Some variations of the Hotelling game

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Hotelling

- In the classical Hotelling model consumers are distributed uniformly on the interval $[0, 1]$.
- Retailers can choose any location in $[0, 1]$ where to set up a shop.
- Consumers shop at one of the closest retailers.
- This defines a (one-shot) game where the players are the retailers, the action set is $[0, 1]$ and the payoff is the amount of consumers that a retailer attracts.
- Depending on the number of retailers, equilibria may or may not exist and may or may not be unique.

Generalizations

- Various generalizations have been considered.
- The space where consumers are distributed can be different from $[0, 1]$.
- The distribution could be non-uniform.
- Retailers could compete not only on location but also on prices.

The model

- Consumers are distributed according to a measure λ on a compact Borel metric space (S, d).
- S could be a compact subset of $\mathbb{R}^2$ or a compact subset of a 2-sphere, but it could also be a (properly metrized) network.
- A finite set $N_n := \{1, \ldots, n\}$ of retailers have to decide where to set shop, knowing that each consumer chooses one of his closest retailers.
- Each retailer wants to maximize her market share.
- The action set of each retailer is a finite subset of S. For instance retailers can set shop only in one of the existing shopping malls in town.

Tessellation

- $K = \{1, \dots, k\}$
- $X_K := \{x_1, \dots, x_k\} \subset S$ is a finite collection of points in S. These are the points where retailers can open a store.
- For every $J \subset K$ call $X_J := \{x_j : j \in J\}$.
- $V(X_J)$ is the Voronoi tessellation of S induced by X_J.
- For each $x_j \in X_J$ the Voronoi cell of x_j is

$$v_J(x_j) := \{y \in S : d(y, x_j) \leqslant d(y, x_\ell) \text{ for all } x_\ell \in X_J\}.$$

- The cell $v_J(x_j)$ contains all points whose distance from x_j is not larger than the distance from the other points in X_J.
- Call

$$V(X_J) := (v_J(x_j))_{j \in J}$$

the set of all Voronoi cells $v_J(x_j)$.
- For $J \subset L \subset K$ we have $v_J(x_j) \supset v_L(x_j)$ for every $j \in J$.

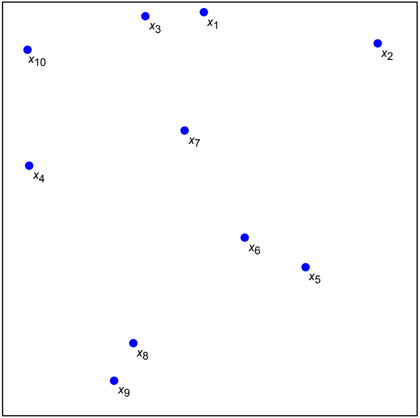


Figure: $X_K \subset [0,1]^2, \; K = \{1, \ldots, 10\}$.

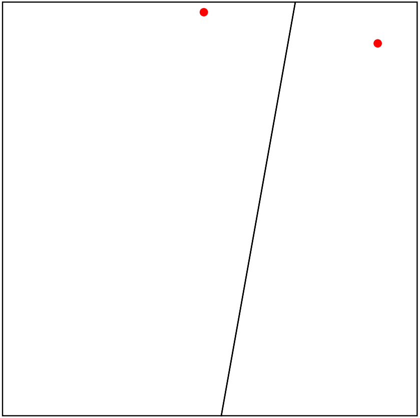


Figure: $V(X_J),\ J = \{1,2\}$.

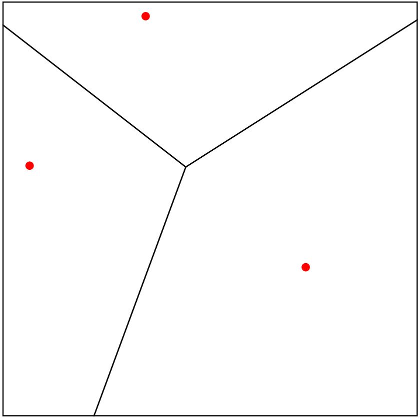


Figure: $V(X_J),\ J = \{3, 4, 5\}$.

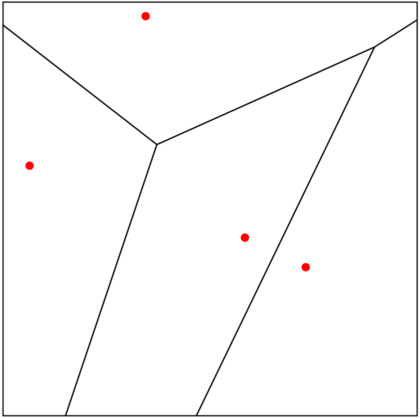


Figure: $V(X_J), \ J = \{3, 4, 5, 6\}$.

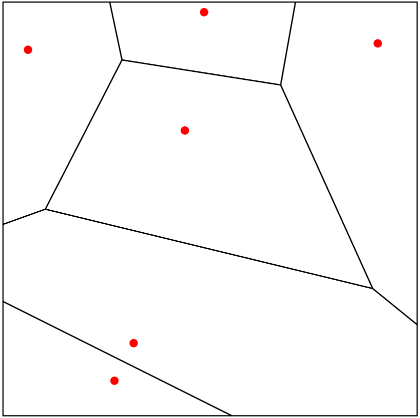


Figure: $V(X_J),\ J = \{1, 2, 7, 8, 9, 10\}$.

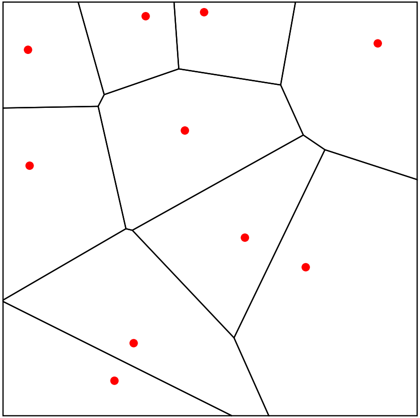


Figure: $V(X_J),\ J = K.$

- $\lambda(v_J(x_j))$ is the mass of consumers who are weakly closer to x_j than to any other point in X_J.
- If price is homogeneous, these consumers will prefer to shop at location x_j rather than at other locations in X_J.
- Consumers that belong to r different Voronoi cells $v_J(x_{j_1}), \dots, v_J(x_{j_r})$, are equally likely to shop at any of the locations $x_{j_1}, \dots, x_{j_r}$.
- λ is absolutely continuous with respect to the Lebesgue measure on this space and

$$\lambda(v_K(x_j)) > 0 \quad \text{for all } x_j \in X_K.$$

- More general situations can be considered but they require more care in handling ties.

The game

- $N_n := \{1, \ldots, n\}$ is the set of players.
- $a_i \in X_K$ is the action of player i.
- $\boldsymbol{a} := (a_i)_{i \in N_n}$ is the profile of actions.
- $\boldsymbol{a}_{-i} := (a_h)_{h \in N_n \setminus \{i\}}$ is the profile of actions of all the players different from i.
- $\boldsymbol{a} = (a_i, \boldsymbol{a}_{-i})$.
- $\boldsymbol{a} := (a_1, \ldots, a_n) \approx X_J$ if for all locations $x_j \in X_J$ there exists a player $i \in N_n$ such that $a_i = x_j$ and for all players $i \in N_n$ there exists a location $x_j \in X_J$ such that $a_i = x_j$.

The payoff

- The payoff of player i is

$$u_i(\boldsymbol{a}) = \frac{1}{\operatorname{card}\{h : a_h = a_i\}} \sum_{J \subset K} \lambda(v_J(a_i)) \mathbb{1}(\boldsymbol{a} \approx X_J),$$

  i.e., the measure of the consumers that are closer to the location that she chooses than to any other location chosen by any other player, divided by the number of retailers that choose the same action as i.
- Some locations may not be chosen by any player, this is why, for every $J \subset K$, we have to consider the Voronoi tessellation $V(X_J)$ with $\boldsymbol{a} \approx X_J$ rather than the finer tessellation $V(X_K)$.

Example

$S = [0,1]$, λ is the Lebesgue measure on $[0,1]$,
$X_K = \{0, 1/2, 1\}$.

$$v_J(0) = \begin{cases} [0,1] & \text{if } X_J = \{0\}, \\ [0,1/2] & \text{if } X_J = \{0,1\}, \\ [0,1/4] & \text{if } X_J = X_K \text{ or } X_J = \{0,1/2\}. \end{cases}$$

$$v_J(1/2) = \begin{cases} [0,1] & \text{if } X_J = \{1/2\}, \\ [1/4,1] & \text{if } X_J = \{0,1/2\} \\ [0,3/4] & \text{if } X_J = \{1/2,1\}, \\ [1/4,3/4] & \text{if } X_J = X_K. \end{cases}$$

$$v_J(1) = \begin{cases} [0,1] & \text{if } X_J = \{1\}, \\ [1/2,1] & \text{if } X_J = \{0,1\}, \\ [3/4,1] & \text{if } X_J = X_K \text{ or } X_J = \{1/2,1\}. \end{cases}$$

Example, continued

$$\lambda(v_J(0)) = \begin{cases} 1 & \text{if } X_J = \{0\}, \\ 1/2 & \text{if } X_J = \{0, 1\}, \\ 1/4 & \text{if } X_J = X_K \text{ or } X_J = \{0, 1/2\}. \end{cases}$$

$$\lambda(v_J(1/2)) = \begin{cases} 1 & \text{if } X_J = \{1/2\}, \\ 3/4 & \text{if } X_J = \{0, 1/2\} \text{ or } X_J = \{1/2, 1\}, \\ 1/2 & \text{if } X_J = X_K. \end{cases}$$

$$\lambda(v_J(1)) = \begin{cases} 1 & \text{if } X_J = \{1\}, \\ 1/2 & \text{if } X_J = \{0, 1\}, \\ 1/4 & \text{if } X_J = X_K \text{ or } X_J = \{1/2, 1\}. \end{cases}$$

Example, continued

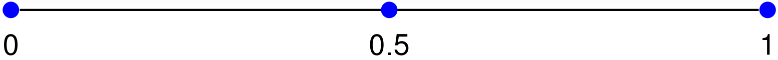
Therefore the payoff for player i, if she chooses location 0 when the rest of the players' pure actions are $\boldsymbol{a}_{-i}$ is

$$u_i(0, \boldsymbol{a}_{-i}) = \frac{1}{\text{card}\{h : a_h = a_i\}} \phi(\boldsymbol{a}_{-i}),$$

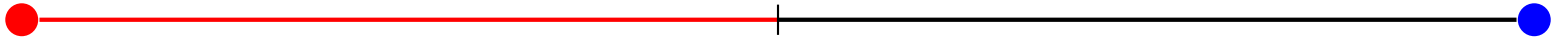
where

$$\phi(\boldsymbol{a}_{-i}) = \begin{cases} 1 & \text{if } \boldsymbol{a} \approx \{0\}, \\ \frac{1}{2} & \text{if } \boldsymbol{a} \approx \{0, 1\}, \\ \frac{1}{4} & \text{if } \boldsymbol{a} \approx X_K \text{ or } \boldsymbol{a} \approx \{0, 1/2\}. \end{cases}$$

The payoffs when she chooses either $1/2$ or 1 can be similarly computed.



$$v_J(0), \quad X_J = \{0\}$$



$$v_J(0), \quad X_J = \{0, 1\}$$

$$v_J(0), \quad X_J = \{0, 0.5, 1\}$$

$$v_J(0), \quad X_J = \{0, 0.5\}$$

- We have defined a game $\mathscr{G}_n = \langle S, \lambda, N_n, X_K, (u_i) \rangle$.
- With an abuse of notation, we use the same symbol $\mathscr{G}_n$ for the mixed extension of the game, where, for a mixed strategy profile $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$, the expected payoff of player i is

$$U_i(\boldsymbol{\sigma}) = \sum_{a_1 \in X_K} \cdots \sum_{a_n \in X_K} u_i(\boldsymbol{a}) \sigma_1(a_1) \ldots \sigma_n(a_n).$$

Equilibria

We consider a sequence $\{\mathscr{G}_n\}$ of games, all of which have the same parameters S, λ, X_K.

Example (A game without pure equilibria)

$\mathscr{G}_n$ with $n = 3$, $S = [0, 1]$, λ the Lebesgue measure, and $X_K = \{i/100 : i = 0, \ldots, 100\}$.
The game does not have a pure equilibrium.
This case is very similar to the classical Hotelling game on $[0, 1]$ with three players.

Dominated strategies

Example (Weakly dominated locations)

Consider a game $\mathscr{G}_n$ with $n = 2$, $S = [0, 1]$, λ the Lebesgue measure, and $X_K = \{0.45, 0.5, 0.55\}$. Then both 0.45 and 0.55 are weakly dominated by 0.5.

Proposition

Consider a sequence of games $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$. There exists $\bar{n}$ such that for all $n \geqslant \bar{n}$ no location in X_K is weakly dominated.

Pure equilibria

Theorem

Consider a sequence of games $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$. There exists $\bar{n}$ such that for all $n \geqslant \bar{n}$ the game $\mathscr{G}_n$ admits a pure equilibrium $\boldsymbol{a}^*$. Moreover, for all $n \geqslant \bar{n}$, any pure equilibrium is such that for every $j, \ell \in K$

$$\frac{n_j(\boldsymbol{a}^*)}{n_\ell(\boldsymbol{a}^*) + 1} \leqslant \frac{\lambda(v_K(x_j))}{\lambda(v_K(x_\ell))} \leqslant \frac{n_j(\boldsymbol{a}^*) + 1}{n_\ell(\boldsymbol{a}^*)}.$$

Mixed equilibria

Theorem

For every $n \in \mathbb{N}$ the game $\mathscr{G}_n$ admits a symmetric mixed equilibrium $\boldsymbol{\gamma}^{(n)} = (\gamma^{(n)}, \dots, \gamma^{(n)})$ such that

$$\lim_{n \to \infty} \gamma^{(n)} = \gamma,$$

with

$$\gamma(x_j) = \frac{\lambda(v_K(x_j))}{\lambda(S)} \quad \text{for all } j \in K.$$

This holds only asymptotically.
Let $S = [0, 1]$ with λ the Lebesgue measure on $[0, 1]$ and $X_K = \{0, 0.5, 1\}$. For each $n > 3$, the game $\mathscr{G}_n$ admits a symmetric mixed equilibrium $\boldsymbol{\gamma}^{(n)}$, where

$$\gamma^{(n)}(0) = \gamma^{(n)}(1) = p_n, \quad \gamma^{(n)}(0.5) = 1 - 2p_n,$$

with p_n as follows:

n	5	7	9	11	16	21
p_n	0.113	0.196	0.225	0.237	0.247	0.249

The probabilities in the symmetric mixed equilibrium converge towards the ones described by the theorem.

- The outcome of pure equilibria mimics the expected outcome of the mixed equilibria.
- The number of players who choose an action in a pure equilibrium is close to the expected number of players who choose the same action in the symmetric mixed equilibrium.
- Obviously no pure equilibrium can be symmetric.

Theorem

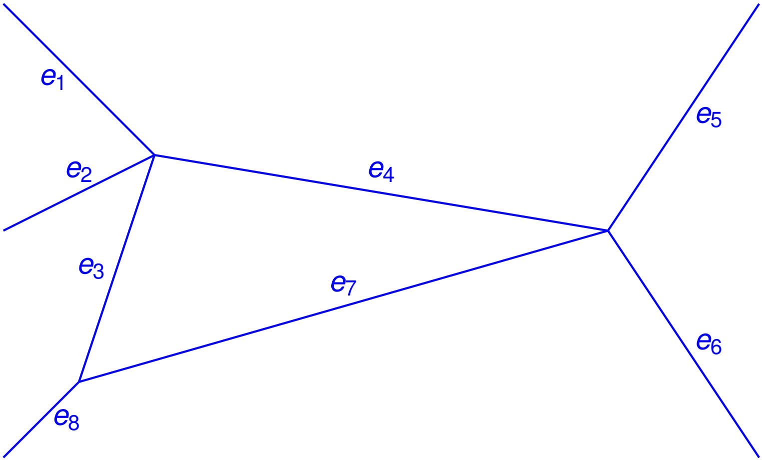
For any equilibrium $\boldsymbol{\alpha}^{(n)}$ of the game $\mathscr{G}_n$ we have for every $j, \ell \in K$

$$\lim_{n\to\infty} \frac{n_j(\boldsymbol{\alpha}^*)}{n_\ell(\boldsymbol{\alpha}^*) + 1} = \frac{\lambda(v_K(x_j))}{\lambda(v_K(x_\ell))}$$

and for every $i, h \in N_n$

$$\lim_{n\to\infty} \frac{U_i(\boldsymbol{\alpha}^{(n)})}{U_h(\boldsymbol{\alpha}^{(\boldsymbol{n})})} = 1.$$

Infinite Hotelling games on graphs



- Continuum of buyers, uniformly distributed on the graph.
- They shop to the closest location.
- A finite number of sellers.
- Sellers can set shop anywhere on the graph.

Theorem

For an arbitrary graph, there exists $\bar{n} \in \mathbb{N}$ such that for every $n \geqslant \bar{n}$, the infinite Hotelling game on the graph admits a pure Nash equilibrium.

Efficiency of equilibria

- Since the game is constant-sum, the issue of efficiency of equilibria for the retailers is not interesting.
- We look at efficiency from the viewpoint of consumers.
- We define the Induced Price of Anarchy and the Induced Price of Stability.

Theorem

For any graph, as the number of sellers tends to infinity,

(a) *the induced price of anarchy is bounded above by 2,*

(b) *the price of stability tends to 1.*

- The above results hold only asymptotically.

Conclusion

- We have considered spatial competition when consumers are arbitrarily distributed on a compact metric space and retailers can choose one of finitely many locations.
- A pure strategy equilibrium exists if the number of retailers is large enough, while it need not exist for a small number of retailers.
- Symmetric mixed equilibria exist for any number of retailers.
- The distribution of retailers tends to agree with the distribution of the consumers both at the pure strategy equilibrium and at the symmetric mixed one.

Variations and extensions

The results are robust to the introduction of

- randomness in the number of retailers, ▸ Poisson
- different ability of the retailers to attract consumers. ▸ types

Open problems

- Three classes of games:

1. finite Hotelling games,
2. infinite Hotelling games,
3. congestion games.

- What are the relations between these three classes of games?

Finite Hotelling and congestion games

- Hotelling games are not congestion games.
- In a Hotelling game (even a finite one) it is not true that the utility of an action depends only on the number of players who choose it.
- But, ...
- When the number of players is large, at equilibrium, finite Hotelling games behave like congestion games.
- This is true only at equilibrium.
- By adding k nonstrategic sellers, a finite Hotelling game can be transformed into a congestion game.

Finite and infinite congestion games

- Take a finite Hotelling game on a graph with uniformly distributed consumers.
- If the set of possible actions for the sellers is made denser and denser, the game looks more and more like an infinite game.
- Nevertheless for n large enough, the efficiency of all equilibria in the finite game is always the same.
- The infinite game has a good equilibrium that is about twice as efficient as a bad equilibrium.

Poisson games

- We consider games where the number of players is random and follows a Poisson distribution.
- Call $\mathscr{P}_n = \langle S, \lambda, N_{\Xi_n}, X_K, (u_i) \rangle$ the game where the cardinality of the players set N_{Ξ_n} is a Poisson random variable Ξ_n, with

$$\mathbb{P}(\Xi_n = k) = \frac{e^{-n} n^k}{k!}.$$

- Just like in game $\mathscr{G}_n$, in game $\mathscr{P}_n$ all players have the same utility function. So the utility function of player i depends only on i' s action and on the number of players who have chosen x_j for all $j \in K$.

- Quoting Myerson (1998), "population uncertainty forces us to treat players symmetrically in our game-theoretic analysis," so each player choses action x_j with probability $\sigma(x_j)$.
- As a consequence, all equilibria are symmetric.
- Properties of the Poisson distribution imply that the number of players choosing action x_j is independent of the number of players choosing action x_ℓ for $j \neq \ell$.

The expected utility of each player, when she chooses action x_j and all the other players act according to the mixed action σ is

$$U(x_j, \sigma) = \sum_{y \in Z(X_K)} \prod_{j \in K} \left(\frac{e^{-n\sigma(x_j)} (n\sigma(x_j))^{y(x_j)}}{y(x_j)} \right) U(x_j, y),$$

where $Z(X_K)$ denotes the set of possible action profiles for the players in a Poisson game.

We consider a sequence $\{\mathscr{P}_n\}$ of games, all of which have the same parameters S, λ, X_K.

Theorem

For every $n \in \mathbb{N}$ the game $\mathscr{P}_n$ admits a symmetric equilibrium $\gamma^{(n)}$ such that

$$\lim_{n\to\infty} \gamma^{(n)}(x_j) = \frac{\lambda(v_K(x_j))}{\lambda(S)} \quad \text{for all } j \in K\,.$$

In general the equilibria of $\mathscr{G}_n$ and $\mathscr{P}_n$ do not coincide.

Example

- Let $S = [0, 1]$ with λ the Lebesgue measure on $[0, 1]$ and $X_K = \{0.1, 0.5, 0.9\}$.
- We consider the equilibria of the games $\mathscr{G}_3$ (static) and $\mathscr{P}_3$ (Poisson).
- In the game $\mathscr{G}_3$, there exists an equilibrium σ^* in which each retailer locates in 0.5.
- Under σ^* the payoff for each retailer equals $1/3$ since they uniformly split the consumers in S.
- σ^* is not an equilibrium in the game $\mathscr{P}_3$.

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Heterogeneous retailers

- Up to now, we have considered a model where all retailers are equally able to attract consumers.
- In many situations some retailers have a comparative advantage due, for instance, to reputation.
- Similar models have been studied in the political competition literature with few strategic parties, see, e.g., Aragones and Palfrey (2002).
- In this literature the term “valence” is used to indicate the competitive advantage of one candidate over another.
- Retailers can be of two types: advantaged (A) and disadvantaged (D).

- When choosing between two retailers of the same type, a consumer takes into account only their distance from her and she prefers the closer of the two.
- When choosing between a retailer of type A located in x^A and a retailer of type D located in x^D, a consumer located in y will prefer the retailer of type A iff

$$d(x^A, y) < d(x^D, y) + \beta, \quad \text{with } \beta > 0.$$

- She will be indifferent between the two retailers iff

$$d(x^A, y) = d(x^D, y) + \beta.$$

- The case $\beta = 0$ corresponds to the homogeneous model.

- $\mathscr{D}_n$ is a game with differentiated retailers.
- For $j \in \{A, D\}$, call N_n^j the set of retailers of type j and define $n^j = \operatorname{card}(N_n^j)$.
- $$N_n = N_n^A \cup N_n^D,$$ $$n = n^A + n^D.$$
- For $j \in \{A, D\}$ and $i \in N_n^j$ call $a_i^j \in X_K$ the action of retailer i.
- The profile of actions is
$$\boldsymbol{a} := (\boldsymbol{a}^A, \boldsymbol{a}^D) := \{(a_i^A)_{i \in N_n^A}, (a_i^D)_{i \in N_n^D}\}.$$
- For any profile $\boldsymbol{a} \in X_K^n$ define
$$n_j^A(\boldsymbol{a}) := \operatorname{card}\{i \in N_n^A : a_i^A = x_j\},$$ $$n_j^D(\boldsymbol{a}) := \operatorname{card}\{i \in N_n^D : a_i^D = x_j\}.$$

- $(\boldsymbol{a}^A, \boldsymbol{a}^D) \approx X_{J^A,J^D}$ if for all locations $x_j \in X_{J^A}$ there exists a player $i \in N_n^A$ such that $a_i^A = x_j$ and for all players $i \in N_n^A$ there exists a location $x_j \in X_{J^A}$ such that $a_i^A = x_j$ and for all locations $x_j \in X_{J^D}$ there exists a player $i \in N_n^D$ such that $a_i^D = x_j$ and for all players $i \in N_n^D$ there exists a location $x_j \in X_{J^D}$ such that $a_i^D = x_j$.
- Fix $\beta > 0$, and, for $J^A, J^D \subset K$, define

$$\begin{aligned}
v^A_{J^A,J^D}(x_j) := \{y \in S : \; & d(y,x_j) \leqslant d(y,x_\ell) \text{ for all } x_\ell \in X_{J^A} \text{ and} \\
& d(y,x_j) \leqslant d(y,x_\ell) + \beta \text{ for all } x_\ell \in X_{J^D}\} \\
v^D_{J^A,J^D}(x_j) := \{y \in S : \; & d(y,x_j) \leqslant d(y,x_\ell) - \beta \text{ for all } x_\ell \in X_{J^A} \text{ and} \\
& d(y,x_j) \leqslant d(y,x_\ell) \text{ for all } x_\ell \in X_{J^D}\}.
\end{aligned}$$

- For $i \in N_n$, the payoff of player i is $u_i : X_K^n \to \mathbb{R}$, defined as follows:

$$u_i(\boldsymbol{a}^A, \boldsymbol{a}^D) =$$
$$\begin{cases} \dfrac{1}{\text{card}\{h : a_h^A = a_i^A\}} \displaystyle\sum_{J^A, J^D \subset K} \lambda(v_{J^A,J^D}^A(a_i^A))\mathbb{1}((\boldsymbol{a}^A, \boldsymbol{a}^D) \approx X_{J^A,J^D}), \\ \dfrac{1}{\text{card}\{h : a_h^D = a_i^D\}} \displaystyle\sum_{J^A, J^D \subset K} \lambda(v_{J^A,J^D}^D(a_i^D))\mathbb{1}((\boldsymbol{a}^A, \boldsymbol{a}^D) \approx X_{J^A,J^D}), \end{cases}$$

- We call $\mathscr{D}_n := \langle S, \lambda, N_n^A, N_n^D, X_K, \beta, (u_i) \rangle$ a Hotelling game with differentiated players.
- In any pure strategy profile of the game $\mathscr{D}_n$, a D-player gets a strictly positive payoff only if she chooses a location that is not chosen by any advantaged players.

The equilibria of a game $\mathscr{G}_n$ and of a game $\mathscr{D}_n$ can be quite different.

Example

Let $S = [0, 1]$ with λ the Lebesgue measure on $[0, 1]$ and $X_K = \{0, 1\}$.
The game $\mathscr{G}_2$ admits pure equilibria.
Any pure or mixed profile is an equilibrium and gives the same payoff $1/2$ to both players.
Consider now the game $\mathscr{D}_2$ with one advantaged and one disadvantaged players.
In the unique equilibrium of $\mathscr{D}_2$ both players randomize with probability $1/2$ over the two possible locations.

Pure equilibria

Theorem

Consider a sequence of games $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$. There exists $\bar{n}$ such that for all $n^A \geqslant \bar{n}$ the game $\mathscr{D}_n$ admits a pure equilibrium $\boldsymbol{a}^*$. Moreover, for all $n^A \geqslant \bar{n}$, any pure equilibrium satisfies

$$\frac{n_j^A(\boldsymbol{a}^*)}{n_\ell^A(\boldsymbol{a}^*)+1} \leqslant \frac{\lambda(v_K(x_j))}{\lambda(v_K(x_\ell))} \leqslant \frac{n_j^A(\boldsymbol{a}^*)+1}{n_\ell^A(\boldsymbol{a}^*)}. \tag{1}$$

Mixed equilibria

Given a game $\mathscr{D}_n$, an equilibrium profile $(\boldsymbol{\gamma}^{A,n}, \boldsymbol{\gamma}^{D,n})$ is called (A, D)-symmetric if

$$\boldsymbol{\gamma}^{A,n} = (\gamma^{A,n}, \dots, \gamma^{A,n}), \tag{2}$$

$$\boldsymbol{\gamma}^{D,n} = (\gamma^{D,n}, \dots, \gamma^{D,n}). \tag{3}$$

Theorem

For every $n \in \mathbb{N}$ the game $\mathscr{D}_n$ admits an (A, D)-symmetric equilibrium $(\boldsymbol{\gamma}^{A,n}, \boldsymbol{\gamma}^{D,n})$ such that

$$\lim_{n^A \to \infty} \gamma^{A,n}(x_j) = \frac{\lambda(v^A_{K,J^D}(x_j))}{\lambda(S)} = \frac{\lambda(v_K(x_j))}{\lambda(S)}$$

$$\text{for all } x_j \in S, \quad \text{for all } J^D \subset K.$$

Moreover, in this equilibrium,

$$\lim_{n^A \to \infty} \sum_{i \in N^D} U^D_i(\boldsymbol{\gamma}^{A,n}, \boldsymbol{\gamma}^{D,n}) = 0. \tag{4}$$

- As the number n^A of advantaged players grows, they behave as if the disadvantaged players did not exist, so they play the same mixed strategies as in the game $\mathscr{G}_{n^A}$.
- The disadvantaged players in turn get a zero payoff whatever they do.

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