The Splitting Game: value and optimal strategies

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Second Workshop on ADGO

January 26, 2016

Universidad de Santiago

General context

Game = interdependent strategic interaction between players

- Nature of the interaction
    - Cooperative
    - Evolutionary
    - Non-cooperative

- Number of players
    - Infinitely many (non-atomic)
    - $N > 2$ players
    - 2 players

- Players' preferences
    - Structure (potential)
    - Identical (coordination, mean field, congestion)
    - Opposite (zero-sum)

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Zero-sum games

A zero-sum game is a triplet (S, T, g), where

- S is the set of actions of player 1
- T is the set of actions of player 2
- $g : S \times T \to \mathbb{R}$ is the payoff function

The game is said to be finite when $S = \Delta(I)$ and $T = \Delta(J)$ are probabilities on finite sets (g is a matrix and actions are mixed strategies)

It admits a value when

$$\sup_{s \in S} \inf_{t \in T} g(s, t) = \inf_{t \in T} \sup_{s \in S} g(s, t)$$

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We are interested in the following two questions:

(a) Existence and description of the value

(b) Existence and description of optimal strategies (or ε-optimal)

Zero-sum games with incomplete information

– Consider a finite family of matrix games $(G^k)_{k \in K}$, where $G^k = (I, J, g^k)$ corresponds to the state of the world occurring with probability p^k

– The state of the world stands for the player's types, their beliefs about the opponents' types, and so on

– Each player has an information set, i.e. a partition of the state of world

Example: three states and information sets $\{1\}, \{2, 3\}$ and $\{1, 2\}, \{3\}$

$$p^1 \qquad\qquad p^2 \qquad\qquad p^3$$

$$G^1 \qquad\qquad G^2 \qquad\qquad G^3$$
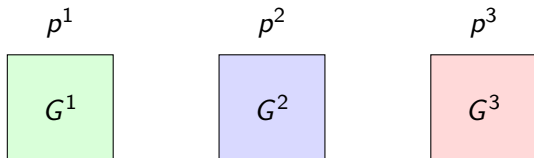
– A state of the world occurs according to $p \in \Delta(K)$; player 1 knows whether it is $\{1\}$ or $\{2, 3\}$, and player 2 knows whether it is $\{1, 2\}$ or $\{3\}$

An equivalent formulation

Alternatively, the players' information structure (i.e. the set of states, the information sets and the probability p) can represented as follows:

- The set of possible types is a product set $K \times L$ and the payoff function depend on the pair of types, i.e. $G^{k\ell} : I \times J \to \mathbb{R}$
- $\pi \in \Delta(K \times L)$ is a probability measure on the set of types
- A couple of types (k, ℓ) is drawn according to π. Player 1 is informed of k and player 2 of ℓ

In the previous example: $K = L = \{1, 2\}$ and

$$\pi = \begin{array}{|c|c|} \hline p^1 & p^3 \\ \hline p^2 & 0 \\ \hline \end{array}$$

Remarks. – The players have private, dependent information
– If L is a singleton, the incomplete information is **on one side**

Repeated games with incomplete information

- Aumann and Maschler consider repetition of games with incomplete information to analyze the strategic use of private information
- A **repeated game with incomplete information** is described by a 6-tuple (I, J, K, L, G, π) where I and J are the sets of actions, K and L the set of types, $G = (G^{k\ell})_{k,\ell}$ the payoff function and π a probability on $K \times L$
- The game is played as follows. First, a couple (k, ℓ) is drawn according to π and each player is informed of one coordinate. Then, the game $G^{k\ell}$ is played over and over: at each stage $m \geq 1$, knowing the past actions, the players choose actions (i_m, j_m)

Strategies and evaluation of the payoff

- Strategies are functions from histories to mixed actions. Here $\sigma = (\sigma_m)_m$ where $\sigma_m : K \times (I \times J)^{m-1} \to \Delta(I)$ and similarly τ stands for strategy of player 2

- Let $\mathbb{P}^{\pi}_{\sigma,\tau}$ be the unique probability distribution on finite plays $h_m = (k, \ell, i_1, j_1, \ldots, i_{m-1}, j_{m-1})$ induced by π, σ and τ

- Player 1 maximizes $\gamma_\theta(\pi, \sigma, \tau) = \mathbb{E}^{\pi}_{\sigma,\tau}[\sum_{m \geq 1} \theta_m G^{k\ell}(i_m, j_m)]$ where $\theta_m \geq 0$ is the weight of stage m

- Two important cases: the *n-stage game* and the *λ-discounted game* which correspond to weights:

$$\left(\frac{1}{n}\mathbb{1}_{\{m \leq n\}}\right)_{m \geq 1} \quad \text{and} \quad \left(\lambda(1-\lambda)^{m-1}\right)_{m \geq 1}$$

Approches: Horizon, Value and Strategies

- **Fixed duration** (fixed evaluation θ)

  (a) ...

  (b) ...

- **Asymptotic approach** ($\sup_{m\geq 1}\theta_m \rightarrow 0$)

  (a) ...

  (b) ...

- **Uniform approach** (the weights are "sufficiently small")

  (a) ...

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Approches: Horizon, Value and Strategies

- **Fixed duration** (fixed evaluation θ)

    (a) Description of the values

    (b) Description of optimal strategies

- **Asymptotic approach** ($\sup_{m \geq 1} \theta_m \to 0$)

    (a) Convergence of the values and caracterization of the limit

    (b) Description of asymptotically optimal strategies

- **Uniform approach** (the weights are "sufficiently small")

    (a) Existence of the uniform value

    (b) Description of robust optimal strategies

Main results on RGII (one or two sides)

Info / Horizon	Asymptotic	Uniform
One side	$\lim_{\theta \to 0} V_\theta = \mathrm{Cav}\, u$ Aumann - Maschler 67	$v_\infty = \mathrm{Cav}\, u$ Aumann - Maschler 67
Two sides	$\lim_{\theta \to 0} V_\theta = MZ(u)$ Mertens-Zamir 71	v_∞ does not exist

The benefit of private information

The use of private information has two effects during the play

(1) Transmits information about the true types. Indeed, let π_m be the conditional probability on $K \times L$ given h_m under $\mathbb{P}^\pi_{\sigma,\tau}$. The players jointly generate the martingale of posteriors $(\pi_m)_m$

(2) Provides an instantaneous benefit

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(2) Provides an instantaneous benefit $\Longrightarrow$ irrelevant in the long run:

$$\left| \gamma_\theta(\pi, \sigma, \tau) - \mathbb{E}^\pi_{\sigma,\tau} \big[\sum_{m \geq 1} \theta_m u(\pi_m) \big] \right| \leq C \big(\sup_{m \geq 1} \theta_m \big)^{1/2}$$

where $u(\pi)$ is the value of the non-revealing game

$$u(\pi) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} \sum_{k,\ell} \pi^{k\ell} G^{k\ell}(x, y)$$

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- The splitting game is introduced by Laraki 2001 and Sorin 2003 motivated by the previous remark

The splitting game (one side)

- Consider the case $|L| = 1$ (i.e. player 1 is informed and player 2 is not)
- The initial probability can be seen as $p \in \Delta(K)$ and the possible games as $(G^k)_{k \in K}$
- Let $u(p) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} \sum_{k \in K} p^k G^k(x, y)$
- $\left| V_\theta(p) - \sup_{(p_m)_{m \geq 1}} \mathbb{E}[\sum_{m \geq 1} \theta_m u(p_m)] \right| \leq C \left(\sup_{m \geq 1} \theta_m \right)^{1/2}$

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Taking the limit, we obtain a martingale optimization problem:

$$v(p) = \sup_{\mathbf{p} \in \mathcal{M}(p)} \mathbb{E} \left[\int_0^1 u(p_t) dt \right]$$

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What is the value ? What about optimal martingales?

The splitting game (two sides, independent case)

- In the independent case, $\pi = p \otimes q$, with $p \in \Delta(K)$ and $q \in \Delta(L)$
- The initial probability can be writen as (p, q)
- Let $u(p, q) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} \sum_{k,\ell} p^k q^\ell G^{k\ell}(x, y)$
- $|V_\theta(p, q) - W_\theta(p, q)| \leq C \left(\sup_{m \geq 1} \theta_m \right)^{1/2}$ where

$$W_\theta(p, q) = \sup_{(p_m)_m} \inf_{(q_m)_m} \mathbb{E}[\sum_{m \geq 1} \theta_m u(p_m, q_m)]$$

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What is the value ? What about optimal martingales?

The independent SG is defined and studied by Laraki 2001 replacing $\Delta(K)$ and $\Delta(L)$ by convex sets of an euclidean space

Main results

    (1) Existence of the value $W_\lambda(p, q)$

    (2) Convergence of $W_\lambda(p, q)$ to $\lim_{\lambda \to 0} V_\lambda = MZ(u)$

    (3) Variational characterization of $MZ(u)$

Further results

Two sides SG

- Cardaliaguet, Laraki and Sorin 2011 prove the convergence of $W_\theta(p, q)$ to $MZ(u)$ as $\theta \to 0$

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One side, time-dependent SG

- In the framework of continuous-times games, Cardaliaguet and Rainer 2009 study the splitting game

$$v(t_0, p) = \sup_{\mathbf{p} \in \mathcal{M}(p)} \mathbb{E}\left[\int_{t_0}^1 u(t, p_t)dt\right]$$

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Two sides, time-dependent SG

- CLS 11 prove the convergence of $W_\theta(t, p, q)$ as $\theta \to 0$ and characterize the limit

Contributions of the paper

The splitting game: uniform value and optimal strategies

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The splitting game: uniform value and optimal strategies

- Definition of dependent splitting game, existence of the value $W_\theta(\pi)$
- Convergence $W_\theta(\pi)$ to $MZ(u)$ as $\theta \to 0$, for general π
- A comparison principle for the the uniqueness of a solution to MZ

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The splitting game: uniform value and optimal strategies

- Definition of dependent splitting game, existence of the value $W_\theta(\pi)$
- Convergence $W_\theta(\pi)$ to $MZ(u)$ as $\theta \to 0$, for general π
- A comparison principle for the the uniqueness of a solution to MZ
- Existence of the uniform value in the SG and exhibition of a couple of optimal strategies with the additional property that the martingale $(\pi_m)_m$ is constant after stage 2

The splitting game

- The splitting game is a stochastic game played on $\Delta(K \times L)$, and where the actions are splittings
- It is described by a 7-tuple $(S, A, B, u, \Phi, \pi, \theta)$ where
    - $S = \Delta(K \times L)$ is the set of states
    - A and B are the sets of splittings
    - $u : S \to \mathbb{R}$ is the payoff function
    - $\Phi : S \times A \times B \to \Delta(S)$ is the transition function
    - $\pi \in S$ is the initial state
    - $\theta = (\theta_m)_m$ is the sequence of weights for the stages
- Strategies are functions from finite histories into splittings
- Player 1 maximizes $\mathbb{E}^{\pi}_{\sigma,\tau}[\sum_{m \geq 1} \theta_m u(\pi_m)]$ where $\mathbb{P}^{\pi}_{\sigma,\tau}$ is the unique probability distributions on finite histories induced by π, σ, τ
- We denote the maxmin and minmax by $W^{-}_{\theta}(\pi)$ and $W^{+}_{\theta}(\pi)$

The splittings

- For any $\pi \in \Delta(K \times L)$ let
    - Let $\pi^K \in \Delta(K)$ be its marginal on K
    - Let $\pi^{L|K} \in \Delta(L)^K$ be the matrix of conditionals on L given $k \in K$
    - Let $\pi^L \in \Delta(L)$ be its marginal on L
    - Let $\pi^{K|L} \in \Delta(K)^L$ be the matrix of conditionals on K given $\ell \in L$

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- For any $p \in \Delta(K)$ let
  - $\Delta_p(\Delta(K))$ be the set of probabilities on $\Delta(K)$ with expectation p

- The set of splittings at π are $A(\pi) := \Delta_p(\Delta(K))$, with $p = \pi^K$ and $B(\pi) := \Delta_q(\Delta(L))$, with $q = \pi^L$

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- $\Phi(\pi, a, b)$ is the unique probability distribution on S induced by π, a and b. It is a splitting of $\Delta_\pi(S)$

- In the independent case, every player controls a separate martingale and $\Phi(\pi, a, b) = a \otimes b$

Notation

For any $f : \Delta(K \times L) \to \mathbb{R}$, $Q \in \Delta(L)^K$ and $P \in \Delta(K)^L$ we set

- $f_K(\,\cdot\,, Q) : \Delta(K) \to \mathbb{R}, \quad p \mapsto f(p \otimes Q)$
- $f_L(\,\cdot\,, P) : \Delta(L) \to \mathbb{R}, \quad q \mapsto f(q \otimes P)$

f is K-concave if f_K is concave on $\Delta(K)$
f is L-convex if f_L is convex on $\Delta(L)$

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f is *K-concave* if f_K is concave on $\Delta(K)$
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Mertens-Zamir system of equations:

$$f_K(p, Q) = \mathrm{Cav}_{\Delta(K)} \min\{u_K, f_K\}(p, Q), \quad \forall p, Q$$

$$f_L(q, P) = \mathrm{Vex}_{\Delta(L)} \max\{u_L, f_L\}(q, P), \quad \forall q, P$$

The unique solution is denoted $v = MZ(u)$

Main results

Theorem 1. The SG has a value $W_\theta(\pi)$. Moreover

- $\pi \mapsto W_\theta(\pi)$ is K-concave, L-convex and Lipschitz
- $W_\theta(\pi) = \max_{a \in A(\pi)} \min_{b \in B(\pi)} \mathbb{E}[\theta_1 u(\pi') + W_{\theta^+}(\pi')]$

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Elements of the proof
(1) $(\pi, a, b) \mapsto \Phi(\pi, a, b)$ is continuous and bi-linear
(2) Define the dependent splitting operator

$$f \mapsto \varphi(f)(\pi) = \max_{a \in A(\pi)} \min_{b \in B(\pi)} \mathbb{E}_{\Phi(\pi, a, b)}[f(\pi')]$$

(3) Establish a recurrence formula for W_θ^- and W_θ^+

Define the following 4 properties for real functions on $\Delta(K \times L)$

$(P1)$ f is L-convex

$(P2)$ $f_K(p, Q) \leq \mathrm{Cav}_{\Delta(K)} \min\{u_K, f_K\}(p, Q)$ for all p, Q

$(Q1)$ f is K-concave

$(Q2)$ $f_L(q, P) \leq \mathrm{Vex}_{\Delta(L} \max\{u_L, f_L\}(q, P)$ for all q, P

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(Q2) $f_L(q, P) \leq \mathrm{Vex}_{\Delta(L} \max\{u_L, f_L\}(q, P)$ for all q, P

Theorem 2. Let $f, g : \Delta(K \times L) \to \mathbb{R}$ be such that f satisfies (P1)-(P2) and g satisfies (Q1)-(Q2). Then

$$f \leq W_\infty^- \leq W_\infty^+ \leq g$$

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Elements of the proof

(1) For any f satisfying (P1)-(P2) define a strategy $\sigma(\varepsilon, f)$, $\pi_m \mapsto a_m$ such that

$$\mathbb{E}_{a_m}[\min\{u_K, f_K\}(p, Q_m) \geq f(\pi_m) - \varepsilon/2^m$$

(2) Define the auxiliary steps $\pi_{m+1/2}$ and work with $(\pi_{m/2})_{m \geq 1}$

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– There exists optimal strategies such that $(\pi_m)_{m \geq 2}$ is constant

– The strategy for player 1 is as follows:

(i) If $u(\pi) \geq v(\pi)$, play δ_p

(ii) If $u(\pi) < v(\pi)$, play $a = \sum_{r \in R} \lambda^r \delta_{p^r}$ where
$u(\pi^r) \geq v(\pi^r)$ for all $r \in R$ and $\sum_{r \in R} \lambda^r \min\{u, v\} v(\pi_r) = v(\pi)$

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Elements of the proof

(1) The MZ system has a unique solution (Mertens-Zamir 71)

(2) Use the above strategy with $\sigma(\varepsilon, v)$, where $v = MZ(u)$

(3) Use the characterization of v from Mertens-Zamir

Corollary. $W_\theta \to v = MZ(u)$, as $\sup_m \theta_m \to 0$

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Proof: The existence of the uniform value implies this statement

Corollary. $V_\theta \to v = MZ(u)$, as $\sup_m \theta_m \to 0$, where V_θ is the value of the repeated games with incomplete information

Proof: $|W_\theta - V_\theta| \le C(\sup_m \theta_m)^{1/2}$ for all evaluations θ

Remarks

- In repeated games with incomplete information the uniform value does not exist : each players prefers the other to reveal first
- Although asymptotically equivalent, **a crucial (and surprising) difference is that the Splitting Game has a uniform value**. Observing the other player's use of information makes the game strategically very stable: under optimal play $\implies$ at most one splitting.

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- The optimal uniform strategy is very simple and "trivializes the game"
- Recently, economists are looking at *commitment strategies* for games with incomplete information, i.e. assume the players can commit to playing some strategy $(\sigma^k)_{k \in K}$. We are then in the splitting game and uniform equilibrium exists

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- Use the splitting game to study non-zero-sum repeated games with incomplete information

Moltes gràcies !

Thanks for your attention