})
\end{aligned}$$

Therefore, firm j does not have incentives to reduce its bonus below $B_j = 1/2$ when the equilibrium in the second period is a double sided poaching equilibrium characterized by Case II.

Recapitulating, we have shown that any feasible deviations from the pair of strategies $(B_i, B_j) = (1/2, 1/2)$ re-

[24]See Claim 7 in the Appendix for details.
[25]See Claim 8 in the Appendix for details
[26]See Claim 9 in the Appendix for details.

duces the payoffs along the equilibrium path. Therefore the strategies described in the Theorem constitute an equilibrium in which the firms have the following payoff,

$$\pi_{1i} = \frac{1}{2}$$
$$\pi_{1j} = \frac{1}{2}$$

■

We can see that, independent of which continuation equilibrium characterized in Infante et al. [2007] we consider, firms have incentives to impose bonus levels greater than zero. In other words, the Bertrand strategies which involve no bonuses and zero prices are not an equilibrium. Moreover, it is also possible to see that when firms choose bonuses $(B_i, B_j) = (\frac{1}{2}, \frac{1}{2})$, there are no profitable deviations and firms share the market while having positive expected payoff. In the next subsection we analyze the possible existence of other equilibria with symmetric bonuses, and show that even when they exist, the ones where firms obtain lower profits (including zero profits) are not likely to be played, since they are dominated (from the point of view of firms) by the equilibria we just presented, which has the same bonuses but higher first period prices. [27]

4.4 Other Symmetric Equilibria

In this subsection we provide a numerical proof that that all equilibria which are symmetric in bonuses ($B_i = B_j = B$) are dominated (in terms of firms' profits) by the equilibrium found in the previous subsection.[28] We first eliminate the possibility of equilibria in which in the second period there is double sided poaching. In particular, this implies that there are no symmetric equilibria with $B \in [0, \frac{1}{4}]$, since for these bonuses only double sided equilibria are possible (see figure 3 and Case III in figure 4). Then, we characterize the one-sided poaching equilibria with $B \in (\frac{1}{4}, \frac{1}{2})$. We show analytically that these equilibria are dominated by the one found in the previous section (which involves no poaching). Finally, we show the existence of a continuum of equilibria with no poaching, similar to that of theorem 4.3 but with lower first period prices. Obviously, these are also dominated.

In the next proposition, we show that if firms choose identical bonuses and they compete in a double sided poaching equilibria in the second period, they have incentives to deviate by raising their bonuses. Since the expected payoffs of the firms in the second period are difficult to handle analytically, we show numerically that the expected payoff of a firm that deviates from $B_i = B_j$ is higher.

Proposition 4.4. *There is no subgame perfect equilibrium where $B_i = B_j = B < \frac{1}{2}$ and firms compete in a double sided poaching equilibrium in the second period. This implies that there is no equilibrium with $B_i = B_j = B \leq \frac{1}{4}$.*

Proof:
(Numerical) When $B_i = B_j = B$, in the second period firms can compete in a double sided poaching equi-

[27] Note that throughout this paper we have focused on symmetric equilibria, and therefore we have not discarded the possibility of an equilibrium in which one firm has the entire market with a positive payoff.

[28] This result is not totally satisfactory, since we do not provide an analytic proof of the result.

libria characterized by Case I (if bonus levels are relatively high) or in a double sided poaching equilibria characterized by Case III (if bonus levels are relatively small).[29] We note that firms impose first period prices defined by equations (4.4) and (4.5), which give firms a positive payoff. In effect, since the bonuses are identical, we have that $d = 0$ and therefore $p_{1i} = p_{1j} = 2V - B$. Payoffs are given by

$$\pi_{1i} = \pi_{1j} = 2V - B > 0$$

which is the minimum price imposed by firms in the second period and therefore positive[30].

Now we show numerically that firms do have incentives to modify their bonuses. We consider a partition of 1/1000 of the bonus space (B_i, B_j), and compare firm i's expected payoff for every $B_i = B_j < \frac{1}{2}$ with firm i's expected payoff using a deviation bonus of $B_i = B_j + 1/1000$. We then plot the positive gain in utility that firm i receives by its deviation, which can be seen in Figure 7 for Case I and Figure 7 for Case III.

We find that when firms compete in the second period in an equilibria characterized by Case III, firm i's gain from the deviation is independent of the value of the bonus initially imposed. When firms compete in the second period in a equilibria characterized by Case I, we see that firm i's utility gain from a deviation decreases as the bonuses become larger. The increase in profits from deviations converges to zero when $B = \frac{1}{2}$, i.e., in the equilibria characterized in Theorem 4.3.

Finally, since from proposition 3.6 we know that only double poaching equilibria are a possible continuation if $B_i = B_j \leq \frac{1}{4}$, we conclude that there is no equilibrium with these bonuses. ■

Now we characterize the equilibria that involve symmetrical bonuses $B_i = B_j = B \in (\frac{1}{4}, \frac{1}{2})$ and single sided poaching in the second period.

Theorem 4.5. *There exist subgame perfect equilibria where firms choose bonuses $B_i = B_j = B \in (\frac{1}{4}, \frac{1}{2})$ and in the second period firms compete in a single sided poaching equilibrium, where both firms have positive profits. These equilibria yield the firms strictly lower profits than the equilibrium with $B_i = B_j = \frac{1}{2}$ described in equilibrium 4.3.*

Proof:

We know that the pricing strategies characterized in subsection 4.2 for the firms' first period price may be part of an equilibrium if the expected payoffs of the firms are nonnegative. We first compute the expected payoffs when using these strategies in order to check this property. From 3.3, the expected payoffs of the firms in the second period are,[31]

$$V_i = \frac{B_i}{2} + \frac{1}{4}, \quad V_j = \frac{1}{2}$$

For $B_i = B_j = B$ we have that $c_i < c_j + d$[32], therefore from equations (4.4) and (4.5) the first period prices are [33],

$$p_{1i} = c_i = \frac{1}{2}, \quad p_{1j} = c_i - d = \frac{3}{2} + B_j - \frac{B_i + 1}{2(1 - B_i)} - \ln(2(1 - B_i))$$

[29] Recall that Case II double poaching equilibria are associated to asymmetrical bonuses, see figure 4.

[30] See Infante et al. [2007] for more details.

[31] In this analysis we will assume that firm j poaches from its rival.

[32] See Claim 10 in Appendix for details

[33] Remember that $B_i + B_j > \frac{1}{2}$ and $B_i < \frac{1}{2}$ for firms to be in a single sided poaching equilibrium, therefore $B \in (\frac{1}{4}, \frac{1}{2})$.

Giving firms the total expected payoff,

$$\pi_{1i} = \frac{1}{2} > 0 \tag{4.6}$$
$$\pi_{1j} = \frac{5}{4} - \frac{1}{2}\Big[\frac{B_i + 1}{2(1-B_i)} + \ln(2(1-B_i))\Big] > 0 \tag{4.7}$$

Before verifying that these strategies are in fact optimal, note that $\pi_{1j} < \frac{1}{2}$, the payoff that firms receive in the equilibrium with $B_i = B_j = \frac{1}{2}$.

Now, to check that these strategies are an equilibrium it just remains to prove that there are no profitable deviations in the bonus stage of the game. For small bonus deviations (such that $c_i < c_j + d$), firm i has a constant payoff and firm j has a payoff that is independent of its own choice of bonus, so there are no profitable deviations in that range. Now we analyze deviations that involve changes in first period prices, i.e., when $c_i \geq c_j + d$. In that case, the firms payoffs are

$$\pi_{1i} = -B_j + \frac{1}{4} + \frac{1}{2}\Big[\frac{B_i + 1}{2(1-B_i)} + \ln(2(1-B_i))\Big] \tag{4.8}$$
$$\pi_{1j} = 1 - B_j \tag{4.9}$$

Note that in this case i's payoff is increasing in B_i[34] and j's payoff is decreasing in B_j.[35] Now consider $g(B_i, B_j) := c_i - [c_j + d]$ and note that $g(B_i, B_j)$ is decreasing in B_i, increasing in B_j[36], and that if $g(B_i, B_j) = 0$ both expressions for π_i are equal ((4.6) and (4.8) are the equal, and (4.7) and (4.9) are equal for firm j). If firm i decides to reduce its own bonus, thus making g positive, its payoff is characterized by (4.8). The new payoff is lower for firm i, since the two expressions for profits are equal when $g = 0$ and we know that π_{1i} given by (4.8) is increasing in B_i. An analogous argument shows that there are no profitable deviations for firm j.

In the case in which the reduction in firm i's bonus leads to negative expected payoffs (i.e., (4.8) is negative), the optimal choice for the first period price is a price that leaves it out of the market with a payoff of zero. Again, in this case the payoff is lower than before the deviation, so that firm i does not choose these bonuses. Thus, in all cases, firms do not have incentives to change the size of their bonus and thus we are in a Nash equilibrium. ■

Finally, we show that there is a continuum of other equilibria, similar to those of theorem 4.3, but with lower prices in the first period. Obviously, they are also dominated.

Proposition 4.6. *For each equilibrium path $B_i^*, B_j^*, p_{1i}^*, p_{1j}^*, p_{2i}^*$, and p_{2j}^*, where p_{1i}^* and p_{1j}^* are characterized by equations 4.4 and 4.5, there exists a continuum of equilibria in which firms first period are,*

$$p_{1i} = p_{1i}^* - K \text{ and } p_{1j} = p_{1j}^* - K$$

with K a constant such that the firms´ payoffs are nonnegative.

Proof:

For any pair of potential equilibrium bonuses (B_i, B_j) we need to consider equilibrium prices such that

[34] See Claim 11 in the Appendix for details.

[35] In what follows we use the expressions (4.6), (4.8), (4.7), (4.9) derived in the previous proof.

[36] See Claim 12 in the Appendix for details.

equations 4.2, 4.3, 4.1 are satisfied, and such that the firm's expected payoffs are non negative. The prices described in subsection 4.2 (denoted p^*_{1i} and p^*_{1j}) satisfy the above conditions at a level that maximizes firms profits and these are the prices that we have considered up to now.

Note however that the following prices: $p_{1i} = p^*_{1i} - K$ and $p_{1j} = p^*_{1j} - K$ also satisfy the aforementioned conditions as long as the firm's payoffs remain greater or equal to zero. Therefore, for every equilibrium we find there is also a continuum of equilibria with lower profits for firms. ■

4.5 Observations

Zero profit equilibria The preceding results shows the existence of equilibria with zero profits (for example $B_i = B_j = \frac{1}{2}$, $p_{1i} = p_{1j} = -\frac{1}{2}$, and $p_{2i} = p_{2j} = 1$). However, these equilibria appear unreasonable, because there is no game-theoretic advantage to these equilibria and profits are lower than in the equilibria described in theorem 4.3. We expect firms to coordinate in the equilibrium that gives the highest profits for a given pair of bonuses (B_i, B_j), i.e., the one described in theorem 4.3.

Robustness to changes in the timing of actions Theorem 4.2, which shows that the strategies with no bonuses and zero prices do not constitute an equilibrium, is robust to a natural change in the timing of the game. Assume that bonuses and prices in the first period are chosen simultaneously, rather than sequentially. In this modified game, the strategy profile with zero prices and zero bonuses again do not constitute an equilibrium. To see this, consider the situation where both firms are playing zero bonuses and prices in the first period, which leads to zero profits in the whole game. Firm i can chose a deviation such that consumers are still indifferent between both firms, using equation 2.1, this corresponds to choosing (p_{1i}, B_i) such that

$$p_{1i} - B_i + E(min\{p_{2i}, p_{2j} + B_i\}) = E(min\{p_{2j}, p_{2i}\})$$

Note that for any deviation with $B_i > 0$, the right hand side is strictly positive (since firms play with positive probability prices above 0). Thus, it suffices for firm i to choose p_{1i} (for the given deviation B_i) such that the above equality holds. In such a case, there is a profitable deviation.

Robustness to perturbations Assume that in the last period there is a small mass of consumers which are unattached, i.e., they do not have to pay a bonus if they buy from any provider in the second period. This modification introduces some changes in the equilibrium of the game, but they do not eliminate the possibility of equilibria with positive profits. Consider, for example, the last period of the game with symmetric bonuses of $\frac{1}{2}$. The existence of non-committed consumers makes a small deviation from the second period price of 1 profitable for both firms, and the equilibrium will be in mixed strategies. However, the support of such a strategy is concentrated in a small interval $[\frac{1}{2} - \delta, \frac{1}{2}]$, $0 < \delta \ll \frac{1}{2}$, since neither firm finds it convenient to sacrifice much revenue from locked-in consumers in an attempt to capture the small mass of unattached consumers. This implies that profits will be close to the profits in theorem 4.3.

However, the potential entry of a third firm in the second period will lead to zero profit equilibria, as the new firm will enter if there are profits, and it will compete to attract consumers, since it does not have market share to protect.

5 Price Discrimination

Now consider that firms are able to offer different prices to their consumers in the second period[37]. Specifically, firm i imposes a price p_{2i}^I, which we denote *insider price*, to those consumers that purchased from it in the previous period and a price p_{2i}^O, denoted *outsider price*, to those consumers that were captured by firm j in the first period.

Under this framework, firm i's demand function in the second period is [38],

$$D_i(p_{2i}^I, p_{2i}^O, p_{2j}^I, p_{2j}^O) = \begin{cases} 0 & \text{if } p_{2i}^I > p_{2j}^O + B_i \text{ and } p_{2i}^O \geq p_{2j}^I - B_j \\ \mu & \text{if } p_{2i}^I \leq p_{2j}^O + B_i \text{ and } p_{2i}^O \geq p_{2j}^I - B_j \\ 1-\mu & \text{if } p_{2i}^I > p_{2j}^O + B_i \text{ and } p_{2i}^O < p_{2j}^I - B_j \\ 1 & \text{if } p_{2i}^I \leq p_{2j}^O + B_i \text{ and } p_{2i}^O < p_{2j}^I - B_j \end{cases}$$

The nature of this demand function depends on the price difference between the firm's *insider price* and their rival's *outsider price*. In the first case, firm i loses all the customers. In the second case, firm i maintains its customer base. In the third case, firm i loses its customer base but is able to poach its rival's consumers. Finally, in the last case, prices are set so that all the market buys from firm i. This demand function leads to a simple pure strategy equilibrium in the second period.

Lemma 5.1. *If price discrimination is allowed in the second period, then for any pair of bonuses B_i, B_j and any market structure, the continuation equilibrium is given by*

$$p_{2i}^I = B_i, p_{2i}^O = 0, p_{2j}^I = B_j, p_{2j}^O = 0$$

, which gives firms the following payoffs,

$$\pi_{2i} = \mu B_i, \quad \pi_{2j} = (1-\mu)B_j$$

Proof:
Consider deviations from this equilibrium. If firm i decides to raise the price it charges its customer base (p_{2i}^I) then j will capture all of i's customer base. Firm i does not have incentives to reduce p_{2i}^I since that will give it a lower payoff. If i decides to reduce the price it charges to its rival's customers (p_{2i}^O), it will capture all the market but have a negative payoff, and therefore will not do so. On the other hand it does not have incentives to raise p_{2i}^O, since there will be no change in demand and it will still have the same payoff. The logic in analogous for firm j.

■

In this equilibrium, both firms retain their market share and charge the bonus originally given in the first period. With the equilibrium characterized by Lemma 5.1 we can see what the firms' first period payoffs are,

[37] Firms have no incentives to charge differentiated prices in the first period since from their point of view all consumers are completely identical.

[38] We assume that if the consumer is indifferent between both firms, he does not switch.

$$\begin{aligned}
\pi_{1i} &= \mu(p_{1i} - B_i) + \mu B_i = \mu p_{1i} \\
\pi_{1j} &= (1-\mu)(p_{1j} - B_j) + (1-\mu)B_j = (1-\mu)p_{1j}.
\end{aligned}$$

Note that the firms' total payoff does not depend on the bonuses. The intuition is that in the second period firms can only recover the bonus, since the competitor is offering customers that switch a price of zero. Recall that μ depends on the total price that consumers pay, which takes on the following form,

$$p_{1i} - B_i + \min\{p^I_{2i}, p^O_{2j} + B_i\} = p_{1i} - B_i + B_i = p_{1i}$$
$$p_{1j} - B_j + \min\{p^I_{2j}, p^O_{2i} + B_j\} = p_{1j} - B_j + B_j = p_{1j}$$

Therefore if $p_{1i} = p_{1j}$, the market is divided into two equal halves. If the first period prices are different, then the firm with the smallest price will gain the entire market. From this observation we can see that in this context competition in the first period is a classical Bertrand game, where prices necessarily have to be equal to their marginal costs. Therefore, we have the following path for the equilibria of the game,

Theorem 5.2. *The following expressions describe the equilibrium paths when firms can price discriminate in the second period,*

$$B_i, B_j \in [0,1]$$

$$p_{1i} = p_{1j} = 0$$

$$p^I_{2i} = B_i, \quad p^I_{2j} = B_j, \quad p^O_{2i} = p^O_{2j} = 0$$

and the firms' payoffs are

$$\pi_{1i} = \pi_{1j} = 0$$

Theorem 5.2 shows that price discrimination eliminates the possibility of a strictly positive payoff for the firms. The intuition is that firms do not gain anything by giving up market share to their rival since their rival will still price aggressively in the second period with their *outsider price*. This phenomena did not occur in the absence of price discrimination because, as there was only one price, firms preferred to "milk" their captured market rather than price aggressively to attract their rival´s clients.[39]

6 Conclusions

The Bertrand Equilibrium, in which homogenous consumers and goods result in a zero rent equilibrium is a very strong result. In order to avoid this seemingly paradoxical result, the literature has introduced heterogeneity, collusion, asymmetric information or exogenous switching costs.

[39] Note that the possibility of price discrimination leads to the same result in Banerjee and Summers' framework, i.e., the equilibrium when firms can discriminate between customers is the same as the one found in Theorem 5.2. Formally, if firms choose their prices sequentially, the firm that places the final price of the game will poach the first mover's customer base if the price difference between their insider and outsider price is larger than the switching cost, and if the poaching firm's price is positive. Therefore, the first firm to act will set the prices described in Lemma 5.1. The intuition for the rest of the game is the same, leading to the equilibrium described in this section.

In this paper, however, we approach this problem while keeping the basic assumptions that lead to the Bertrand Paradigm: perfect competition, homogeneous goods and firms, and no exogenous switching costs. We show that if firms can offer bonuses to loyal clients, which must be returned if consumers switch firms, they are able to avoid rent dissipation. In particular the standard strategies of zero prices and bonuses do not constitute an equilibrium. In the equilibria we find, firms compete fiercely in the first period and dissipate rents, but are able to charge the monopoly price to their customer base in the second period. The natural equilibrium allows firms to break even in the first period and therefore extract rents equal to the one period monopoly rent overall. There still exist equilibria with lower and even zero rents, which correspond to strategies that involve a negative payoff in the first period, but these are dominated for the firm by the previous equilibrium.

If firms are also allowed to charge different prices to loyal and and new customers, the possibility of rent extraction disappears. This is so because firms can use low prices to tempt the clients belonging to the rival, while keeping a high price for its own customers. The strong competition induced by price discrimination leads to a zero rent equilibrium.

These two results have important policy prescriptions. By themselves, either price discrimination or bonuses can be used to reduce competition. However, If bonuses are allowed, the antitrust agency should let firms engage in price discrimination.

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7 Appendix

7.1 Proof of Proposition 4.1 :

To calculate these expected values we must first characterize the distribution of the minimum between the two equilibrium strategies in the second period.

$$\begin{aligned} G(z) &= \mathbb{P}(\min\{p_{2i}, p_{2j} + B_i\} \leq z) \\ &= \mathbb{P}(p_{2i} \leq z) + \mathbb{P}(p_{2j} \leq z - B_i) - \mathbb{P}(p_{2i} \leq z)\mathbb{P}(p_{2j} \leq z - B_i) \\ &= F_i(z) + F_j(z - B_i) - F_i(z)F_j(z - B_i) \end{aligned}$$

With this we can rewrite the expectation as[40],

$$
\begin{aligned}
\mathbb{E}\Big(min\{p_{2i}, p_{2j}+B_i\}\Big) &= \int_0^1 z dG(z) \\
&= 1-\int_0^1 F_i(z)dz - \int_0^1 F_j(z-B_i)dz + \int_0^1 F_i(z)F_j(z-B_i)dz \qquad (7.1)
\end{aligned}
$$

Single Sided Poaching Equilibrium

For this case the equilibrium strategies in the second period are given by Proposition 3.3[41], therefore from 7.1 we have that,

$$
\begin{aligned}
\int_0^1 F_i(z)dz &= \int_{B_i+\frac{1}{2}}^1 2\Big(1-\frac{V_j}{z-B_i}\Big)dz \\
&= 2\Big\{1-(B_i+\frac{1}{2})-V_j[\ln(1-B_i)-\ln(\frac{1}{2})]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 F_j(z-B_i)dz &= \int_{B_i+\frac{1}{2}}^1 \Big(1-\frac{2V_i-B_i}{z-B_i}\Big)dz \\
&= 1-(B_i+\frac{1}{2})-(2V_i-B_i)[\ln(1-B_i)-\ln(\frac{1}{2})]
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 F_i(z)F_j(z-B_i)dz &= \int_{\frac{1}{2}+B_i}^1 2\Big(1-\frac{V_j}{z-B_i}\Big)\Big(1-\frac{2V_i-B_i}{z-B_i}\Big)dz \\
&= 2\Big\{1-(\frac{1}{2}+B_i)-(2V_i-B_i+V_j)[\ln(1-B_i)-\ln(\frac{1}{2})] \\
&\quad -V_j(2V_i-B_i)\Big[\frac{1}{1-B_i}-2\Big]\Big\}
\end{aligned}
$$

Replacing these integrals in equation 7.1 we have,

$$
\mathbb{E}\Big(min\{p_{2i}, p_{2j}+B_i\}\Big) = \frac{3}{2}+B_i-\frac{1}{2(1-B_i)}-\frac{1}{2}\Big[\ln(1-B_i)-\ln(\frac{1}{2})\Big]
$$

To calculate $\mathbb{E}\Big(\min\{p_{2j}, p_{2i}+B_j\}\Big)$, again we use the strategies characterized in Proposition 3.3[42], therefore we have that,

$$
\begin{aligned}
\int_0^1 F_j(z)dz &= \int_{\frac{1}{2}}^{1-B_i} \Big(1-\frac{2V_i-B_i}{z}\Big)dz + \int_{1-B_i}^1 \Big(1-\frac{2V_i-B_i}{1-B_i}\Big)dz \\
&= (1-B_i)-\frac{1}{2}-(2V_i-B_i)[\ln(1-B_i)-\ln(\frac{1}{2})] \\
&\quad +\Big(1-\frac{2V_i-B_i}{1-B_i}\Big)[1-(1-B_i)]
\end{aligned}
$$

[40] Recall that according to Infante et al. [2007] we know that $\underline{p}_i$ is always positive, which implies that $F_i(0)=0$ and therefore $G(0)=0$.

[41] Remember we are considering the case in which firm j can poach consumers from firm i.

[42] In this case we use equation 7.1 with the sub indexes interchanged.

$$\int_0^1 F_i(z-B_j)dz = 0$$

$$\int_0^1 F_j(z)F_i(z-B_j)dz = 0$$

Replacing these integrals in equation 7.1 we have,

$$\mathbb{E}\Big(min\{p_{2j},p_{2i}+B_j\}\Big) = \frac{1}{2}+\frac{B_i}{2(1-B_i)}+\frac{1}{2}\Big[\ln(1-B_i)-\ln(\frac{1}{2})\Big]$$

Double Sided Poaching Equilibrium: Case I

For this case both firms' strategies are characterized by Lemmas 3.4, therefore we have,

$$\begin{aligned}\int_0^1 F_i(z)dz &= \int_{2V_j-B_j}^{1-B_j}\Big(1-\frac{2V_j-B_j}{z}\Big)dz+\int_{1-B_j}^{2V_i}\Big(1-\frac{2V_j-B_j}{1-B_j}\Big)dz \\ &\quad +\int_{2V_i}^{1}2\Big(1-\frac{V_j}{z-B_i}\Big)dz \\ &= (1-B_j)-(2V_j-B_j)-(2V_j-B_j)[\ln(1-B_j)-\ln(2V_j-B_j)] \\ &\quad +\Big(1-\frac{2V_j-B_j}{1-B_j}\Big)[2V_i-(1-B_j)]+2(1-2V_i)-2V_j[\ln(1-B_i)-\ln(2V_i-B_i)]\end{aligned}$$

$$\begin{aligned}\int_0^1 F_j(z-B_i)dz &= \int_{2V_i}^1\Big(1-\frac{2V_i-B_i}{z-B_i}\Big)dz \\ &= (1-2V_i)-(2V_i-B_i)[\ln(1-B_i)-\ln(2V_i-B_i)]\end{aligned}$$

$$\begin{aligned}\int_0^1 F_i(z)F_j(z-B_i)dz &= \int_{2V_i}^1 2\Big(1-\frac{V_j}{z-B_i}\Big)\Big(1-\frac{2V_i-B_i}{z-B_i}\Big)dz \\ &= 2\Big\{(1-2V_i)-(2V_i-B_i+V_j)[\ln(1-B_i)-\ln(2V_i-B_i)] \\ &\quad -V_j(2V_i-B_i)\Big[\frac{1}{1-B_i}-\frac{1}{2V_i-B_i}\Big]\Big\}\end{aligned}$$

Replacing these integrals in equation 7.1 we have,

$$\begin{aligned}\mathbb{E}\Big(\min\{p_{2i},p_{2j}+B_i\}\Big) &= 2V_j-(2V_i-B_i)\Big[\frac{2V_j}{1-B_i}+\ln(1-B_i)-\ln(2V_i-B_i)\Big] \\ &\quad +(2V_j-B_j)\Big[\frac{2V_i}{1-B_j}+\ln(1-B_j)-\ln(2V_j-B_j)\Big].\end{aligned}$$

To obtain the expression for $\mathbb{E}\Big(\min\{p_{2j},p_{2i}+B_j\}\Big)$ just interchange i for j due symmetry. Therefore,

$$
\begin{aligned}
\mathbb{E}\Big(\min\{p_{2j}, p_{2i}+B_j\}\Big) &= 2V_i-(2V_j-B_j)\Big[\frac{2V_i}{1-B_j}+\ln(1-B_j)-\ln(2V_j-B_j)\Big]\\
&\quad +(2V_i-B_i)\Big[\frac{2V_j}{1-B_i}+\ln(1-B_i)-\ln(2V_i-B_i)\Big]
\end{aligned}
$$

Double Sided Poaching: Case II

For this case the j's strategy is characterized by Lemma 3.4 and i's strategy is characterized by Lemma 3.5, [43]

$$
\begin{aligned}
\int_0^1 F_i(z)dz &= \int_{2V_j-B_j}^{2V_i}\Big(1-\frac{2V_j-B_j}{z}\Big)dz+\int_{2V_i}^1 2\Big(1-\frac{V_j}{z-B_i}dz\Big)dz\\
&= 2V_i-(2V_j-B_j)-(2V_j-B_j)[\ln(2V_i)-\ln(2V_j-B_j)]\\
&\quad +2(1-2V_i)-2V_j[\ln(1-B_i)-\ln(2V_i-B_i)]
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 F_j(z-B_i)dz &= \int_{2V_i}^1\Big(1-\frac{2V_i-B_i}{z-B_i}\Big)dz\\
&= (1-2V_i)-(2V_i-B_i)[\ln(1-B_i)-\ln(2V_i-B_i)]
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 F_i(z)F_j(z-B_i)dz &= \int_{2V_i}^1 2\Big(1-\frac{V_j}{z-B_i}\Big)\Big(1-\frac{2V_i-B_i}{z-B_i}\Big)dz\\
&= 2\Big\{(1-2V_i)-(2V_i-B_i+V_j)[\ln(1-B_i)-\ln(2V_i-B_i)]\\
&\quad -V_j(2V_i-B_i)\Big[\frac{1}{1-B_i}-\frac{1}{2V_i-B_i}\Big]\Big\}
\end{aligned}
$$

Replacing these integrals in equation 7.1 we have,

$$
\begin{aligned}
\mathbb{E}\Big(\min\{p_{2i}, p_{2j}+B_i\}\Big) &= 4V_j-B_j-(2V_i-B_i)\Big[\frac{2V_j}{1-B_i}+\ln(1-B_i)-\ln(2V_i-B_i)\Big]\\
&\quad +(2V_j-B_j)\Big[\ln(2V_i)-\ln(2V_j-B_j)\Big]
\end{aligned}
$$

Now we calculate $\mathbb{E}\Big(\min\{p_{2j}, p_{2i}+B_j\}\Big)$, [44]

[43] Remember we are considering the case in which $B_j < B_i$.
[44] In this case we use equation 7.1 with the sub indexes interchanged.

$$
\begin{aligned}
\int_0^1 F_j(z)dz &= \int_{2V_i-B_i}^{1-B_i}\Big(1-\frac{2V_i-B_i}{z}\Big)dz+\int_{1-B_i}^{2V_j}\Big(1-\frac{2V_i-B_i}{1-B_i}\Big)dz \\
&\quad +\int_{2V_j}^{2V_i+B_j}2\Big(1-\frac{V_i}{z-B_j}\Big)dz+\int_{2V_i+B_j}^{1}1dz \\
&= (1-B_i)-(2V_i-B_i)-(2V_i-B_i)[\ln(1-B_i)-\ln(2V_i-B_i)] \\
&\quad +\Big(1-\frac{2V_i-B_i}{1-B_i}\Big)[2V_j-(1-B_i)]+2(2V_i+B_j-2V_j)-2V_i[\ln(2V_i)-\ln(2V_j-B_j)] \\
&\quad +(1-2V_j-B_j)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 F_i(z-B_j)dz &= \int_{2V_j}^{2V_i+B_j}\Big(1-\frac{2V_j-B_j}{z-B_j}\Big)dz+\int_{2V_i+B_j}^{1}2\Big(1-\frac{V_j}{z-B_i-B_j}\Big)dz \\
&= (2V_i+B_j-2V_j)-(2V_j-B_j)[\ln(2V_i)-\ln(2V_j-B_j)] \\
&\quad +2(1-2V_i-B_j)-2V_j[\ln(1-B_i-B_j)-\ln(2V_i-B_i)]
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 F_j(z)F_i(z-B_j)dz &= \int_{2V_j}^{2V_i+B_j}2\Big(1-\frac{V_i}{z-B_j}\Big)\Big(1-\frac{2V_j-B_j}{z-B_j}\Big)dz \\
&\quad +\int_{2V_i+B_j}^{1}2\Big(1-\frac{V_j}{z-B_i-B_j}\Big)dz \\
&= 2\Big\{(2V_i+B_j-2V_j)-(2V_j-B_j+V_i)[\ln(2V_i)-\ln(2V_j-B_j)] \\
&\quad -V_i(2V_j-B_j)\Big[\frac{1}{2V_i}-\frac{1}{2V_j-B_j}\Big]+1-(2V_i+B_j) \\
&\quad -V_j[\ln(1-B_i-B_j)-\ln(2V_i-B_i)]\Big\}
\end{aligned}
$$

Replacing these integrals in equation 7.1 we have,

$$
\begin{aligned}
\mathbb{E}\Big(\min\{p_{2j},p_{2i}+B_j\}\Big) &= 2V_i-2V_j+B_j-(2V_j-B_j)\Big[\ln(2V_i)-\ln(2V_j-B_j)\Big] \\
&\quad +(2V_i-B_i)\Big[\frac{2V_j}{1-B_i}+\ln(1-B_i)-\ln(2V_i-B_i)\Big]
\end{aligned}
$$

Double Sided Poaching: Case III

For this case the both firms' strategies are characterized by Lemma 3.5, therefore we have,

$$
\begin{aligned}
\int_0^1 F_i(z)dz &= \int_{2V_j-B_j}^{2V_i}\Big(1-\frac{2V_j-B_j}{z}\Big)dz+\int_{2V_i}^{2V_j+B_i}2\Big(1-\frac{V_j}{z-B_i}dz\Big)dz \\
&\quad \int_{2V_j+B_i}^{1}1dz \\
&= 2V_i-(2V_j-B_j)-(2V_j-B_j)[\ln(2V_i)-\ln(2V_j-B_j)] \\
&\quad +2(2V_j+B_i-2V_i)-2V_j[\ln(2V_j)-\ln(2V_i-B_i)]+(1-2V_j-B_i)
\end{aligned}
$$

$$\begin{aligned}\int_0^1 F_j(z-B_i)dz &= \int_{2V_i}^{2V_j+B_i}\Big(1-\frac{2V_i-B_i}{z-B_i}\Big)dz+\int_{2V_j+B_i}^{1}2\Big(1-\frac{V_i}{z-B_i-B_j}\Big)dz\\ &= (2V_j+B_j-2V_i)-(2V_i-B_i)[\ln(2V_j)-\ln(2V_i-B_i)]\\ &\quad +2(1-2V_j-B_i)-2V_i[\ln(1-B_i-B_j)-\ln(2V_j-B_j)]\end{aligned}$$

$$\begin{aligned}\int_0^1 F_i(z)F_j(z-B_i)dz &= \int_{2V_i}^{2V_j+B_i}2\Big(1-\frac{V_j}{z-B_i}\Big)\Big(1-\frac{2V_i-B_i}{z-B_i}\Big)dz\\ &\quad +\int_{2V_j+B_i}^{1}2\Big(1-\frac{V_i}{z-B_i-B_j}\Big)dz\\ &= 2\Big\{(2V_j+B_i-2V_i)-(2V_i-B_i+V_j)[\ln(2V_j)-\ln(2V_i-B_i)]\\ &\quad -V_j(2V_i-B_i)\Big[\frac{1}{2V_j}-\frac{1}{2V_i-B_i}\Big]\\ &\quad +1-(2V_j+B_i)-V_i[\ln(1-B_i-B_j)-\ln(2V_j-B_j)]\Big\}\end{aligned}$$

Replacing these integrals in equation 7.1 we have,

$$\begin{aligned}\mathbb{E}\Big(\min\{p_{2i},p_{2j}+B_i\}\Big) &= B_i-B_j+4V_j-2V_i-(2V_i-B_i)\Big[\ln(2V_j)-\ln(2V_i-B_i)\Big]\\ &\quad +(2V_j-B_j)\Big[\ln(2V_i)-\ln(2V_j-B_j)\Big]\end{aligned}$$

As in Case I, to obtain the expression for $\mathbb{E}\Big(\min\{p_{2j},p_{2i}+B_j\}\Big)$ just interchange i for j due symmetry. Therefore,

$$\begin{aligned}\mathbb{E}\Big(min\{p_{2j},p_{2i}+B_j\}\Big) &= B_j-B_i+4V_i-2V_j-(2V_j-B_j)\Big[\ln(2V_i)-\ln(2V_j-B_j)\Big]\\ &\quad +(2V_i-B_i)\Big[\ln(2V_j)-\ln(2V_i-B_i)\Big]\end{aligned}$$

which completes the proof for Proposition 4.1. ■

7.2 Proof of Theorem 4.2:

In this subsection we shall prove the inequality cited in Theorem 4.2.

Claim 1: $c_i > c_j + d$ for a double sided poaching equilibrium in Case III when $(B_i, B_j) = (0, \epsilon)$.

Using the expression for c_i, c_j, and d we have

$$
\begin{aligned}
c_i - [c_j + d] &= 2(V_j - V_i) - \Big[(2V_i - B_i)[\ln(2V_j) - \ln(2V_i - B_i)] - (2V_j - B_j)[\ln(2V_i) - \ln(2V_j - B_j)]\Big] \\
&= \frac{\sqrt{5}-1}{2}\epsilon - \Big[\ln\Big(\frac{1+\sqrt{5}}{2}\Big)\epsilon - \Big(\frac{\sqrt{5}-1}{2}\Big)\ln\Big(\frac{2}{\sqrt{5}-1}\Big)\epsilon\Big] \\
&= \Big[\frac{\sqrt{5}-1}{2} - \frac{3-\sqrt{5}}{2}\ln\Big(\frac{1+\sqrt{5}}{2}\Big)\Big]\epsilon > 0
\end{aligned}
$$

where the second equality comes from replacing $B_i = 0, B_j = \epsilon, V_i = \frac{\epsilon}{2}$, and $V_j = \frac{(1+\sqrt{5})\epsilon}{4}$.[45] •

7.3 Proof of Theorem 4.3:

In this subsection we shall prove all of the equalities and inequalities cited in Theorem 4.3.

Claim 2: $c_j > c_i - d$ for a single sided poaching equilibrium when $(B_i, B_j) = (\frac{1}{2}, \frac{1}{2} - \epsilon)$.

Using the expression for c_i, c_j, and d we have

$$
\begin{aligned}
c_j - \Big[c_i - d\Big] &= 2V_j - B_j - \Big[2V_i - B_i - \Big(B_i - B_j - \Big[\frac{1}{2} + \frac{1}{2}(\ln(1-B_j) - \ln(\frac{1}{2})) + \frac{B_j}{2(1-B_j)}\Big] \\
&\quad + \Big[\frac{3}{2} + B_j - \frac{1}{2}(\ln(1-B_j) - \ln(\frac{1}{2})) - \frac{1}{2(1-B_j)}\Big]\Big)\Big] \\
&= 2V_j - 2B_j - \Big[2V_i - 2B_i - 1 + \ln(1-B_j) - \ln(\frac{1}{2}) - B_j + \frac{1+B_j}{2(1-B_j)}\Big] \\
&= 1 - \epsilon - 1 + 2\epsilon - \Big[1 - 1 - 1 + \ln(\frac{1}{2} + \epsilon) - \ln(\frac{1}{2}) - \frac{1}{2} + \epsilon + \frac{\frac{3}{2} - \epsilon}{2(\frac{1}{2} + \epsilon)}\Big] \\
&= \frac{4\epsilon}{1 + 2\epsilon} - \ln(1 + 2\epsilon) > 0
\end{aligned}
$$

where the third equality comes from replacing $B_i = \frac{1}{2}, B_j = \frac{1}{2} - \epsilon, V_i = \frac{1}{2}$, and $V_j = \frac{B_j}{2} + \frac{1}{4} = \frac{1-\epsilon}{2}$.[46] The fact that this last expression is positive is due to the well known bound $\ln(x) \leq x - 1$. •

Claim 3: If $(B_i, B_j) = (\frac{1}{2}, \frac{1}{2} - \epsilon)$ and firms are in a double sided poaching equilibrium in Case I then, $V_i = \frac{1}{2} - \frac{\epsilon^2}{2(1+\epsilon)}$ and $V_j = \frac{1-\epsilon}{2}$

We know from Proposition 3.6 that,

$$
V_i = \frac{1}{4}\left(\frac{3B_i + B_j - (B_i + B_j)^2 + \xi(B_i, B_j)}{2 - B_i - B_j}\right)
$$

[45] See 3.6 for details
[46] See 3.3 for details

and

$$V_j = \frac{1}{4}\left(\frac{3B_j + B_i - (B_i + B_j)^2 + \xi(B_i, B_j)}{2 - B_i - B_j}\right)$$

where $\xi(\frac{1}{2}, \frac{1}{2} - \epsilon)$ is,

$$\xi(\frac{1}{2}, \frac{1}{2} - \epsilon) = \frac{5}{4} - (\frac{1}{2} - \epsilon)^2 = 1 + \epsilon - \epsilon^2$$

replacing in the above expressions $B_i = \frac{1}{2}$ and $B_j = \frac{1}{2} - \epsilon$ the result follows. •

Claim 4: $c_i < c_j + d$ for a double sided poaching equilibrium in Case I when $(B_i, B_j) = (\frac{1}{2}, \frac{1}{2} - \epsilon)$.

Using the expression for c_i, c_j, and d we have

$$\begin{aligned} c_i - [c_j + d] &= B_j - B_i - \Bigg[(2V_i - B_i)\Big\{\frac{2V_j}{1 - B_i} + \ln(1 - B_i) - \ln(2V_i - B_i)\Big\} \\ &\quad -(2V_j - B_j)\Big\{\frac{2V_i}{1 - B_j} + \ln(1 - B_j) - \ln(2V_j - B_j)\Big\}\Bigg] \\ &= -\epsilon + \Bigg[\underbrace{\frac{-4\epsilon^4 + 4\epsilon^3 + 2\epsilon^2 - \epsilon}{(1+\epsilon)(1+2\epsilon)}}_{\Lambda_1} \\ &\quad \underbrace{+\frac{-2\epsilon^2 + \epsilon + 1}{2(1+\epsilon)}\ln\Big(\frac{-2\epsilon^2 + \epsilon + 1}{1+\epsilon}\Big) + \frac{1}{2}\ln(1 + 2\epsilon)}_{\Lambda_2}\Bigg] \end{aligned} \tag{7.2}$$

using a well known inequality for $\ln(x) \leq x - 1$ we get the following bound for $\Lambda_1 + \Lambda_2$,

$$\begin{aligned} \Lambda_1 + \Lambda_2 &< \frac{-4\epsilon^4 + 4\epsilon^3 + 2\epsilon^2 - \epsilon}{(1+\epsilon)(1+2\epsilon)} + \frac{-2\epsilon^2 + \epsilon + 1}{2(1+\epsilon)}\Big(\frac{-2\epsilon^2 + \epsilon + 1}{1+\epsilon} - 1\Big) \\ &\quad + \frac{1}{2}(1 + 2\epsilon - 1) \\ &= \frac{3\epsilon^3 - \epsilon}{(1+\epsilon)^2(1+2\epsilon)} + \epsilon \end{aligned}$$

which makes expression 7.2 negative since $3\epsilon^3 - \epsilon$ is negative for all $\epsilon \in (0, \frac{1}{2})$. •

Claim 5: $\pi_{1j} = \frac{1}{2}(2V_i - B_i - d - B_j) + V_j > 0$ for a double sided poaching equilibrium in Case I when $(B_i, B_j) = (\frac{1}{2}, \frac{1}{2} - \epsilon)$.

In this type of equilibrium, when $B_i = \frac{1}{2}$ and $B_j = \frac{1}{2} - \epsilon$, j is better off imposing a price to split the market rather than not participating at all,

$$\begin{aligned}
2V_i + 2V_j - B_i - B_j - d &= 2\Big[2V_j - B_i - (2V_i - B_i)\Big[\frac{2V_j}{1-B_i} + \ln(1-B_i) - \ln(2V_i - B_i)\Big] \\
&\quad + (2V_j - B_j)\Big[\frac{2V_i}{1-B_j}\ln(1-B_j) - \ln(2V_j - B_j)\Big]\Big] \\
&= 1 - \underbrace{\Big[\frac{8\epsilon^4 - 4\epsilon^3 + 2\epsilon^2 + 4\epsilon}{(1+\epsilon)(1+2\epsilon)} + \frac{1+\epsilon-2\epsilon^2}{1+\epsilon}\ln\Big(\frac{1+\epsilon}{-2\epsilon^2+\epsilon+1}\Big) + \ln\Big(\frac{1}{1+2\epsilon}\Big)\Big]}_{=\Lambda}
\end{aligned}$$

where the second equality comes from replacing $B_i = \frac{1}{2}, B_j = \frac{1}{2} - \epsilon, V_i = \frac{1}{2} - \frac{\epsilon^2}{2(1+\epsilon)}$ and $V_j = \frac{1-\epsilon}{2}$[47]. Using a well known inequality for $\ln(x) \le x - 1$ we get the following bound for Λ,

$$\begin{aligned}
\Lambda &< \frac{8\epsilon^4 - 4\epsilon^3 + 2\epsilon^2 + 4\epsilon}{(1+\epsilon)(1+2\epsilon)} + \frac{1+\epsilon-2\epsilon^2}{1+\epsilon}\Big(\frac{1+\epsilon}{1+\epsilon-2\epsilon^2} - 1\Big) + \frac{1}{1+2\epsilon} - 1 \\
&= \frac{4\epsilon^3 - 2\epsilon^2 + 2\epsilon}{1+\epsilon} < 1
\end{aligned}$$

which proves the claim for all $\epsilon \in (0\frac{1}{2}]$. •

Claim 6: If $(B_i, B_j) = (\frac{1}{2}, \frac{1}{2} - \epsilon)$ and firms are in a double sided poaching equilibrium in Case II then, $V_i = \frac{1}{4}\Big(1 - \epsilon + (\epsilon^2 - 2\epsilon + 2)^{\frac{1}{2}}\Big), \quad V_j = \frac{1-\epsilon}{2}$.

We know from Proposition 3.6 that,

$$V_i = \frac{1}{4}\left(3B_i + B_j - 1 + \alpha(B_i, B_j)\right)$$

and

$$V_j = \frac{1}{4B_i}\left(1 + B_j - B_i + (2B_i - 1)\alpha(B_i, B_j)\right)$$

where $\alpha\Big(\frac{1}{2}, \frac{1}{2} - \epsilon\Big)$ is,

$$\alpha\Big(\frac{1}{2}, \frac{1}{2} - \epsilon\Big) = (\epsilon^2 - 2\epsilon + 2)^{\frac{1}{2}}$$

replacing in the above expressions $B_i = \frac{1}{2}$ and $B_j = \frac{1}{2} - \epsilon$ the result follows •

Claim 7: If firms are in a double sided poaching equilibrium in Case II, then $\epsilon \ge \frac{\sqrt{17}-1}{8}$.

We impose that j's maximum price is smaller than 1,

[47]See Claim 3 for details

$$
\begin{array}{rcl}
2V_i + B_j & \leq & 1 \\
\iff \quad (\epsilon^2 - 2\epsilon + 2)^{\frac{1}{2}} & \leq & 3\epsilon \\
\iff \quad -4\epsilon^2 - \epsilon + 1 & \leq & 0
\end{array}
$$

which holds for all $\epsilon \geq -\frac{1}{8} + \frac{\sqrt{17}}{8}$. •

Claim 8: $c_i < c_j + d$ for a double sided poaching equilibrium in Case II when $(B_i, B_j) = (\frac{1}{2}, \frac{1}{2} - \epsilon)$.

Using the expression for c_i, c_j, and d we have

$$
\begin{array}{rcl}
c_i - [c_j + d] & = & 4V_j - 2B_i - \left[\dfrac{4V_j(2V_i - B_i)}{1 - B_i} - 2(2V_j - B_j)[\ln(2V_i) - \ln(2V_j - B_j)] \right. \\
& & \left. +2(2V_i - B_i)[\ln(1 - B_i) - \ln(2V_i - B_i)] \right] \\
& = & \underbrace{1 - 2\epsilon - 2(1-\epsilon)(\tilde{\alpha} - \epsilon) + \ln(1 + \tilde{\alpha} - \epsilon) + (\tilde{\alpha} - \epsilon)\ln(\tilde{\alpha} - \epsilon)}_{\Lambda}
\end{array} \tag{7.3}
$$

where the second inequality comes from replacing $B_i = \frac{1}{2}, B_j = \frac{1}{2} - \epsilon, V_i = \frac{1}{4}\left(1 - \epsilon + (\epsilon^2 - 2\epsilon + 2)^{\frac{1}{2}}\right), V_j = \frac{1-\epsilon}{2}$, and $\tilde{\alpha} = \alpha(\frac{1}{2}, \frac{1}{2} - \epsilon)$. Using a well known inequality for $\ln(x) \leq x - 1$ get the following bound for Λ,

$$
\begin{array}{rcl}
\Lambda & < & 1 - 2\epsilon - 2(1-\epsilon)(\tilde{\alpha} - \epsilon) + (\tilde{\alpha} - \epsilon) + (\tilde{\alpha} - \epsilon)^2 - (\tilde{\alpha} - \epsilon) \\
& = & 1 - 2\tilde{\alpha} - \epsilon^2 + \tilde{\alpha}^2 < 0
\end{array}
$$

therefore we get that equation 7.3 is negative since $\epsilon \geq -\frac{1}{8} + \frac{\sqrt{17}}{8}$. •

Claim 9: $\pi_{1j} = \frac{1}{2}(2V_i - B_i - d - B_j) + V_j > 0$ for a double sided poaching equilibrium in Case II when $(B_i, B_j) = (\frac{1}{2}, \frac{1}{2} - \epsilon)$.

We prove that in this type of equilibrium when $B_i = \frac{1}{2}$ and $B_j = \frac{1}{2} - \epsilon$, j is better off imposing a price to split the market rather than not participating at all,

$$
\begin{array}{rcl}
2V_i + 2V_j - B_i - B_j - d & = & 2\Big[2V_i + 4V_j - B_i - B_j + (2V_j - B_j)[\ln(2V_i) - \ln(2V_j - B_j)] \\
& & +(2V_i - B_i)\Big[\dfrac{2V_j}{1 - B_i} - [\ln(2V_i) - \ln(2V_j - B_j)] \Big] \Big] \\
& = & \underbrace{-4 + 4\epsilon - 4(1-\epsilon)(\tilde{\alpha} - \epsilon) + 2\ln\left(\dfrac{1}{1 + \tilde{\alpha} - \epsilon}\right) + 2(\tilde{\alpha} - \epsilon)\ln\left(\dfrac{1}{\tilde{\alpha} - \epsilon}\right)}_{\Lambda}
\end{array} \tag{7.4}
$$

where the second inequality comes from replacing $B_i = \frac{1}{2}, B_j = \frac{1}{2}-\epsilon, V_i = \frac{1}{4}\Big(1-\epsilon+(\epsilon^2-2\epsilon+2)^{\frac{1}{2}}\Big), \quad V_j = \frac{1-\epsilon}{2}$, and $\tilde{\alpha} = \alpha(\frac{1}{2}, \frac{1}{2}-\epsilon)$. Using a well known inequality for $\ln(x) \leq x-1$ get the following bound for Λ,

$$
\begin{aligned}
\Lambda &< -4+4\epsilon-4(1-\epsilon)(\tilde{\alpha}-\epsilon)+2\Big(\frac{1}{1+\tilde{\alpha}-\epsilon}-1\Big)+2(\tilde{\alpha}-\epsilon)\Big(\frac{1}{\tilde{\alpha}-\epsilon}-1\Big) \\
&= -4+4\epsilon+2(\tilde{\alpha}-\epsilon)-4\epsilon(\tilde{\alpha}-\epsilon)+\frac{2}{1+\tilde{\alpha}-\epsilon} < 0
\end{aligned}
$$

which is negative since $\epsilon \geq -\frac{1}{8}+\frac{\sqrt{17}}{8}$. •

7.4 Proof of Theorem 4.5 :

In this subsection we shall prove the inequality cited in Theorem 4.5.

Claim 10: If $B_i = B_j = B$ and firms enter a single sided poaching equilibrium in the second period, then $c_i < c_j + d$.

Using the expression for c_i, c_j, and d; and considering $B_i = B_j = B$ we have

$$
\begin{aligned}
c_i - [c_j + d] &= 2V_i - B_i - \Big[2V_j - B_j - 1 - B_j + \frac{B_i+1}{2(1-B_i)} + \ln(2(1-B_i))\Big] \\
&= \frac{1}{2} + 2B - \frac{B+1}{2(1-B)} - \ln(2(1-B))
\end{aligned}
$$

Using a well known inequality for $\ln(x) \leq x-1$ we get the following bound,

$$
c_i - [c_j + d] \leq 4B - \frac{1}{2} - \frac{1+B}{2(1-B)}
$$

which is negative for all $B < \frac{1}{2}$. •

Claim 11: $\pi_{i1} = -B_j + \frac{1}{4} + \frac{1}{2}\Big[\frac{B_i+1}{2(1-B_i)} + \ln(2(1-B_i))\Big]$ is increasing in B_i.

$$
\begin{aligned}
\frac{\partial \pi_{1i}}{\partial B_i} &= \frac{1}{2}\Big[\frac{1}{2(1-B_i)} + \frac{2(B_i+1)}{2(1-B_i)^2} - \frac{2}{2(1-B_i)}\Big] \\
&= \frac{3B_i+1}{4(1-B_i)^2} > 0
\end{aligned}
$$

therefore π_{1i} is increasing in B_i. •

Claim 12: $g(B_i, B_j) = c_i - [c_j + d]$ is decreasing in B_i and increasing in B_j.

Using the expression for c_i, c_j, and d we have,

$$
\begin{aligned}
g(B_i, B_j) &= 2V_i - B_i - \left[2V_j - B_j - 1 - B_j + \frac{B_i + 1}{2(1-B_i)} + \ln(2(1-B_i))\right] \\
&= \frac{1}{2} + 2B_j - \frac{1+B_i}{2(1-B_i)} - \ln(2(1-B_i))
\end{aligned}
$$

It is clear to see that $g(B_i, B_j)$ is increasing in B_j. Let us see what occur when we modify B_i,

$$
\begin{aligned}
\frac{\partial g}{\partial B_i} &= -\frac{1}{2(1-B_i)} - \frac{2(B_i+1)}{2(1-B_i)^2} + \frac{2}{2(1-B_i)} \\
&= -\frac{3B_i + 1}{2(1-B_i)^2} < 0
\end{aligned}
$$

therefore g is decreasing in B_i. $\bullet$

A Figures

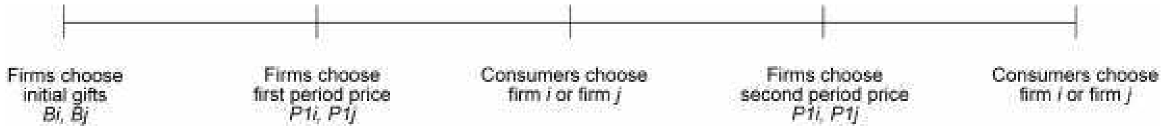


Figure 1: Time Line

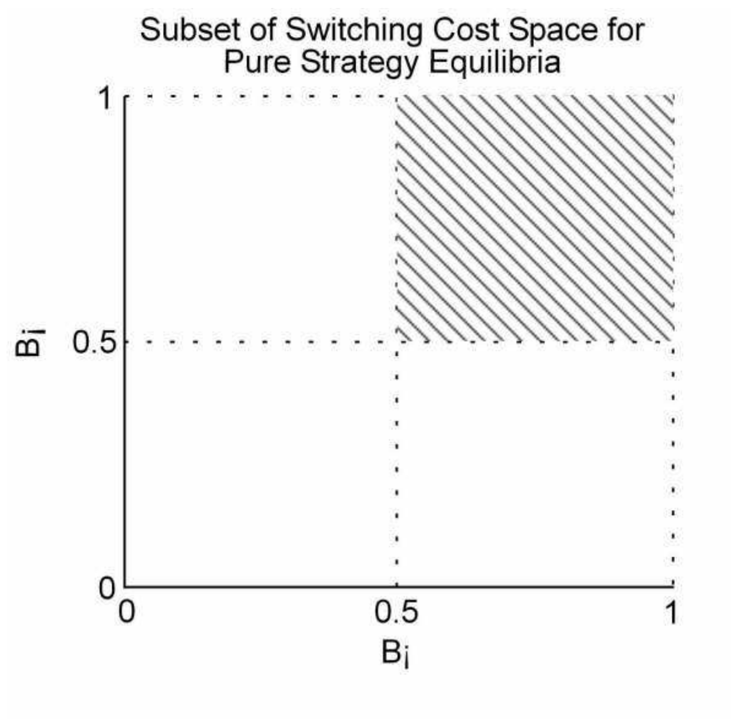


Figure 2: Partition of Gift Space for Pure Strategy Competition

medskip

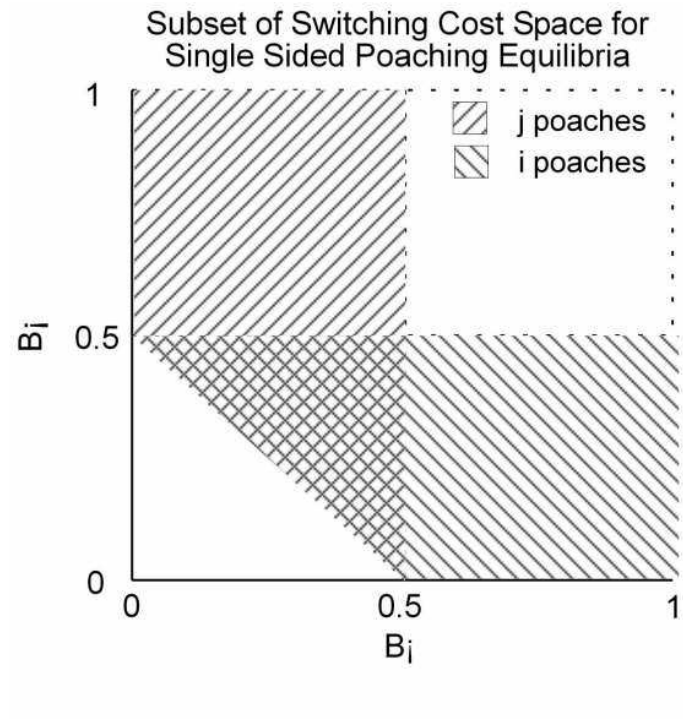


Figure 3: Partition of Gift Space for Single Sided Poaching Competition

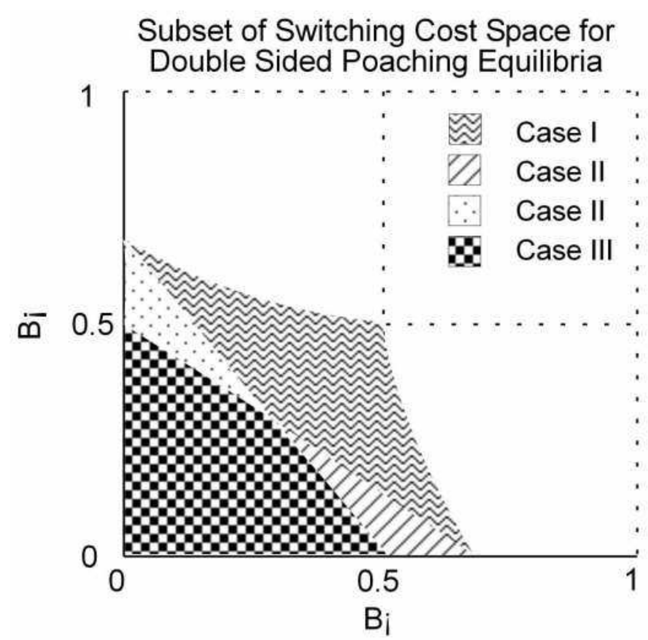
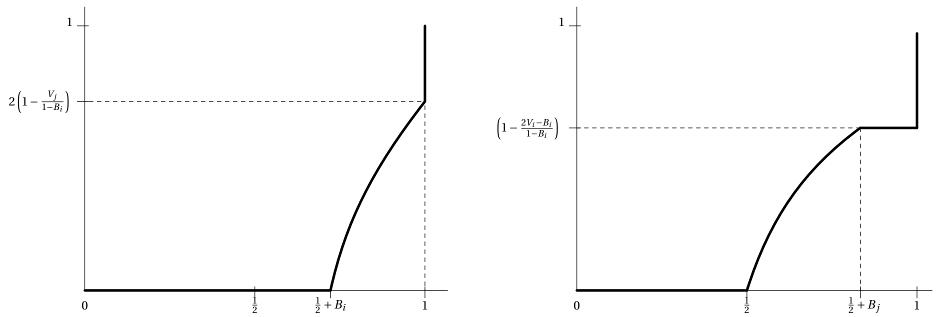


Figure 4: Partition of Gift Space for Double Sided Poaching Competition



(a) Poached firm

(b) Poaching firm

Figure 5: Single sided poaching price distributions

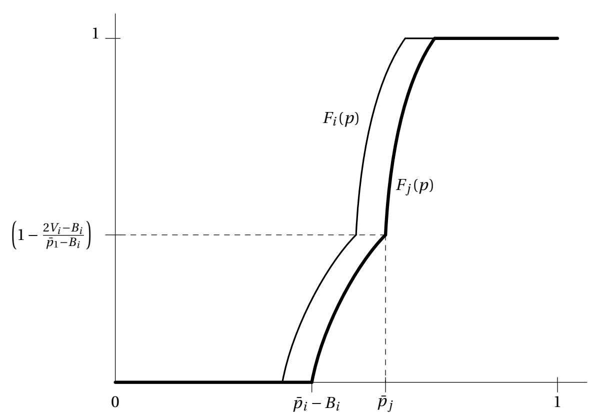


Figure 6: Double sided poaching asymmetric price distributions

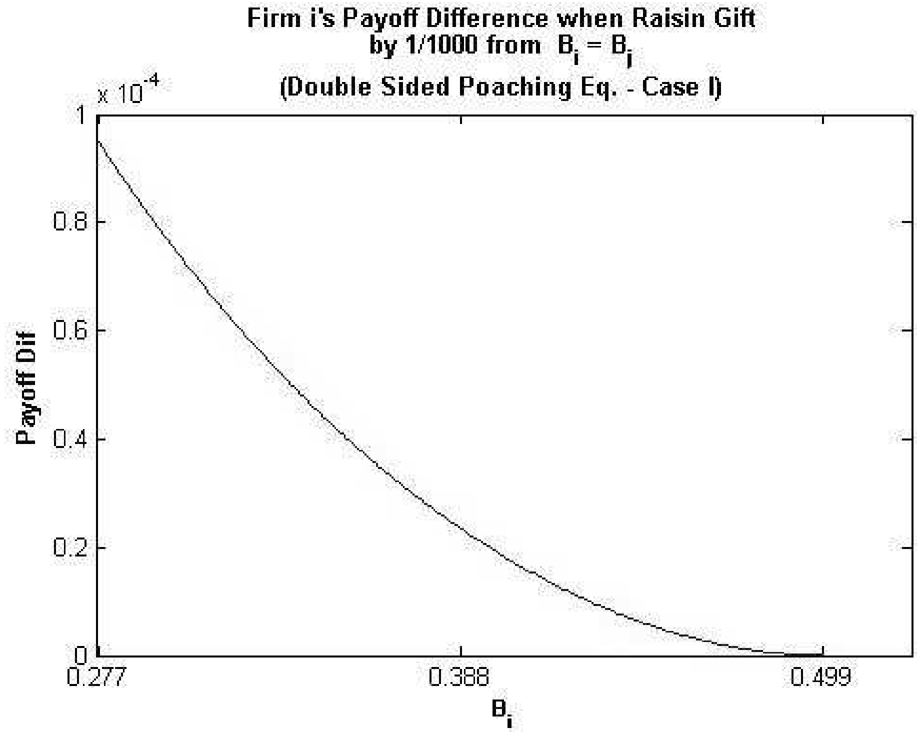


Figure 7: Payoff Difference for Firm i Deviation: Double Sided Poaching Case I

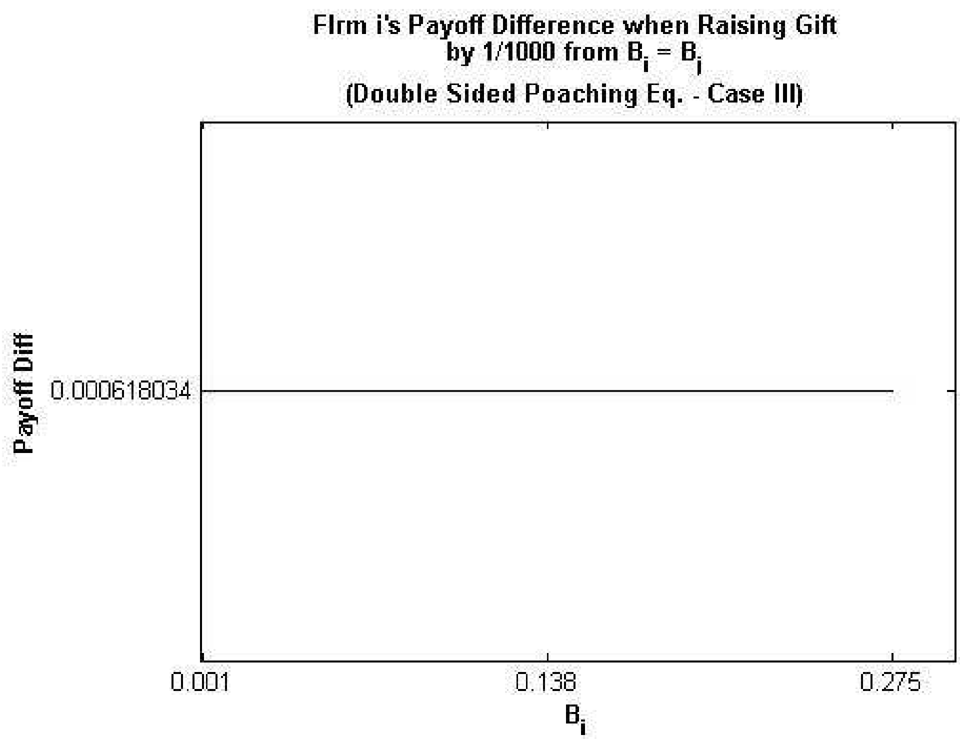


Figure 8: Payoff Difference for Firm i Deviation: Double Sided Poaching Case III