TWO OPEN QUESTIONS

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Abstract. These are some questions that I haven't been able to find an answer for, all motivated from the theory of stochastic approximation. Maybe they admit a trivial answer and I just haven't been looking at the right place, maybe they don't. Any feedback welcome!

1. Weighted/Convolved sums of martingale differences

Let $\{X_i\}_{i=1}^{\infty}$ be a martingale difference sequence, i.e. $\mathbb{E}[X_i] < \infty$ and $\mathbb{E}[X_i \mid \mathcal{F}_{i-1}] = 0$ where $\mathcal{F}$ denotes the history (natural filtration) of X.[1] If $\mathbb{E}[|X_i|^2 \mid \mathcal{F}_{i-1}] < \infty$, the strong law of large numbers states that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j = 0 \quad (a.s.). \tag{LLN}$$

The question is what happens if, instead of the standard Cesàro mean above, we consider a weighted/convolved average of the form

$$S_n^{\gamma} = \sum\nolimits_{j=1}^{n} \gamma_{j,n} X_j \tag{1}$$

for some (double) sequence $\gamma_{j,n} > 0$ which is non-increasing in j. In particular:

Question 1. *Let $t_n = \sum_{j=1}^{n} \gamma_{j,n}$. Under what conditions for X and γ do we have $S_n^{\gamma} = o(t_n)$ (a.s.)?*

Partial answer: in the "pure weights" case ($\gamma_{j,n} \equiv \gamma_j$) this is a classical result (modulo a finite variance requirement). But what happens when the sum is a convolution of the X_j's?

Question 2. *What can be said about the sum-of-squares $\sum_{j=1}^{n} \gamma_{j,n}^2 X_j^2$? Under what conditions for X and γ is it finite (a.s.)?*

2. Continuous Images of Stochastic Approximations

Let M be a metric space and let Θ be a *flow* on M, i.e. a continuous map $\Theta\colon M \times \mathbb{R}_+ \to M$ such that, for all $x \in M$ and for all $t, s \geq 0$:

a) $\Theta(x, 0) = x$.
b) $\Theta(x, t+s) = \Theta(\Theta(x,t), s)$.

Then, an *asymptotic pseudotrajectory* (APT) of Θ is any continuous path $X\colon \mathbb{R}_+ \to M$ that shadows the trajectories of Θ with arbitrary accuracy over arbitrarily large intervals for large enough times; formally:

$$\lim_{t\to\infty} \sup_{0\leq h\leq T} \operatorname{dist}(X(t+h), \Theta(X(t), h)) = 0. \tag{APT}$$

[1] If you prefer, you can think of a sequence of i.i.d. zero-mean random variables.

If $A \subseteq M$ is a global attractor of Θ then, roughly speaking, any asymptotic pseudotrajectory X either converges to A or "escapes to infinity" (if M is not compact). The question is whether similar conclusions can be drawn from information on an *image* of Θ. Specifically:

Question 3. *Let M, N be metric spaces, let Θ be a flow on M, and let $F\colon M \to N$ be a continuous map. Suppose further that B is a compact subset of N such that* $\operatorname{dist}(F(\Theta, x(t)), B) \to 0$ *as $t \to \infty$, uniformly on $x \in N$ (i.e. $F \circ \Theta$ is globally attracted to B). Under what conditions can we conclude that $F(X(t)) \to B$ if X is an asymptotic pseudotrajectory of Θ?*