Network Disconnection Games

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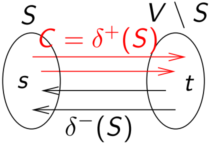
A two-person zero-sum game

- Let $G = (V, A)$ be a directed graph with two distinguished nodes s and t.
- The first player called the *attacker* chooses a set of arcs to intercept every path from s to t.
- The second player called the *inspector* inspects an arc trying to find the attacker.

Washburn and Wood (1995) study the game where the *evader* chooses a path from s to t and the *inspector* chooses an arc to find the evader.

We assume that the attacker concentrate on intercepting the arcs in an st-cut C, he will choose it with probability y_C.



The inspector inspects an arc a with probability x_a. Moreover, if the inspector is at arc a , there is a probability p_a of detecting the attacker if he is at this arc a.

- D the matrix whose columns are the incidence vectors of all st-cuts.
- P diagonal matrix that contains the probabilities $\{p_a\}$.
- Dy is a column whose component associated with an arc a is the probability that the attacker will be at a.
- $xPDy$ is the probability that the attacker will be detected.

Thus we concentrate on the following two-person game:

$$\max_x \min_y xPDy$$
$$\sum x_a = 1,$$
$$\sum y_C = 1,$$
$$x \geq 0, y \geq 0.$$

We fix y,

$$\max_x xPDy$$
$$\sum x_a = 1,$$
$$x \geq 0.$$

Its dual is

$$\min_\mu \mu$$
$$\mu \geq p_a \sum \{y_C \mid a \in C\} \quad \forall a.$$

Then our problem is

$$\min_{\mu, y} \mu \tag{1}$$
$$\sum \{y_C \mid a \in C\} \leq \frac{\mu}{p_a} \quad \forall a, \tag{2}$$
$$\sum y_C = 1, \tag{3}$$
$$y \geq 0. \tag{4}$$

Look for the following problem

$$\max \sum y_C$$
$$\sum \{y_C \mid a \in C\} \leq \frac{1}{p_a} \quad \forall a,$$
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$$\min \sum_a \frac{1}{p_a} x_a$$
$$x(C) \geq 1 \quad \forall st\text{-cut } C,$$
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- The first problem is a maximum packing of *st*-cuts and the second problem is the shortest path problem with weights $\frac{1}{p_a}$.
- Let λ be the value of the shortest path with weights $\frac{1}{p_a}$ and $\bar{y}$ is the solution of the maximum packing of *st*-cuts. Then $\mu = \frac{1}{\lambda}$ and $\hat{y} = (\frac{1}{\lambda})\bar{y}$ is a solution of (1)-(4).

- So $\hat{y}$ is the strategy of the attacker.
- Let $\mathcal{P}$ be a shortest *st*-path with arc weights $\{\frac{1}{p_a}\}$.
- Then the strategy of the inspector is $\hat{x}(a) = \frac{1}{(\lambda p_a)}$ if $a \in \mathcal{P}$, otherwise $\hat{x}(a) = 0$.

The complementary slackness conditions imply

$$\sum\{\hat{y}_C \mid a \in C\} = \frac{\mu}{p_a} \quad \forall a \in \mathcal{P}.$$

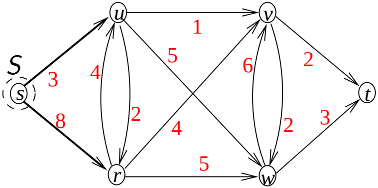
Thus

$$\begin{aligned} \hat{x}PD\hat{y} \quad &= \textstyle\sum_{a\in\mathcal{P}} \frac{1}{\lambda p_a} p_a \sum\{\hat{y}_C \mid a \in C\} \\ &= \textstyle\sum_{a\in\mathcal{P}} \frac{1}{\lambda p_a} p_a \frac{\mu}{p_a} = \mu. \end{aligned}$$

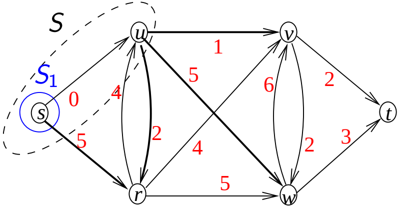
Theorem

Optimal strategies for both players can be computed in polynomial time. The inspector strategy can be obtained from a shortest st-path with arc weights $\{\frac{1}{p_a}\}$. *The attacker strategy can be obtained from a maximum packing of st-cuts with arc capacities* $\{\frac{1}{p_a}\}$.

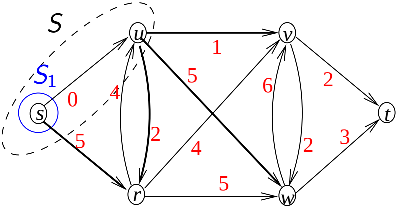
Dijkstra's Algorithm



$\bar{y}_C = 0$ for each st-cut C.
$S = \{s\}$.



$C_1 = \delta^+(S_1)$.
$\bar{y}_{C_1} = 3$.
$S = \{s, u\}$.



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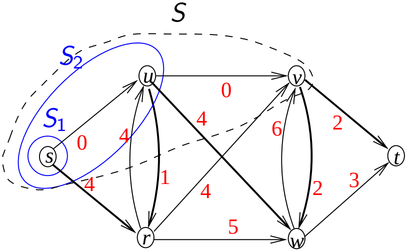
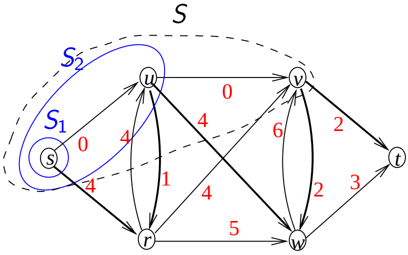


$C_2 = \delta^+(S_2)$.
$\bar{y}_{C_2} = 1$.
$S = \{s, u, v\}$.



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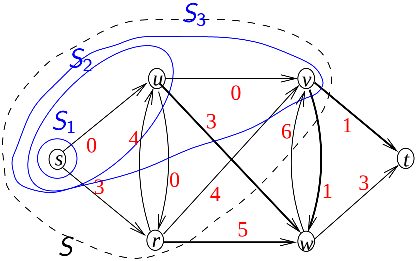
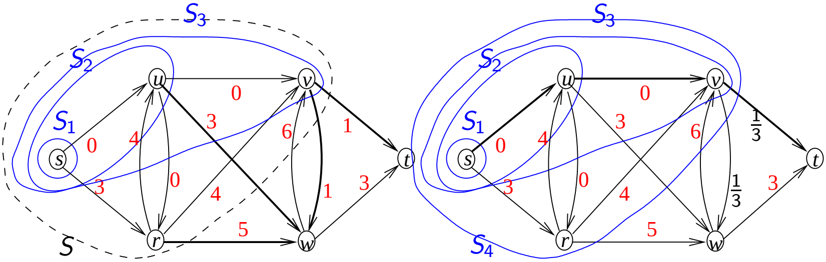


$C_3 = \delta^+(S_3)$.

$\bar{y}_{C_3} = 1$.

$S = \{s, u, v, r\}$.



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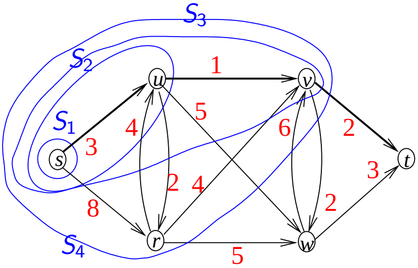
$S = \{s, u, v, r\}$.

$C_4 = \delta^+(S_4)$.
$\bar{y}_{C_4} = 1$.
$S = \{s, u, v, r, t\}$.



$\hat{y}_{C_1} = \frac{1}{2}$; $\hat{y}_{C_2} = \hat{y}_{C_3} = \hat{y}_{C_4} = \frac{1}{6}$.

$\hat{x}(s,u) = \frac{1}{2}$; $\hat{x}(u,v) = \frac{1}{6}$; $\hat{x}(v,t) = \frac{1}{3}$.

A Cooperative Game

Let $G = (V, A)$ a directed graph with two distinguished nodes s and t. In the *Network Disconnection Game* (A, σ) each player owns an arc in A. The characteristic function of the game σ gives for each coalition S, the maximum number of disjoint st-cuts.

The core.

$$\begin{aligned} x(A) &= \sigma(A), \\ x(S) &\geq \sigma(S), \quad \forall S \subseteq A. \end{aligned}$$

Lemma

Let k be the length of an st-path of minimum cardinality. Then the core is determined by

$$\begin{aligned} x(A) &= k, \\ x(C) &\geq 1, \quad \textit{for each st-cut } C, \\ x &\geq 0. \end{aligned}$$

So the extreme points of the core are the shortest paths. Therefore the core is also defined by the following *network flow formulation*:

$$x(\delta^-(v)) - x(\delta^+(v)) = \begin{cases} -1 & \text{if } v = s, \\ 0 & \text{if } v \neq s, t, \\ 1 & \text{if } v = t, \end{cases}$$

$$x(u,v) \geq 0 \quad \text{for all } (u,v) \in A,$$

$$x(u,v) = 0 \quad \text{for all } (u,v) \in A_0.$$

Here A_0 is the set of arcs that do not belong to any shortest *st*-path.

Theorem

For the Network Disconnection Game:

1. *The core is non-empty if and only if there is a path from s to t.*
2. *Given a vector $\bar{x}$, we can test whether $\bar{x}$ belongs to the core in polynomial time.*

The nucleolus.

- For a coalition S and a vector x in the core, the *excess* is $e(x, S) = x(S) - \sigma(S)$.
- The *nucleolus* is the vector in the core that lexicographically maximizes the vector of non-decreasingly ordered excesses Schmeidler (1969).

The nucleolus can be computed with a sequence of linear programs, Kopelowitz (1967):

$$\max \epsilon$$
$$x(S) \geq \sigma(S) + \epsilon, \quad \forall S \subseteq A,$$
$$x(A) = \sigma(A).$$

Let ϵ_1 be the optimal value, then $P_1(\epsilon_1)$ will denote the set of the optimal solutions.
For a polytope $P \subset \mathbb{R}^A$ let

$$\mathcal{F}(P) = \{S \subseteq A \mid x(S) = y(S), \forall x, y \in P\}.$$

In general given ϵ_{r-1} we solve

$$\max \epsilon \tag{5}$$
$$x(S) \geq \sigma(S) + \epsilon, \quad \forall S \notin \mathcal{F}(P_{r-1}(\epsilon_{r-1})), \tag{6}$$
$$x \in P_{r-1}(\epsilon_{r-1}). \tag{7}$$

Lemma

Instead of solving (5)-(7), we can solve

$$\begin{aligned} &\max \epsilon \\ &x(C) \geq 1 + \epsilon, \quad \textit{for each st-cut } C \notin \mathcal{F}(P_{r-1}(\epsilon_{r-1})), \\ &x(a) \geq \epsilon, \quad \textit{for each arc } a \notin \mathcal{F}(P_{r-1}(\epsilon_{r-1})) \\ &x \in P_{r-1}(\epsilon_{r-1}). \end{aligned}$$

Lemma

For x in the core, if $x(a) \geq \epsilon$ for each $a \notin \mathcal{F}(P_{r-1}(\epsilon_{r-1}))$ and $x(a) = l(a)$ for $a \in \mathcal{F}(P_{r-1}(\epsilon_{r-1}))$, then $x(\delta^+(S)) \geq 1 + \epsilon$ for $\delta^+(S) \notin \mathcal{F}(P_{r-1}(\epsilon_{r-1}))$.

Proof.

Using the network flow formulation we get $x(\delta^+(S)) = 1 + x(\delta^-(S))$. □

Thus we have to look for the maximum value λ such that the system below has a solution:

$$x(\delta^-(v)) - x(\delta^+(v)) = \begin{cases} -1 & \text{if } v = s, \\ 0 & \text{if } v \neq s, t, \\ 1 & \text{if } v = t, \end{cases}$$

$$x(u,v) = l(u,v) \qquad \forall (u,v) \in \mathcal{F}(P_{r-1}(\epsilon_{r-1})),$$

$$x(u,v) \geq l(u,v) + \lambda \quad \forall (u,v) \in V_r = V \setminus \mathcal{F}(P_{r-1}(\epsilon_{r-1}))$$

which may be reduced to the following system :

$$\begin{aligned} & x'(\delta^-(v)) - x'(\delta^+(v)) = b(v) + \lambda d(v), \quad \forall v \in V \\ & x' \geq 0. \end{aligned}$$

with $x'(u,v) = x(u,v) - l(u,v) - \lambda$ for each arc $(u,v) \in A_r$. Here $\sum b(v) = 0$ and $\sum d(v) = 0$.

How to decide if the system below has a solution or not.

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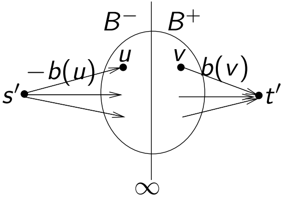

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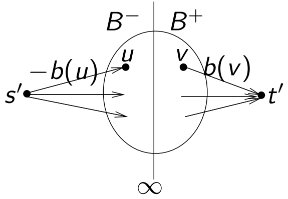

$\exists$ a solution iff the value of the maximum $s't'$-flow is

$$\alpha = -b(B^-) = b(B^+)$$

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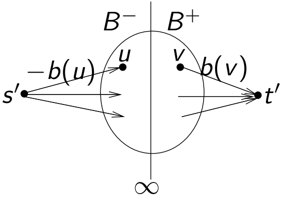

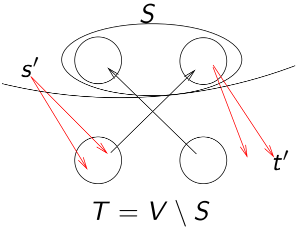

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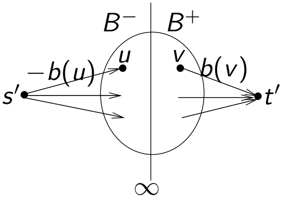

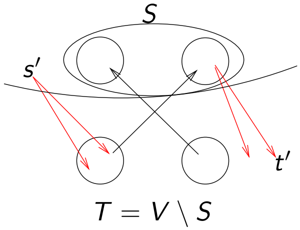

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The capacity of the cut $\delta^+((S \cup \{s'\})$ is

$$
\begin{aligned}
& = -b(B^- \cap T) + b(B^+ \cap S) \\
& = \alpha + b(B^- \cap S) + b(B^+ \cap S) \\
& = \alpha + b(S).
\end{aligned}
$$

- Finding the max $s't'$-flow reduces to minimizing $b(S)$ for $S \subseteq V$ and $\delta^+(S) = \emptyset$.
- When there is no solution, $b(S) < 0$.
- For our problem, when $\lambda = 0$ we have a solution.
- If the system is infeasible for $\bar{\lambda} > 0$, then there is $\delta^+(S) = \emptyset$ with $b(S) + \bar{\lambda} d(S) < 0$ and $d(S) < 0$.

To have feasibility we should impose $b(S) + \lambda d(S) \geq 0$. Therefore

$$\lambda = \min \frac{b(S)}{-d(S)}.$$

We solve this with Newton's method.

1. Set $\lambda = \lambda_M$.
2. $\bar{S} = \text{argmin}\{b(S) + \lambda d(S)\}$, $\delta^+(S) = \emptyset$ and $d(S) < 0$.
3. If $b(\bar{S}) + \lambda d(\bar{S}) < 0$, then update

$$\lambda = \frac{b(\bar{S})}{-d(\bar{S})},$$

and go to 2. Otherwise $b(\bar{S}) + \lambda d(\bar{S}) = 0$, and we stop.

Theorem

Computing the nucleolus of the Network Disconnection Game requires $O(|A|^2)$ min-cut problems.