Composite games: strategies, equilibria, dynamics and applications

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Cheng Wan, University of Oxford.

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We consider finite games :
there are finitely many “participants", $i \in I$
each of them has finitely many “choices", $p \in S^i$.

The basic variable describing the interaction is thus a profile $x = \{x^i, i \in I\}$, where each $x^i = \{x^{ip}, p \in S^i\}$ is an element of the simplex $X^i = \Delta(S^i)$ on S^i. Let $X = \prod_{i \in I} X^i$.

We consider three frameworks with the following types of participants:
(I) populations of nonatomic players,
(II) atomic splittable players,
(III) atomic non splittable players.

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We compare and unify the basic properties, expressed through variational inequalities, concerning equilibria, potential games and dissipative games, and we study the associated evolutionary dynamics.
We further extend the analysis to composite games.

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Replicator dynamics for one population

S is the set of "types", x_t^p is the proportion of type $p \in S$ in the population at time t, $A = ((A_{pq}))$ is the fitness matrix ($p, q \in S$)

$$\dot{x}_t^p = x_t^p [e^p A x_t - x_t A x_t], \qquad p \in S$$

Replicator dynamics for two populations (cross matching)

$$\dot{x}_t^{1p} = x_t^{1p} [e^{1p} A^1 x_t^2 - x_t^1 A^1 x_t^2], \qquad p \in S^1$$

and similarly for x^2.

Replicator dynamics for I populations

$$\dot{x}_t^{ip} = x_t^{ip} [A^i(e^{ip}, x_t^{-i}) - A^i(x_t^i, x_t^{-i})], \qquad p \in S^i, i \in I$$

natural interpretation: $x_t^{ip}, p \in S^i$, is a mixed strategy of player i.

Unilateral replicator dynamics for one participant

$$\dot{x}_t^{ip} = x_t^{ip} [U_t^{ip} - \langle x_t^i, U_t^i \rangle], \qquad p \in S^i$$

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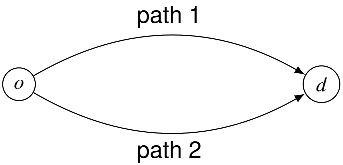
Routing game



- Population games : each participant $i \in I$ corresponds to a nonatomic set of agents (with a given mass m^i) having all the same characteristics. x^{ip} is the proportion of agents of choosing path p in population i.

Two kinds of I-player games where each participant $i \in I$ stands for an atomic player (with a given mass m^i) :
- Splittable case: x^{ip} is the ratio that player i allocates to path p. (The set of pure moves of player i is X^i.)
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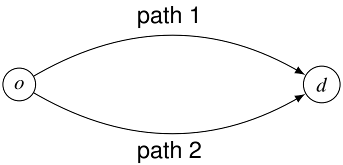

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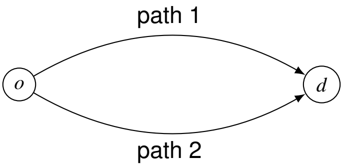

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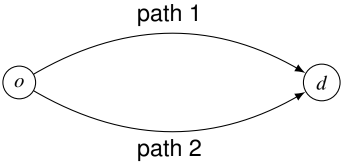

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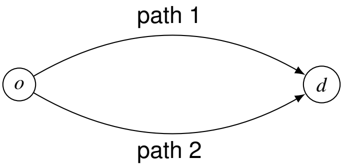

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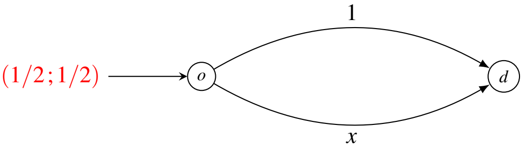

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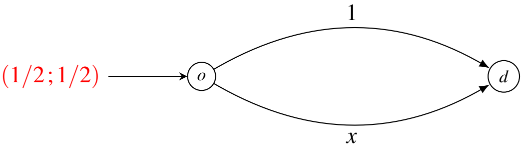

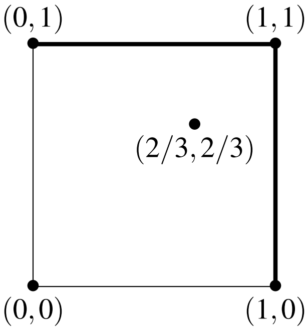

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Framework $\mathbf{I}$: population games

The payoffs are defined by a family of continuous functions $\{F^{ip}, i \in I, p \in S^i\}$, all from X to $\mathbb{R}$, where $F^{ip}(x)$ is the outcome of a member in population i choosing p, when the environment is given by x.

An equilibrium is a point $x \in X$ satisfying:

$$x^{ip} > 0 \Rightarrow F^{ip}(x) \geq F^{iq}(x), \quad \forall p, q \in S^i, \forall i \in I. \tag{1}$$

This corresponds to a Wardrop equilibrium.

An equivalent characterization of (1) is through the variational inequality:

$$\langle F^i(x), x^i - y^i \rangle \geq 0, \quad \forall y^i \in X^i, \forall i \in I, \tag{2}$$

or alternatively:

$$\langle F(x), x - y \rangle = \sum_{i \in I} \langle F^i(x), x^i - y^i \rangle \geq 0, \quad \forall y \in X. \tag{3}$$

(Smith, Dafermos ...)

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Framework II: atomic splittable

In this case each participant $i \in I$ corresponds to an atomic player with action set X^i. Given functions F^{ip} as introduced above, his gain is defined by:

$$H^i(x) = \langle x^i, F^i(x) \rangle = \sum_{p \in S^i} x^{ip} F^{ip}(x).$$

An equilibrium is as usual a profile $x \in X$ satisfying:

$$H^i(x) \geq H^i(y^i, x^{-i}), \quad \forall y^i \in X^i, \forall i \in I. \tag{4}$$

Suppose that for all $p \in S^i$, $F^{ip}(\cdot)$ is of class $\mathscr{C}^1$ on a neighborhood Ω of X, then any solution of (4) satisfies

$$\langle \nabla H(x), x - y \rangle = \sum_{i \in I} \langle \nabla^i H^i(x), x^i - y^i \rangle \geq 0, \qquad \forall\, y \in X. \tag{5}$$

where ∇^i is the gradient w.r.t. x^i. Moreover, if each H^i is concave with respect to x^i, there is equivalence.

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Framework III: atomic non splittable

We consider here an I-player game where the payoff is defined by a family of functions $\{G^i, i \in I\}$ from (the finite set) $S = \prod_{i \in I} S^i$ to $\mathbb{R}$.

We still denote by G the multilinear extension to X where each $X^i = \Delta(S^i)$ is considered as the set of mixed actions of player i.

An equilibrium is a profile $x \in X$ satisfying:

$$G^i(x^i, x^{-i}) \geq G^i(y^i, x^{-i}), \quad \forall y^i \in X^i, \forall i \in I. \tag{6}$$

Let VG^i denote the vector payoff associated to G^i. Explicitly, $VG^{ip} : X^{-i} \to \mathbb{R}$ is defined by $VG^{ip}(x^{-i}) = G^i(p, x^{-i})$, for all $p \in S^i$.

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Equilibrium and variational inequality

Note that F, ∇H and VG play similar roles in the three frameworks.
We call them evaluation functions and denote them by Φ with for each (i,p), $\Phi^{ip} : X \longrightarrow \mathbb{R}$.
The corresponding game is $\Gamma(\Phi)$.

Definition

$NE(\Phi)$ is the set of $x \in X$ satisfying:

$$\langle \Phi(x), x - y \rangle \geq 0, \qquad \forall y \in X. \tag{8}$$

$NE(\Phi)$ = equilibria of $\Gamma(\Phi)$.

Let $C \subset \mathbb{R}^d$ be a closed convex set and Ψ a map from C to $\mathbb{R}^d$. Consider the variational inequality:

$$\langle \Psi(x), x - y \rangle \geq 0, \qquad \forall y \in C. \tag{9}$$

Four equivalent representations are given by:

$$\Psi(x) \in N_C(x), \tag{10}$$

where $N_C(x)$ is the normal cône to C at x;

$$\Psi(x) \in [T_C(x)]^{\perp}, \tag{11}$$

where $T_C(x)$ is the tangent cône to C at x and $[T_C(x)]^{\perp}$ its polar;

$$\Pi_{T_C(x)} \Psi(x) = 0, \tag{12}$$

where Π is the projection operator on a closed convex subset; and

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Definition

A real function W, of class $\mathscr{C}^1$ on a neighborhood Ω of X, is a potential for Φ if for each $i \in I$, there exists a strictly positive function $\mu^i(x)$ defined on X such that

$$\left\langle \nabla^i W(x) - \mu^i(x)\Phi^i(x), y^i \right\rangle = 0, \quad \forall x \in X, \forall y^i \in X_0^i, \forall i \in I, \tag{14}$$

where $X_0^i = \{y \in \mathbb{R}^{|S^i|}, \sum_{p \in S^i} y_p = 0\}$ is the tangent space to X^i.

The game $\Gamma(\Phi)$ is then called a potential game and one says that Φ derives from W.

Monderer and Shapley, Sandholm

Theorem

Let $\Gamma(\Phi)$ be a game with potential W.

1. Every local maximum of W is an equilibrium of $\Gamma(\Phi)$.

2. If W is concave on X, then any equilibrium of $\Gamma(\Phi)$ is a global maximum of W on X.

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Dissipative games

Definition

The game $\Gamma(\Phi)$ is dissipative if Φ satisfies:

$$\langle \Phi(x) - \Phi(y), x - y \rangle \leq 0, \qquad \forall\, (x,y) \in X \times X.$$

In the framework of population games, Hofbauer and Sandholm studied this class under the name "stable games".

Let $SNE(\Phi)$ be the set of $x \in X$ satisfying:

$$\langle \Phi(y), x - y \rangle \geq 0, \qquad \forall y \in X.$$

Proposition

If $\Gamma(\Phi)$ is dissipative

$$SNE(\Phi) = NE(\Phi).$$

in particular $NE(\Phi)$ is convex.

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$$\dot{x}_t = \mathscr{B}_\Phi(x_t), \quad x \in X,$$

where for each $i \in I$, $\mathscr{B}^i_\Phi(x) \in X^i_0$ and X is invariant.

Replicator dynamics (RD) (Taylor and Jonker)

$$\dot{x}^{ip}_t = x^{ip}_t[\Phi^{ip}_t(x_t) - \overline{\Phi}^i(x_t)], \quad p \in S^i, i \in I,$$

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General properties

We define here properties expressed in terms of Φ.

The dynamics $\mathscr{B}_\Phi$ satisfies:

i) positive correlation (PC)(Sandholm) if:

$$\langle \mathscr{B}^i_\Phi(x), \Phi^i(x)\rangle > 0, \quad \forall i \in I, \forall x \in X \text{ s.t. } \mathscr{B}^i_\Phi(x) \neq 0.$$

This corresponds to MAD (myopic adjustment dynamics, Swinkels)

ii) Nash stationarity if:
for $x \in X$, $\mathscr{B}_\Phi(x) = 0$ if and only if $x \in NE(\Phi)$.

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All previous dynamics (RD), (BNN), (Smith), (LP), (GP) and (BR) satisfy (PC).

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Potential games

Proposition

Consider a potential game $\Gamma(\Phi)$ with potential function W. If the dynamics $\dot{x} = \mathscr{B}_\Phi(x)$ satisfies (PC), then W is a strict Lyapunov function for $\mathscr{B}_\Phi$. Besides, all ω-limit points are rest points of $\mathscr{B}_\Phi$.

$$\frac{d}{dt}W(x_t) = \sum_i \langle \nabla^i W(x_t), \dot{x}_t^i \rangle = \sum_i h^i(x_t) \langle \Phi^i(x_t), \dot{x}_t^i \rangle > 0$$

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It follows that, with the appropriate definitions, the convergence results established for several dynamics and potential games in framework $\mathbf{I}$ can be extended. Explicitly:

Proposition

Consider a potential game $\Gamma(\Phi)$ with potential function W.
If the dynamics is (RD), (BNN), (Smith), (LP), (GP) or (BR), W is a strict Lyapunov function for $\mathscr{B}_{\Phi}$.
In addition, except for (RD), all ω-limit points are equilibria of $\Gamma(\Phi)$.

Similar results hold for dissipative games with ad hoc Lyapunov functions.

Proposition

Consider a dissipative game $\Gamma(\Phi)$*.*
(1) RD: Let $x^* \in NE(\Phi)$*. Define:*

$$H(x) = \sum_{i \in I} \sum_{p \in supp(x^{i*})} x_p^{i*} \ln \frac{x_p^{i*}}{x_p^i}.$$

Then H *is a local Lyapunov function.*
If $\Gamma(\Phi)$ *is strictly dissipative, then* H *is a local strict Lyapunov function.*
(2) BNN: Assume Φ $\mathscr{C}^1$ *on a neighborhood* Ω *of* X*. Define:*

$$H(x) = \frac{1}{2} \sum_{i \in I} \sum_{p \in S^i} \hat{\Phi}_p^i(x)^2.$$

Then H *is a strict Lyapunov function which is minimal on* $NE(\Phi)$*.*

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(3) Smith: Assume Φ $\mathscr{C}^1$ on a neighborhood Ω of X. Define :

$$H(x) = \sum_{i \in I} \sum_{p,q \in S^i} x_p^i \left\{ [\Phi_q^i(x) - \Phi_p^i(x)]^+ \right\}^2.$$

Then H is a strict Lyapunov function which is minimal on $NE(\Phi)$.

(4) LP: Let $x^* \in NE(\Phi)$. Define:

$$H(x) = \frac{1}{2} \|x - x^*\|^2.$$

Then H is a Lyapunov function.
If $\Gamma(\Phi)$ is strictly dissipative, then H is a strict Lyapunov function.

(5) GP: Assume Φ $\mathscr{C}^1$ on a neighborhood Ω of X. Define :

$$H(x) = \sup_{y \in X} \langle y - x, \Phi(x) \rangle - \frac{1}{2} \|y - x\|^2.$$

Then H is a Lyapunov function.
If $\Gamma(\Phi)$ is strongly dissipative, then H is a strict Lyapunov function.

(6) BR: Assume Φ $\mathscr{C}^1$ on a neighborhood Ω of X. Define:

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Then H is a strict Lyapunov function which is minimal on $NE(\Phi)$.

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Congestion games and composite games

In a network congestion game, or routing game, the underlying network is a finite directed graph $G=(V,A)$, where V is the set of nodes, A the set of links.

$\mathbf{l}=(l_a)_{a\in A}$ denotes a family of cost functions from $\mathbb{R}$ to $\mathbb{R}^+$: if the aggregate weight on arc a is m, the cost per unit (of weight) is $l_a(m)$.

The set I of participants is finite. A participant i is characterized by his weight m^i and an origin/destination pair $(o^i,d^i)\in V\times V$ such that the constraint is to send a quantity m^i from o^i to d^i.

The set of choices of participant $i\in I$ is S^i: a family of directed acyclic paths linking o^i to d^i. Let $P=\cup_{i\in I}S^i$.

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In each of the three frameworks considered, a configuration x induces a (random) flow f on the arcs. This defines the cost on each arc then for each path and finally the payoff of each participant.

Congestion games are thus natural settings where each kind of participants appears.

Moreover one can even consider a game where participants of different natures coexist: some of them being of type $\mathbf{I}$, $\mathbf{II}$ or $\mathbf{III}$. This leads to the notion of composite game.

Composite congestion games with participants of type $\mathbf{I}$ and $\mathbf{II}$ have been studied by Harker; Boulogne, Altman, Pourtallier and Kameda; Yang and Zhang; Cominetti, Correa and Stier-Moses, etc... under the name "mixed equilibria".

In addition, congestion games are a natural example of aggregative games (Selten) where the payoff of a participant i depends only on $x^i \in X^i$ and on some fixed dimensional function $\alpha^i(\{x^j\}_{j\in I})$.

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Composite games

We have seen that the properties of equilibrium and dynamics in the three frameworks all depend on the evaluation function Φ and the variational inequalities associated to it. One can define a more general class of games called composite games, which exhibit different types of players.
Explicitly consider a finite set I_1 of populations composed of nonatomic players, a finite set I_2 of atomic splittable players and a finite set I_3 of atomic non splittable players. Let $I = I_1 \cup I_2 \cup I_3$.
All the analysis of the previous sections extend to these configurations where $x = \{x^i\}_{i \in I_1 \cup I_2 \cup I_3}$ and $\Phi^{ip}(x)$ depends upon the type of participant i:
- expression of equilibria trough variational inequalities,
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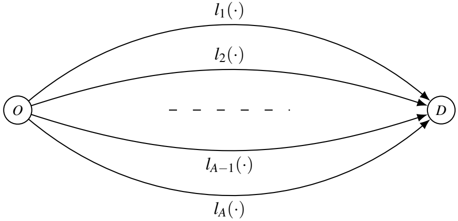
One example of a composite potential game

Consider a composite congestion game, with three types of participants $i \in I = I_1 \cup I_2 \cup I_3$, of mass m^i each, taking place in a network composed of two nodes o and d connected by a finite set A of parallel arcs.

Figure: Example of a composite potential game



Denote by $s = (s^k)_{k \in I_3} \in S_3 = A^{I_3}$ a pure strategy profile of participants in I_3 and let $z = ((x^i)_{i \in I_1}, (x^j)_{j \in I_2}, (s^k)_{k \in I_3})$. Let $f(z)$ be the aggregate flow induced by the pure-strategy profile z. Namely: $f_a(z) = \sum_{i \in I_1} m^i x_a^i + \sum_{j \in I_2} m^j x_a^j + \sum_{k \in I_3} m^k \mathbf{I}_{\{s^k = a\}}$.

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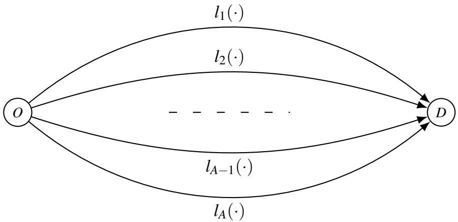
Figure: Example of a composite potential game



Denote by $s = (s^k)_{k \in I_3} \in S_3 = A^{I_3}$ a pure strategy profile of participants in I_3 and let $z = ((x^i)_{i \in I_1}, (x^j)_{j \in I_2}, (s^k)_{k \in I_3})$. Let $f(z)$ be the aggregate flow induced by the pure-strategy profile z. Namely: $f_a(z) = \sum_{i \in I_1} m^i x_a^i + \sum_{j \in I_2} m^j x_a^j + \sum_{k \in I_3} m^k \mathbf{I}_{\{s^k = a\}}$.

Theorem

Assume that for all $a \in A$, the per-unit cost function is affine, i.e. $l_a(u) = b_a u + d_a$, with $b_a > 0$ and $d_a \geq 0$. Then a composite congestion game on this network is a potential game.

A potential function defined on X is given by:

$$W(x) = -\sum_{s \in S_3} \Big(\prod_{k \in I_3} x^k_{s^k} \Big) \Big\{ \frac{1}{2} \sum_{a \in A} b_a \big[(f_a(z))^2 + \sum_{j \in I_2} (m^j x^j_a)^2 + \sum_{k \in I_3} (m^k)^2 \mathbf{I}_{\{s^k = a\}} \big] + \sum_{a \in A} d_a f_a(z) \Big\},$$

with $\mu^i(x) \equiv m^i$ for all $i \in I = I_1 \cup I_2 \cup I_3$ and all $x \in X$.

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Related topics

Asymptotic analysis for aggregative games (Haurie and Marcotte)
Replace one participant (atomic) i of size m_i by n participants with same characteristics and weight m_i/n. Accumulation points of a sequence of equilibria as n goes to ∞ are equilibria in the game where participant i is a population.

Composite players
A composite (atomic) player of weight m^i is described by a splittable component of weight $m^{i,0}$ and non splittable components of weight $m^{i,l}$, thus represented by a vector $\underline{m}^i = (m^{i,0}, m^{i,1}, \ldots, m^{i,n^i})$, where $n^i \in \mathbb{N}^*$, $m^{i,0} \geq 0$, $m^{i,l} > 0$ and $m^{i,0} + \sum_{l=1}^{n^i} m^{i,l} = m^i$.
Player i may allocate proportions of the splittable component to different choices and also allocate different non splittable components to different choices. However, a non splittable component cannot be divided.

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Delegation games

In the splittable case (or more generally for a composite player) a player i can delegate his mass among several players and get as payoff the sum of the payoff of the delegates (Sorin and Wan).

- conditions to have simple best reply strategies
- dynamical stability

Reinforcement and learning

Starting from a discrete time random adjustment process, tools from stochastic approximation may allow to to work with a continuous time deterministic dynamics

However the state variable may change:

in fictitious play $x_{n+1} \in BR(\bar{x}_n)$ leads to $\dot{z}_t \in BR(z_t) - z_t$ but now the variable z^i still in the simplex X^i corresponds to the time average behavior of participant i.

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Structure of the set of equilibria

Fix an evaluation Φ, then on $\Phi + \mathbb{R}^n$ the set of equilibria is homeomorphic to a graph,where $n^i = \#S^i$ and $n = \sum_i n^i$.

Index of Nash vector fields

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SORIN S., WAN C. *Finite composite games: equilibria and dynamics*, ArXiv:1503.07935v1, 2015.