# Minimum makespan scheduling: LP and SDP integrality gaps

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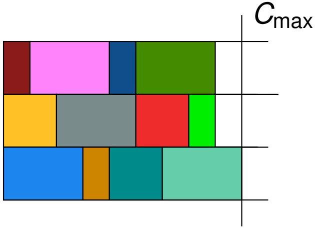
# Scheduling identical machines

- A set $M$ of identical machines,
- a set $J$ of jobs,
- a processing time $p_j$ for each $j \in J$.

Goal: to find an assignment from jobs to machines in order to minimize the maximum load.



- The problem is strongly NP-hard.
- There exists a PTAS, that is, for each $\varepsilon > 0$ there exists an algorithm returning a schedule of cost at most $(1+\varepsilon)\cdot$ opt.

*Is it possible to obtain a polytime $(1+\epsilon)$-approximation algorithm based on known LP/SDP relaxations?*

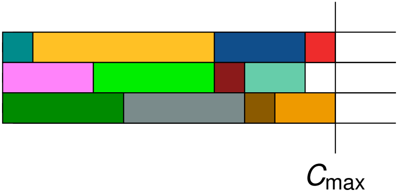
# 0-1 formulation

$x_{ij}$ indicates whether $j$ goes to machine $i$.

$$\sum_{i \in M} x_{ij} = 1 \qquad \text{for each } j \in J,$$

$$x_{ij} \in \{0, 1\} \qquad \text{for each } j \in J, i \in M,$$

Minimize: $$C_{\max} = \max_{i \in M} \underbrace{\sum_{j \in J} p_j x_{ij}}_{\text{load of machine } i}$$



# Assignment LP

$$\begin{aligned} \sum_{i \in M} x_{ij} &= 1 && \text{for each } j, \\ \sum_{j \in J} x_{ij} p_j &\leq T && \text{for each } i, \\ x_{ij} &= 0 && \text{if } p_j > T, \\ x_{ij} &\geq 0 && \text{for each } i \in M, j \in J. \end{aligned}$$

Bad news: for each $\varepsilon > 0$ there exists an instance $I_\varepsilon$ such that

$$\frac{\mathrm{opt}(I_\varepsilon)}{\min\{T : \mathrm{LP}(T, I_\varepsilon) \neq \emptyset\}} \geq 2 - \varepsilon.$$

*[Lenstra, Shmoys & Tardos]*

# An LP based on configurations

Configurations: ways of scheduling a single machine.
$n_p$ : number of jobs with procesing time equal to $p$ .
$m(p, C)$ : multiplicity of $p$ in the multiset $C$.

$$\mathcal{C} = \left\{ C : \sum_p p \cdot m(p, C) \leq T \right\}$$

Configuration LP:

$$\sum_{C \in \mathcal{C}} y_{iC} = 1 \quad \text{for every } i \in M,$$

$$\sum_{i \in M} \sum_{C \in \mathcal{C}} m(p, C) y_{iC} = n_p \quad \text{for every } p \in \{p_j : j \in J\},$$

$$y_{iC} \geq 0 \quad \text{for every } \; i \in M, C \in \mathcal{C}.$$

**Theorem**

*For each $n \in \mathbb{N}$ there exists an instance with $n$ jobs and $O(n)$ machines such that the configuration LP has an integrality gap of at least $1 + 1/1023$.*

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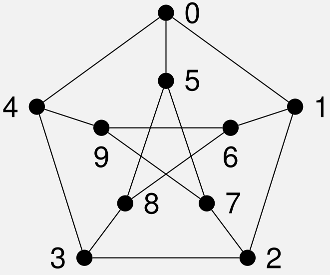
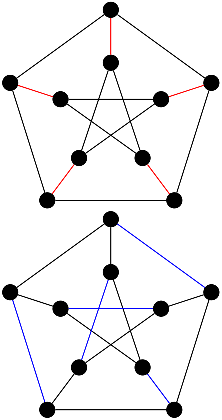



*The Petersen Graph*

Key fact: It admits a fractional 1-factorization, but *not an integral one*, i.e., there is no 0-1 vector $\alpha$ st

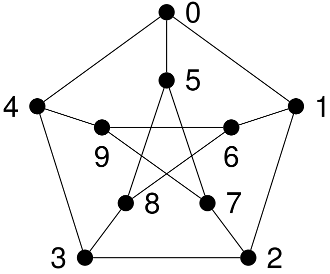
$$\sum_{\substack{M \in \mathrm{PM}(G):\\ e \in M}} \alpha_M = 1, \text{ for all } e \in E(G).$$

# Perfect Matchings (PM) of Petersen: $M_1, \ldots, M_6$



Each edge appears **in exactly two** of the six matchings!
$\alpha_M = 1/2$ for each perfect matching $M$.

# Instance $I_k$




- Let $k$ be odd.
- For each edge $\{u, v\}$, we have $k$ copies of a job with processing time $2^u + 2^v$. That is a total of $15k$ jobs.
- There are $3k$ machines.

# First step: CLP is feasible for $T = 1023$

For each perfect matching $M_\ell$ in Petersen, construct a configuration $C_\ell$ having one copy of a job $j_e$ for each $e \in M_\ell$.

$$\text{load}(C_\ell) = \sum_{j \in [9]} 2^j = 1023, \text{ for all } \ell \in \{1, 2, \ldots, 6\}.$$

machine $i$

The $k$ copies for each size are scheduled,

$$3k \cdot 2 \cdot 1/6 = k.$$

Fractional solution:

$$y_{iC_\ell} = 1/6,$$

for each
$i \in \{1, \cdots, 3k\}$ and
$\ell \in \{1, \cdots, 6\}$.

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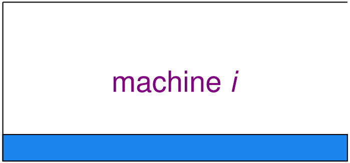


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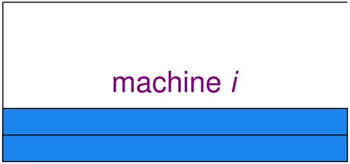


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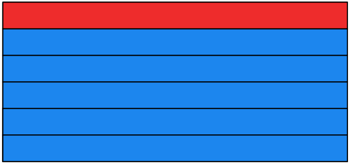

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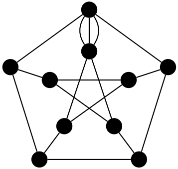
# Second step: optimal makespan is at least 1024

$$\text{Total load of the 15 job types } = \frac{1023 \cdot 6}{2} = 3 \cdot 1023,$$

$$\text{Total load in } I_k = 3k \cdot 1023.$$

Any integral solution of makespan equal to 1023 induces a 1-factorization of the Petersen multigraph ($k$ copies of each edge) ... contradiction! ($k$ odd is used here)



$$\frac{\mathrm{opt}(I_k)}{\min\{T : \mathrm{clp}(T, I_k) \neq \emptyset\}} \geq \frac{1024}{1023}.$$

**How to strengthen? ... let's try to Lift & Project.**

# LP/SDP hierarchies

- Systematic way for strengthening relaxation $P \subseteq [0,1]^n$.
- Determines a sequence of relaxations satisfying

$$P \supseteq P_1 \supseteq P_2 \supseteq \cdots \supseteq P_n = \mathrm{conv}(P \cap \{0,1\}^n).$$

- It is possible to optimize over $P_t$ in time $n^{O(t)}$.

| | |
|---|---|
| Sherali & Adams '90 (SA) | LP |
| Lovász & Schrijver '91 (LS/LS$_+$) | LP/SDP |
| Sum-of-Squares '00 (Parrillo, Lasserre) | SDP |

# Some related results

Lower bounds: Min-sum tardy jobs: unbounded gap after $O(\sqrt{n})$ rounds of SoS. (Kurpisz, Leppänen, Mastrolilli, 2015)
Upper bounds:

- 1st round of $LS_+$ yields a $(3/2 - c)$-apx for minimizing the weighted sum of completion times in unrelated machines. (Bansal, Srinivasan, Svensson, 2015)
- For a fixed number of machines, the $r = (\log n)^{\Theta(\log \log n)}$ round of SA gives a $(1 + \varepsilon)$-apx for scheduling parallel machines under precedence constraints and unit size jobs to minimize makespan. (Levey, Rothvoss, 2015)

# Sherali & Adams hierarchy

Let $y$ be a 0-1 solution. If $H, L \subseteq M \times \mathcal{C}$,

$$\sum_{C \in \mathcal{C}} y_{iC} \prod_{q \in H} y_q \prod_{q \in L} (1 - y_q) = \prod_{q \in H} y_q \prod_{q \in L} (1 - y_q),$$

$$\sum_i \sum_{C \in \mathcal{C}} m(p, C) y_{iC} \prod_{q \in H} y_q \prod_{q \in L} (1 - y_q) = n_p \prod_{q \in H} y_q \prod_{q \in L} (1 - y_q),$$

are valid. They can be linearized using Inclusion-Exclusion,

$$\prod_{q \in H} y_q \sim y_H.$$

# Sherali & Adams hierarchy

At level $r$ there is one variable for each subset $H \in M \times \mathcal{C}$ with cardinality at most $r+1$.

SA$^r$:

$$\begin{aligned}
\sum_{C \in \mathcal{C}} y_{H \cup \{(i,C)\}} &= y_H && \text{for every } i,\ |H| \leq r,\\
\sum_i \sum_{C \in \mathcal{C}} m(p,C) y_{H \cup \{(i,C)\}} &= n_p y_H && \text{for every } p,\ |H| \leq r,\\
y &\geq 0,\\
y_\emptyset &= 1.
\end{aligned}$$

## Theorem

*After applying $r = \Omega(n)$ rounds of the SA hierarchy to the configuration LP, the obtained relaxation has an integrality gap of at least $1 + 1/1023$.*

*Proof idea.* Consider $r = 1$ and $H = \{(1, C_1)\}$.

$$n_p = \sum_{i \neq 1} \sum_{C \in \mathcal{C}} m(p, C) \frac{y_{\{(1,C_1),(i,C)\}}}{y_{\{(1,C_1)\}}} + \sum_{C \in \mathcal{C}} m(p, C) \frac{y_{\{(1,C_1),(1,C)\}}}{y_{\{(1,C_1)\}}}.$$

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**Lemma:** If $C \neq C_1$ then $y_{\{(1,C_1),(1,C)\}} = 0$.

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Define $z^H_{(i,C)} = \frac{y_{\{(1,C_1),(i,C)\}}}{y_{\{(1,C_1)\}}}$. Then $z^H_{\{(1,C_1)\}} = 1$ and

$$\sum_{i \neq 1} \sum_{C \in \mathcal{C}} m(p, C) z^H_{\{(i,C)\}} = n_p - m(p, C_1).$$

That is, after scheduling $i$ in configuration $C_1$, the vector $z^H$ is a fractional solution for the reduced instance. For example,

$$y_{\{(1,C_1),(2,C_2)\}} = \frac{1}{6} \cdot \frac{1}{6} \quad \text{and} \quad y_{\{(1,C_1),(2,C_1)\}} = \frac{1}{6} \cdot \frac{k/2 - 1}{3k - 1}.$$

# SDP relaxations

The same holds for the SDP hierarchy $LS_+$.

**Theorem**

*After applying $r = \Omega(n)$ rounds of the $LS_+$ hierarchy to the configuration LP the obtained relaxation has an integrality gap of at least $1 + 1/1023$.*

# Conclusions/Open problems

- SA and $LS_+$ fail to to schedule identical machines within a factor of $1+\varepsilon$.
- What about unrelated machines? machine-dependent processing times $p_{ij}$. Best apx factor is 2.

Gracias!