A Comparison of Integer Programming Formulations for Stackelberg Games

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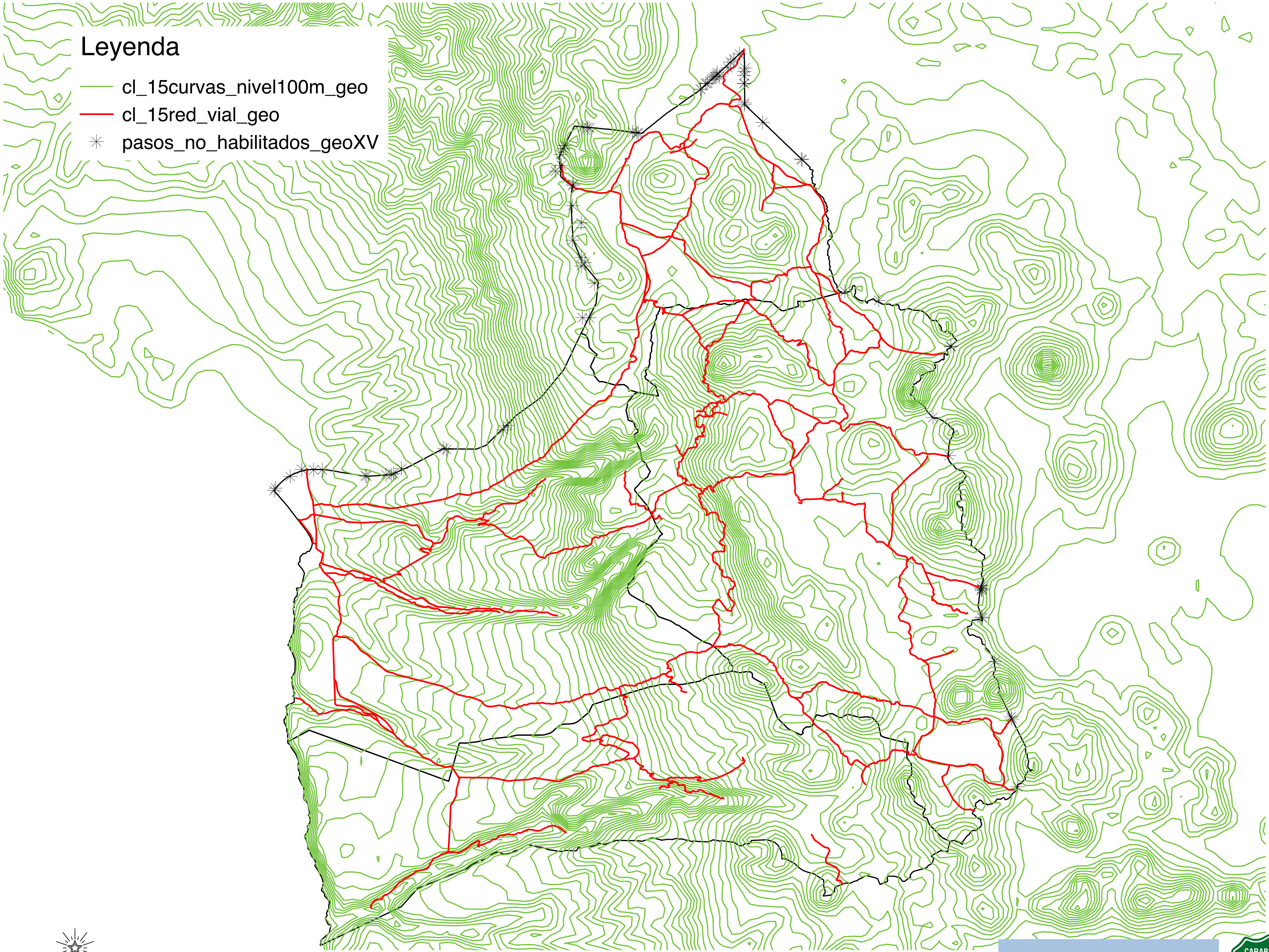
Datos


Leyenda
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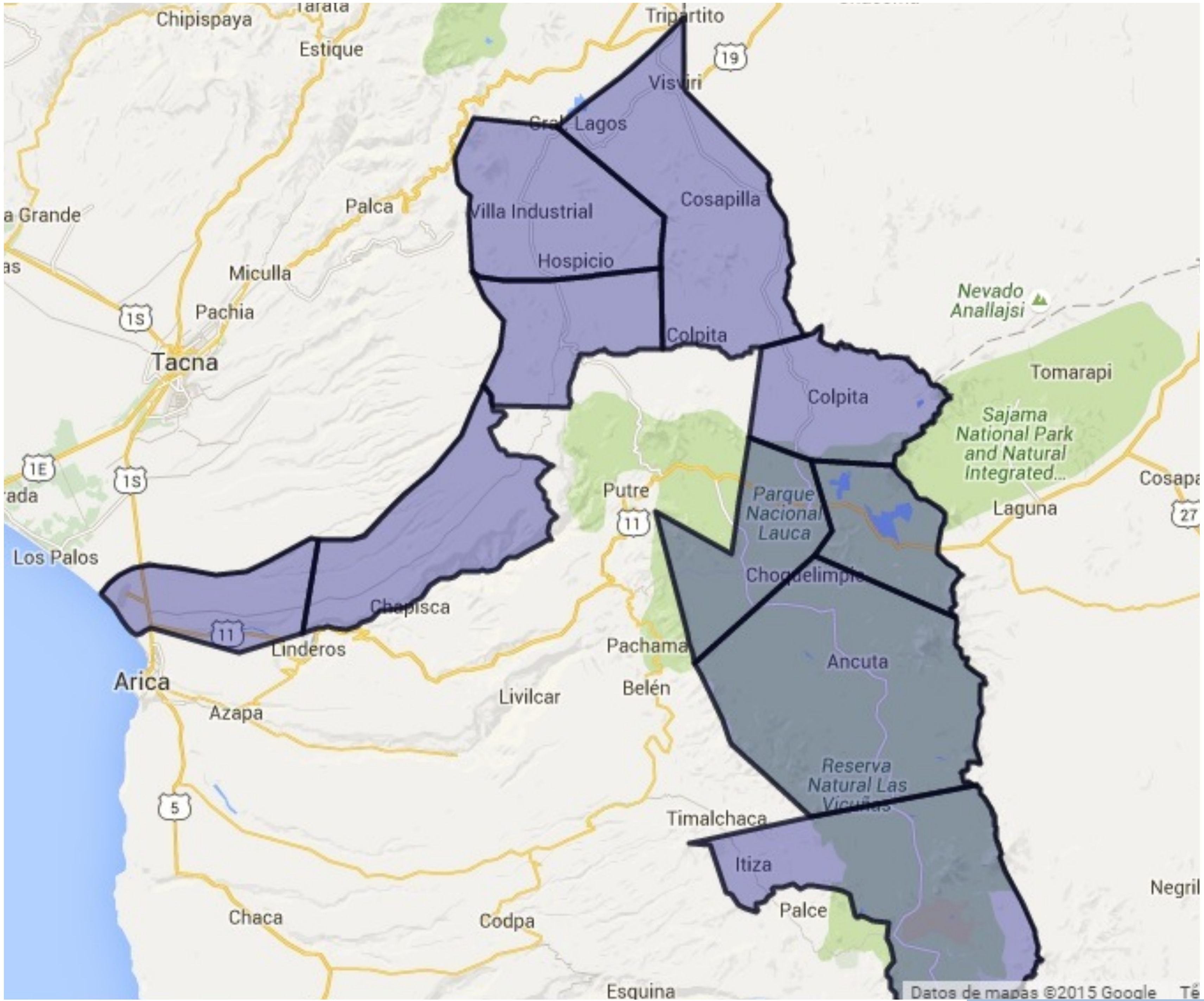
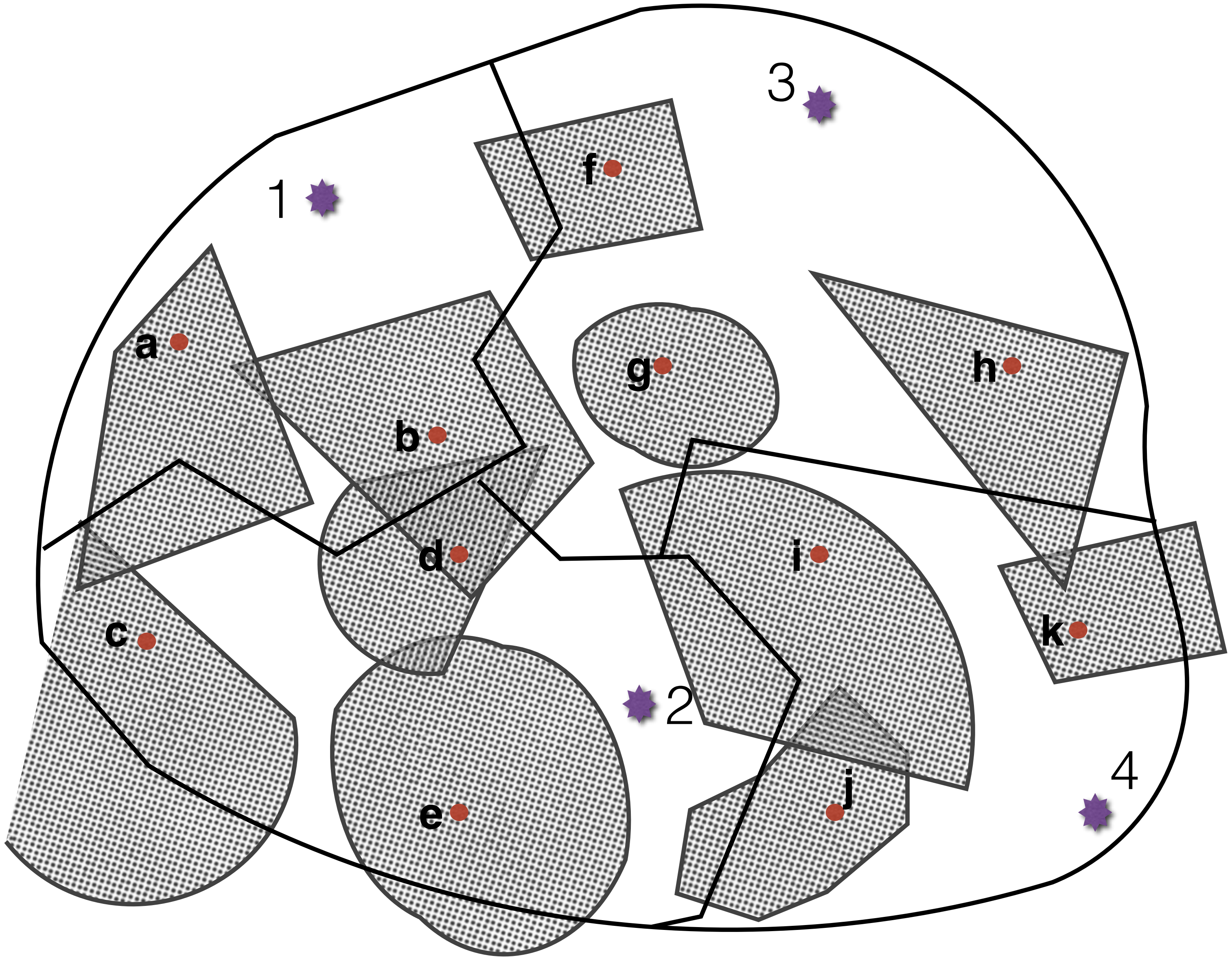
pasos_no_habilitados_geoXV

Datos

Distritos policiales



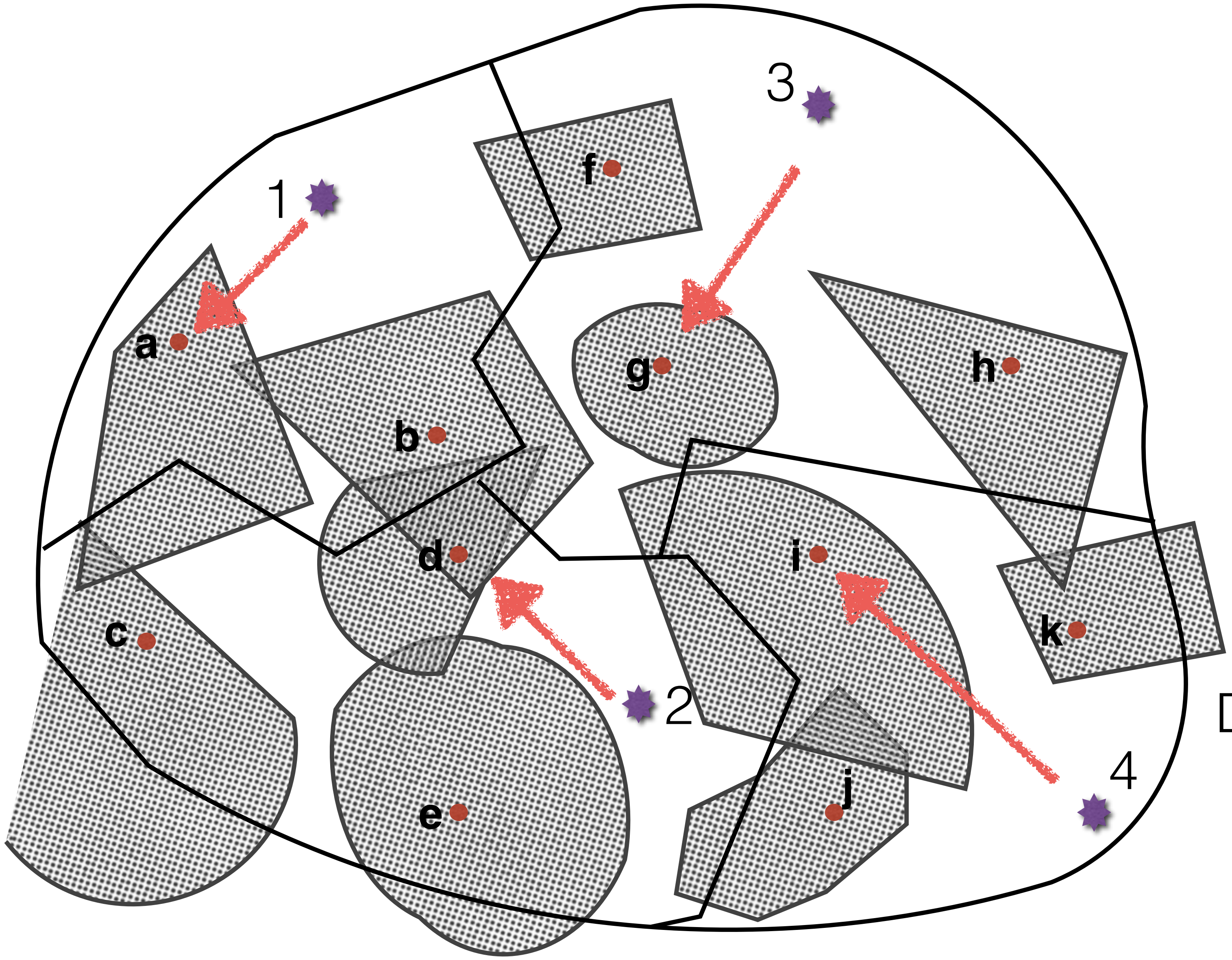
Patrullaje Nocturno



Turnos nocturnos: 22:00-04:00

Planificación quincenal

Patrullaje Nocturno

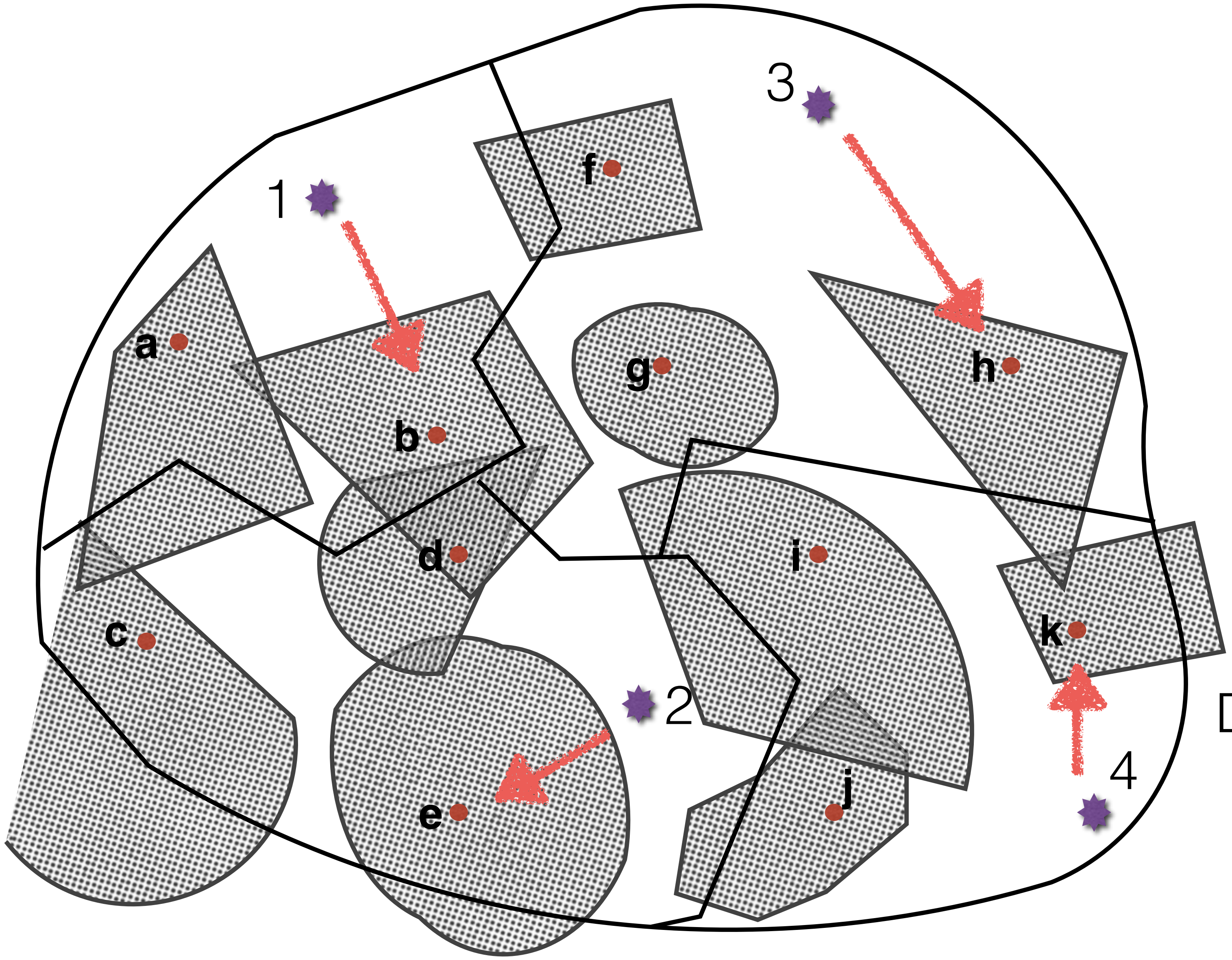


	1		2			3			4		
	a	b	c	d	e	f	g	h	i	j	k
Día 1	1	0	0	1	0	0	1	0	1	0	0
Día 2											
Día 3											
⋮											
Día 15											

Turnos nocturnos: 22:00-04:00

Planificación quincenal

Patrullaje Nocturno



	1		2			3			4		
	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**	**j**	**k**
Día 1	1	0	0	1	0	0	1	0	1	0	0
Día 2	0	1	0	0	1	0	0	1	0	0	1
Día 3											
⋮											
Día 15											

Turnos nocturnos: 22:00-04:00

Planificación quincenal

Stackelberg Security Games: Applications.

ARMOR: LAX (2007)

IRIS: FAMS (2009)

GUARDS: TSA (2010)

PROTECT: USCG (2011)

Outline

- MIP Formulations for Stackelberg Games

- Stackelberg Security Game
  - Formulations
  - What we know

- Computational Results

Bayesian Stackelberg Games

- Perfect information game
- Adversary observes policy of leader before deciding response
- If indiferent, adversary will favor leader (inducible)
- Multiple adversaries $k \in K$, with a-priori probability π^k
- Data:
  - R_{ij}^k benefit to leader of playing i if adversary k plays j
  - C_{ij}^k benefit to adversary k of playing j if leader plays i

Stackelberg Games

Leader: maximize reward under optimal adversary response

$$\begin{array}{ll} \max_{x,q,a} & \displaystyle\sum_{i\in X}\sum_{k\in K}\sum_{j\in J}\pi^k R_{ij}^k x_i q_j^k \\ \text{s.t.} & \sum_{i\in X} x_i = 1 \\ & x_i \in [0,1] \\ & q^k = \text{argmax}_y\{\displaystyle\sum_{i\in X} C_{ij}^k x_i y_j \mid \sum_{j\in J} y_j = 1,\ y \geq 0\} \end{array}$$

k-th follower problem optimality conditions

$$\begin{array}{l} \sum_{j\in J} q_j^k = 1,\ q_j^k \geq 0 \\ a^k - \sum_{i\in X} C_{ij}^k x_i \geq 0 \\ q_j^k(a^k - \sum_{i\in X} C_{ij}^k x_i) = 0 \end{array}$$

Stackelberg Games

$$
(SG) \quad
\begin{array}{rl}
\max_{x,q,a} & \displaystyle\sum_{i\in X}\sum_{k\in K}\sum_{j\in J}\pi^k R_{ij}^k x_i q_j^k \\
\text{s.t.} & \sum_{i\in X} x_i = 1,\ \ x_i \in [0,1] \\
 & \sum_{j\in J} q_j^k = 1,\ \ q_j^k \in [0,1] \\
 & 0 \le a^k - \sum_{i\in X} C_{ij}^k x_i \\
 & q_j^k(a^k - \sum_{i\in X} C_{ij}^k x_i) = 0 \\
 & a \in \Re^{|K|}.
\end{array}
$$

Stackelberg Games

If follower optimal solution has $q_j^k > 0$ then action j has optimal value a^k and $q_j^k = 1$ is an optimal pure strategy.

Optimality conditions, restricted to pure follower strategies

$$\begin{aligned}
\textstyle\sum_{j\in J} q_j^k &= 1 \\
a^k - \textstyle\sum_{i\in X} C_{ij}^k x_i &\geq 0 \\
a^k - \textstyle\sum_{i\in X} C_{ij}^k x_i &\leq (1 - q_j^k)M \\
q_j^k &\in \{0, 1\}
\end{aligned}$$

(Paruchuri et al., 2008)

$$
\begin{array}{lllr}
\text{(D2)} & \text{Max}_{x,q,a,d} & \displaystyle\sum_{k\in K} \pi^k d^k & \\
 & \text{s.t.} & \displaystyle d^k \leq \sum_{i\in I} R^k_{ij} x_i + (1 - q^k_j) M_1 & \forall j \in J, \forall k \in K, \\
 & & \displaystyle\sum_{i\in I} x_i = 1, & \\
 & & x_i \in [0, 1] & \forall i \in I, \\
 & & \displaystyle\sum_{j\in J} q^k_j = 1 & \forall k \in K, \\
 & & q^k_j \in \{0, 1\} & \forall j \in J, \forall k \in K, \\
 & & \displaystyle 0 \leq a^k - \sum_{i\in I} C^k_{ij} x_i \leq (1 - q^k_j) M_2 & \forall j \in J, \forall k \in K, \\
 & & a^k, d^k \in \mathbb{R} & \forall k \in K.
\end{array}
$$

(Paruchuri et al., 2008)

$$
\begin{array}{llll}
\text{(D2)} & \text{Max}_{x,q,a,d} & \displaystyle\sum_{k\in K} \pi^k d^k & \\
& \text{s.t.} & \displaystyle d^k \leq \sum_{i\in I} R^k_{ij} x_i + (1 - q^k_j) M_1 & \forall j \in J, \forall k \in K, \\
& & \displaystyle\sum_{i\in I} x_i = 1, & \\
& & x_i \in [0,1] & \forall i \in I, \\
& & \displaystyle\sum_{j\in J} q^k_j = 1 & \forall k \in K, \\
& & q^k_j \in \{0,1\} & \forall j \in J, \forall k \in K, \\
& & \displaystyle 0 \leq a^k - \sum_{i\in I} C^k_{ij} x_i \leq (1 - q^k_j) M_2 & \forall j \in J, \forall k \in K, \\
& & a^k, d^k \in \mathbb{R} & \forall k \in K.
\end{array}
$$

$$
\begin{array}{llll}
\text{(FMD2)} & \text{Max}_{x,q,a,d} & \displaystyle\sum_{k\in K} \pi^k d^k & \\
& \text{s.t.} & \displaystyle d^k \le \sum_{i\in I} R^k_{ij} x_i + (1-q^k_j)M_1 & \forall j \in J, \forall k \in K, \\
& & \displaystyle\sum_{i\in I} x_i = 1, & \\
& & x_i \in [0,1] & \forall i \in I, \\
& & \displaystyle\sum_{j\in J} q^k_j = 1 & \forall k \in K, \\
& & q^k_j \in \{0,1\} & \forall j \in J, \forall k \in K, \\
& & \displaystyle\sum_{i\in I} (C^k_{i\ell} - C^k_{ij}) x_i \ge (q^k_\ell - 1)M_2 & \forall j, \ell \in J, \forall k \in K, \\
& & a^k, d^k \in \mathbb{R} & \forall k \in K.
\end{array}
$$

Stackelberg Games

SG leader problem with pure adversary strategies

$$\begin{array}{ll} \text{max}_{x,q,a} & \displaystyle\sum_{i \in X} \sum_{k \in K} \sum_{j \in J} \pi^k R^k_{ij} x_i q^k_j \\ \text{s.t.} & \sum_{i \in X} x_i = 1,\ x_i \in [0,1] \\ & \sum_{j \in Q} q^k_j = 1,\ q^k_j \in \{0,1\} \\ & 0 \leq a^k - \sum_{i \in X} C^k_{ij} x_i \leq (1 - q^k_j) M \\ & a \in \Re^{|K|}. \end{array}$$

(Paruchuri et al., 2008)

$$
\begin{aligned}
\text{(DOBSS)} \quad \text{Max}_{x,q} \quad & \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \pi^k R^k_{ij} z^{kext}_{ij} & \\
\text{s.t.} \quad & \sum_{i\in I}\sum_{j\in J} z^{kext}_{ij} = 1 & \forall k \in K, \\
& \sum_{j\in J} z^{kext}_{ij} \le 1 & \forall i \in I, \forall k \in K, \\
& q^k_j \le \sum_{i\in I} z^{kext}_{ij} \le 1 & \forall j \in J, \forall k \in K, \\
& \sum_{j\in J} q^k_j = 1 & \forall k \in K, \\
& 0 \le a^k - \sum_{i\in I}\sum_{j'\in J} C^k_{ij} z^{kext}_{ij'} \le (1 - q^k_j)M & \forall j \in J, \forall k \in K, \\
& \sum_{j\in J} z^{kext}_{ij} = \sum_{j\in J} z^{1ext}_{ij} & \forall i \in I, \forall k \in K, \\
& z^{kext}_{ij} \in [0, 1] & \forall i \in I, \forall j \in J, \forall k \in K, \\
& q^k_j \in \{0, 1\} & \forall j \in J, \forall k \in K, \\
& a^k \in \mathbb{R} & \forall k \in K.
\end{aligned}
$$

(Paruchuri et al., 2008)

$$
\begin{array}{llll}
\text{(DOBSS)} & \text{Max}_{x,q} & \displaystyle\sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\pi^k R_{ij}^k z_{ij}^{kext} & \\
& \text{s.t.} & \displaystyle\sum_{i\in I}\sum_{j\in J} z_{ij}^{kext} = 1 & \forall k \in K, \\
& & \displaystyle\sum_{j\in J} z_{ij}^{kext} \leq 1 & \forall i \in I, \forall k \in K, \\
& & \displaystyle q_j^k \leq \sum_{i\in I} z_{ij}^{kext} \leq 1 & \forall j \in J, \forall k \in K, \\
& & \displaystyle\sum_{j\in J} q_j^k = 1 & \forall k \in K, \\
& & \displaystyle 0 \leq a^k - \sum_{i\in I}\sum_{j'\in J} C_{ij}^k z_{ij'}^{kext} \leq (1-q_j^k)M & \forall j \in J, \forall k \in K, \\
& & \displaystyle\sum_{j\in J} z_{ij}^{kext} = \sum_{j\in J} z_{ij}^{1ext} & \forall i \in I, \forall k \in K, \\
& & z_{ij}^{kext} \in [0,1] & \forall i \in I, \forall j \in J, \forall k \in K, \\
& & q_j^k \in \{0,1\} & \forall j \in J, \forall k \in K, \\
& & a^k \in \mathbb{R} & \forall k \in K.
\end{array}
$$

$$
\begin{aligned}
\text{(FMDOBSS) Max}_{x,q} \quad & \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \pi^k R_{ij}^k z_{ij}^{kext} & \\
\text{s.t.} \quad & \sum_{i \in I} \sum_{j \in J} z_{ij}^{kext} = 1 & \forall k \in K, \\
& \sum_{j \in J} z_{ij}^{kext} \leq 1 & \forall i \in I, \forall k \in K, \\
& q_j^k \leq \sum_{i \in I} z_{ij}^{kext} \leq 1 & \forall j \in J, \forall k \in K, \\
& \sum_{j \in J} q_j^k = 1 & \forall k \in K, \\
& \sum_{i \in I} \sum_{j' \in J} (C_{i\ell}^k - C_{ij}^k) z_{ij'}^{kext} \geq (q_\ell^k - 1)M & \forall j, \ell \in J, \forall k \in K, \\
& \sum_{j \in J} z_{ij}^{kext} = \sum_{j \in J} z_{ij}^{1ext} & \forall i \in I, \forall k \in K, \\
& z_{ij}^{kext} \in [0, 1] & \forall i \in I, \forall j \in J, \forall k \in K, \\
& q_j^k \in \{0, 1\} & \forall j \in J, \forall k \in K, \\
& a^k \in \mathbb{R} & \forall k \in K.
\end{aligned}
$$

$$\sum_{i \in I} (C_{ij}^k - C_{i\ell}^k) x_i \leq (1 - q_\ell^k) M_2 \ \forall j, \ell \in J, \forall k \in K$$

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La liberté de chercher

$$\sum_{i\in I}(C^k_{ij} - C^k_{i\ell})x_i \leq (1 - q^k_\ell)M_2 \;\forall j, \ell \in J, \forall k \in K$$

Teoría RLT

(Sherali, H.D., Adams, W.P., 1999)

$\times q^k_\ell$

- $q^k_\ell \cdot (1 - q^k_\ell) = 0$

$$\sum_{i\in I}(C^k_{ij} - C^k_{i\ell})z^{kext}_{ij} \geq 0 \qquad \forall j, \ell \in J, \forall k \in K.$$

$$\sum_{i\in I}(C^k_{ij} - C^k_{i\ell})x_i \leq (1 - q^k_\ell)M_2 \ \forall j, \ell \in J, \forall k \in K$$

Teoría RLT

(Sherali, H.D., Adams, W.P., 1999)

$\times q^k_\ell$

$\times (1 - q^k_\ell)$

- $q^k_\ell \cdot (1 - q^k_\ell) = 0$
- $(1 - q^k_\ell)^2 = (1 - q^k_\ell)$

$$\sum_{i\in I}(C^k_{ij} - C^k_{i\ell})z^{kext}_{ij} \geq 0 \qquad \forall j, \ell \in J, \forall k \in K.$$

$$
\begin{aligned}
(\text{MIP-}p\text{-G}) \quad \text{Max}_{x,q} \quad & \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \pi^k R_{ij}^k z_{ij}^{kext} \\
\text{s.t.} \quad & \sum_{i\in I}\sum_{j\in J} z_{ij}^{kext} = 1, && \forall k \in K \\
& \sum_{i\in I} (C_{ij}^k - C_{i\ell}^k) z_{ij}^{kext} \geq 0 && \forall j, \ell \in J, \forall k \in K, \\
& z_{ij}^{kext} \geq 0 && \forall i \in I, \forall j \in J, \forall k \in K, \\
& \sum_{i\in I} z_{ij}^{kext} \in \{0,1\} && \forall j \in J, \forall k \in K, \\
& \sum_{j\in J} z_{ij}^{kext} = \sum_{j\in J} z_{ij}^{1ext} && \forall i \in I, \forall k \in K.
\end{aligned}
$$

What we know

- $Proj_{x,q,f,s}\mathcal{P}(\overline{f\text{-DOBSS}}) \subsetneq \mathcal{P}(\overline{\text{D2}})$
- $\mathcal{P}(\overline{\text{MIP-}p\text{-G}}) \subsetneq \mathcal{P}(\overline{\text{FMDOBSS}}) = Proj_{x,q,z}\mathcal{P}(\overline{\text{DOBSS}})$
- $z_{LP}(\text{MIP-}p\text{-G}) \leq z_{LP}(f\text{-DOBSS}) = z_{LP}(\text{DOBSS}) \leq z_{LP}(\text{D2})$
- $\mathcal{P}(\overline{\text{MIP-}p\text{-G}})$ is an integer polyhedron when $k = 1$

Stackelberg Games: complexity

- If $|K| = 1$ then solve (SG) by, for every $j \in J$, fixing $q_j = 1$ and solving

$$\begin{aligned} v_k = \text{max}_x \quad & \textstyle\sum_{i \in X} R_{ij} x_i \\ \text{s.t.} \quad & \textstyle\sum_{i \in X} x_i = 1, \;\; x_i \in [0,1] \\ & \textstyle\sum_{i \in X} C^k_{il} x_i \leq \sum_{i \in X} C^k_{ij} x_i \qquad \forall l \in J \end{aligned}$$

- Multiple adversaries (SG) is NP-hard

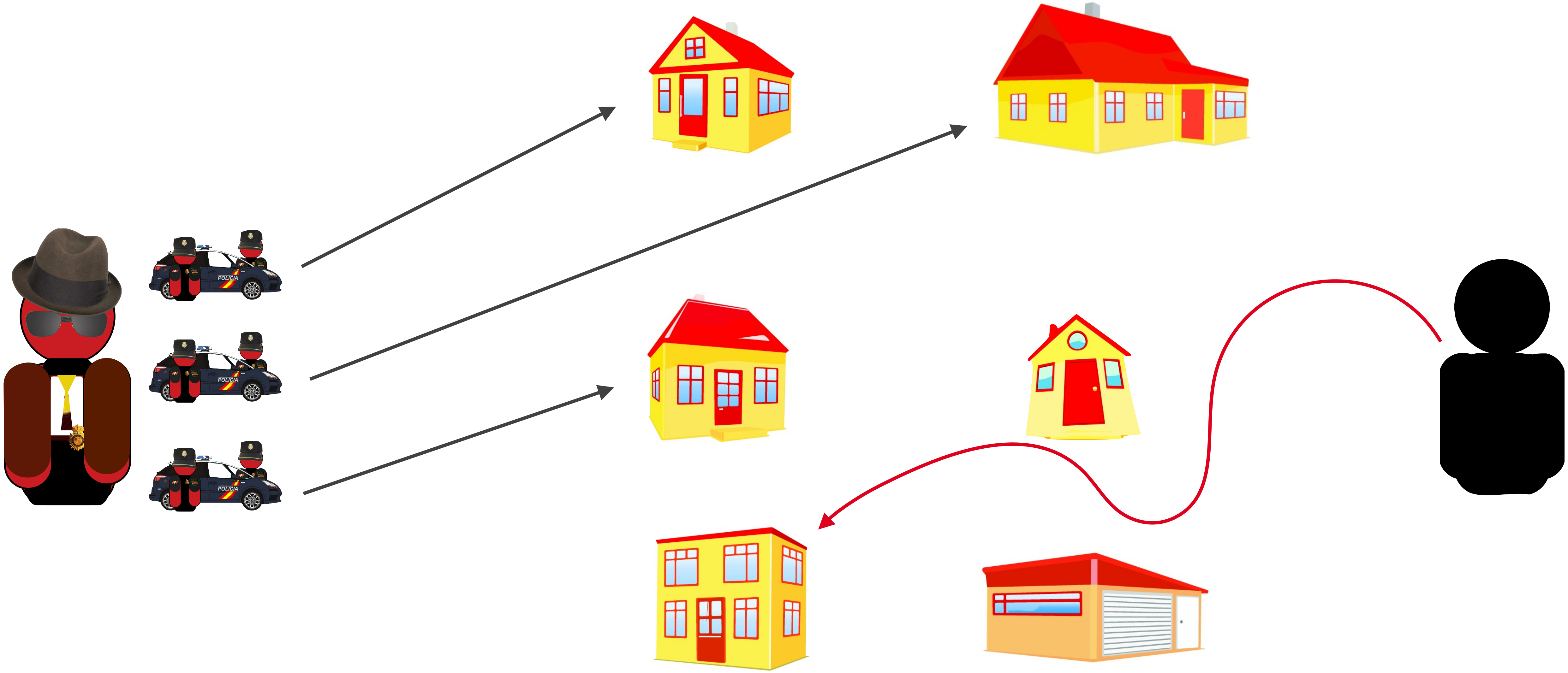
Conitzer and Sandholm (2006)

Outline

- MIP Formulations for Stackelberg Games
- Stackelberg Security Game
    - Formulations
    - What we know
- Computational Results

Stackelberg Security Games

Stackelberg Security Games



Stackelberg Security Games

Place n security resources to cover $|Q| = m$ targets

$|X|$ is large $\sim \binom{m}{n}$ but rewards are structured

X	action	var
1	1,2,3	x_1
2	1,2,4	x_2
1	1,2,5	x_3
$\vdots$	$\vdots$	$\vdots$
120	8,9,10	x_{120}

action	$q_1 = 1$	$q_2 = 1$	...	$q_{10} = 1$
1,2,3	5, -10	4, -8	...	-20, 5
1,2,4	5, -10	4,-8	...	-20, 5
$\vdots$				
1,3,5	5, -10	-12, 9	...	-20, 5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$R^k_{ij} = \begin{cases} D^k(j|c) & \text{if } i \text{ protects } j \\ D^k(j|u) & \text{it does not} \end{cases} \qquad C^k_{ij} = \begin{cases} A^k(j|c) & \text{if } i \text{ protects } j \\ A^k(j|u) & \text{it does not} \end{cases}$$

$$
\begin{aligned}
\text{Max}_{x,q} \quad & \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \pi^k R_{ij}^k x_i q_j^k \\
\text{s.t.} \quad & \sum_{i\in I} x_i = 1, \\
& x_i \in [0,1] && \forall i \in I, \\
& 0 \leq (s^k - \sum_{i\in I} C_{ij}^k x_i) \leq (1 - q_j^k) M && \forall j \in J, \forall k \in K, \\
& q_j^k \in \{0,1\} && \forall j \in J, \forall k \in K, \\
& \sum_{j\in J} q_j^k = 1 && \forall k \in K, \\
& s^k \in \mathbb{R} && \forall k \in K.
\end{aligned}
$$

$$
\begin{aligned}
\text{Max}_{x,q} \quad & \sum_{k \in K} \sum_{j \in J} \pi^k q_j^k D^k(j|c) \sum_{\substack{i \in I \\ j \in i}} x_i + D^k(j|u) \sum_{\substack{i \in I: \\ j \notin i}} x_i \\
\text{s.t.} \quad & \sum_{i \in I} x_i = 1, \\
& x_i \in [0,1] & \forall i \in I, \\
0 \leq (s^k - A^k(j|c) & \sum_{i \in I: j \in i} x_i - A^k(j|u) \sum_{i \in I: j \notin i} x_i) \leq (1 - q_j^k) \cdot M & \forall j \in J, \forall k \in K, \\
& q_j^k \in \{0,1\} & \forall j \in J, \forall k \in K, \\
& \sum_{j \in J} q_j^k = 1 & \forall k \in K, \\
& s^k \in \mathbb{R} & \forall k \in K.
\end{aligned}
$$

$$
\begin{aligned}
\text{Max}_{x,q} \quad & \sum_{k\in K}\sum_{j\in J}\pi^k q_j^k D^k(j|c)\underbrace{\boxed{\sum_{\substack{i\in I\\ j\in i}} x_i}}_{c_j} + D^k(j|u)\underbrace{\boxed{\sum_{\substack{i\in I:\\ j\notin i}} x_i}}_{1-c_j} \\
\text{s.t.} \quad & \sum_{i\in I} x_i = 1, \\
& x_i \in [0,1] && \forall i \in I, \\
0 \le (s^k - A^k(j|c) & \overbrace{\boxed{\sum_{i\in I:j\in i} x_i}}^{c_j} - A^k(j|u)\overbrace{\boxed{\sum_{i\in I:j\notin i} x_i}}^{1-c_j}) \le (1-q_j^k)\cdot M && \forall j\in J, \forall k \in K, \\
& q_j^k \in \{0,1\} && \forall j\in J, \forall k \in K, \\
& \sum_{j\in J} q_j^k = 1 && \forall k \in K, \\
& s^k \in \mathbb{R} && \forall k \in K.
\end{aligned}
$$

$$
\begin{aligned}
\text{Max} \quad & \sum_{k\in K}\sum_{j\in J}\pi^k q_j^k\{D^k(j|c)c_j + D^k(j|u)(1-c_j)\} & \\
\text{s.t.} \quad & \sum_{i\in I} x_i = 1, & \\
& x_i \geq 0 & \forall i \in I, \\
& 0 \leq (s^k - A^k(j|c)c_j - A^k(j|u)(1-c_j)) \leq (1-q_j^k)\cdot M & \forall j \in J, \forall k \in K, \\
& \sum_{j\in J} q_j^k = 1 & \forall k \in K, \\
& q_j^k \in \{0,1\} & \forall j \in J, \forall k \in K, \\
& \sum_{\substack{i\in I:\\ j\in i}} x_i = c_j & \forall j \in J, \\
& s^k \in \mathbb{R} & \forall k \in K.
\end{aligned}
$$

Juegos de Stackelberg de seguridad

$$
\begin{aligned}
\text{Max} \quad & \sum_{k\in K}\sum_{j\in J}\pi^k q_j^k\{D^k(j|c)c_j + D^k(j|u)(1-c_j)\} && \\
\text{s.t.} \quad & \boxed{\sum_{i\in I} x_i = 1,} && \\
& \boxed{x_i \geq 0} && \forall i \in I, \\
& 0 \leq (s^k - A^k(j|c)c_j - A^k(j|u)(1-c_j)) \leq (1-q_j^k)\cdot M && \forall j \in J, \forall k \in K, \\
& \sum_{j\in J} q_j^k = 1 && \forall k \in K, \\
& q_j^k \in \{0,1\} && \forall j \in J, \forall k \in K, \\
& \boxed{\sum_{\substack{i\in I:\\ j\in i}} x_i = c_j} && \forall j \in J, \\
& s^k \in \mathbb{R} && \forall k \in K.
\end{aligned}
$$

Juegos de Stackelberg de seguridad

Formulación Compacta

$$
\begin{array}{lll}
\text{Max} & \displaystyle\sum_{k\in K}\sum_{j\in J}\pi^k q_j^k\{D^k(j|c)c_j + D^k(j|u)(1-c_j)\} & \\
\text{s.t.} & \boxed{\begin{array}{l}\displaystyle\sum_{j\in J} c_j \le m \\ c_j \in [0,1]\end{array}} & \forall j \in J, \\
 & 0 \le (s^k - A^k(j|c)c_j - A^k(j|u)(1-c_j)) \le (1-q_j^k)\cdot M & \forall j \in J, \forall k \in K, \\
 & \displaystyle\sum_{j\in J} q_j^k = 1 & \forall k \in K, \\
 & q_j^k \in \{0,1\} & \forall j \in J, \forall k \in K, \\
 & & \forall j \in J, \\
 & s^k \in \mathbb{R} & \forall k \in K.
\end{array}
$$

Juegos de Stackelberg de seguridad

Formulación Compacta

$$
\begin{array}{llll}
\text{(ERASER)} & \text{Max} & \displaystyle\sum_{k\in K} \pi^k f^k & \\
& \text{s.t.} & f^k \leq D^k(j|c)c_j + D^k(j|u)(1-c_j) + (1-q_j^k)\cdot M & \forall j \in J, \forall k \in K, \\
& & \displaystyle\sum_{j\in J} c_j \leq m, & \\
& & c_j \in [0,1] & \forall j \in J, \\
& & 0 \leq (a^k - A^k(j|c)c_j - A^k(j|u)(1-c_j)) \leq (1-q_j^k)\cdot M & \forall j \in J, \forall k \in K, \\
& & q_j^k \in \{0,1\} & \forall j \in J, \forall k \in K, \\
& & \displaystyle\sum_{j\in J} q_j^k = 1 & \forall k \in K, \\
& & a^k, f^k \in \mathbb{R} & \forall k \in K.
\end{array}
$$

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Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$
\begin{array}{llll}
\text{(DOBSS)} & \text{Max} & \displaystyle\sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\pi^k R_{ij}^k z_{ij}^k & \\
& \text{s.t.} & \displaystyle\sum_{j\in J} z_{ij}^k = \sum_{j\in J} z_{ij}^1 & \forall i\in I, \forall k\in K, \\
& & \displaystyle\sum_{i\in I}\sum_{j\in J} z_{ij}^k \leq 1 & \forall k\in K, \\
& & \displaystyle\sum_{j\in J} z_{ij}^k \leq 1 & \forall i\in I, \forall k\in K, \\
& & \displaystyle q_j^k \leq \sum_{i\in I} z_{ij}^k \leq 1 & \forall j\in J, \forall k\in K, \\
& & z_{ij}^k \in [0,1] & \forall i\in I, \forall j\in J, \forall k\in K, \\
& & \displaystyle 0 \leq s^k - \sum_{i\in I}\sum_{j'\in J} C_{ij}^k z_{ij'}^k \leq (1-q_j^k)\cdot M & \forall j\in J, \forall k\in K, \\
& & \displaystyle\sum_{j\in J} q_j^k = 1 & \forall k\in K, \\
& & q_j^k \in \{0,1\} & \forall j\in J, \forall k\in K, \\
& & s^k \in \mathbb{R} & \forall k\in K.
\end{array}
$$

Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$
\begin{aligned}
\text{(DOBSS)}\quad \text{Max}\quad & \sum_{i\in I}\sum_{j\in J}\sum_{k\in K}\pi^k R_{ij}^k z_{ij}^k & \\
\text{s.t.}\quad & \sum_{j\in J} z_{ij}^k = \sum_{j\in J} z_{ij}^1 & \forall i\in I, \forall k\in K,\\
& \sum_{j\in J} z_{ij}^k \leq 1 & \forall i\in I, \forall k\in K,\\
& q_j^k \color{red}{=} \sum_{i\in I} z_{ij}^k & \forall j\in J, \forall k\in K,\\
& z_{ij}^k \geq 0 & \forall i\in I, \forall j\in J, \forall k\in K,\\
& 0 \leq s^k - \sum_{i\in I}\sum_{j'\in J} C_{ij}^k z_{ij'}^k \leq (1-q_j^k)\cdot M & \forall j\in J, \forall k\in K,\\
& \sum_{j\in J} q_j^k = 1 & \forall k\in K,\\
& q_j^k \in \{0,1\} & \forall j\in J, \forall k\in K,\\
& s^k \in \mathbb{R} & \forall k\in K.
\end{aligned}
$$

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Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$\text{(DOBSS)} \quad \text{Max} \quad \sum_{j\in J}\sum_{k\in K}\pi^k\Big(D^k(j|c)\sum_{\substack{i\in I:\\ j\in i}} z_{ij}^k + D^k(j|u)\sum_{\substack{i\in I:\\ j\notin i}} z_{ij}^k\Big)$$

$$\text{s.t.} \quad \sum_{j\in J} z_{ij}^k = \sum_{j\in J} z_{ij}^1 \qquad \forall i\in I, \forall k\in K,$$

$$\sum_{j\in J} z_{ij}^k \leq 1 \qquad \forall i\in I, \forall k\in K,$$

$$q_j^k = \sum_{i\in I} z_{ij}^k \qquad \forall j\in J, \forall k\in K,$$

$$z_{ij}^k \geq 0 \qquad \forall i\in I, \forall j\in J, \forall k\in K,$$

$$0 \leq s^k - A^k(j|c)\sum_{j'\in J}\sum_{\substack{i\in I:\\ j\in i}} z_{ij'}^k - A^k(j|u)\sum_{j'\in J}\sum_{\substack{i\in I:\\ j\notin i}} z_{ij'}^k \leq (1-q_j^k)\cdot M \quad \forall j\in J, \forall k\in K,$$

$$\sum_{j\in J} q_j^k = 1 \qquad \forall k\in K,$$

$$q_j^k \in \{0,1\} \qquad \forall j\in J, \forall k\in K,$$

$$s^k \in \mathbb{R} \qquad \forall k\in K.$$

Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$\text{(DOBSS)} \quad \text{Max} \quad \sum_{j\in J}\sum_{k\in K}\pi^k\Big(D^k(j|c)\sum_{\substack{i\in I:\\ j\in i}} z_{ij}^k + D^k(j|u)\sum_{\substack{i\in I:\\ j\notin i}} z_{ij}^k\Big)$$

$$\text{s.t.} \quad \sum_{j\in J} z_{ij}^k = \sum_{j\in J} z_{ij}^1 \qquad \forall i\in I, \forall k\in K,$$

$$\boxed{\sum_{i\in I:\ell\in i} z_{ij}^k = y_{\ell j}^k}$$

$$\sum_{j\in J} z_{ij}^k \leq 1 \qquad \forall i\in I, \forall k\in K,$$

$$q_j^k = \sum_{i\in I} z_{ij}^k \qquad \forall j\in J, \forall k\in K,$$

$$z_{ij}^k \geq 0 \qquad \forall i\in I, \forall j\in J, \forall k\in K,$$

$$0 \leq s^k - A^k(j|c)\sum_{j'\in J}\sum_{\substack{i\in I:\\ j\in i}} z_{ij'}^k - A^k(j|u)\sum_{j'\in J}\sum_{\substack{i\in I:\\ j\notin i}} z_{ij'}^k \leq (1-q_j^k)\cdot M \quad \forall j\in J, \forall k\in K,$$

$$\sum_{j\in J} q_j^k = 1 \qquad \forall k\in K,$$

$$q_j^k \in \{0,1\} \qquad \forall j\in J, \forall k\in K,$$

$$s^k \in \mathbb{R} \qquad \forall k\in K.$$

Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$\text{(DOBSS)} \quad \text{Max} \quad \sum_{j\in J}\sum_{k\in K} \pi^k \left(D^k(j|c) \boxed{\sum_{\substack{i\in I:\\ j\in i}} z^k_{ij}} + D^k(j|u) \boxed{\sum_{\substack{i\in I:\\ j\notin i}} z^k_{ij}}\right)$$

$$q^k_j - y^k_{jj}$$

$$y^k_{jj}$$

$$\text{s.t.} \quad \sum_{j\in J} z^k_{ij} = \sum_{j\in J} z^1_{ij} \qquad \forall i\in I, \forall k\in K,$$

$$\boxed{\sum_{i\in I:\ell\in i} z^k_{ij} = y^k_{\ell j}}$$

$$\sum_{j\in J} z^k_{ij} \le 1 \qquad \forall i\in I, \forall k\in K,$$

$$q^k_j = \sum_{i\in I} z^k_{ij} \qquad \forall j\in J, \forall k\in K,$$

$$y^k_{jj'} \qquad q^k_{j'} - y^k_{jj'}$$

$$z^k_{ij} \ge 0 \qquad \forall i\in I, \forall j\in J, \forall k\in K,$$

$$0 \le s^k - A^k(j|c) \sum_{j'\in J} \boxed{\sum_{\substack{i\in I:\\ j\in i}} z^k_{ij'}} - A^k(j|u) \sum_{j'\in J} \boxed{\sum_{\substack{i\in I:\\ j\notin i}} z^k_{ij'}} \le (1-q^k_j)\cdot M \quad \forall j\in J, \forall k\in K,$$

$$\sum_{j\in J} q^k_j = 1 \qquad \forall k\in K,$$

$$q^k_j \in \{0,1\} \qquad \forall j\in J, \forall k\in K,$$

$$s^k \in \mathbb{R} \qquad \forall k\in K.$$

Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$
\begin{aligned}
\text{(DOBSS)} \quad \text{Max} \quad & \sum_{j\in J}\sum_{k\in K} \pi^k (D^k(j|c) y^k_{jj} + D^k(j|u)(q^k_j - y^k_{jj})) & \\
\text{s.t.} \quad & \sum_{j\in J} z^k_{ij} = \sum_{j\in J} z^1_{ij} & \forall i \in I, \forall k \in K, \\
& q^k_j = \sum_{i\in I} z^k_{ij} & \forall j \in J, \forall k \in K, \\
& z^k_{ij} \geq 0 & \forall i \in I, \forall j \in J, \forall k \in K, \\
& 0 \leq s^k - A^k(j|c) \sum_{j'\in J} y^k_{jj'} - & \\
& A^k(j|u)(\sum_{j'\in J} q^k_{j'} - \sum_{j'\in J} y^k_{jj'}) \leq (1 - q^k_j) \cdot M & \forall j \in J, \forall k \in K, \\
& \sum_{j\in J} q^k_j = 1 & \forall k \in K, \\
& \sum_{i\in I:\ell\in j} z^k_{ij} = y^k_{\ell j} & \forall j, \ell \in J, \forall k \in K, \\
& s^k \in \mathbb{R} & \forall k \in K. \\
& q^k_j \in \{0,1\} \quad \forall j \in J, \forall k \in K &
\end{aligned}
$$

Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$
\begin{array}{llll}
\text{(DOBSS)} & \text{Max} & \displaystyle\sum_{j\in J}\sum_{k\in K}\pi^k(D^k(j|c)y^k_{jj}+D^k(j|u)(q^k_j-y^k_{jj})) & \\
 & \text{s.t.} & \boxed{\displaystyle\sum_{j\in J} z^k_{ij}=\sum_{j\in J} z^1_{ij}} & \forall i\in I,\forall k\in K, \\
 & & \boxed{\displaystyle q^k_j=\sum_{i\in I} z^k_{ij}} & \forall j\in J,\forall k\in K, \\
 & & \boxed{z^k_{ij}\geq 0} & \forall i\in I,\forall j\in J,\forall k\in K, \\
 & & \displaystyle 0\leq s^k-A^k(j|c)\sum_{j'\in J} y^k_{jj'}- & \\
 & & \displaystyle A^k(j|u)(\sum_{j'\in J} q^k_{j'}-\sum_{j'\in J} y^k_{jj'})\leq(1-q^k_j)\cdot M & \forall j\in J,\forall k\in K, \\
 & & \displaystyle\sum_{j\in J} q^k_j=1 & \forall k\in K, \\
 & & \boxed{\displaystyle\sum_{i\in I:\ell\in j} z^k_{ij}=y^k_{\ell j}} & \forall j,\ell\in J,\forall k\in K, \\
 & & s^k\in\mathbb{R} & \forall k\in K. \\
 & & q^k_j\in\{0,1\}\quad\forall j\in J,\forall k\in K &
\end{array}
$$

Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$\sum_{j \in J} z_{ij}^{k} = \sum_{j \in J} z_{ij}^{1} \ \forall i \in I, \forall k \in K,$$

Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$\sum_{j \in J} z_{ij}^{k} = \sum_{j \in J} z_{ij}^{1} \ \forall i \in I, \forall k \in K,$$

$$\sum_{i \in I : \ell \in i} \sum_{j \in J} z_{ij}^{k} = \sum_{i \in I : \ell \in i} \sum_{j \in J} z_{ij}^{1} \ \forall \ell \in J, \forall k \in K,$$

Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$\sum_{j \in J} z_{ij}^{k} = \sum_{j \in J} z_{ij}^{1} \ \forall i \in I, \forall k \in K,$$

$$\sum_{i \in I : \ell \in i} \sum_{j \in J} z_{ij}^{k} = \sum_{i \in I : \ell \in i} \sum_{j \in J} z_{ij}^{1} \ \forall \ell \in J, \forall k \in K,$$

$$\sum_{j \in J} y_{\ell j}^{k} = \sum_{j \in J} y_{\ell j}^{1}, \ \forall \ell \in J, \forall k \in K$$

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Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$
\begin{aligned}
\text{Max} \quad & \sum_{j\in J}\sum_{k\in K}\pi^k(D^k(j|c)y^k_{jj} + D^k(j|u)(q^k_j - y^k_{jj})) && \\
\text{s.t.} \quad & \sum_{j\in J} y^k_{\ell j} = \sum_{j\in J} y^1_{\ell j} && \forall \ell \in J, \forall k \in K \\
& q^k_j = \sum_{i\in I} z^k_{ij} && \forall j \in J, \forall k \in K, \\
& z^k_{ij} \geq 0 && \forall i \in I, \forall j \in J, \forall k \in K, \\
& 0 \leq s^k - A^k(j|c)\sum_{j'\in J} y^k_{jj'} - && \\
& A^k(j|u)(\sum_{j'\in J} q^k_{j'} - \sum_{j'\in J} y^k_{jj'}) \leq (1 - q^k_j)\cdot M && \forall j \in J, \forall k \in K, \\
& \sum_{j\in J} q^k_j = 1 && \forall k \in K, \\
& \sum_{i\in I:\ell\in j} z^k_{ij} = y^k_{\ell j} && \forall j, \ell \in J, \forall k \in K, \\
& s^k \in \mathbb{R} && \forall k \in K.
\end{aligned}
$$

Juegos de Stackelberg de seguridad

Formulación extendida basada en (DOBSS)

$$\begin{aligned}
\text{Max} \quad & \sum_{j\in J}\sum_{k\in K} \pi^k(D^k(j|c)y^k_{jj} + D^k(j|u)(q^k_j - y^k_{jj})) && \\
\text{s.t.} \quad & \sum_{j\in J} y^k_{\ell j} = \sum_{j\in J} y^1_{\ell j} && \forall \ell \in J, \forall k \in K \\
& \boxed{q^k_j = \sum_{i\in I} z^k_{ij}} && \forall j \in J, \forall k \in K, \\
& \boxed{z^k_{ij} \geq 0} && \forall i \in I, \forall j \in J, \forall k \in K, \\
& 0 \leq s^k - A^k(j|c)\sum_{j'\in J} y^k_{jj'} - && \\
& A^k(j|u)(\sum_{j'\in J} q^k_{j'} - \sum_{j'\in J} y^k_{jj'}) \leq (1 - q^k_j)\cdot M && \forall j \in J, \forall k \in K, \\
& \sum_{j\in J} q^k_j = 1 && \forall k \in K, \\
& \boxed{\sum_{i\in I:\ell\in j} z^k_{ij} = y^k_{\ell j}} && \forall j, \ell \in J, \forall k \in K, \\
& s^k \in \mathbb{R} && \forall k \in K.
\end{aligned}$$

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Juegos de Stackelberg de seguridad

Formulación compacta basada en (DOBSS)

$$
\begin{aligned}
\text{(SDOBSS)} \quad \text{Max} \quad & \sum_{j\in J}\sum_{k\in K} \pi^k (D^k(j|c) y^k_{jj} + D^k(j|u)(q^k_j - y^k_{jj})) & \\
0 \leq s^k - A^k(j|c)c_j - & & \\
A^k(j|u)(1-c_j) \leq (1-q^k_j)\cdot M & & \forall j \in J, \forall k \in K, \\
\sum_{j\in J} q^k_j = 1 & & \forall k \in K, \\
\sum_{\ell\in J} y^k_{\ell j} \leq m q^k_j & & \forall j \in J, \forall k \in K, \\
y^k_{\ell j} \geq 0 & & \forall \ell, j \in J, \forall k \in K, \\
y^k_{\ell j} \leq q^k_j & & \forall \ell, j \in J, \forall k \in K, \\
\sum_{j\in J} y^k_{\ell j} = \sum_{j\in J} y^1_{\ell j} & & \forall \ell \in J, \forall k \in K, \\
q^k_j \in \{0,1\} & & \forall j \in J, \forall k \in K, \\
s^k \in \mathbb{R} & & \forall k \in K.
\end{aligned}
$$

Juegos de Stackelberg de seguridad

Formulación compacta basada en (MIP-p-G)

$$
\begin{aligned}
\text{(MIP-}p\text{-S)} \quad \text{Max} \quad & \sum_{k\in K}\sum_{j\in J}\pi^k\left(D^k(j|c)y^k_{jj}+D^k(j|u)(q^k_j-y^k_{jj})\right) & \\
\text{s.t.} \quad & \sum_{j\in J} y^k_{\ell j} = \sum_{j\in J} y^1_{\ell j} & \forall \ell\in J, \forall k\in K, \\
& \sum_{j\in J} q^k_j = 1, & \forall k\in K \\
& 0\le y^k_{\ell j}\le q^k_j & \forall j,\ell\in J, \forall k\in K, \\
& \sum_{\ell\in J} y^k_{\ell j}\le m q^k_j & \forall j\in J, \forall k\in K, \\
& A^k(j|c)y^k_{jj}+A^k(j|u)(q^k_j-y^k_{jj})- & \\
& A^k(\ell|c)y^k_{\ell j}-A^k(\ell|u)(q^k_j-y^k_{\ell j})\ge 0 & \forall j,\ell\in J, \forall k\in K, \\
& q^k_j\in\{0,1\} & \forall j\in J, \forall k\in K.
\end{aligned}
$$

What we know

- $Proj_{c,q,f,s}\mathcal{P}(\overline{\text{D2}}) = \mathcal{P}(\overline{\text{ERASER}})$
- $Proj_{y,q,s}\mathcal{P}(\overline{\text{DOBSS}}) \subseteq \mathcal{P}(\overline{\text{SDOBSS}})$
- $Proj_{y,q}\mathcal{P}(\overline{\text{MIP-}p\text{-G}}) \subseteq \mathcal{P}(\overline{\text{MIP-}p\text{-S}})$

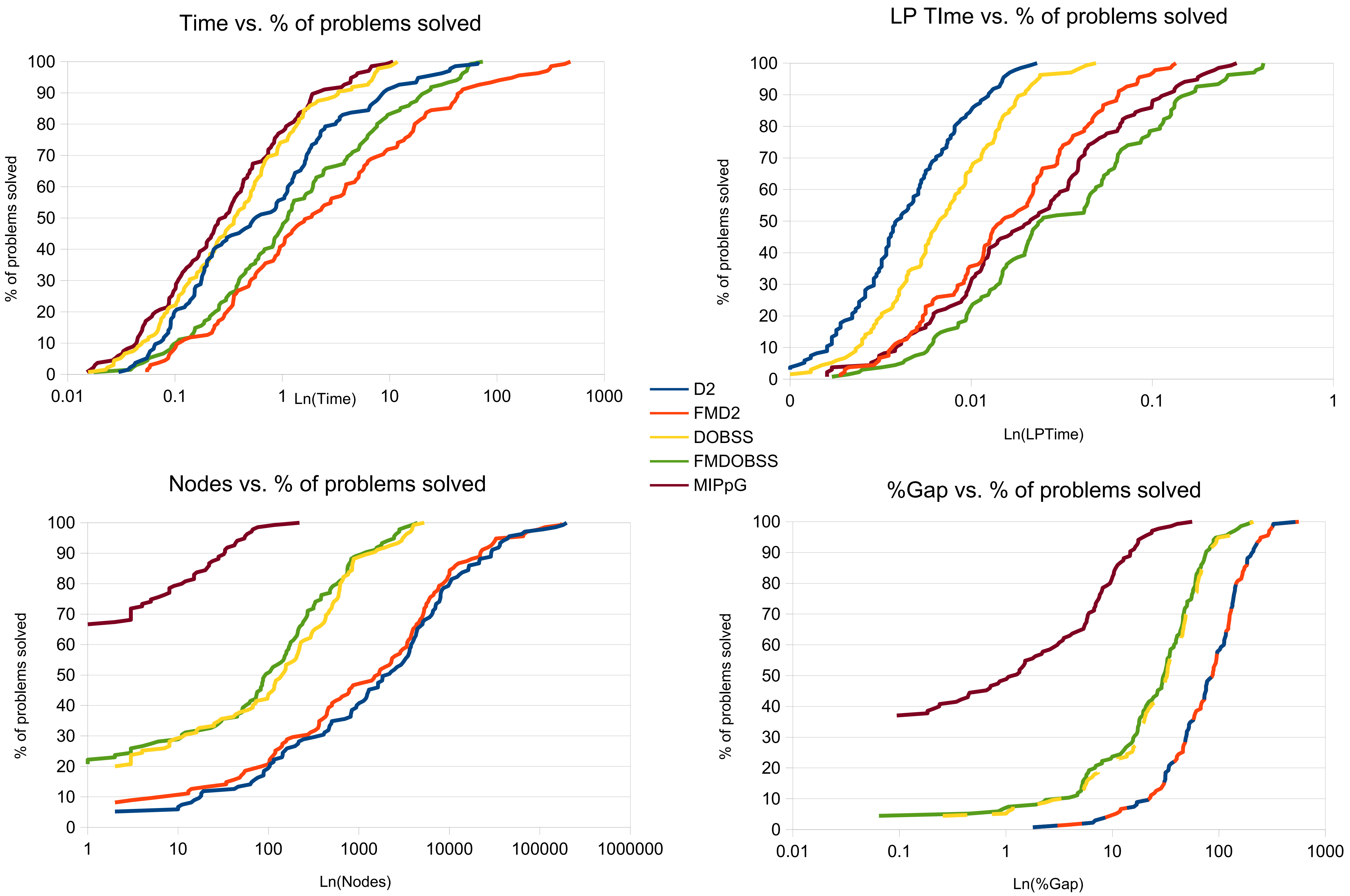
What we know

- $Proj_{c,q,f,s}(\overline{f\text{-SDOBSS}}) \subsetneq \mathcal{P}(\overline{\text{ERASER}})$
- $\mathcal{P}(\overline{\text{MIP-}p\text{-S}}) \subsetneq \mathcal{P}(\overline{\text{FMSDOBSS}}) = Proj_{c,q,y}\mathcal{P}(\overline{\text{SDOBSS}})$
- $v(\text{MIP-}p\text{-S}) \leq v(\text{SDOBSS}) = v(f\text{-SDOBSS}) \leq v(\text{ERASER})$
- $\mathcal{P}(\overline{\text{MIP-}p\text{-S}})$ integer polyhedron when $k = 1$

Outline

- MIP Formulations for Stackelberg Games
- Stackelberg Security Game
  - Formulations
  - What we know
- Computational Results

Estudio computacional



Tiempo máx.: 10800 sec.

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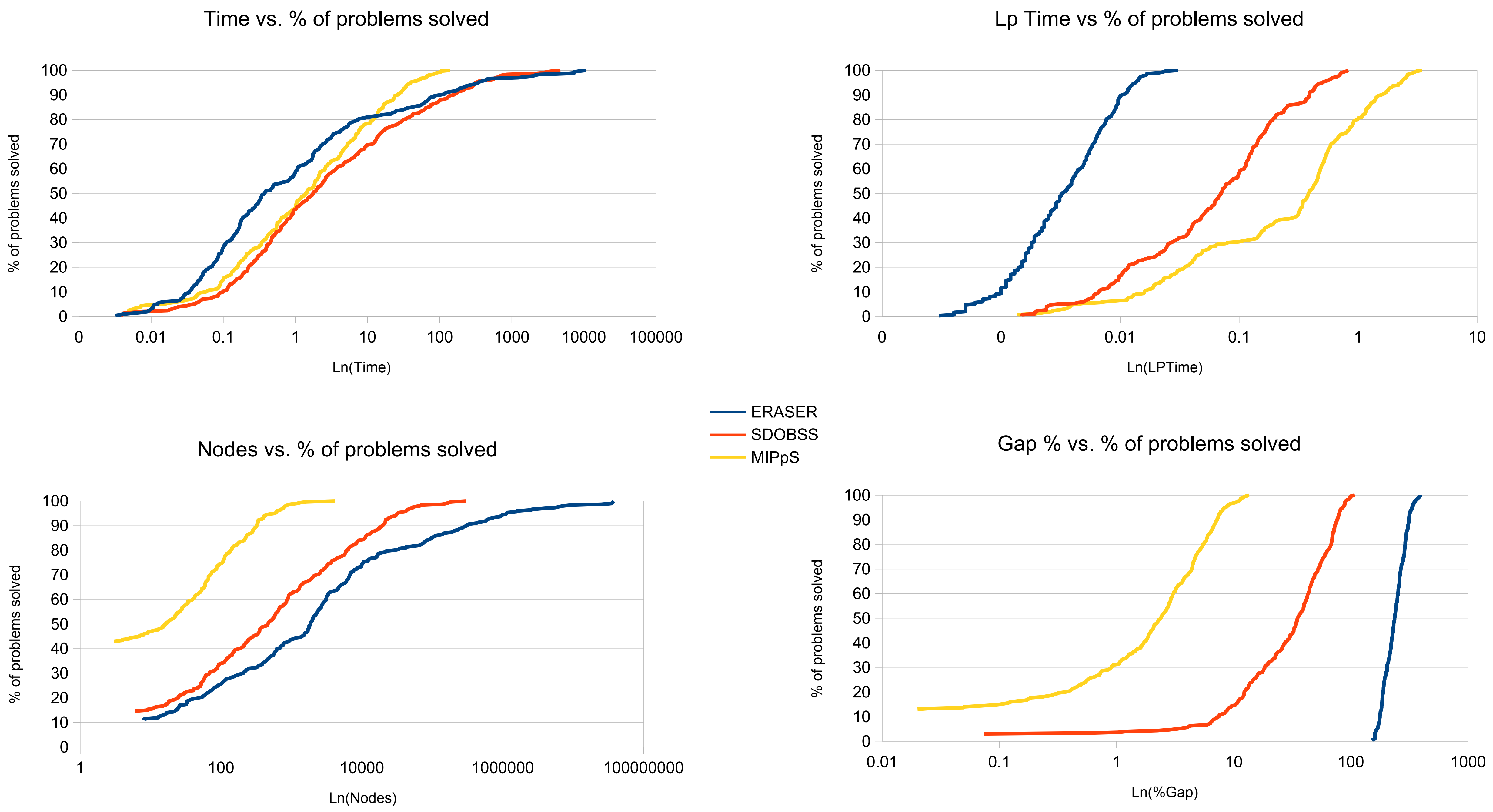
Estudio computacional


Time vs. % of problems solved

Lp Time vs % of problems solved

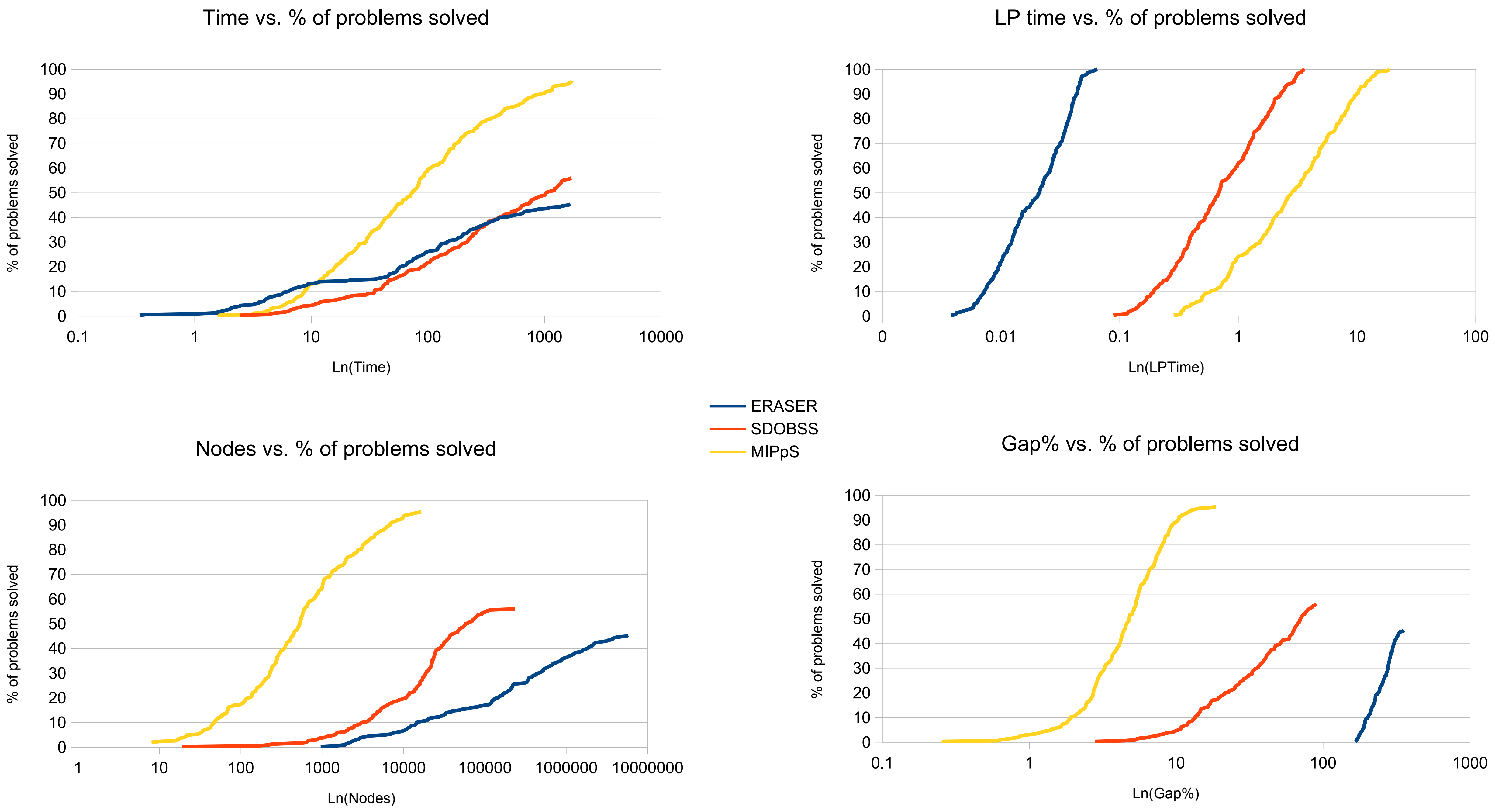
Nodes vs. % of problems solved

Gap % vs. % of problems solved

$$K = \{2, 4, 6, 8\}, J = \{10, 20, 30, 40, 50\}, m = \{25\%, 50\%, 75\%\}$$

Tiempo máx.: 10800 sec.

Estudio computacional



Time vs. % of problems solved

LP time vs. % of problems solved

Nodes vs. % of problems solved

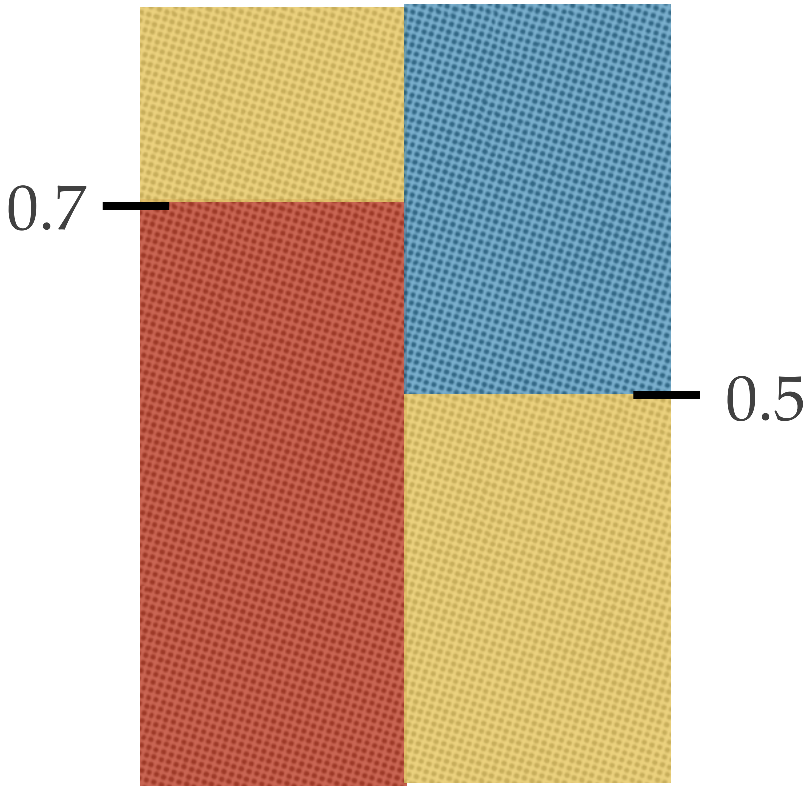
Gap% vs. % of problems solved

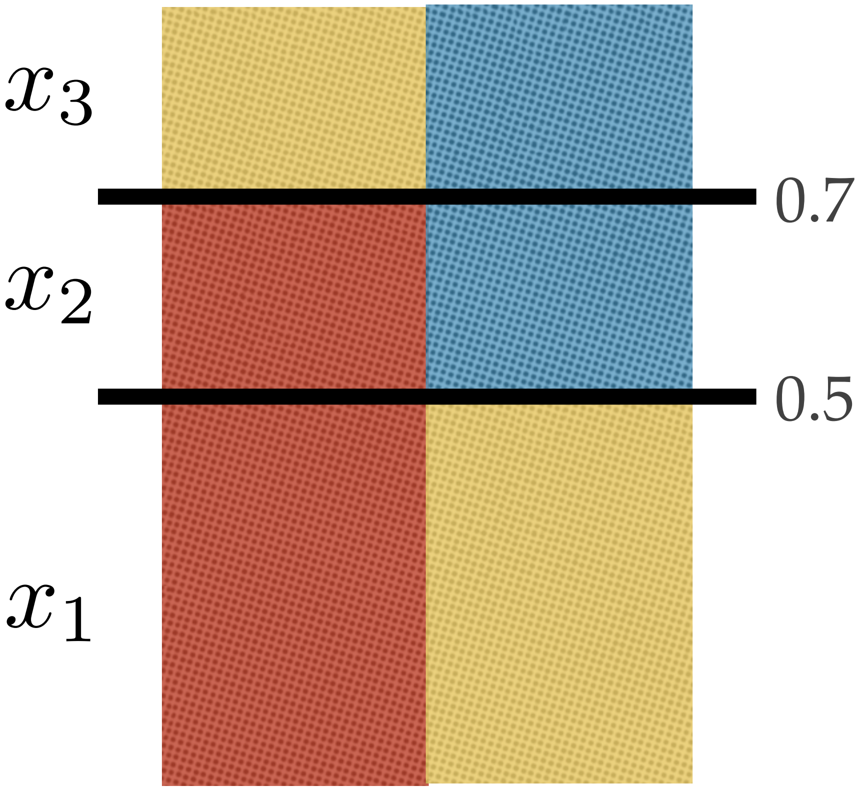
$$K = \{6, 8, 10, 12\}, J = \{30, 40, 50, 60, 70\}, m = \{25\%, 50\%, 75\%\}$$

Tiempo máx.: 1800 sec.

Retrieving mixed strategies
Example: m=2

Optimal sol.: $C_1=0.7$ $C_2=0.8$ $C_3=0.5$





Stackelberg Security Games

Using

$$\sum_{i \in X} R^k_{ij} x_i = \sum_{i \mid j \in i} R^k_{Dj} x_i + \sum_{i \mid j \notin i} \pi^k_{Dj} x_i = R^k_{Dj} y_j + \pi^k_{Dj}(1 - y_j)$$

$$(SSGM) \qquad \begin{array}{rl} \max_{x,q,a,\gamma} & \sum\limits_{k \in K} \pi^k \gamma^k \\ \text{s.t.} & \gamma^k \leq \pi^k_{Dj} + y_j(R^k_{Dj} - \pi^k_{Dj}) + (1 - q^k_j)M \\ & \sum_{j \in Q} y_j = m, \;\; y_j \in [0, 1] \\ & \sum_{j \in Q} q^k_j = 1, \;\; q^k_j \in \{0, 1\} \\ & 0 \leq a^k - R^k_{Aj} - y_j(P^k_{Aj} - R^k_{Aj}) \leq (1 - q^k_j)M \\ & a, \gamma \in \Re^{|L|} \\ & y_j = \sum_{\{i \in X \mid j \in i\}} x_i, \;\; \sum_{i \in X} x_i = 1, \;\; x_i \in [0, 1] \end{array}$$

Novel Formulations

- “$z_{ij}^k = x_i q_j^k$”
- $\sum_i C_{ij}^k x_i \geq \sum_i C_{ik}^k x_i \;\Leftrightarrow\; \sum_i (C_{ij}^k - C_{ik}^k) z_{ij}^k \geq 0$

$$
(MIPSG) \quad
\begin{array}{ll}
\max_z & \displaystyle\sum_{i \in X} \sum_{k \in K} \sum_{j \in J} \pi^k R_{ij}^k z_{ij}^k \\
\text{s.t.} & \sum_{i \in X} \sum_{j \in Q} z_{ij}^k = 1, \;\; z_{ij}^k \in [0,1] \\
& \sum_{j \in Q} z_{ij}^k = \sum_{j \in Q} z_{ij}^1 \\
& \sum_{i \in X} (C_{ij}^k - C_{ik}^k) z_{ij}^k \geq 0 \\
& \sum_{i \in X} z_{ij}^k \in \{0,1\}
\end{array}
$$

Novel Formulations

When $|L| = 1$, the LP relaxation of $(MIPSG)$ is integral.

Proof Sketch:
Let z be a vertex of the LP relaxation. Let $q_k = \sum_{i \in X} z_{ik}$ and assume some q_k is fractional.

For any $q_k > 0$ (at least 2, since $\sum_k q_k = 1$) define

$$z_{ij}^k = \begin{cases} \frac{z_{ik}}{q_k} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

z^k is feasible for the LP relaxation for any k such that $q_k > 0$

and $z = \sum_k q_k z^k$, which is a contradiction

Novel Formulation for SSG

$(MIPSSG)$

$$\begin{aligned}
\max_{z,w,q} \quad & \sum_{k\in K}\sum_{j\in J} \pi^k \left(R^k_{Dj} - \pi^k_{Dj}\right) w^k_{jj} + \pi^k \pi^k_{Dj} q^k_j \\
\text{s.t.} \quad & 0 \le w^k_{kj} \le q^k_j \\
& y_k - 1 + q^k_j \le w^k_{kj} \le y_k + 1 - q^k_j \\
& \textstyle\sum_{j\in Q} y_j = m, \quad 0 \le y \le 1 \\
& \left(\pi^k_{Aj} - R^k_{Aj}\right) w^k_{jj} + R^k_{Aj} q^k_j \ge \left(\pi^k_{Ak} - R^k_{Ak}\right) w^k_{kj} + R^k_{Ak} q^k_j \\
& \textstyle\sum_{j\in Q} q^k_j = 1, \quad q^k_j \in \{0,1\} \\
& w^k_{kj} = \textstyle\sum_{\{i\in X | k\in i\}} z^k_{ij}, \;\; \sum_{i\in X} z^k_{ij} = q^k_j, \;\; z^k_{ij} \in [0,1]
\end{aligned}$$

Novel Formulations

Logarithmic representation of integer choice

Theorem 1 *(Vielma 2014) Given* $\{b^i\}_{i=1}^k \subset \{0,1\}^{\lceil \log_2(k) \rceil}$*, such that* $b^i \neq b^j$ *if* $i \neq j$*. Then the set* $S = \{y \in \{0,1\}^k \mid \sum_{i=1}^k y_i = 1\}$ *is equivalent to*

$$\left\{ y \in \Re_+^k \mid \sum_{i=1}^k y_i = 1, \;\; \sum_{i=1}^k b^i y_i = w \in \{0,1\}^{\lceil \log_2(k) \rceil} \right\}$$

Novel Logarithmic Formulations

$$(LSGM)\qquad \begin{aligned} \text{max}_{x,q,a,\gamma,w} \quad & \sum_{k\in K} \pi^k \gamma^k \\ \text{s.t.} \quad & \gamma^k \leq \textstyle\sum_{i\in X} R^k_{ij} x_i + (1 - q^k_j) M \\ & \textstyle\sum_{i\in X} x_i = 1,\ \ x_i \in [0,1] \\ & \textstyle\sum_{j\in Q} q^k_j = 1,\ \ q^k_j \in [0,1] \\ & \textstyle\sum_{j\in Q} b^j q^k_j = w^k \in \{0,1\}^{\lceil \log_2(|Q|) \rceil} \\ & 0 \leq a^k - \textstyle\sum_{i\in X} C^k_{ij} x_i \leq (1 - q^k_j) M \\ & a, \gamma \in \Re^{|L|}. \end{aligned}$$

Novel Logarithmic Formulations

$$
(LMIPSG) \qquad
\begin{array}{ll}
\text{max}_{z,w} & \displaystyle\sum_{i\in X}\sum_{k\in K}\sum_{j\in J}\pi^k R^k_{ij} z^k_{ij} \\
\text{s.t.} & \sum_{i\in X}\sum_{j\in Q} z^k_{ij} = 1,\;\; z^k_{ij} \in [0,1] \\
 & \sum_{j\in Q} z^k_{ij} = \sum_{j\in Q} z^1_{ij} \\
 & \sum_{i\in X}(C^k_{ij} - C^k_{ik}) z^k_{ij} \geq 0 \\
 & \sum_{j\in Q}\sum_{i\in X} b^j z^k_{ij} = w^k \in \{0,1\}^{\lceil \log_2(|Q|) \rceil}
\end{array}
$$

Computational Results

These results compare

- Tighter second level optimality constraints: (SGM) vs $(MIPSG)$
- This effect for SSG : $(SSGM)$ vs $(MIPSSG)$
- The effect of the logarithmic formulation: (SGM) vs $(LSGM)$ and $(MIPSG)$ vs $(LMIPSG)$

Random instances, average of 30 instances, increasing $|X|$, $|Q|$, or $|L|$.

Computational Results

			MIPSG			DOBBS			SGM		
$\lvert X\rvert$	$\lvert Q\rvert$	$\lvert L\rvert$	Δt	gap	bnb	Δt	gap	bnb	Δt	gap	bnb
10	10	2	0.05	0.05	0	0.17	0.23	0	0.11	1.65	4
10	20	2	0.17	0.05	0	0.62	0.23	24	0.50	1.51	509
10	30	2	0.28	0.05	0	1.29	0.20	144	0.80	1.35	1115
20	10	2	0.08	0.03	0	0.19	0.12	0	0.16	1.48	73
20	20	2	0.30	0.03	1	0.69	0.13	103	0.88	1.41	694
20	30	2	0.61	0.03	2	1.64	0.11	172	1.23	1.32	1604
30	10	2	0.09	0.02	0	0.18	0.09	23	0.22	1.43	127
30	20	2	0.42	0.03	4	0.94	0.09	92	1.03	1.37	741
30	30	2	1.01	0.02	4	2.43	0.09	170	1.61	1.32	1693
10	10	4	0.35	0.15	24	0.90	0.47	314	1.55	2.20	2292
10	20	4	1.65	0.14	49	4.76	0.40	1140	6.68	1.80	1.3+4
10	30	4	4.77	0.16	67	15.94	0.39	3082	15.65	1.70	3.0+4
20	10	4	0.95	0.15	49	1.72	0.33	478	3.31	1.91	5333
20	20	4	5.90	0.13	99	10.39	0.29	2012	25.17	1.72	4.3+4
20	30	4	20.88	0.14	170	56.48	0.29	5588	80.91	1.65	1.2+5
30	10	4	1.65	0.13	69	2.43	0.28	501	5.55	1.84	7999
30	20	4	14.15	0.14	188	24.17	0.27	3077	49.72	1.74	7.1+4
30	30	4	52.69	0.13	291	114.01	0.25	6684	196.45	1.62	2.4+5
10	10	6	2.14	0.31	266	5.28	0.70	2967	13.13	2.70	2.5+4
10	20	6	12.30	0.27	466	44.77	0.59	1.6+4	71.74	2.19	1.5+5
10	30	6	40.51	0.26	742	187.50	0.55	3.4+4	211.86	1.98	3.3+5
20	10	6	5.97	0.23	366	13.35	0.47	4709	65.07	2.27	1.2+5
20	20	6	68.81	0.25	1286	301.46	0.46	4 .5+4	1085.62	2.08	1.7+6

Computational Results

			MIPSG			DOBBS			SGM		
$\lvert Q\rvert$	m	$\lvert L\rvert$	Δt	gap	bnb	Δt	gap	bnb	Δt	gap	bnb
20	5	2	0.67	26.72	48	0.72	50.04	51	0.13	92.61	198
20	10	2	0.91	0.68	28	0.77	1.41	26	0.14	4.11	144
30	8	2	2.60	2.16	93	2.15	3.33	113	0.22	6.88	470
30	15	2	1.80	0.40	26	1.43	0.84	25	0.19	2.79	260
40	10	2	4.89	2.06	103	3.63	3.08	136	0.33	6.22	707
40	20	2	3.47	0.39	33	3.16	0.80	43	0.26	2.67	437
20	10	4	8.59	1.09	571	7.54	1.92	646	2.17	4.80	3893
30	8	4	47.82	5.02	1853	37.48	7.49	2879	9.46	13.50	6272
30	15	4	26.93	0.86	826	24.34	1.42	949	7.31	3.77	5216
40	10	4	144.27	4.27	3182	155.60	6.30	5401	14.44	11.13	8138
40	20	4	70.62	0.80	1236	60.81	1.34	1358	12.99	3.55	6481
20	10	6	142.60	1.47	5995	155.82	2.43	8037	26.39	5.78	1.9+4
30	8	6	961.21	12.31	1.4+4	1433.32	16.48	3.7+4	94.50	28.08	5.4+4

Computational Results

				MIPSG					SGM				
$\|X\|$	$\|Q\|$	$\|L\|$		nvar	nvari	ncost	Δt	gap	nvar	nvari	ncost	Δt	gap
20	10	4		840	40	468	1.65	0.14	68	40	125	4.04	1.93
20	20	4		1680	80	1668	7.95	0.14	108	80	245	28.83	1.76
20	30	4		2518	119	3666	21.75	0.12	148	120	365	82.05	1.61
20	10	4	Log	816	16	440	2.42	0.14	84	16	141	4.75	1.93
20	20	4	Log	1620	20	1604	12.03	0.14	128	20	265	31.25	1.76
20	30	4	Log	2420	20	3564	37.69	0.12	168	20	385	73.74	1.61
30	10	4		1240	40	498	2.51	0.12	78	40	125	7.19	1.80
30	20	4		2480	80	1698	16.56	0.14	118	80	245	58.04	1.76
30	30	4		3720	120	3698	53.32	0.13	158	120	365	217.72	1.63
30	10	4	Log	1216	16	470	3.83	0.12	94	16	141	6.68	1.80
30	20	4	Log	2420	20	1634	24.59	0.14	138	20	265	54.35	1.76
30	30	4	Log	3620	20	3594	90.93	0.13	178	20	385	155.09	1.63
10	10	6		644	58	647	2.80	0.27	82	60	187	15.08	2.58
10	10	6	Log	617	24	613	3.80	0.27	106	24	211	22.81	2.58