Insights into the nucleolus of the assignment game

Javier Martínez de Albéniz[1], Carles Rafels[1] and Neus Ybern[2]

[1]Universitat de Barcelona

[2]Universitat Politècnica de Catalunya

ADGO'16 Santiago de Chile, January 29, 2016

Outline

We want to address the following questions:

- How to compute the nucleolus of the assignment game?
- Are there many matrices with the same nucleolus?
- Which vectors can be a nucleolus?
- Which is the structure of ass. games with the same nucleolus?

Why?

- The nucleolus for the general case is computed by a series of linear programs.
- No formulas are known. Only an adapted algorithm (Solymosi & Raghavan, 1994) is available.

• A procedure based on giving equal 'dividends' to the agents, until some agents leave, and then giving to the rest of agents in an ordered manner. Oper Res Lett, 2013

• Necessary and sufficient conditions for a vector to be a nucleolus.

• The family of matrices with the same nucleolus is a join-semilattice with one maximal element.

• Its unique maximum element is a valuation matrix and we give its explicit form.

• It is a path-connected set, and we give the precise path. We construct some minimal elements of the family,

• We give a rule to compute the nucleolus in some specific cases.

What is an assignment market?

Introduced by Shapley and Shubik (1972):
There are two sides: sellers and buyers.

- M finite set of buyers,
- M' finite set of sellers,
- A non-negative matrix of profits:

a_{ij} joint profit if $i \in M, j \in M'$ trade.

$$(M, M', A)$$

Each buyer demands exactly one unit of an indivisible good (houses, horses), and each seller supplies one unit of the good.

From the valuations of the buyers and the reservation prices of the sellers, a non-negative matrix can be obtained that represents the joint profit that each buyer-seller pair can achieve.

A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection between a subset of M and a subset of M'.

We write $(i, j) \in \mu$ as well as $j = \mu(i)$ or $i = \mu^{-1}(j)$. The set of all maximal matchings is denoted by $\mathcal{M}(M, M')$.

✓ A *matching* $\mu \in \mathcal{M}(M, M')$ is *optimal* for the assignment market (M, M', A) if for all $\mu' \in \mathcal{M}(M, M')$ we have

$$\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij},$$

and we denote the set of optimal matchings by $\mathcal{M}_A^*(M, M')$.

Cooperative games with transferable utility

A TU- cooperative game in coalitional form is described by a pair (N, v)

$N = \{1, 2, \ldots, n\}$ is the set of players,

$v(S)$ is the worth of the coalition $S \subseteq N$, with $v(\varnothing) = 0$.

By Shapley and Shubik (1972):

- players: $N = M \cup M'$, and
- characteristic function w_A, defined by:

for $S \subseteq M$ and $T \subseteq M'$,

$$w_A(S \cup T) = \max \left\{ \sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T) \right\}.$$

Coalitions of buyers only or sellers only get zero.

The best that a coalition can do is to find the best pairs and pool the profit.

Consider the following matrix:

$$A = \begin{pmatrix} \mathbf{4} & 5 & 5 \\ 4 & \mathbf{5} & 1 \\ 4 & 1 & \mathbf{5} \end{pmatrix}$$

An optimal matching is set in boldface

$$\mu_A^* = \{(1, 1'), (2, 2'), (3, 3')\}$$

and the worth of the grand coalition is:

$$w_A(N) = 14.$$

✓ How to allocate this total worth $w_A(N)$?
In a way such that no coalition has incentives to block the formation of the grand coalition: the core.

$$Core\,(w_A) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = w_A\,(N)\,, \sum_{i \in S} x_i \geq w_A\,(S)\,, \text{ for all } S \subseteq N \right\}.$$

✓ In the case of assignment games, it is proved that the core coincides with the set of stable allocations: efficient and such that no buyer-seller pair would do better by rejecting the assigned partner and the proposed payoff and being matched together.
This guarantees that third-party payments are excluded in the core of the assignment market.

For the core of the assignment game it is enough to impose coalitional rationality for one-player coalitions and mixed-pair coalitions:

$$Core\left(w_A\right)=\left\{(u,v)\in\mathbb{R}_+^M\times\mathbb{R}_+^{M'}\ \middle|\ \begin{array}{c}\sum_{i\in M}u_i+\sum_{j\in M'}v_j=w_A\left(N\right),\\ u_i+v_j\geq a_{ij},\ \text{for all } (i,j)\in M\times M'\end{array}\right\}$$

Then, $(u,v)\in Core(w_A)$ if and only if for any optimal assignment $\mu_A^*\in\mathcal{M}_A^*\left(M,M'\right)$ the following holds true:

1. $u_i+v_j=a_{ij}$ if $(i,j)\in\mu_A^*$
2. $u_i+v_j\geq a_{ij}$ if $(i,j)\notin\mu_A^*$
3. any player who is not assigned receives a payoff equal to 0, i.e.
   $u_i=0$ if $(i,j)\notin\mu_A^*$ $\forall j\in M'$,
   $v_j=0$ if $(i,j)\notin\mu_A^*$ $\forall i\in M$.

Some properties

✓ The core is always **non-empty** (Shapley and Shubik, 1972)

Some properties

✓ The core is always **non-empty** (Shapley and Shubik, 1972)

✓ The core of the assignment game has a **lattice structure** with two opposite extreme points:
the **buyers-optimal core allocation**, where each buyer receives her maximum core payoff,
and the **sellers-optimal core allocation** where each seller does.

Some properties

✓ The core is always **non-empty** (Shapley and Shubik, 1972)

✓ The core of the assignment game has a **lattice structure** with two opposite extreme points:
the **buyers-optimal core allocation**, where each buyer receives her maximum core payoff,
and the **sellers-optimal core allocation** where each seller does.

✓ Demange (1982) and Leonard (1983) prove that, if the buyers-optimal core allocation is implemented, it is a dominant strategy for each buyer to reveal her true valuations. Similarly, truth-telling is a dominant strategy for the sellers under a mechanism that assigns to each market its sellers-optimal core allocation.

• The **nucleolus** (Schmeidler, 1969) is the unique core element that lexicographically minimizes the vector of non-increasingly ordered excesses of coalitions.

If $x \in C(w_A)$, define for each coalition $S \subseteq M \cup M'$ its excess as

$$e\left(S, x\right) := w_A\left(S\right) - \sum_{i \in S} x_i.$$

✓ For assignment games (see Solymosi and Raghavan, 1994) the only coalitions that matter are the individual and mixed-pair ones.

Define the vector $\theta\left(x\right)$ of excesses of individual and mixed-pair coalitions arranged in a non-increasing order.

The nucleolus of the assignment game

The **nucleolus** of the game $(M \cup M', w_A)$ is the **unique** allocation

$$\nu(w_A) \in C(w_A)$$

which minimizes $\theta(x)$ with respect to the lexicographic order over the set of core allocations.

The *lexicographic order* $\geq_{lex}$ on $\mathbb{R}^d$, is defined in the following way: $x \geq_{lex} y$, where $x, y \in \mathbb{R}^d$, if $x = y$ or if there exists $1 \leq t \leq d$ such that $x_k = y_k$ for all $1 \leq k < t$ and $x_t > y_t$.

Llerena and Núñez (2011) characterize the nucleolus of a square assignment game from a geometric point of view.
The nucleolus is the unique core allocation that is the midpoint of some well-defined segments inside the core.

Let $\emptyset \neq S \subseteq M$, and $\emptyset \neq T \subseteq M'$, with $|S| = |T|$

$$\delta^A_{S,T}(u,v) := \min_{i \in S,\ j \in M' \setminus T} \{u_i, u_i + v_j - a_{ij}\},$$
$$\delta^A_{T,S}(u,v) := \min_{j \in T,\ i \in M \setminus S} \{v_j, u_i + v_j - a_{ij}\},$$

for any core allocation $(u,v) \in C(w_A)$.
This is the **largest amount that can be transferred** from players in S to players in T with respect to the core allocation (u,v) while remaining in the core.

The nucleolus is the unique core allocation $(u,v) \in C(w_A)$ such that

$$\delta^A_{S,T}(u,v) = \delta^A_{T,S}(u,v)$$

for any $\emptyset \neq S \subseteq M$ and $\emptyset \neq T \subseteq M'$ with $|S| = |T|$.

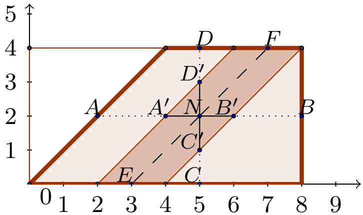
Notice that if $T \neq \mu(S)$ for some $\mu \in \mathcal{M}^*_A(M,M')$, then $\delta^A_{S,T}(u,v) = \delta^A_{T,S}(u,v) = 0$. Then, for this characterization we only check the case $T = \mu(S)$ for all optimal matchings.

[bisection property]

Consider

$$A = \begin{pmatrix} 8 & 6 \\ 4 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 8 & 4 \\ 0 & 4 \end{pmatrix}.$$

The worth to share is $v^* = 12$, and their nucleolus are in both cases $(5, 2, 3, 2) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$.
We depict the core of the associated assignment games and their nucleolus. We depict the projection on the buyers' (first) coordinates of the core of both games. The core of the first one $C(w_A)$ is in dark shading and the second one $C(w_B)$ in light shading.



When a vector may be a nucleolus?

Notice that **not** any vector is a candidate to be a nucleolus. For example,
$(3, 2, 1, 4) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$
can never be the nucleolus of any 2×2 assignment game,

Nevertheless, and curiously enough, for the non-square case, that is $|M| \neq |M'|$, the vector $(3, 2, 1, 4, 0) \in \mathbb{R}^2_+ \times \mathbb{R}^3_+$ may be the nucleolus of an assignment game, for example, for the assignment game associated to

$$\begin{pmatrix} 4 & 6 & 0 \\ 0 & 6 & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 7 & 2 \\ 3 & 5 & 0 \end{pmatrix}.$$

1. For square markets

$(x,y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$, with $|M| = |M'|$.

The vector (x,y) is the nucleolus of a square assignment game if and only if

$$\min_{i \in M} \{x_i\} = \min_{j \in M'} \{y_j\}\,.$$

When a vector may be a nucleolus?

1. For square markets

$(x, y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$, with $|M| = |M'|$.

The vector (x, y) is the nucleolus of a square assignment game if and only if

$$\min_{i \in M} \{x_i\} = \min_{j \in M'} \{y_j\}.$$

2. For non-square markets

$(x, y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$, with $|M| < |M'|$, and

$Z_0 = \{j \in M' \mid y_j = 0\}$.

The vector (x, y) is the nucleolus of a non-square assignment game if and only if

1. There exists $Z_0' \subseteq Z_0$ with $|Z_0'| = |M'| - |M|$, and
2. $\min_{i \in M} \{x_i\} \geq \min_{j \in M' \setminus Z_0'} \{y_j\}$.

Which are the matrices with the same nucleolus?

Proposition
Let $A, A' \in \mathrm{M}^{+}_{m\times m'}$ two matrices such that they share the same nucleolus. Then, $A \vee A'$ has the same nucleolus.

- Notice that the nucleolus is not preserved by taking the minimum operator.

The family of matrices with the same nucleolus is a $\vee-$semilattice or join-semilattice.
It is a compact set.

Theorem

Let $(x, y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ be a vector, and $\mathcal{F}_\nu(x, y)$ be the family of matrices in $\mathrm{M}^+_{m\times m'}$ such that (x, y) is the nucleolus of the associated assignment game.

If $\mathcal{F}_\nu(x, y) \neq \emptyset$, there exists a unique maximum element, the valuation matrix $V \in \mathcal{F}_\nu(x, y)$ such that $A \leq V$ for all $A \in \mathcal{F}_\nu(x, y)$.

Matrix V is given by

$$v_{ij} = \begin{cases} x_i + y_j & \text{if } \; i \in M, \text{ and } j \in M' \setminus Z_0', \\ x_i - \min\limits_{j\in M'\setminus Z_0'} \{y_j\} & \text{if } \; i \in M, \text{ and } j \in Z_0', \end{cases}$$

where Z_0' is any subset of $Z_0 = \{j \in M' \mid y_j = 0\}$ with cardinality $|Z_0'| = |M'| - |M|$.

- A matrix $A \in \mathrm{M}^+_{m\times m'}$ is a valuation if for any $i, i' \in \{1, \dots m\}$ and $j, j' \in \{1, \dots m'\}$ we have $a_{ij} + a_{i'j'} = a_{ij'} + a_{i'j}$.

Let $\mathcal{F}_\nu(x, y)$ be a nonempty family of matrices with a given nucleolus, where $(x, y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}, |M| \leq |M'|$, and $V \in \mathcal{F}_\nu(x, y)$ be its maximum.

Proposition

There is a continuous piecewise linear path (maybe not unique) between any matrix in $\mathcal{F}_\nu(x, y)$ and its maximum element V. From here it is clear that the family $\mathcal{F}_\nu(x, y)$ is a path-connected set.

There are many minimal elements. Basically we obtain a minimal matrix each time we fix an appropriate optimal matching, but not any optimal matching can be used.

For instance, take the nucleolus $(x,y)=(0,3,2,0)\in\mathbb{R}^2_+\times\mathbb{R}^2_+$. Note that $\mathcal{F}_\nu(x,y)\neq\emptyset$. and $\min\{x_1,x_2\}=0=\min\{y_1,y_2\}$. The valuation matrix

$$V=\begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}$$

has two optimal matchings.

The first one, $\mu_1=\{(1,1),(2,2)\}$ cannot be preserved if we look for minimality, but the second one $\mu_2=\{(1,2),(2,1)\}$ can. Indeed,

$$C=\begin{pmatrix} 0 & 0 \\ 5 & 1 \end{pmatrix}$$

is the desired minimal matrix.

Minimal elements in $\mathcal{F}_{\nu}(x, y)$

We say that an optimal matching $\mu \in \mathcal{M}_V^*(M, M')$ is a *minimal-matrix compatible matching (m2-compatible)* if $\min_{j \in \mu(M)} \{y_j\} = 0$ then there exists a buyer $i^* \in M$ such that $x_{i^*} = \min_{i \in M} \{x_i\}$ and his optimal partner receives $y_{\mu(i^*)} = \min_{j \in \mu(M)} \{y_j\} = 0.$

The set of all m2-compatible matchings is denoted by $\mathcal{M}_m(V)$.

Notice that in the square case, if $\min_{i \in M} \{x_i\} = \min_{j \in M'} \{y_j\} > 0$, all matchings are m2-compatible. As a consequence, $m!$ minimal matrices may appear.

Theorem

For any minimal-matrix compatible matching $\mu \in \mathcal{M}_m(V)$ there exists matrix $C \in \mathcal{F}_\nu(x,y)$ with $\mu \in \mathcal{M}^*_C(M,M')$ and C is minimal in $(\mathcal{F}_\nu(x,y), \leq)$. Moreover, if $|M| \geq 3$ then $C \neq V$ whenever (x,y) is not the null vector.

As a direct consequence we obtain an interesting result on the cardinality of the family $\mathcal{F}_\nu(x, y)$.

Corollary

For any vector $(x, y) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ either

(a) $\mathcal{F}_\nu(x, y) = \emptyset$,

(b) $\mathcal{F}_\nu(x, y)$ is a singleton, or

(c) $\mathcal{F}_\nu(x, y)$ has a continuum of elements.

Equal-split Smallest Entry Rule

For any valuation square assignment matrix, divide equally the smallest entry(ies) of the matrix between the two involved agents and complete the payoff by solving the adequate core equalities. This is the nucleolus.

Example:

$$A = \begin{pmatrix} 6 & 8 & 8 & 11 \\ 4 & 6 & 6 & 9 \\ 1 & 3 & 3 & 6 \\ 2 & 4 & 4 & 7 \end{pmatrix}.$$

Its nucleolus is:

$$(5.5,\ 3.5,\ 0.5,\ 1.5;\ 0.5, 2.5,\ 2.5,\ 5.5).$$

Assortative markets

A square assignment matrix is called assortative if it satisfies two properties:

1. The matrix has increasing rows and columns, i.e.
$$\begin{aligned} a_{i,k} \leq a_{i,k+1} &\quad \text{for} \quad k = 1, 2, \ldots, m-1. \\ a_{k,j} \leq a_{k+1,j} &\quad \text{for} \quad k = 1, 2, \ldots, m-1, \end{aligned}$$
for all i and j, and
2. The matrix satisfies the inverse Monge property, i.e.
$$a_{ij}+a_{kl} \geq a_{il}+a_{kj} \quad \text{for all } 1 \leq i < k \leq m, \text{ and } 1 \leq j < l \leq m'.$$

This is equivalent to saying that any 2×2 submatrix has an optimal matching in its main diagonal.
Becker, 1973, or Eriksson et al., 2000 analyze assignment markets where agents can be ordered by some trait, and it is preferable to match with "better" agents, because they produce a larger output, that is, the *mating of the likes*.

Example:

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 3 & 6 & 8 \\ 4 & 7 & 10 \end{pmatrix}.$$

Its nucleolus is:

$$(1,\ 2.5,\ 4;\ 1, 3.5,\ 6).$$

The formula:

$$\begin{aligned} x_i(w_A) &= \frac{1}{2}\, a_{ii} + \frac{1}{2}\sum_{k=1}^{i-1} a_{k+1,k} - \frac{1}{2}\sum_{k=1}^{i-1} a_{k,k+1} \qquad \text{for } i \in M, \\ y_j(w_A) &= \frac{1}{2}\, a_{jj} + \frac{1}{2}\sum_{k=1}^{j-1} a_{k,k+1} - \frac{1}{2}\sum_{k=1}^{j-1} a_{k+1,k} \qquad \text{for } j \in M'. \end{aligned}$$

The procedure to compute the nucleolus

- Main idea:

1. Distribute some "dividends" to the players in such a way that we retain an assignment market, whose nucleolus gives the remaining worth to the agents.
2. In it, we lower the entries in the matrix, until at least one optimal entry of some optimal matching is set to zero. Players involved in this(these) entry(ies) will not receive any more dividends.
3. In this way we associate a new game with, at least, one player less on each side.

Notation

Given a square assignment matrix $A \in \mathrm{M}_m^+$ we define the set of all entries that belong to some optimal matching,

$$H^A = \left\{(i,j) \in M \times M' \mid (i,j) \text{belongs to some optimal matching in} A\right\}.$$

Consider now the minimum entry in matrix A that is in some optimal matching, and define

$$\alpha^A := \min\left\{\frac{a_{ij}}{2} \mid (i,j) \in H^A\right\}.$$

The procedure to compute the nucleolus

For $t \geq 0$, we introduce the following matrix A^t.

$$a_{ij}^t = \begin{cases} \max\{0, a_{ij} - 2t\} & \text{for} \quad (i,j) \in H^A, \\ \max\{0, a_{ij} - t\} & \text{for} \quad (i,j) \notin H^A. \end{cases}$$

Now for each non-optimal matching,
$\mu \in \mathcal{M}\left(M, M'\right) \setminus \mathcal{M}_A^*\left(M, M'\right)$, consider the following equation,
in $t \geq 0$:

$$f_\mu^A(t) = w_A(M \cup M') - 2mt - \sum_{(i,j) \in \mu} a_{ij}^t = 0,$$

and denote $t_\mu^A \geq 0$ its unique solution.

Define

$$\beta^A := \min\left\{t_\mu^A \mid \mu \in \mathcal{M}\left(M, M'\right) \setminus \mathcal{M}_A^*\left(M, M'\right)\right\}.$$

The procedure to compute the nucleolus

In each step, for

$$\varepsilon = \min\{\alpha^A, \beta^A\},$$

either we obtain at least one entry of an optimal matching equal to zero, and/or at least one more optimal matching.

If $\alpha^A < \beta^A$ we obtain, for $\varepsilon = \alpha^A$, that at least one entry in the optimal matching has been dropped to zero.

If $\alpha^A > \beta^A$ we obtain, for $\varepsilon = \beta^A$, that at least we have another optimal matching.

The iterated application of the procedure increases the number of optimal matchings and/or reduces the number of players. In a finite number of steps we finish the procedure.

Consider the following assignment market: $M = \{1, 2, 3, 4\}$ and $M' = \{1', 2', 3', 4', 5'\}$, and matrix

$$A = \begin{pmatrix} 6 & 7 & 4 & 5 & 9 \\ 4 & 3 & 7 & 8 & 3 \\ 0 & 1 & 3 & 6 & 4 \\ 2 & 2 & 5 & 7 & 8 \end{pmatrix}.$$

[Section 7 in Solymosi and Raghavan, 1994]

In the first place, we add a dummy buyer, buyer 5, whose row is filled with zeroes. The optimal matching is denoted by the boxes around the entries.

Therefore, the square matrix that we begin with is the following one:

$$A^{[0]} = \begin{pmatrix} 6 & \boxed{\mathbf{7}} & 4 & 5 & 9 \\ 4 & 3 & \boxed{\mathbf{7}} & 8 & 3 \\ 0 & 1 & 3 & \boxed{\mathbf{6}} & 4 \\ 2 & 2 & 5 & 7 & \boxed{\mathbf{8}} \\ \boxed{\mathbf{0}} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 1:

Since there is one entry equal to zero in one optimal matching, $a_{51} = 0$, players 5 and 1' leave the market.

The new assignment market is $M = \{1, 2, 3, 4\}$ and $M' = \{2', 3', 4', 5'\}$ and its matrix is:

$$A^{[1]} = \begin{pmatrix} \boxed{\mathbf{1}} & 0 & 0 & 3 \\ 0 & \boxed{\mathbf{3}} & 4 & 0 \\ 1 & 3 & \boxed{\mathbf{6}} & 4 \\ 0 & 3 & 5 & \boxed{\mathbf{6}} \end{pmatrix}.$$

Recall that $a^{[1]}_{ij} = \max\{0, a_{ij} - a_{i1} - a_{5j}\}$, for $i = 1, 2, 3, 4$ and $j = 2, 3, 4, 5$.

Step 2:
Since there is no entry equal to zero in some optimal matching, we distribute the players $\varepsilon = \frac{1}{2}$ which is exactly one half of the minimum entry in the unique optimal matching.

In fact, in this case $\beta^{A^{[1]}} = 1$ and $\min\{\alpha^{A^{[1]}}, \beta^{A^{[1]}}\} = \frac{1}{2}$. The new assignment market is $M = \{1, 2, 3, 4\}$ and $M' = \{2', 3', 4', 5'\}$ and its matrix is:

$$A^{[2]} = \begin{pmatrix} \boxed{\mathbf{0}} & 0 & 0 & 2\frac{1}{2} \\ 0 & \boxed{\mathbf{2}} & 3\frac{1}{2} & 0 \\ \frac{1}{2} & 2\frac{1}{2} & \boxed{\mathbf{5}} & 3\frac{1}{2} \\ 0 & 2\frac{1}{2} & 4\frac{1}{2} & \boxed{\mathbf{5}} \end{pmatrix}.$$

Notice that the optimal entries reduce their worth by $2\,\varepsilon = 2\,\frac{1}{2} = 1$, whilst the non-optimal entries reduce their worth by $\varepsilon = \frac{1}{2}$.

Step 3:
Since there is one entry equal to zero in one optimal matching, players 1 and 2' leave the market.

The new assignment market is $M = \{2, 3, 4\}$ and $M' = \{3', 4', 5'\}$ and its matrix is:

$$A^{[3]} = \begin{pmatrix} \boxed{\mathbf{2}} & 3\frac{1}{2} & 0 \\ 2 & \boxed{\mathbf{4\frac{1}{2}}} & \frac{1}{2} \\ 2\frac{1}{2} & 4\frac{1}{2} & \boxed{\mathbf{2\frac{1}{2}}} \end{pmatrix}.$$

Step 4:

Since there is no entry equal to zero in some optimal matching, we must compute α and β.

In this case $\alpha^{A^{[3]}} = 1$ and $\beta^{A^{[3]}} = \frac{1}{2}$. Players receive $\frac{1}{2}$ and we obtain another optimal matching.

The new assignment market is $M = \{2, 3, 4\}$ and $M' = \{3', 4', 5'\}$ and its matrix is:

$$A^{[4]} = \begin{pmatrix} \boxed{\mathbf{1}} & \boxed{\mathbf{3}} & 0 \\ \boxed{\mathbf{1\frac{1}{2}}} & \boxed{\mathbf{3\frac{1}{2}}} & 0 \\ 2 & 4 & \boxed{\mathbf{1\frac{1}{2}}} \end{pmatrix}.$$

Notice that this matrix has two optimal matchings.

Step 5:
Since there is no entry equal to zero in some optimal matching, we obtain $\alpha^{A^{[4]}} = \frac{1}{2}$ and $\beta^{A^{[4]}} = \frac{1}{6}$. Therefore we distribute $\frac{1}{6}$ to the players and we obtain several additional optimal matchings.

The new assignment market is $M = \{2, 3, 4\}$ and $M' = \{3', 4', 5'\}$ and its matrix is:

$$A^{[5]} = \begin{pmatrix} \boxed{\mathbf{\frac{2}{3}}} & \boxed{\mathbf{2\frac{2}{3}}} & \boxed{\mathbf{0}} \\ \boxed{\mathbf{1\frac{1}{6}}} & \boxed{\mathbf{3\frac{1}{6}}} & 0 \\ \boxed{\mathbf{1\frac{5}{6}}} & \boxed{\mathbf{3\frac{5}{6}}} & \boxed{\mathbf{1\frac{1}{6}}} \end{pmatrix}.$$

Notice that we have obtained a new matrix with four optimal matchings.

Step 6:
Since there is one entry equal to zero in one optimal matching, players 2 and 5' leave the market.

The new assignment market is $M = \{3, 4\}$ and $M' = \{3', 4'\}$ and its matrix is:

$$A^{[6]} = \begin{pmatrix} \boxed{\frac{1}{2}} & \boxed{\frac{1}{2}} \\ \boxed{\mathbf{0}} & \boxed{\mathbf{0}} \end{pmatrix}.$$

Step 7:
Since there are two entries equal to zero, one in each optimal matching, all remaining players leave the market.

An example

Player	**1**	**2**	**3**	**4**	**5**	**1'**	**2'**	**3'**	**4'**	**5'**
Step 1	6	4	0	2	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	0	0	0	0
Step 2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$			$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Step 3	$\boxed{\mathbf{0}}$	0	$\frac{1}{2}$	0			$\boxed{\mathbf{0}}$	0	0	$2\frac{1}{2}$
Step 4		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$				$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Step 5		$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$				$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
Step 6		$\boxed{\mathbf{0}}$	0	$\frac{7}{6}$				$\frac{2}{3}$	$2\frac{2}{3}$	$\boxed{\mathbf{0}}$
Step 7			$\boxed{\frac{1}{2}}$	$\boxed{\mathbf{0}}$				$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	
TOTAL	$6\frac{1}{2}$	$5\frac{1}{6}$	$2\frac{1}{6}$	$4\frac{1}{3}$	0	0	$\frac{1}{2}$	$1\frac{5}{6}$	$3\frac{5}{6}$	$3\frac{2}{3}$

$$\nu(w_A) = \left(6\frac{1}{2},\ 5\frac{1}{6},\ 2\frac{1}{6},\ 4\frac{1}{3};\ 0,\ \frac{1}{2},\ 1\frac{5}{6},\ 3\frac{5}{6},\ 3\frac{2}{3}\right).$$

Thanks!!!