Fixed-charge transportation problems on trees

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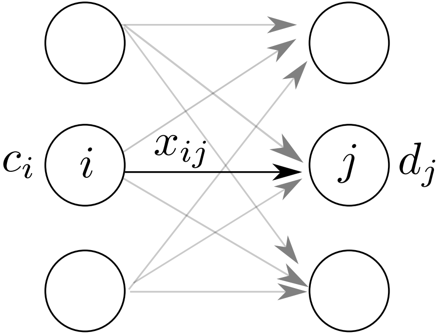
Fixed-Charge Transportation Problem (FCTP)

- A set N of n warehouses with capacities $c_i \in \mathbb{Z}_+$
- A set M of m clients with demands $d_j \in \mathbb{Z}_+$
- For each pair (i,j): a fixed cost $q_{ij} > 0$ and a variable cost p_{ij}

GOAL: find amounts x_{ij} to be transported from i to j that minimizes overall cost:

$$
\begin{aligned}
\text{(IP)} \quad \min \quad & p^\top x + q^\top y \\
\text{s.t.} \quad & \sum_{j=1}^{m} x_{ij} \le c_i \quad i \in N && (1) \\
& \sum_{i=1}^{n} x_{ij} = d_j \quad j \in M && (2) \\
& 0 \le x_{ij} \le \min\{c_i, d_j\} y_{ij} \quad i \in N,\ j \in M && (3) \\
& y_{ij} \in \{0,1\} \quad i \in N,\ j \in M. && (4)
\end{aligned}
$$

Fixed-Charge Transportation Problem (FCTP)



What is known about FCTP

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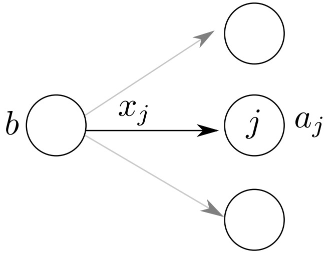
Generalizes the single-node flow set: (x, y) such that

$$\sum_{j=1}^{n} x_j \leq b$$

$$0 \leq x_j \leq a_j y_j \quad j \in N$$

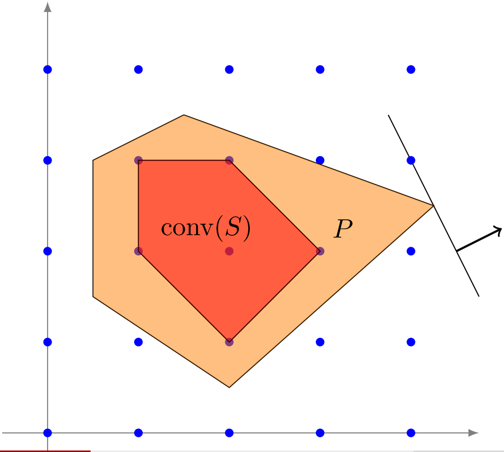
$$y_j \in \{0, 1\} \quad j \in N$$



$\rightarrow$ FCTP is (at least) weakly NP-hard.

Solving integer programs

S: feasible set (integral points)
P: linear relaxation (formulation)
$\mathrm{conv}(S)$: convex hull of S



Results for 30 x 30 instances

B: upper bound on arc capacities
r: total demand to total supply ratio

		(IP)		
B	r	Gap [%]	Time [s]	Nodes [#]
20	0.90	0.00	167	29033
	0.95	0.17	853	114655
	1.00	2.31	2905	308104
40	0.90	0.00	626	106839
	0.95	0.87	2419	329429
	1.00	8.66	3600	427371
60	0.90	0.00	290	58686
	0.95	1.89	2585	327116
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Aggarwal and Aneja (OR, 2012): valid inequalities involving binary variables only + B&C.

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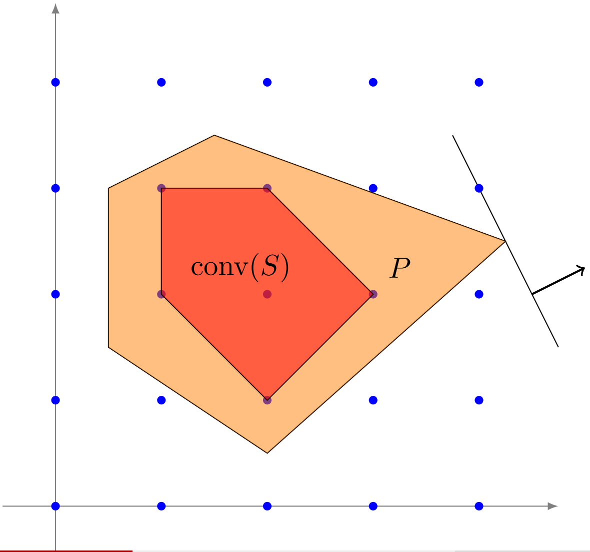
Complexity? (In)Approximability?

Looking for better (tighter) formulations



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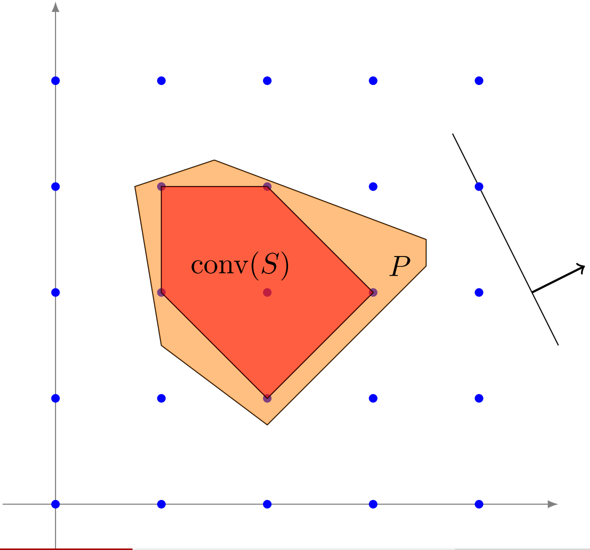
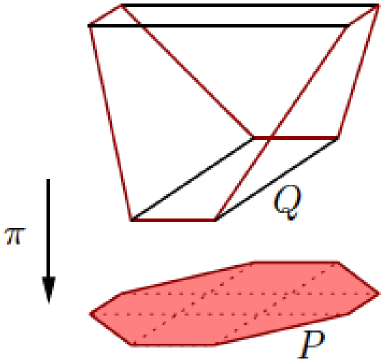


Extended formulation of P

Higher dimensional polyhedron Q that linearly projects onto P.



[S. Pokutta]

Projection can imply a large (exponential) number of inequalities.

A unary expansion-based formulation

$$
\begin{aligned}
(\mathrm{IP}+z) \quad \min \quad & p^\top x + q^\top y \\
\text{s.t.} \quad & (1) - (4), \\
& \sum_{l=0}^{a_{ij}} l * z_{ijl} = x_{ij} \quad (i,j) \in E \\
& \sum_{l=1}^{a_{ij}} z_{ijl} \le y_{ij} \quad (i,j) \in E \\
& \sum_{l=0}^{a_{ij}} z_{ijl} = 1 \quad (i,j) \in E \\
& z_{ijl} \in \{0,1\} \quad (i,j) \in E,\ 0 \le l \le a_{ij}.
\end{aligned}
$$

where the intended meaning is that $z_{ijl} = 1$ if $x_{ij} = l$ and 0 otherwise.

A short proof that I'm being stupid

Theorem

The LP relaxation of (IP+z) is NOT stronger than that of (IP).

Proof.

Given (x, y) in the linear relaxation of (IP), for each arc (i, j) let:

- $z_{ij(a_{ij})} = x_{ij}/a_{ij}$,
- $z_{ij0} = 1 - z_{ija_{ij}}$
- $z_{ijl} = 0$ for $0 < l < a_{ij}$.

Then (x, y, z) belongs to the linear relaxation of $(\mathrm{IP}+z)$. $\square$

Or not?? Results for 30 x 30 instances

		(IP)			(IP+z)				
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	1.00	2.31	2905	308104	2.16	-0.22	0.00	68	789
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	0.95	0.87	2419	329429	0.86	-0.03	0.00	42	475
	1.00	8.66	3600	427371	5.75	-0.32	2.93	2824	13022
60	0.90	0.00	290	58686	0.00	0.00	0.00	15	84
	0.95	1.89	2585	327116	1.82	-0.13	0.00	184	1323
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WHY?

Overview

1. Complexity results.
2. Extended formulations.
3. Computational Results.

FCTP is strongly NP-hard

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3-Partition: given $3n$ nonnegative integers $a_1, \ldots, a_{3n}$ such that $\sum_i a_i = nb$ and $\frac{b}{4} < a_i < \frac{b}{2}$ $\forall i$.

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Consider an instance of FCTP with

n suppliers with capacity b each,
$3n$ clients with demands $a_1, \ldots, a_{3n}$,
no variable cost and unit fixed cost $q_{ij} = 1$ for all (i, j).

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Consider an instance of FCTP with

- n suppliers with capacity b each,
- $3n$ clients with demands $a_1, \ldots, a_{3n}$,
- no variable cost and unit fixed cost $q_{ij} = 1$ for all (i, j).

optimal value of $\mathrm{FCTP} = 3n \Leftrightarrow$ answer to 3-Partition is YES.

For notational convenience, let's change the problem

Given a graph $G = (V, E)$, consider

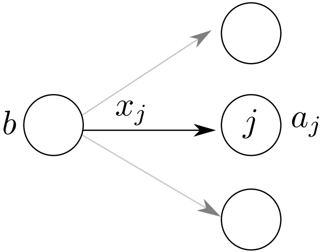
$$\begin{aligned} \text{(IP)} \quad \min \quad & p^\top x + q^\top y \\ \text{s.t.} \quad & \sum_{j \in V:\ (i,j) \in E} x_{ij} \leq b_i \quad i \in V \\ & 0 \leq x_{ij} \leq a_{ij} y_{ij} \quad (i,j) \in E \\ & y_{ij} \in \{0,1\} \quad (i,j) \in E, \end{aligned}$$

so that there is no distinction between suppliers and consumers.

The single-node flow set

Pairs of vectors (x, y) such that

$$\sum_{j=1}^{n} x_j \leq b$$
$$0 \leq x_j \leq a_j y_j \quad j = 1, \ldots, n$$
$$y_j \in \{0, 1\} \quad j = 1, \ldots, n$$

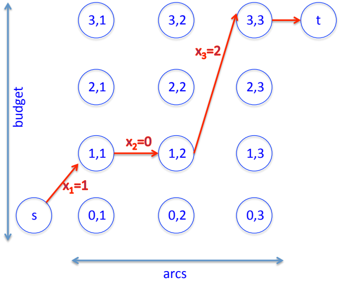


The single-node flow set

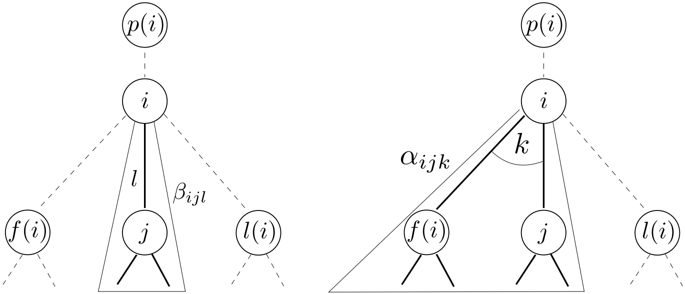
$$\sum_{k' \le k} f_{jk'k} = \sum_{k' \ge k} f_{(j+1)kk'} \quad j = 1, \ldots, n-1, \; k = 0, \ldots, b$$

$$x_j = \sum_{k-k' \ge 0} (k - k') * f_{jk'k} \quad j = 1, \ldots, n$$

$$y_j \ge \sum_{k-k' > 0} f_{jk'k} \quad j = 1, \ldots, n$$



FCTP is pseudo-polynomially solvable on a tree



$$\alpha_{ijk} = \begin{cases} \beta_{ijk} & j = f(i) \\ \min\limits_{k' \geq 0:\ 0 \leq k - k' \leq a_{ij}} \left\{ \alpha_{i(j-1)k'} + \beta_{ij(k-k')} \right\} & j > f(i) \end{cases} \qquad \forall i, j, k$$

$$\beta_{ijl} = \begin{cases} c_{ijl} & j \text{ is a leaf node} \\ \min\limits_{0 \leq k \leq b_j - l} \left\{ \alpha_{jl(j)k} \right\} + c_{ijl} & j \text{ is a nonleaf node} \end{cases} \qquad \forall i, j, l$$

$$\beta_{010} = \min_{0 \leq k \leq b_1} \left\{ \alpha_{1l(1)k} \right\}$$

Writing the DP as an LP

$$\max \beta_{010}$$

$$\alpha_{ijk} \leq \begin{cases} \beta_{ijk} & j = f(i) \\ \alpha_{i(j-1)k'} + \beta_{ij(k-k')} & j > f(i),\ 0 \leq k - k' \leq a_{ij} \end{cases} \qquad \forall i, j, k$$

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$$\beta_{010} \leq \alpha_{1l(1)k} \quad 0 \leq k \leq b_1$$

$$\alpha, \beta \in \mathbb{R}^{|E|B}.$$

The DP has complexity $\mathcal{O}(|E|B^2)$

By duality, this yields an ugly LP extended formulation of the same size:

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in E} \sum_{kl} c_{ijl} v_{ijlk} \\
\text{s.t.} \quad & \sum_{0\le k-k'\le a_{ij}} u_{ijk'k} = \begin{cases} \displaystyle\sum_{k'\le b_i:\ 0\le k'-k\le a_{i(j+1)}} u_{i(j+1)kk'} & f(i)\le j < l(i) \\ \displaystyle\sum_{0\le l\le b_i-k} v_{p(i)ilk} & j = l(i) \end{cases} \\
& \sum_{0\le k\le b_j-l} v_{ijlk} = \sum_{0\le k'\le k\le b_i:\ k-k'=l} u_{ijk'k} (i,j)\in E,\ 0\le l\le a_{ij} \\
& \sum_{0\le k\le b_1} v_{010k} = 1 \\
& u, v \ge 0.
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This does not seem to be useful for other things than trees...

A unary expansion-based formulation

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\text{s.t.} \quad & x = \ldots, y = \ldots \\
& \sum_j \sum_{l=0}^{a_{ij}} l * z_{ijl} \leq b_i \quad i \in V \\
& \sum_{l=0}^{a_{ij}} z_{ijl} = 1 \quad (i,j) \in E \\
& z_{ijl} \in \{0,1\} \quad (i,j) \in E, 0 \leq l \leq a_{ij}.
\end{aligned}
$$

Each node i is essentially a single-node flow set for which we can write an identical DP and an extended formulation of size $O(d(i)B^2)$.

Another ugly extended formulation

$$\sum_{k' \geq 0:\ 0 \leq k-k' \leq a_{ij}} f_{ijk'k} = \begin{cases} \displaystyle\sum_{k' \leq b_i:\ 0 \leq k'-k \leq a_{i(j+1)}} f_{i(j+1)kk'} & f(i) \leq j < l(i) \\ \displaystyle\sum_{k' \leq b_i:\ 0 \leq k'-k \leq a_{ip(i)}} f_{ip(i)kk'} & j = l(i) \end{cases}$$

$$\sum_{0 \leq k \leq b_i} \sum_{k' \geq 0:\ 0 \leq k-k' \leq a_{ip(i)}} f_{ip(i)k'k} = 1, \quad i \in V$$

$$\sum_{k} \sum_{k' \geq 0:\ k-k'=l} f_{ijk'k} = z_{ijl} \quad (i,j) \in E,\ 0 \leq l \leq a_{ij}$$

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$$f \geq 0.$$

$f_{ijkk'} = 1$ iff looking at node i, considering edges $j, \ldots, l(i)$, budget of k is used and $x_{ij} = k - k'$.

A unary expansion-based extended formulation (SNF)

$$\sum_{k'\geq 0:\ 0\leq k-k'\leq a_{ij}} f_{ijk'k} = \begin{cases} \displaystyle\sum_{k'\leq b_i:\ 0\leq k'-k\leq a_{i(j+1)}} f_{i(j+1)kk'} & f(i)\leq j < l(i) \\ \displaystyle\sum_{k'\leq b_i:\ 0\leq k'-k\leq a_{ip(i)}} f_{ip(i)kk'} & j = l(i) \end{cases}$$

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Theorem

This formulation is tight as well (for trees).

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Theorem

This formulation is not tight (even for trees).

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For trees: is tightening single-node flow sets enough?
NO, in the original variable space.
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Very much in the spirit of Bodur, Dash, Günlük (2015):
There is an extended formulation Q of P, i.e. $\mathrm{proj}_{x,y}(Q) = P$, such that

$$\mathrm{proj}_{x,y}(Q + \mathrm{cuts}) = \mathrm{conv}(S) \subsetneq (P + \mathrm{cuts}).$$

But Q needs an exponential number of additional variables.

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B^2 is too large for practical purposes. But let's let the solver do the single-node tightening in variable space z_{ijl} (only linear in B).

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	1.00	2.31	2905	308104	2.16	-0.22	0.00	68	789
	0.90	0.00	626	106839	0.00	0.00	0.00	16	180
40	0.95	0.87	2419	329429	0.86	-0.03	0.00	42	475
	1.00	8.66	3600	427371	5.75	-0.32	2.93	2824	13022
	0.90	0.00	290	58686	0.00	0.00	0.00	15	84
60	0.95	1.89	2585	327116	1.82	-0.13	0.00	184	1323
	1.00	10.92	3600	456224	6.51	7.81	11.90	3600	12197

Results for 30 x 30 instances

		(IP)				(IP+z)							
		Root	B&C			Root			B&C				
B	r	Time	Gap	Time	Nodes	ΔLB	ΔUB	Time	ΔLB	ΔUB	Gap	Time	Nod
20	0.90	1	0.00	167	29033	2.06	-1.95	4	0.00	0.00	0.00	4	[illegible]
	0.95	2	0.17	853	114655	3.87	-4.89	7	0.17	0.00	0.00	8	[illegible]
	1.00	3	2.31	2905	308104	5.49	69.52	15	2.16	-0.22	0.00	68	78
40	0.90	2	0.00	626	106839	3.58	-2.36	10	0.00	0.00	0.00	16	18
	0.95	2	0.87	2419	329429	4.76	-1.68	17	0.86	-0.03	0.00	42	47
	1.00	3	8.66	3600	427371	8.08	353.10	31	5.75	-0.32	2.93	2824	130
60	0.90	2	0.00	290	58686	3.60	-2.67	14	0.00	0.00	0.00	15	[illegible]
	0.95	3	1.89	2585	327116	4.63	231.29	19	1.82	-0.13	0.00	184	13
	1.00	3	10.92	3600	456224	8.77	468.14	46	6.51	7.81	11.90	3600	121

Results for 40 x 40 instances

		(IP)			(IP+z)				
B	r	Gap	Time	Nodes	ΔLB	ΔUB	Gap	Time	Nodes
20	0.90	0.01	402	39345	0.00	0.00	0.00	8	2
	0.95	1.53	3303	246765	1.38	-0.18	0.00	20	123
	1.00	5.27	3600	257034	4.22	-1.14	0.15	1193	5751
40	0.90	0.57	2221	195118	0.53	-0.05	0.00	22	73
	0.95	4.32	3600	260138	3.46	-1.02	0.00	175	1070
	1.00	10.93	3600	223986	6.64	5.31	9.65	3360	7815
60	0.90	0.57	1725	150346	0.51	-0.07	0.00	48	308
	0.95	4.61	3600	265815	3.89	-0.91	0.01	1023	4430
	1.00	13.31	3602	216712	6.71	12.08	17.39	3600	6045

Results for 40 x 40 instances

		(IP)				(IP_z)							
		Root	B&C			Root			B&C				
B	r	Time	Gap	Time	Nodes	ΔLB	ΔUB	Time	ΔLB	ΔUB	Gap	Time	Nod
	0.90	3	0.01	402	39345	2.15	-2.23	9	0.00	0.00	0.01	8	
20	0.95	4	1.53	3303	246765	3.80	-6.49	16	1.38	-0.18	0.00	20	1
	1.00	5	5.27	3600	257034	5.83	240.27	33	4.22	-1.14	0.15	1193	57
	0.90	3	0.57	2221	195118	3.14	-4.13	21	0.53	-0.05	0.00	22	
40	0.95	5	4.32	3600	260138	4.98	61.81	34	3.46	-1.02	0.00	175	10
	1.00	5	10.93	3600	223986	8.51	408.39	66	6.64	5.31	9.65	3360	78
	0.90	3	0.57	1725	150346	2.89	-3.26	25	0.51	-0.07	0.00	48	30
60	0.95	4	4.61	3600	265815	4.73	257.09	40	3.89	-0.91	0.01	1023	44
	1.00	5	13.31	3602	216712	8.68	492.87	96	6.71	12.08	17.39	3600	60

Conclusion

Binarization of continuous variables is not a ridiculous idea!

Conclusion

Binarization of continuous variables is not a ridiculous idea!

Possible workarounds for the size of these formulations:

- power-of-2 binarization of continuous variable does not do the job
- approximate binarization (each binary variable represents an interval of values) does not do the job
- but maybe we should build the discretization "dynamically" (depending on the objective).
- Derive strong valid inequalities in the original variable space (hard!)

Thanks!!