Consistency of Intertemporal Decisions: Approaches through Robust and Stochastic Optimization.

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The Problem

- Optimization has been used for long to solve many relevant problems.
- But many times we face uncertainty, or data is not known exactly.
- We also face changing conditions through time.

- What happens when we use optimization models for decision making and things change with time?

The Problem

- We consider questions related to decision making in different time horizons
- One of the most typical examples is production planning decisions in different stages:
- Strategic, Tactical, Operational.

- Tactical decisions
  - Monthly planning decisions (for instance)
  - Aggregate production data and demand
  - Aggregate production decisions and processes
  - Aggregate decisions on resources and raw materials

- Operational decisions
  - Weekly (or daily) decisions for the first month (say)
  - Detailed use of resources
  - Detailed production plan

The Planning Process

- We say the plans are consistent if feasible operational decisions can be generated, considering the constraints imposed by the tactical decisions.
- But inconsistencies may appear due to:
  - Different degrees of aggregation
  - Uncertainty and variations which are not captured in the tactical plan.
- The plan: to study factors affecting the consistency of decisions and procedures to cope with this.

Outline

- A general Setting of Intertemporal Decisions
- A framework to control inconsistencies
- A practical problem: intertemporal decisions in forest management.
- A Robus Optimization and a 2-stage stochastic approach for the problem.
- Measures of sensitivity and robustness and its connections to the question

- Collaborators: Alfonso Lobos (M.Sc. student), Pamela Alvarez (Ph.D. student, UAB), Ana Batista (Ph.D. student).

A General Setting

- There is a tactical planning problem in T periods (months)
- At period t, production, resources and logistic decisions are made.
- Variables: (x_t, y_t), x: production, y: resources
- Data parameters: ω_t.
- Cost functions: C_t .
- **The tactical problem:**

$$
TP) \quad \begin{array}{ll} \min & \sum_{t=1}^{T} C_t(\omega_t, x_t, y_t) \\ s.t. & G_t(\omega_t, x_t, y_t) \leq b_t \quad t = 1, ..., T \\ & H(\omega, x, y) = 0 \end{array}
$$

A General Setting

- At the operational level, we see subperiods (weeks within the month, for instance):
- $J(t)$: the set of subperiods in period t.
- Operational decisions: (x_{tk}^o, y_{tk}^o), $k \in J(t)$
- Operational parameters: $\bar{\omega}_{tk}$, $k \in J(t)$.
- The operational problem is affected by the tactical planning:

$$
OP) \quad \begin{array}{ll} \min & \displaystyle\sum_{k \in J(t)} \bar{C}_{tk}(\bar{\omega}_{tk}, x_{tk}^o, y_{tk}^o, x_t, y_t) \\ s.t. & \bar{G}_{tk}(\bar{\omega}_{tk}, x_{tk}^o, y_{tk}^o, x_t, y_t) \leq \bar{b}_{tk} \quad k \in J(t) \\ & \bar{H}_t(\bar{\omega}_t, x_{tk}^o, y_{tk}^o, x_t, y_t) = 0 \end{array}
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- Can we guarantee to obtain a reasonable operational plan?

- Old question, in fact, consistency in hierarchical planning was studied initially by Bitran, Hax and Hass[1980] and several others later

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- There is, in some cases, a way to guarantee consistency: There must be a relation between the operational parameters, $\bar{\omega}_{tk}$, and the tactical ones, ω_t.

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- At operational level we decide production of all three products and we "know" demand for the three.
- But at the tactical level, an aggregated planning is done, for total liters of milk to be processed over a longer horizon.
- The tactical planning define aggregated production capacity.
- However, use of production capacity depends on the detailed product, so an inconsistency might be generated between aggregated capacity and actual detailed capacity requirements.

A General Setting

- The solution:
- The aggregated productivity used as parameter in the tactical planning has to be computed as a weighted average of the detailed one, and the weights have to be the relative demand for the three products.
- Bitran, Hax and Hass[1980] prove a theorem about this.
- It is also related to results on aggregation in Linear Programming by Zipkin[1980].

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- Nice result, but it requires exact knowledge of detailed future demand.
- How to get that information?

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- A couple of alternatives:

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- Of course, in practice we could get estimates, but they will have error...

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which measures the "response" of the operational problem.

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- Idea: Do the plan incorporating an estimated feedback of the inconsistencies...
- Suppose we were able to "compute" a value

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  which measures the "response" of the operational problem.
- For instance:
  - $s(\cdot)$ could be cost of not fulfilling operational requirements.
  - $s(\cdot)$ could be the probability of not fulfilling operational requirements.
  - $s(\cdot)$ could be a measure of stability of the operational problem: larger s means a less stable problem.

A General Setting

- Then, we could state the following problems:

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\begin{array}{llll}
 & \min & \sum_{t=1}^{T} C_t(\omega_t, x_t, y_t) & \\
TP_{R1}) & s.t. & G_t(\omega_t, x_t, y_t) \leq b_t & t = 1, ..., T \\
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- These problems tries to compute tactical decisions in such a way that their impact on the operational problem is controlled.

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- To continue the explanation we introduce the specific test problem we have been using.

Sawmill Planning Problem

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- The sawmill produces lumber (boards) in different sizes using a variety of cutting patterns.
- **Tactical Planning:** assign aggregated load to the sawmill and determine monthly log orders.
- **Operational planning:** used weekly (even daily) for production scheduling.
- When the sawmill receives actual shipping of logs, the operational model is executed.
- But what you get is not always what you asked for...
- We "present" now the (simplified) models, which are based on Weintraub and Epstein[2002] and others.

Tactical Model: a compact version

- Variables:
  - X_t: hours of labor for month t.
  - raw_{ct}: logs of type c to be ordered for month t.
  - r_{ct}: logs of type c processed in month t.
  - z_{mt}: inventory of lumber m in the month t.
  - w_{ct}: inventory of the logs c in the month t.

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- Parameters:
  - W_t: cost of labor in month t.
  - ϕ: productivity of labor.
  - $Craw_{ct}$: cost of log type c bought in period t.
  - UX, LX: upper and lower bound on labor.
  - ME_{ct}: upper bound in the amount of logs type c the company can buy in period t.
  - h_{mt}: the storage cost of product m in the month t.
  - hw_{ct}: storage cost for log c in the month t.
  - Y_{cm}: average amount of lumber of type m obtained from a log type c.
  - d_{mt}: demand for product m the company has in the month t.

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- The model cover 4 months.
- C, set of logs types, M: set of products.

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- The model cover 4 months.
- C, set of logs types, M: set of products.
- The model seeks to determine raw material and labor need so that cost is minimized

$$
\begin{array}{ll}
\min & \sum_{t=1}^{4} \left(\sum_{m \in M} (h_{mt} z_{mt}) + \sum_{c \in C} (Craw_{ct} raw_{ct} + h_{ct} w_{ct}) + W_t X_t \right) \\
s.t. & LX \leq X_t \leq UX \quad \forall t \in 1, ..., 4 \\
& raw_{ct} \leq ME_{ct} \quad \forall c \in C, t \in 1, ..., 4 \\
& w_{ct} = w_{c,t-1} + raw_{ct} - r_{ct} \quad \forall c \in C, t = 2, .., 4 \\
& z_{mt} = z_{m,t-1} + \sum_{c \in C} Y_{cm} r_{ct} - d_{mt} \quad \forall m \in M, t = 2, ..., 4 \\
& \sum_{c \in C} r_{ct} \leq \phi X_t \quad \forall t = 1, ..., 4 \\
& z_{mt} \geq 0, raw_{mt} \geq 0, w_{ct} \geq 0, X_t \geq 0
\end{array}
$$

The operational Model: a compact version

- Operational variables
  - r'_{eci}: logs type c processed with cutting pattern e in week i.
  - ex'_i: overtime at week i.
  - z'_{mi}: inventory of product m in week i.
  - w'_{ci}: inventory of logs c in week i.
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- Operational parameters for the first four weeks:
  - α_m: percentage of acceptable shortage for product m.
  - β'_{mi}: backlog cost for product m in week i.
  - h'_{mi}: storage cost of product m in week i.
  - h'_{ci}: storage cost for log c in week i.
  - EW_i: overtime cost in week i.
  - RR'_{ci}: ctual logs of type c received in week i.
  - Y_{ecm}: yield of lumber m from logs c using cutting pattern e.
  - d_{mi}: demand for product m in i.

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- The model covers four weeks of the first month.
- Here E is the set of detailed cutting patterns.
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- There can be backlog from one week to the following one.
- The model seeks to determine detailed operations, given resources and raw material assigned for the first month in the tactical model.

$$
\begin{array}{ll}
\min & \sum_{i=1}^{4}\left(EW_i' ex_i' + \sum_{m\in M} \beta'_{mi} b_{mi} \right) \\
s.t. & w'_{ci} = w'_{c,i-1} + RR'_{ci} - \sum_{e\in E_c} (r'_{eci}) \ \ \forall c \in C, i \in 1,...,4 \\
 & z'_{mi} = z'_{m,0} + \sum_{c\in C}\sum_{e\in E_c} Y_{ecm} r'_{eci} + b'_{mi} - b'_{m,i-1} - d_{mi} \ \ \forall m \in M, \forall i \\
 & \sum_{c\in C}\sum_{e\in E_c} r'_{eci} \le \phi \frac{X_1}{4} \ \ \forall i \\
 & b'_{mi} \le \alpha_m d_{mi} \ \ \forall m \in M, i \in 1,..,4 \\
 & ex_i' \ge 0, b_{mt} \ge 0.
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- Log supply estimated at tactical level is received equally distributed in the four weeks, and demand happens in the same way.

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- But in reality, there can be many changes and variations.

- The represented situation:
- Those proportional weekly quantities are randomly perturbed:

$$RR'_{ci} = \left(\frac{raw_{c1}}{4}\right) + \xi \ , \ \forall c \in C, i \in 1, ..., 4,$$

- where ξ is a random perturbation.

A Robust Optimization Approach

- The first idea was to use Robust Optimization.
- The aggregated yield coefficient Y_{cm} can be used to represent all the "noise" from the aggregation and variation at the operational.
- Hence, robust decisions are made at the tactical level and transferred to the operational first month.
- The performance in the operational model is evaluated using Monte-Carlo simulation.

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- Hence, robust decisions are made at the tactical level and transferred to the operational first month.
- The performance in the operational model is evaluated using Monte-Carlo simulation.

- What are robust solutions?

Robust solutions

- Consider the following Linear Optimization problem

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \ , \end{array}$$

- $\alpha_1, \ldots, \alpha_m$ are the rows of A.

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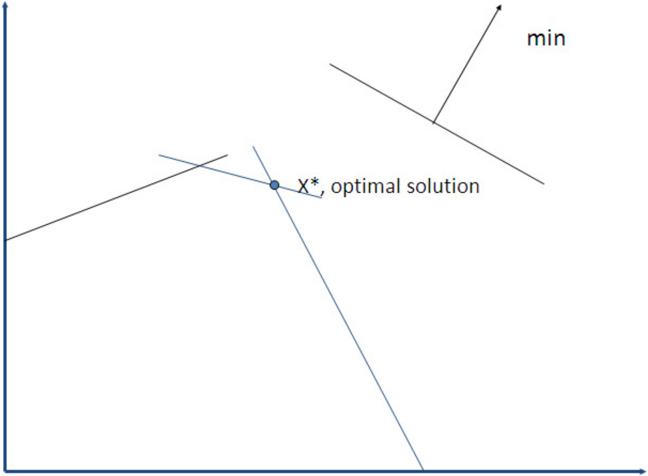
- Γ is the degree of uncertainty (or noise) allowed.
- We look for a solution feasible for all cases of A, that is, a robust solution.

Robust solutions

- Graphically:



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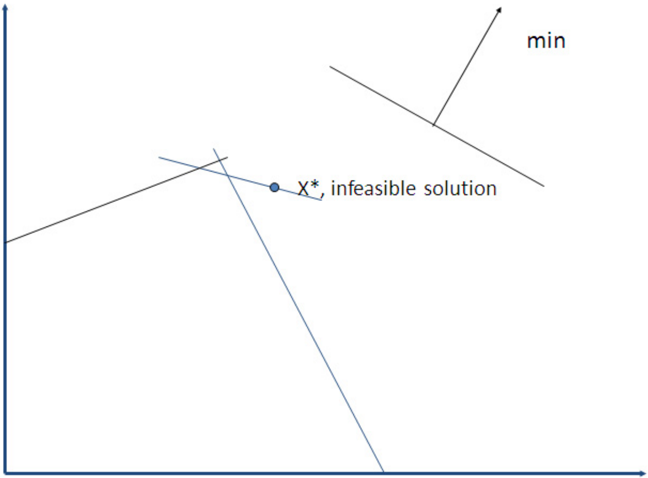

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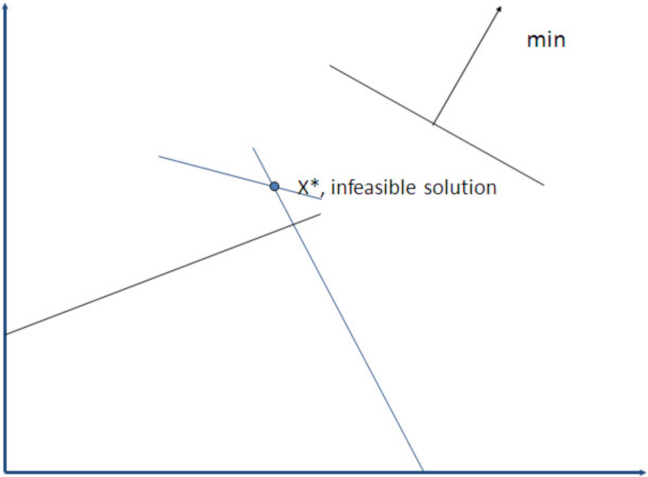

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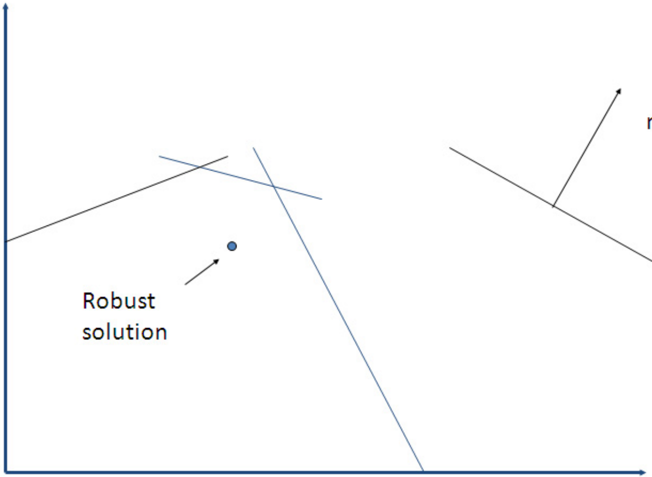

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- The robust problem is:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & \bar{\alpha}_i^T x + \beta_i(x, \Gamma) \leq b_i \quad i = 1, \ldots, m \\ & x \geq 0 \ , \end{array}$$

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- The robust problem is:

$$\min_x \{c^T x : (\bar{A} + U)x \le b, x \ge 0, \text{ para todo } U \in D(\Gamma)\} .$$

- Typically, we assume variability independently by constraint and define sets $\mathcal{D}_i(\Gamma)$ for each constraint.
- The robust problem is:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & \bar{\alpha}_i^T x + \beta_i(x, \Gamma) \le b_i \quad i = 1, \ldots, m \\ & x \ge 0 \ , \end{array}$$

- where, for each $i = 1, \ldots, m$,

$$\begin{array}{rcll} \beta_i(x, \Gamma) & = & \max & x^T u_i \\ & & \text{s.t.} & u_i \in \mathcal{D}_i(\Gamma) \ . \end{array}$$

are the "protection functions" of the constraints.

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- To handle the problem, we need to specify the sets $D_i(\Gamma)$, specify the protection functions and obtain the robust counterpart.

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- The robust counterpart, in this case, takes the worst case scenario,
- This could be too conservative...
- There are other ways to vary the coefficients avoiding the worst case.
- Initial developments: Ben-Tal and Nemirovski[1998-2002], ellipsoidal uncertainty:

$$\sum_{j=1}^{n} \left(\frac{a_{ij} - \bar{a}_{ij}}{s_{ij}} \right)^2 \leq \Gamma$$

Different Approaches: The "Uncertainty Budget"

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- For intermediate values of Γ there is an “uncertainty budget” to distribute among all coefficients as the simultaneous variation is bounded.
- With these definitions we can build the robust counterpart, which is a linear program.
- And, if there are integer variables, the robust counterpart is a mixed integer problem.

Results: Base case

- Simulation of 1,000 scenarios with different value of perturbation ξ.
- ξ is sampled from a uniform distribution, with mean $raw_{c1}/4$ and perturbed in a certain %.
- We registered when the operational problem unmet demand and when it was completely infeasible (due to processing capacity).

% perturbation	% infeasible scenarios	% of feasible scenarios with unmet demand
20	2.60	0.00
25	7.00	0.97
30	14.3	6.67
35	20.8	17.7
40	27.8	24.4

Results: Base case

- We now use Bertsimas and Sim robust optimization formulation for the tactical problem.
- We show the case with a variation of 30% in the log supply.
- We assume that the forest perturb with the same pattern as in the corresponding base case.
- Here are the results for some values of Γ, the uncertainty budget in the B&S formulation, for interval widths of 15%

Value of Γ	% infeasible scenarios	% of feasible scenarios with unmet demand
1	12.60	3.43
3	6.80	0.00
6	1.70	0.00

Conclusions so far...

- A robust tactical plan increases the chances of getting feasible production at the operational level, with an increase in cost.
- From extensive simulations we could infer the right value for Γ for an acceptable feasibility level.

- However, the approach still handles both problems separated.

A 2-Stage Stochastic Approach

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  where the function G represents all the tactical constraints.
- $Q((raw, X), \xi)$ is the optimal value of the second stage, as a function of the first stage decisions (raw, X) and a random perturbation ξ.
- Hence, the problem seeks to optimize tactical decisions in such a way that the cost generated to the second stage is also taken into account.

The second stage problem

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- The random element enters in the perturbation of the disaggregated raw material supply.

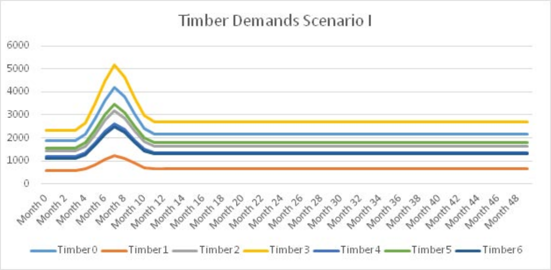
$$Q((raw, X), \xi) = \begin{array}{ll} \min & f_O(ex', b) \\ s.t. & \\ & H_1(ex', b) + H_2(raw, X, ex', b) = h(\xi), \end{array}$$

where the functions H_1 and H_2 represent all the operational constraints, with H_2 depending on the first stage variables.

- $h(\xi)$ represents the right-hand side as a function of the random perturbation.

A Rolling Horizon Framework

- We wanted to test the models in a rolling horizon framework
- We considered 48 months and the planning horizon moves sequentially.
- We defined certain scenarios of timber demand:



- The results we present later are for this demand pattern.

Alternative models for the 2 stages

- We considered different combinations of the tactical and operational models, which represent different views of the hierarchical decisions.

MODEL	First Stage	Second Stage
FMA	Tactical, aggregated yield	Operational month 1
FMD	Tactical, disaggregated yield	Operational month 1
SMA	Simple tactical	Operational month 1 tactical inventory ag. yield month 2-4
SMD	Simple tactical	Operational month 1 tactical inventory disag. yield

- The disaggregated models represent a situation in which more detailed information is used on the planning.
- The second model represents the situation in which the second stage considers inventory decisions as variables that adjust to uncertainty over the whole horizon.

Solving the Problems

- We have a 2-stage Stochastic Linear Problem which we can solve in different ways.

- We used a simplified version of a real industrial problem: only a few products and cutting patterns.
- We solved the problems using an SAA (Stochastic Average Approximation) approach with 96 scenarios.

- The computations were performed in a Dell cluster with Intel E5-2470 processors with a total of 168 cores.
- The programming was developed in Python, using Gurobi for optimization.

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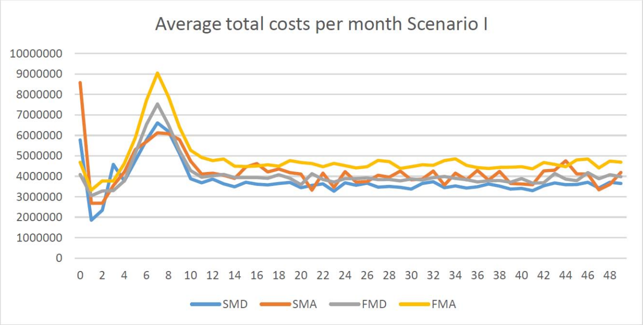
- We make the operational always feasible: if there are not enough logs, we assume we purchase them on spot, with the corresponding extra cost (50% higher than the original).

Some results for demand scenario I

- **Total Operational cost for the first month k for all the horizon** (in monetary units)



- SMD performs better and FMA performs worst, possibly indicating that the simultaneous consideration of more information, and a full horizon in the second stage is beneficial.
- (Note: uncertainty only affects the first month).

Results

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	SMD	SMA	FMD	FMA
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- We see that the disaggregated model can anticipate better.
- The second model, with the inventory in the second stage, seems to perform well only under disaggregated information.
- The basic 2 stages planning model (FMA) performs well anyway.

Consistency with the other measures "s"

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  - $s(\cdot)$ as a measure of stability of the operational problem: larger s means a less stable problem.
- Stability is important as a more stable operational problem will remain feasible even of some data (from the tactical) change.

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- We consider the following Linear Problem as an illustration:

$$P) \quad \begin{array}{rl} z^* = \max & c^T x \\ \text{s.t.} & Ax \le b \ , \end{array}$$

Condition and Ill-posedness Measures

- Suppose the data is $d = (A, b)$.
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- Condition number is $C(d) = \dfrac{\|d\|}{\rho(d)}$
- The notion was studied, for Optimization, by Renegar[1995] and further results by V[1996], Freund and Vera[1999,2003], Ordoñez and Freund[2003], and several others.
- The condition number explains, among others, sensitivity properties of the problem.

Conditioning and Problem Sensitivity

- For the optimization problem:

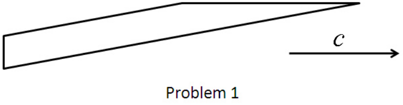
$$P) \quad \begin{array}{rl} z^* = \max & c^T x \\ \text{s.t.} & Ax \leq b \ , \end{array}$$

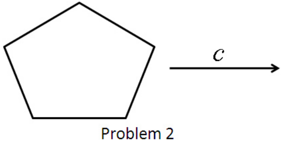
- If we perturb (A, b) to $(A + \Delta A, b + \Delta b)$ and z' is the new optimal value, then

$$|z' - z^*| \leq C(d)^2 \|(\Delta A, \Delta b)\|$$

Sensitivity of solutions: The impact of geometry

- These are two polyhedrons and objective functions:





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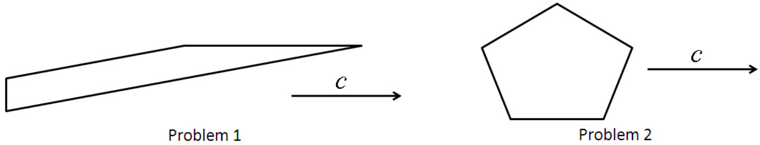

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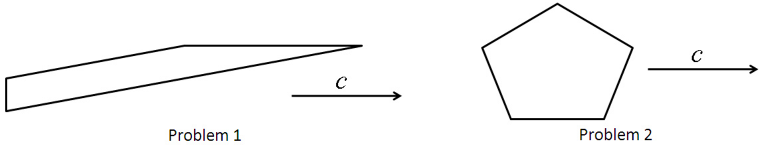

- problem 1 can be much more sensible (less robust) than problem 2 (for the current objective function)
- Intuition: geometric shape has something to say about sensitivity.
- Can we describe the way in which the optimal value of the problem changes with changes in the data, using geometric measures?

Geometric measures of problems

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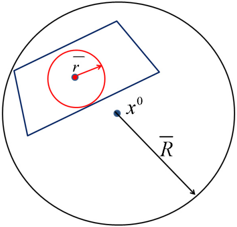
- Let x^0 be a reference point.
- R is such that $\mathcal{F} \subset B(x^0, R)$.
- r is such that $B(\bar{x}, r) \subset \mathcal{F}$, for some $\bar{x}$.



- The number R/r is an "aspect ratio" of the set S.

Changes in the objective function

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Theorem

Let $\Delta d = (\Delta A, \Delta b, 0)$ *be a perturbation of the problem instance. Let* $z(d)$ *and* $z(d + \Delta d)$ *be the corresponding optimal values. Then,*

$$|z(d) - z(d + \Delta d)| \leq 2\|c\| \left(\frac{R}{r}\right) \left(\frac{1}{\gamma(A)}\right) \times (\|\Delta b\| + \|\Delta A\|(R + \|x^0\|))$$

- where $\gamma(A)$ is a number depending on the matrix A.

The question: introducing those measures in the 2-stage problem

- Either $C(d)$ or R/r could be used in connection to the function s we postulated before, but the dependence on the data has to be made explicit.
- Recall the conceptual problem:

$$
TP_{R2}) \quad \begin{array}{lll} \min & \sum_{t=1}^{T} C_t(\omega_t, x_t, y_t) + \sum_{t=1}^{T} s(\bar{\omega}_t, x_t, y_t) & \\ s.t. & G_t(\omega_t, x_t, y_t) \leq b_t & t = 1, ..., T \\ & H(\omega, x, y) = 0 & \end{array}
$$

- The structure of this problem will be complicated, but it could be possible to construct bounds that limit the variation of s in terms of the tactical decisions.
- So far, this is an open problem, at least for me...

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Final Coments

- We have been exploring consistency issues in the use of Optimization models in an intertemporal setting.
- Robust Optimization and the classical 2-stage stochastic approach have been promising on helping to achieve consistency.

- We plan to work on the general formulation considering direct consideration of stability measures.

- Our current "test bed" is a production planning problem in the forest industry.
- We are beginning to work in a second problem related to planning capacity in an hospital, where there are many sources un uncertainty.

THANKS!!