On the dominating set problem

January 26, 2016

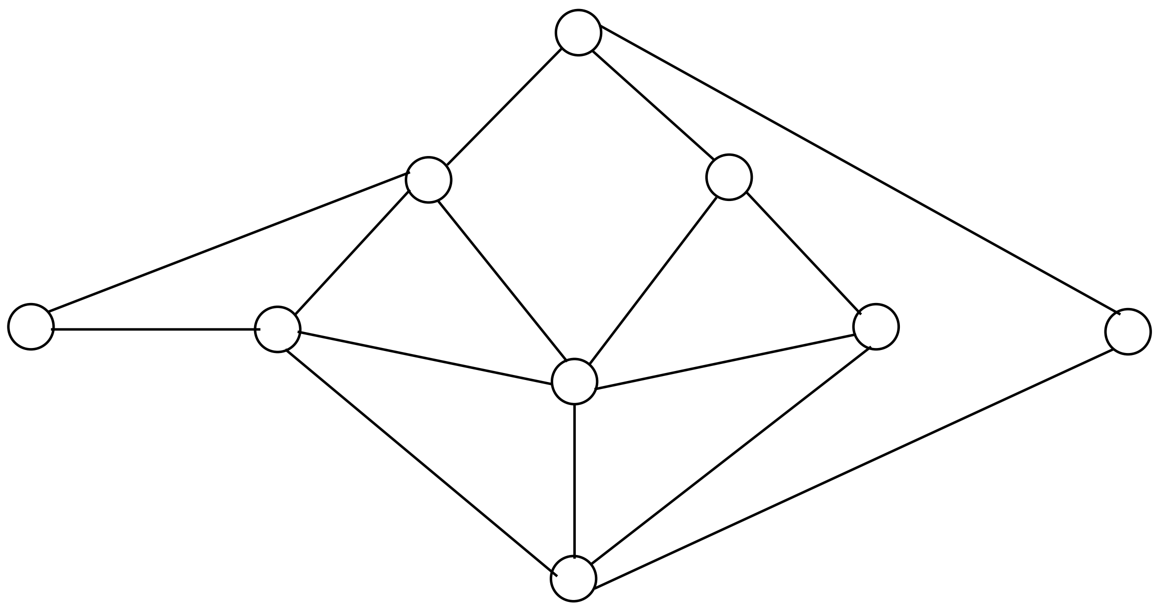
In the following presentation we presented an extended formulation for the dominating set problem. We showed that it defines an integral polytope for cactus graphs and it is valid for any graph. To show the integrality of the polytope in cactus graph, we first gave an extended formulation when the underlying graph is a cycle, then we proved a 1-sum composition theorem. Our 1-sum composition is much more simpler than the one with the original variables, since it is sufficient to put together the two pieces of a certain relaxation. In fact our polytope is a face of this relaxation.

This shows that composing with the extended formulation is easier than composing with the original variables formulation. The question is : how to do a 2-sum with the extended formulation? Now the graph has a 2-node cutset, u and v. That is the deletion of u and v disconnects the graph into two pieces $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$ with $V_1 \cap V_2 = \{u, v\}$, $E_1 \cup E_2 = E$, $E_1 \cap E_2 = \emptyset$. You will remark that now we cannot longer put the two pieces together, since we may have inequalities defining facets that have one part in G_1 and the other part in G_2 as the lifted g-odd inequalities we introduced below. So we have to add some extra graph to both G_1 and G_2, the idea is that this extra graph must as simple as possible.

As a result of doing a 2-sum is a complete characterization of the dominating set polytope on series-parallel graphs. The minimum cardinality (all the weights are equal 1) dominating set problem is polynomial in series-parallel graphs but we do not have have such a result for the weighted case.

Problem definition

$G = (V, E)$ is an undirected graph. A subset $D \subseteq V$ is called a *dominating set* if every node in $V \setminus D$ is adjacent to a node in D.



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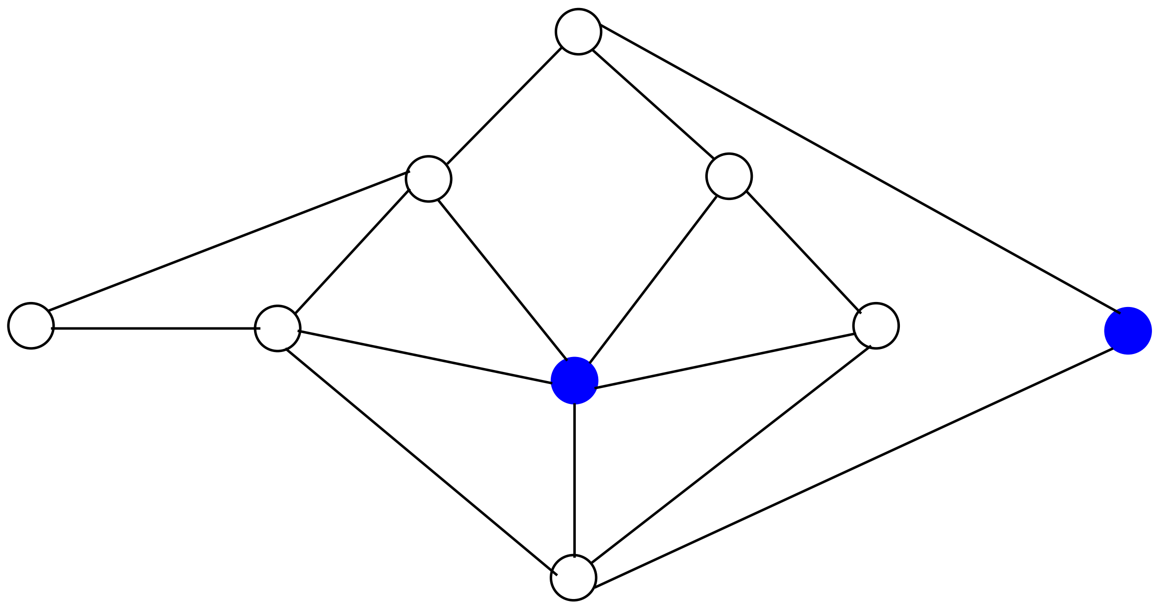

The blue nodes are not a dominating set

Problem definition

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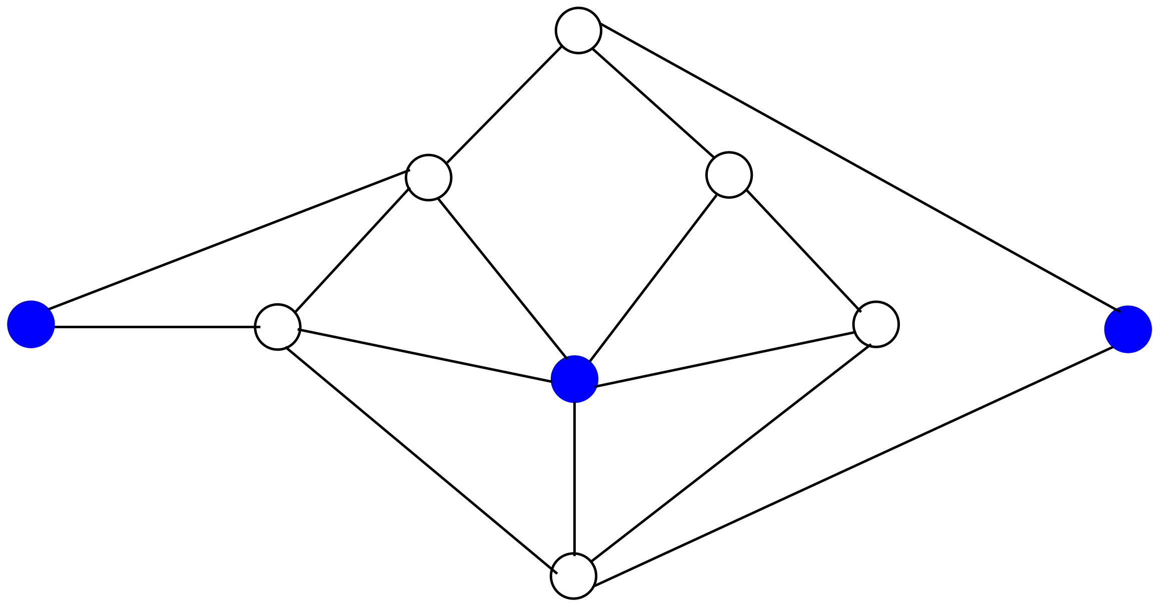


The blue nodes are a dominating set

Problem Input:

- $G = (V, E)$ undirected graph.
- Node weight: w_v for all $v \in V$.

The *minimum weight dominating set problem* **(MWDS):**

Find a minimum weight dominating set

Related problems and known results

When the weights associated with the nodes are all equal to 1, the problem is to find the *minimum dominating set problem* (MDS). If in addition the graph induced by the dominating set must be connected the problem is called *minimum weight (cardinality) connected dominating set problem* (MWCDS (MCDS)).

- **Complexity**
  - The MDS problem is NP-hard.
  - Remains NP-hard for planar graphs with maximum degree 3 and those that are regular of degree 4. (Garey and Johnson ' 79).

- **Polynomiality via cominatorial algorithms**
  - MDS is polynomial for trees (Cockayne et al.' 75), for cactus graphs (Ore'62) and for series-parallel graphs (Kikuno et al' 83), permutation graphs (Farber and Keil' 85), ...
  - WMDS is polynomial for strongly chordal graphs (A primal dual algorithm, (Farber' 82)), Bounded boolean-width (Bui-Xuan et al.' 2011) ($O(n^2 + nk2^{3k})$).
- **Polynomiality via polyhedra**
  - WMDS is polynomial when G is a cycle (Bouchakour' 97, Bouchakour et al' 05).

Polyhedra: The node-variable formulation

The following is the natural linear relaxation of the WMDS problem for a given graph $G = (V, E)$ with w a weight function associated with the nodes of G.

$N[v]$= is the set containing v with its neighbors.
$y(A) = \sum_{e \in A} y_e$.

$$\min \sum_{v \in V} w_v y_v$$

$$RL(G) = \begin{cases} y(N[v]) \geq 1 & \text{for all} \quad v \in V, & (1) \\ 0 \leq y_v \leq 1 & \text{for all} \quad v \in V, & (2) \end{cases}$$

Let $DSP(G)$ be the *dominating set polytope*,

$$DSP(G) = \text{conv } \{y \in RL(G) \cap \{0,1\}^{|V|}\}.$$

Few papers studied $DSP(G)$:

- $DSP(G)$ is characterized for threshold graphs, (Mahjoub' 83).
- If the graph is strongly chordal, then $DSP(G) = RL(G)$, (Farber' 84).
- Composition by 1-sum and facets of $DSP(G)$, (Bouchakour and Mahjoub' 95).
- Complete description of $DSP(G)$ when G is a cycle, (Bouchakour et al' 97). Independently this result has been obtained by (Saxena'04)
- Facial study of $DSP(G)$, (Saxena' 03), (Bianchi et al' 09).

Our main result:

- *An extended formulation via facility location that characterize $DSP(G)$ when G is a cactus, with a polynomial algorithm to solve the WMDS problem in this class of graph.*

This was an open question since the study of 1-sum compositions for $DSP(G)$ by Bouchakour and Mahjoub' 95.

Polyhedra composition

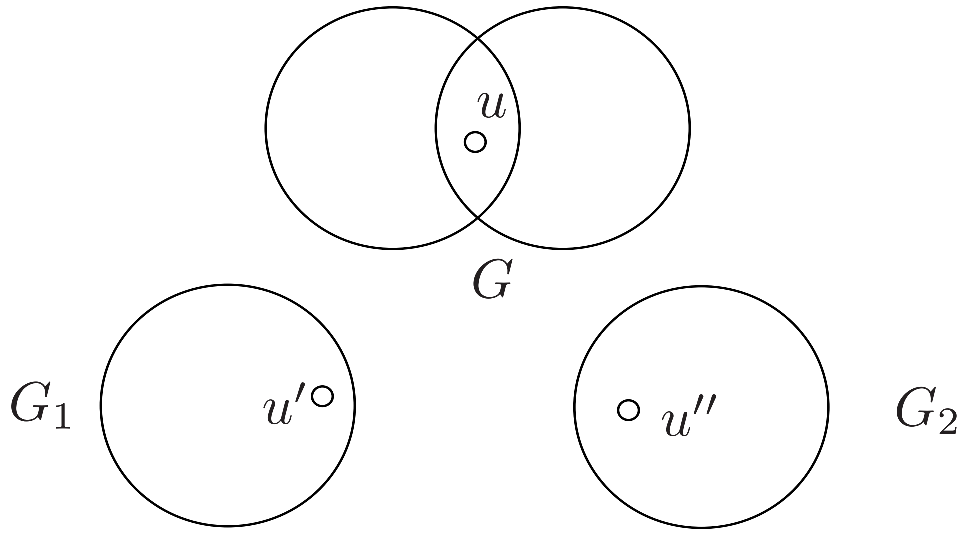


Figure 1: A 1-sum decomposition.

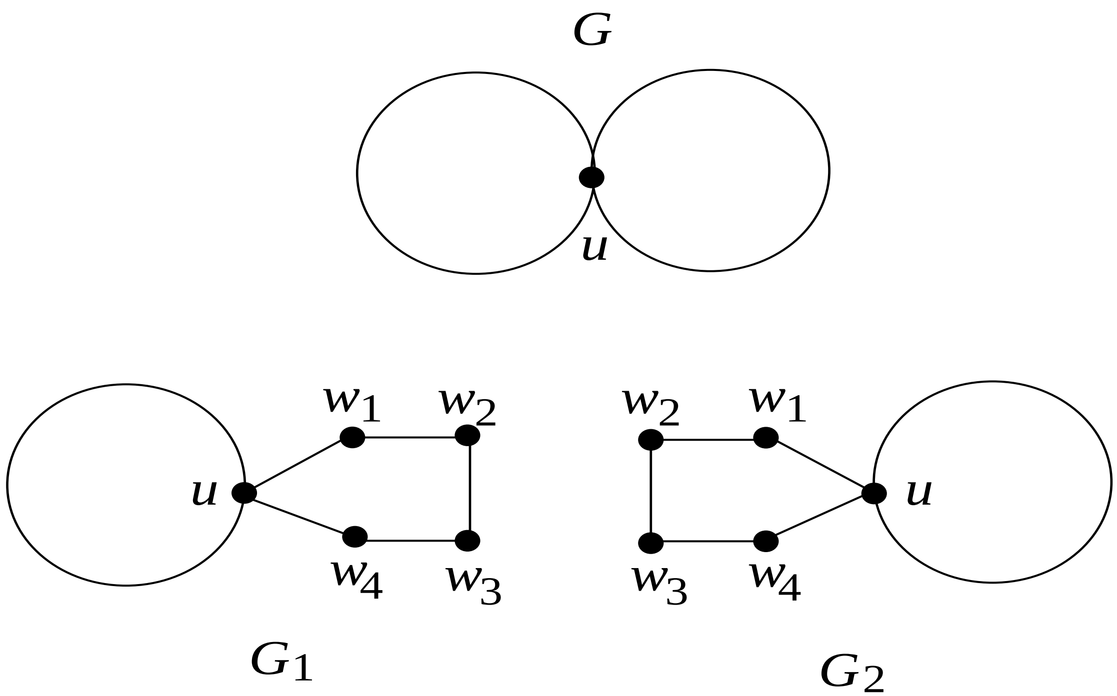
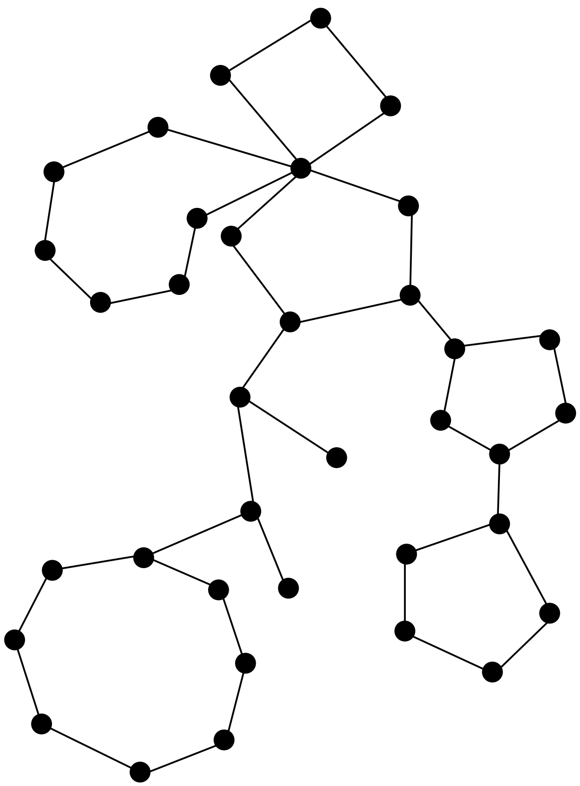


Figure 2: A 1-sum decomposition with auxiliary graphs.

Bouchakour and mahjoub (95) gave a complete characterization of $DSP(G)$ from $DSP(G_1)$ and $DSP(G_2)$

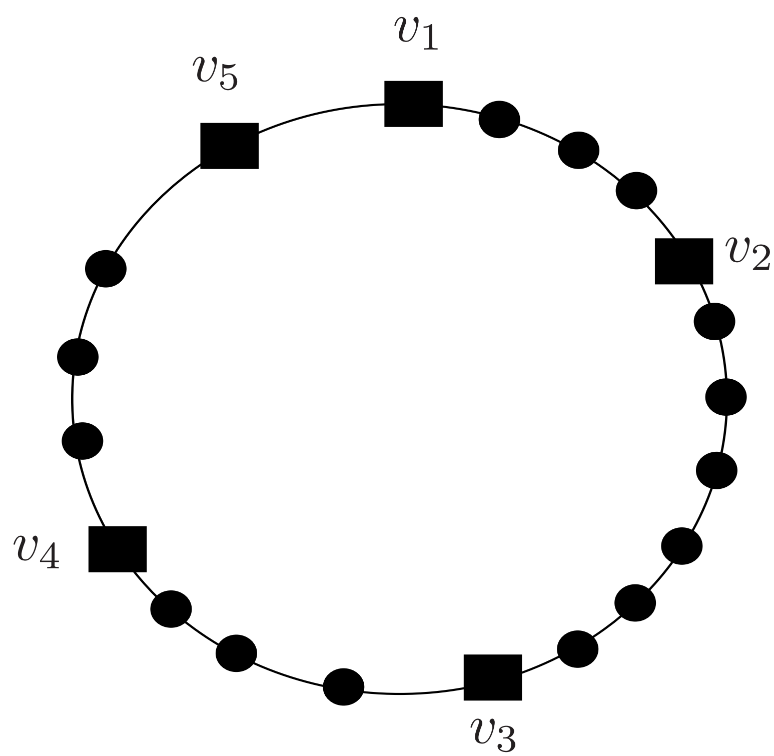
$DSP(G)$ when G is a cactus

- A graph G is a *cactus* if each edge belongs to at mots one cycle f G.



Theorem 1. [Bouchakour, Contenza, Lee and Mahjoub'05] *When* $G = (V, E)$ *is a cycle, DSP(G) is characterized by the following system of linear inequalities*

$$
\begin{cases}
y(N[v]) \geq 1 & \text{for all} \qquad v \in V, \\
y(V) \geq \lceil \frac{|V|}{3} \rceil & \\
2\displaystyle\sum_{v \in W} y_v + \sum_{v \in V \setminus W} y_v \geq \sum_{i=1}^{|W|} k_i + \lceil \frac{|W|}{2} \rceil & \text{for all } W \text{ as defined below} \\
0 \leq y_v \leq 1 & \text{for all} \qquad v \in V,
\end{cases}
$$



The number between two consecutive nodes v_i *and* v_{i+1} *in* W *is* $3k_i$.

The squares are the nodes $v_1, \ldots, v_5$ *in* W.

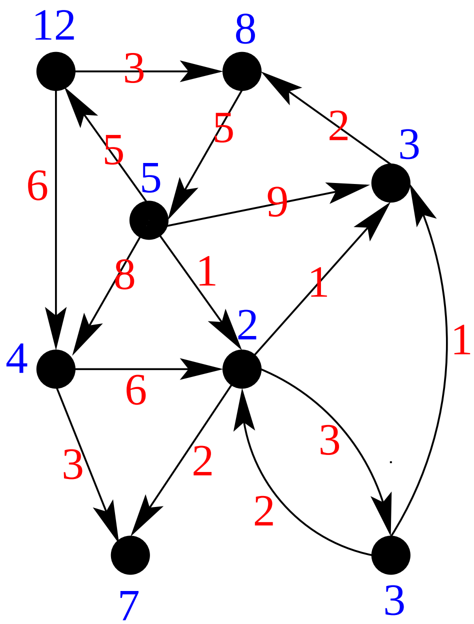
The rhs in this example is 5+3.

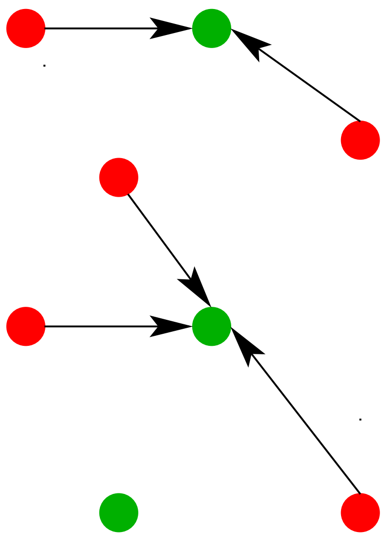
- **Remark 1. [Bouchakour et al.'05]** *Given an integer* $p > 0$, *there exists a cactus graph* G *such that* $DSP(G)$ *has a facet defining inequality with coefficients* $1, \ldots, p$.

Extended formulation for $DSP(G)$ via facility location

Given a direct graph $D = (V, A)$ where each arc (u, v) and node v are associated with a weight w_{uv} and w_v.

The *Uncapacitated Fcaility Location* (UFL) problem is to select some nodes, and assign to them the non-selected one such that the assignment cost + the cost of the selected nodes is minimized.





Selected nodes
Non−Selected nodes

The costof this solution is :

8+2+7+ 2+3+1+6+2

The following is a linear relaxation for the uncapacitated facility location problem:

minimize $\sum_{(u,v)\in E} w_{uv}x_{uv} + \sum_{v\in V} w_v y_v$

$$P(D) = \begin{cases} x(\delta^+(u)) + y_u = 1 & \forall u \in V, \\ x_{uv} \le y_v & \forall (u,v) \in A, \\ x_{uv} \ge 0 & \forall (u,v) \in A, \end{cases}$$

where $\delta^+(u) = \{(u,v) : (u,v) \in A\}$ (the set of arcs leaving u).

Let $UFLP(D)$ denote the uncapacitated facility location polytope,

$$UFLP(D) = \text{conv}\ \{(x,y) \in P(D) \cap \{0,1\}^{|A|+|V|}\}.$$

Let $G = (V, E)$ an undirected graph. Let $\overleftrightarrow{G} = (V, A)$ a *symmetric directed* graph obtained from G by replacing each edge uv of G by two arcs (u, v) and (v, u).

Theorem 2. *The projection of $UFLP(\overleftrightarrow{G})$ onto the y-variables gives $DSP(G)$.*

In the sequel we will give a complete description of $UFLP(\overleftrightarrow{G})$ when G is a cactus.

1-sum composition

Suppose u is an articulation point in G.

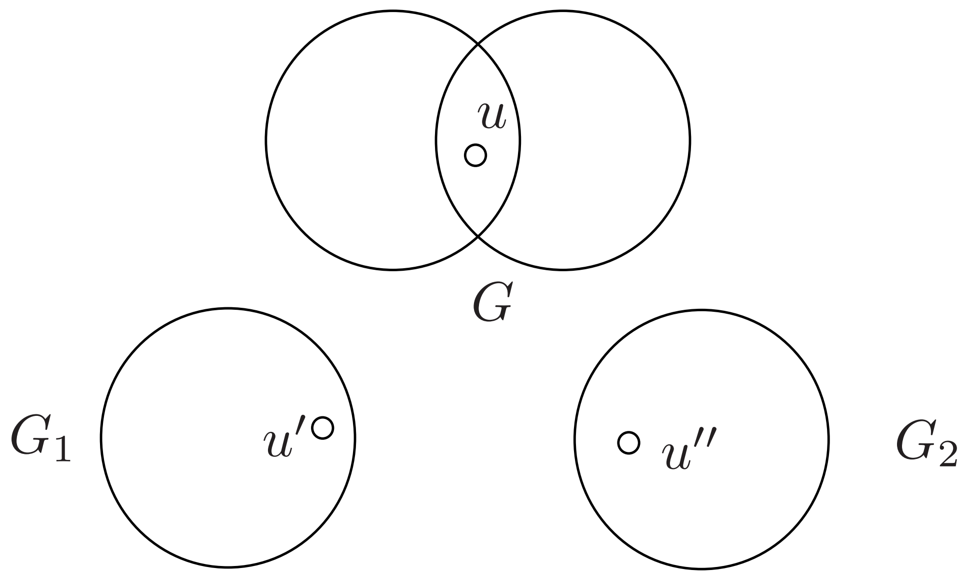
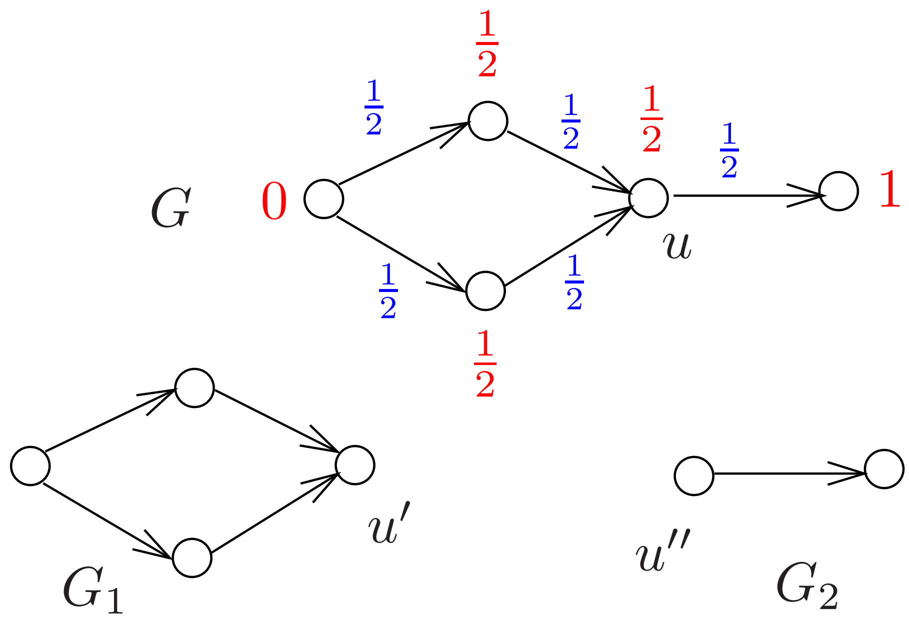


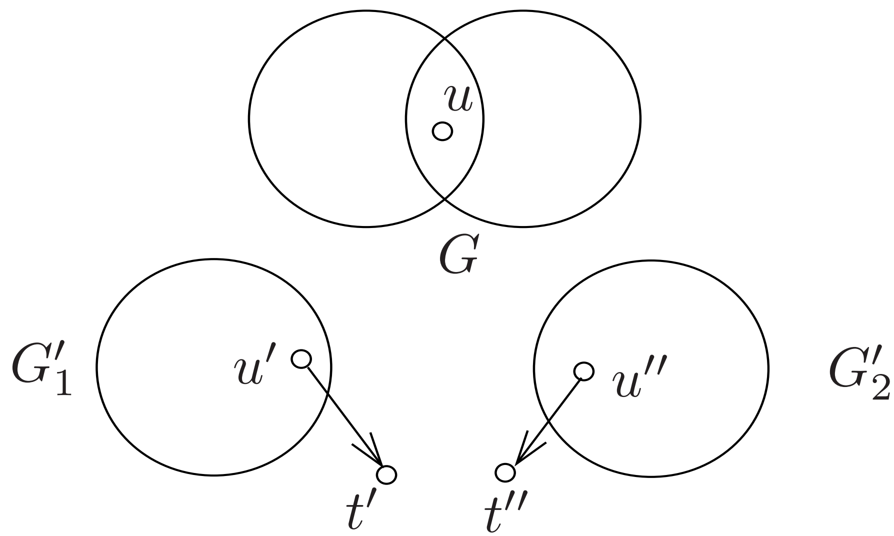
Figure 3: A 1-sum decomposition.

Notice that just a 1-sum does not suffice to describe $UFLP(G)$ from $UFLP(G_1)$ and $UFLP(G_2)$ as shown by the following example.



$P(G_1)$ and $P(G_2)$ are integral but not $P(G)$.

From G_1 and G_2 define G_1' and G_2''.



Assume that:

$UFLP(G_1')$ is described by $A \begin{bmatrix} z_1 \\ \alpha \end{bmatrix} \leq b$, α is associated with (u', t').

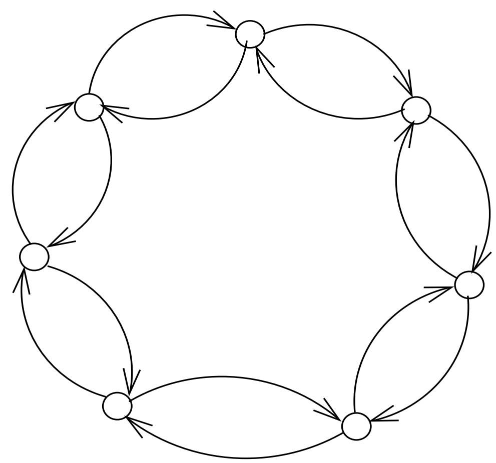
$UFLP(G_2')$ is described by $C \begin{bmatrix} z_2 \\ \beta \end{bmatrix} \leq d$, β is associated with (u'', t'').

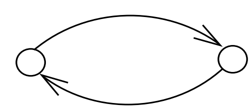
Theorem 3. $UFLP(G)$ *is described by*

$$
\begin{aligned}
A \begin{bmatrix} z_1 \\ \alpha \end{bmatrix} & \leq b, \\
C \begin{bmatrix} z_2 \\ \beta \end{bmatrix} & \leq d, \\
\alpha = z_2(\delta^+(u'')), & \\
\beta = z_1(\delta^+(u')), & \\
z_1(u') = z_2(u''). &
\end{aligned}
$$

Characterization of $UFLP(\overleftrightarrow{G})$ when G is a cactus

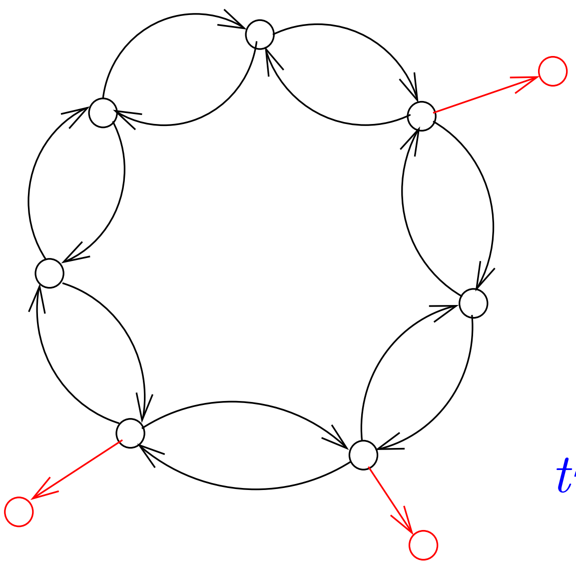
If G is a cactus, then $\overleftrightarrow{G}$ may be decomposed by means of 1-sum into the following pieces:

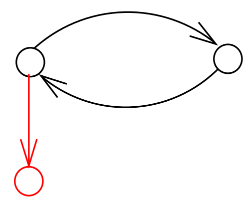




bidirected cycle: BIC_n

From the composition Theorem 1, to characterize $UFLP(\overleftrightarrow{G})$ it suffices to describe $UFLP(D)$ when D is one of the following pieces:





trivial graph

extended bidirected cycle

For a directed graph $D = (V, A)$ let $P'(D)$ be the polytope defined from $P(D)$ by replacing the equalities in $P(D)$ by inequalities. Define $UFLP'(D)$ to be to be convex hull of the 0-1 solutions in $P'(D)$.

It is easy to check the following

Lemma 1. *The characterization of $UFLP(D)$ when D is an extended bidirected cycle reduces to the characterization of $UFLP'(D)$ when D is a bidirected cycle.*

Characterizing $UFLP'(BIC_n)$

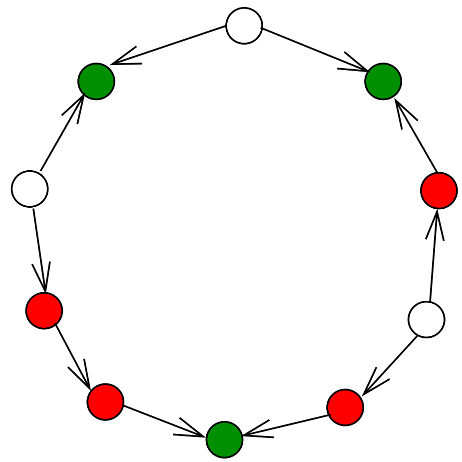
• Valid inequalities for $UFLP'(D)$ for any directed graph D.

- bicycle inequalities. If BIC_r is a bicycle in D (not necessary induced), then

$$\sum_{a \in A(BIC_r)} x(a) \leq \left\lfloor \frac{2|r|}{3} \right\rfloor, \tag{1}$$

is valid for $UFLP'(D)$.

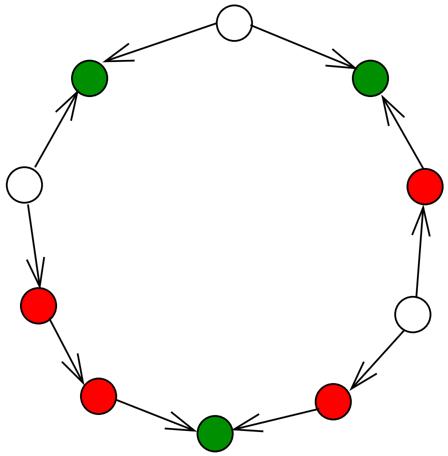
- g-dd cycle inequalities.



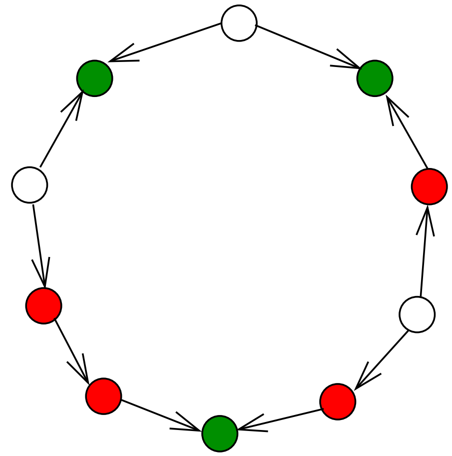
A g-odd cyle C. Its parity is the number of green and red nodes

$$\sum_{a \in A(C)} x(a) - \sum_{\textcolor{green}{\bullet}} y(v) \leq \frac{|\textcolor{green}{\bullet}| + |\textcolor{red}{\bullet}| - 1}{2} \qquad (2)$$

- lifted g-odd inequalities.

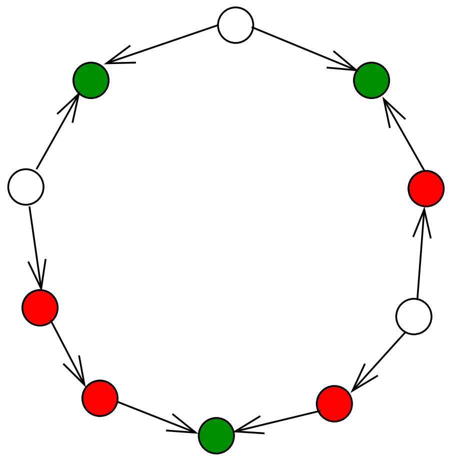


A g-odd cycle

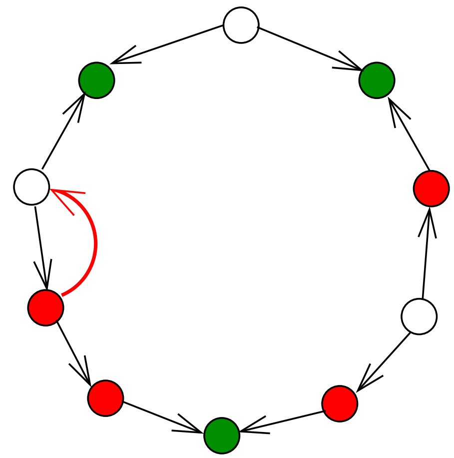


A lifted g-odd cycle

- lifted g-odd inequalities.

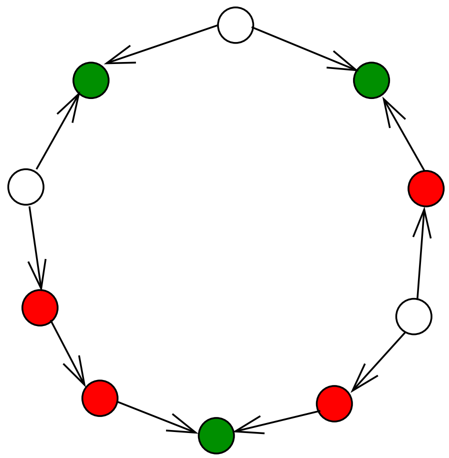


A g-odd cycle



A lifted g-odd cycle

- lifted g-odd inequalities



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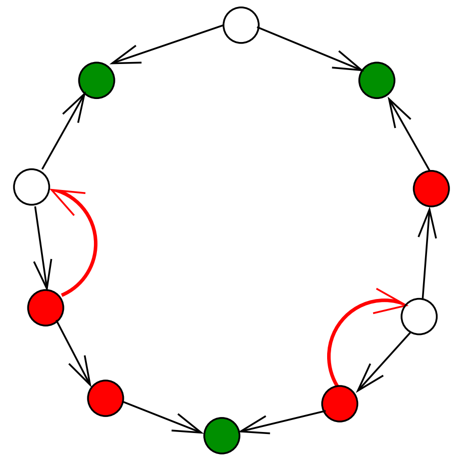
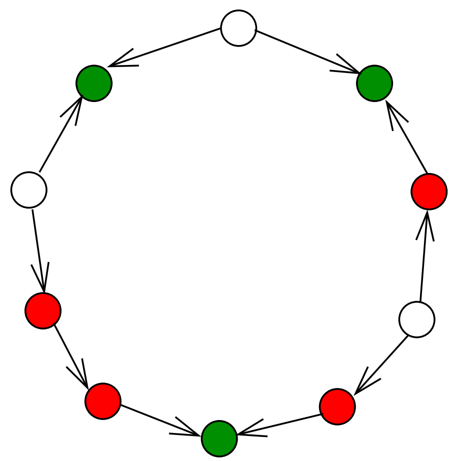
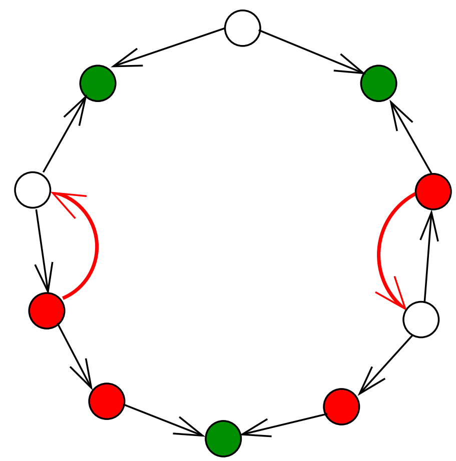


A lifted g-odd cycle

- lifted g-odd inequalities.

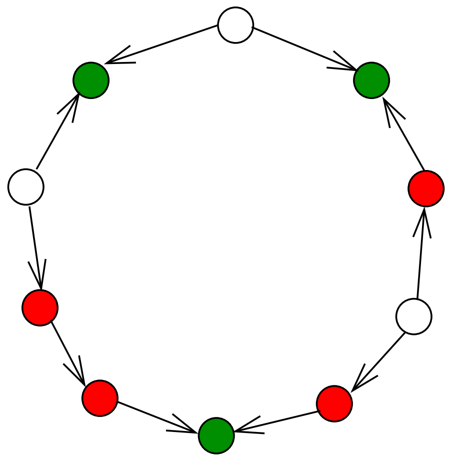


A g-odd cycle

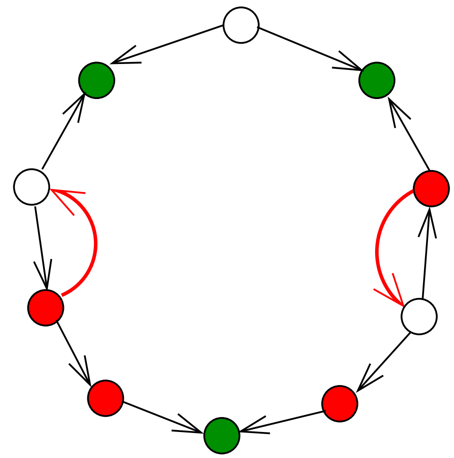


A lifted g-odd cycle

- lifted g-odd inequalities.



A g-odd cycle



A lifted g-odd cycle

$$\sum_{a \in A(C)} x(a) + \sum_{a \in \color{red}{red\ arcs}} x(a) - \sum_{\color{green}{\bullet}} y(v) \leq \frac{|\color{green}{\bullet}| + |\color{red}{\bullet}| - 1}{2} \tag{3}$$

Theorem 4. *$UFLP'(BIC_n)$ is described by*

$$\begin{cases} P(BIC_n): \text{ the linear relaxation,} \\ (1): \text{ the bicycle inequality,} \\ (2): \text{ the lifted g-odd cycle inequalities.} \end{cases}$$

Idea of the proof.

- Let $\alpha^T x + \beta^T y \leq \rho$ be a facet of $UFLP'(BIC_n)$.
- Eisenbrand, Oriolo, Stauffer and Ventura' 05 completely describe the stable set polytope when G is a quasi-line graph. Using their result we show that the coefficients of any facet of $UFLP'(BIC_n)$ are all in $\{0, 1, -1\}$.
- The last part of the proof shows that any inequality defining a facet of

$UFLP'(BIC_n)$ is a bicycle inequality, a g-odd cycle inequality, a lifted g-odd cycle or among the inequalities that define $P(BIC_n)$.

Let us conclude

- Notice that the coefficients of our description are all 0, 1 or -1. And may be reduced to only 0, 1 coefficients.

- We can optimize over $UFLP(\overleftrightarrow{G})$ in polynomial time, when G is a cactus. This is done using a linear time algorithm for the separation problem of the g-odd lifted cycle inequalities. The separation of the inequalities in node-variables formulation may be done in $O(n^2)$.

- Using the extended formulation, we can solve the MWDSP in linear time when G is a cycle. In the original variable-dimension it may be solved in $O(n^2)$.

- If we add $\sum_{v \in V} y_v = p$ to $UFLP(\overleftrightarrow{G})$ when G is a cycle, then the bicycle with the lifted g-odd inequalities are redundant.

First part of the proof uses the following reduction to the stable set problem

• $G = (V, E)$ an undirected graph. A subset $S \subseteq V$ is called a *stable* set of G if no two nodes in S are adjacent.

• When each node v is associated with a weight w'_v, the *maximum weighted stable set* (MWSS) problem is to find a stable set S that maximize $w'(S)$.

• MWSS can be formulated as follows:

$$\text{maximize} \sum_{v \in v} w'_v x_v$$

$$Q(G) = \begin{cases} x(C) \leq 1 & \text{for each maximal clique } C \text{ in } G, \\ x(v) \geq 0 & \forall v \in V, \end{cases}$$

$$x_v \in \{0, 1\} \qquad \forall v \in V.$$

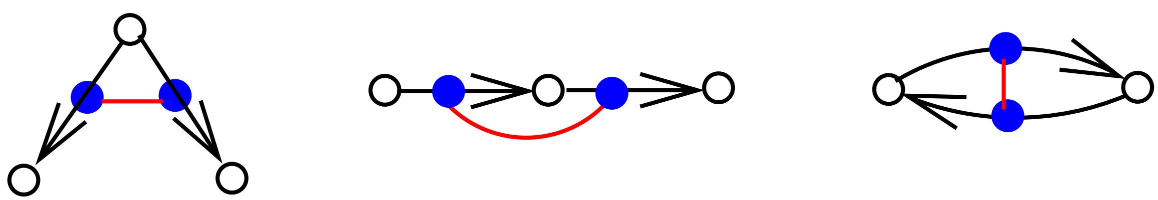
The *stable set polytope* of G is

$$SSP(G) = \mathsf{conv}\{x \in Q(G) \cap \{0,1\}^{|V|}\}.$$

- From a directed graph $D = (V, A)$ define an undirected graph $I(D) = (A, E)$ called the *intersection graph of D*.

- The nodes of $I(D)$ are the arcs of D,

- The edges of $I(D)$ are defined as below:



minimize $\sum_{(u,v)\in A} w_{uv}x_{uv} + \sum_{v\in V} w_v y_v$

$P(D) =$

$$\begin{cases} \sum_{(u,v)\in A} x_{uv} + y_u = 1 & \forall u \\ x_{uv} \le y_v & \forall (u,v) \\ x_{uv} \ge 0 & \forall (u,v) \end{cases}$$

x and y are integer.

maximize $\sum_{(u,v)\in A} w'_{uv}x_{uv}$

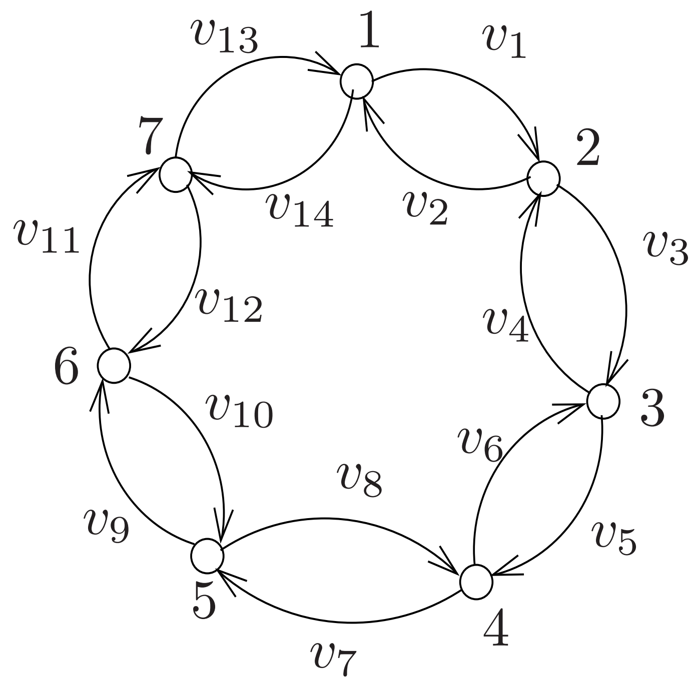
$Proj(D) =$

$$\begin{cases} \sum_{(u,v)\in A} x_{uv} \le 1 & \forall u \\ x_{uv} + \sum_{(v,t)\in A} x_{vt} \le 1 & \forall (u,v) \\ x_{uv} \ge 0 & \forall (u,v) \end{cases}$$

x integer .

Notice that $P(D)$ is integral if and only if $Proj(D)$ is integral.

Remark 2. $Proj(D) = Q(I(D))$.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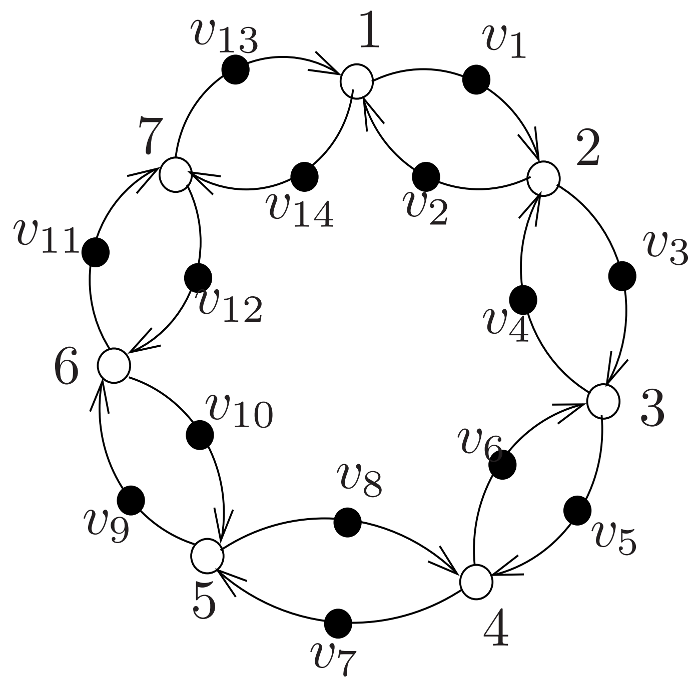
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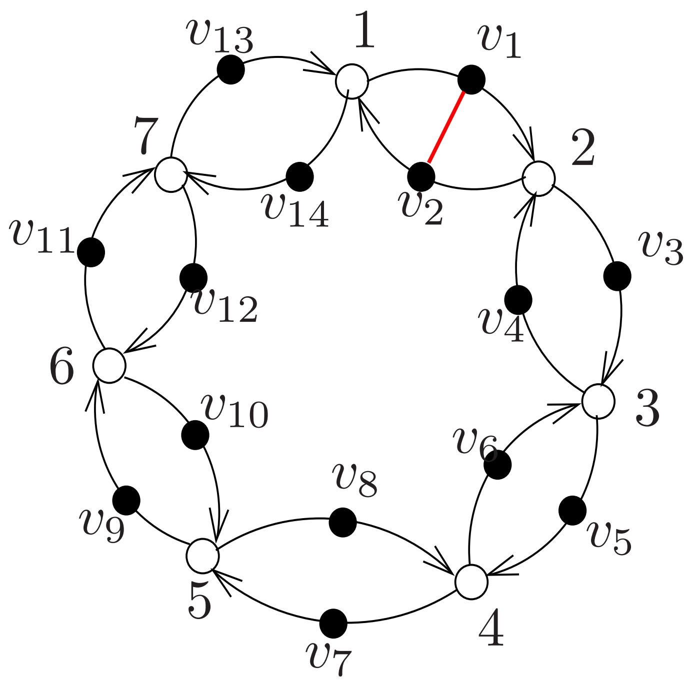
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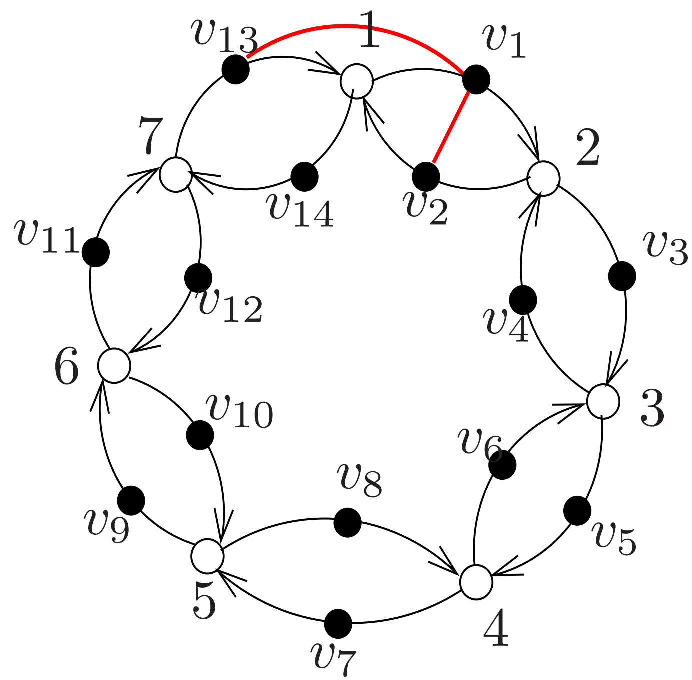
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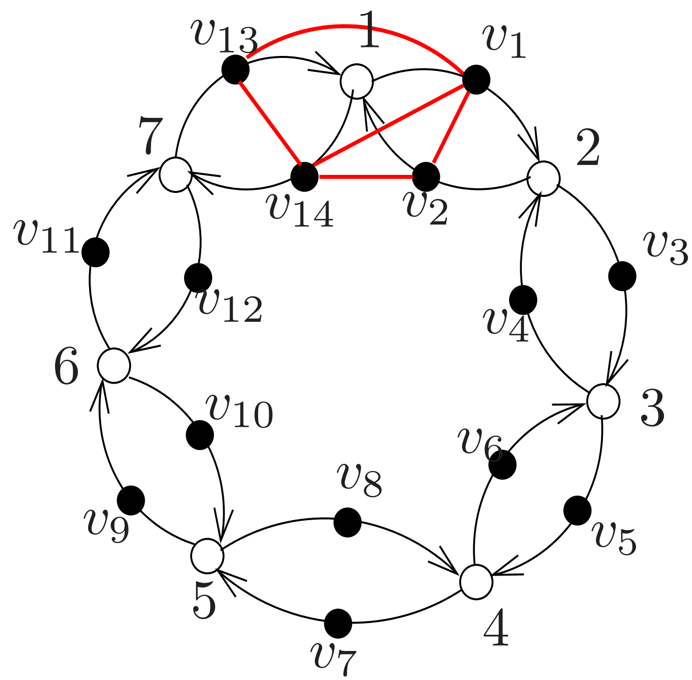
1
2
3
4
5
6
7
v_1
v_2
v_3
v_4
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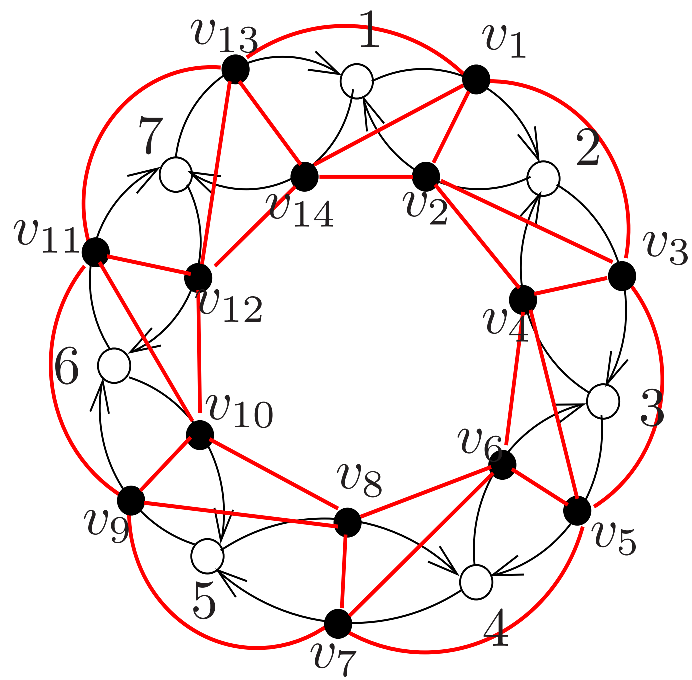
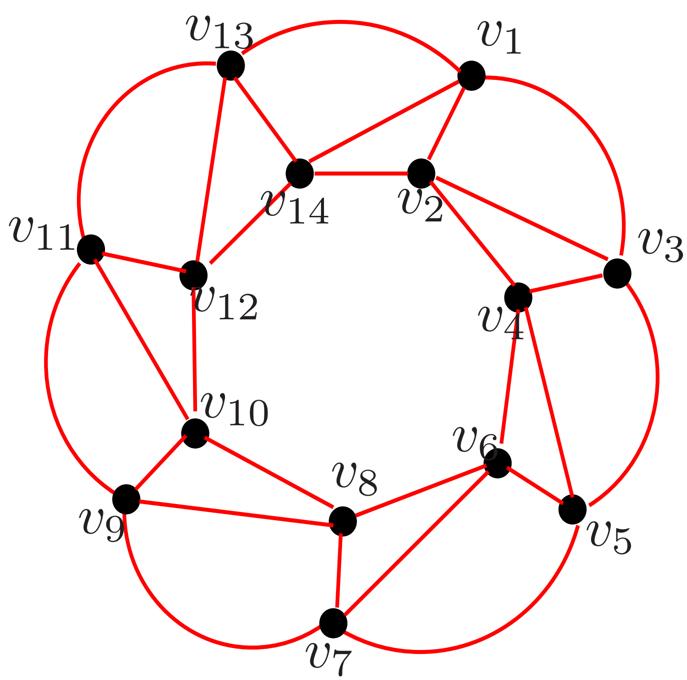
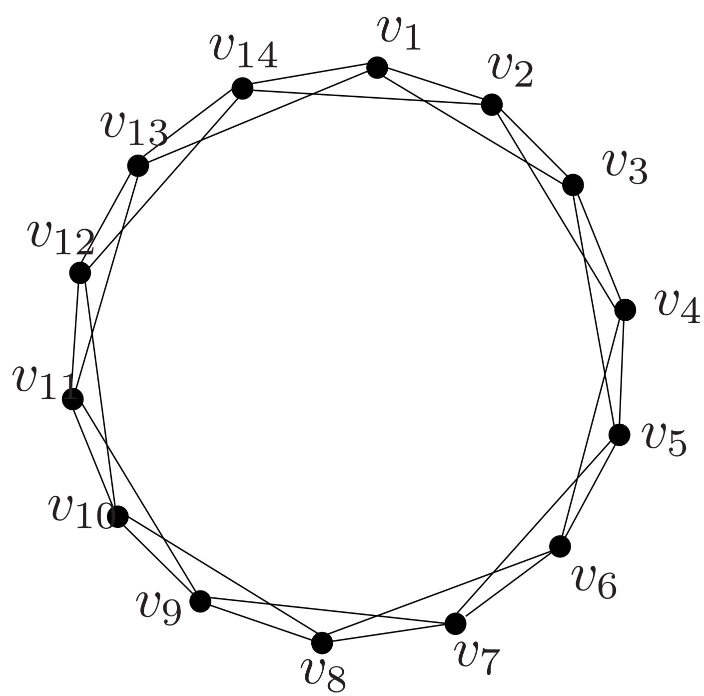
v_{10}
v_{11}
v_{12}
v_{13}
v_{14}



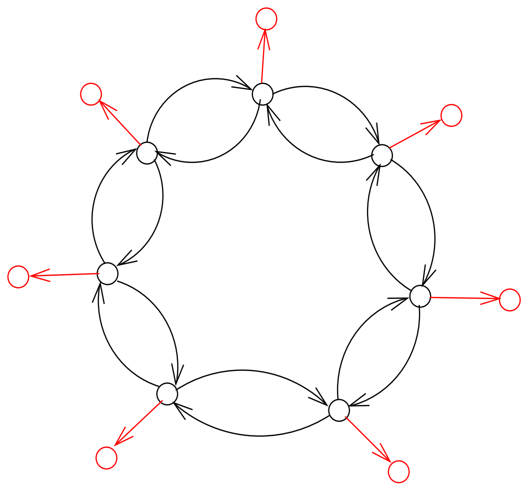
v_{13}
1
v_1
7
2
v_{14}
v_2
v_{11}
v_3
v_{12}
v_4
6
3
v_{10}
v_6
v_8
v_9
v_5
5
4
v_7



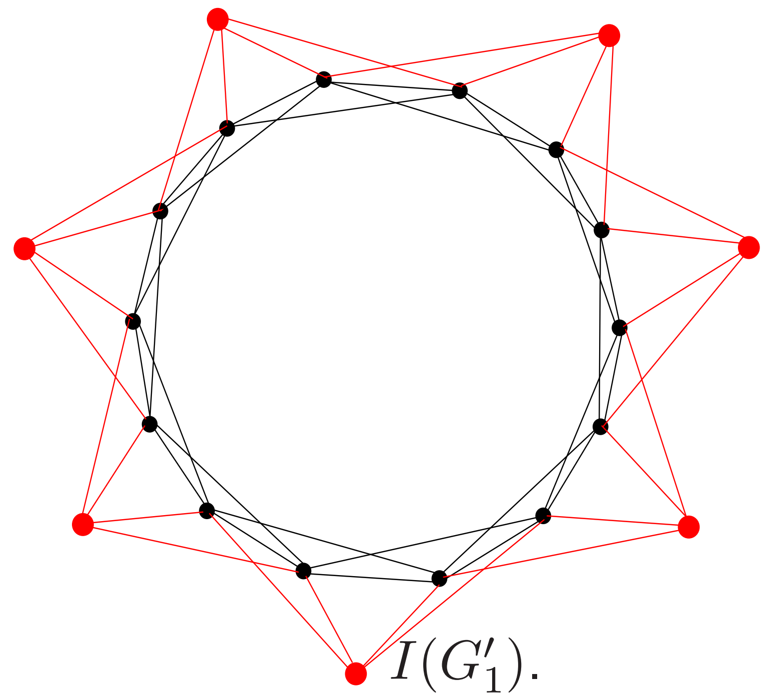
A circulant graph.



A circulant graph.



G'_1



$I(G'_1)$.

Remark 3. *The intersection graph* $I(G'_1)$ *of* G'_1 *is a quasi-line graph. But* $I(\overleftrightarrow{G})$ *when* G *is a cactus is not even a claw-free graph.*

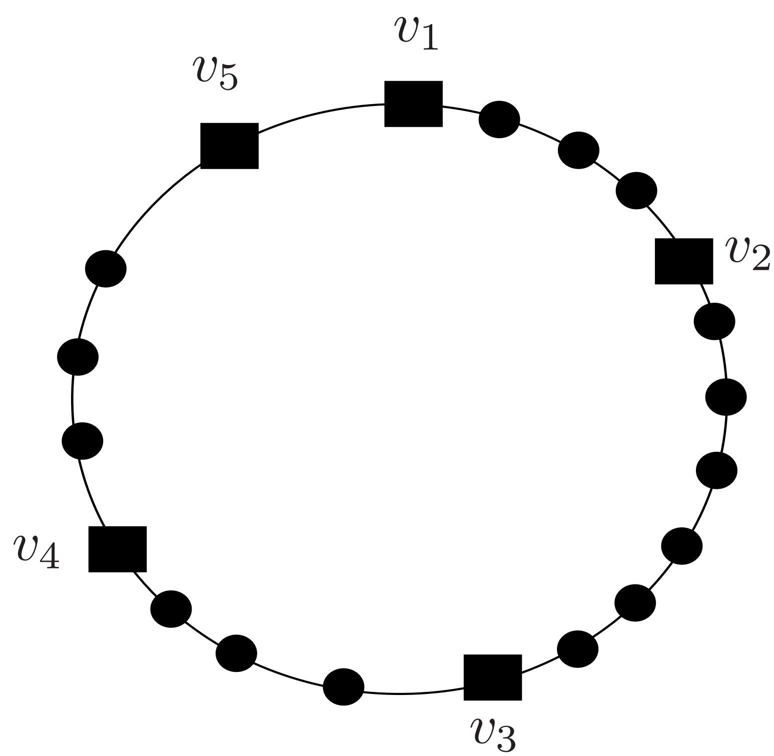
Eisenbrand, Oriolo, Stauffer and Ventura' 05 completely describe $SSP(G)$ when G is a quasi-line graph. Using their result we show that the coefficients of any facet of $SSP(G'_1)$ are all in $\{0, 1, -1\}$.

• The second part of the proof shows that any inequality defining a facet of $SSP(G'_1)$ is a bicycle inequality, a g-odd cycle inequality, a lifted g-odd cycle or among the inequalities that define $P(G'_1)$.

Let us conclude

Theorem 5. [Bouchakour, Contenza, Lee and Mahjoub'05] *When $G = (V, E)$ is a cycle, DSP(G) is characterized by the following system of linear inequalities*

$$
\begin{cases}
y(N[v]) \geq 1 & \textit{for all} \quad v \in V, \\
y(V) \geq \lceil \frac{|V|}{3} \rceil & \\
2 \sum_{v \in W} y_v + \sum_{v \in V \setminus W} y_v \geq \sum_{i=1}^{|W|} k_i + \lceil \frac{|W|}{2} \rceil & \textit{for all } W \textit{ as defined below} \\
0 \leq y_v \leq 1 & \textit{for all} \quad v \in V,
\end{cases}
$$



The number between two consecutive nodes v_i *and* v_{i+1} *in* W *is* $3k_i$.

The squares $v_1, \ldots, v_p$ *are the nodes in* W.

The rhs in this example is 5+3.

- **Remark 4. [Bouchakour et al.'05]** *Given an integer $p > 0$, there exists a cactus graph G such that $DSP(G)$ has a facet defining inequality with coefficients $1, dots, p$.*

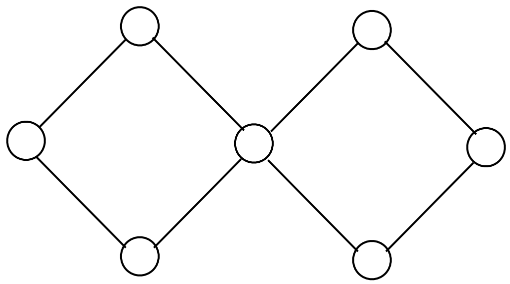
- Notice that the coefficients of our description are all 0, 1 or -1. And may be reduced to only 0, 1 coefficients.

- We can optimize over $UFLP(\overleftrightarrow{G})$ in polynomial time, when G is a cactus. This is done using a linear time algorithm for the separation problem of the g-odd lifted cycle inequalities. The separation of the inequalities in Theorem 3 may be done in $O(n^2)$.

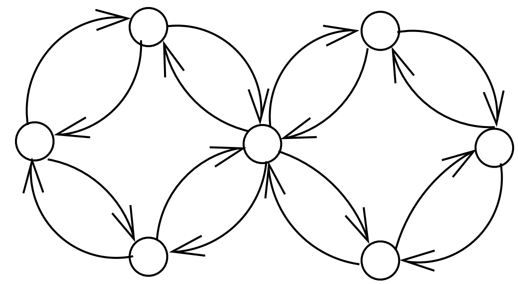
- Using the extended formulation, we can solve the MWDSP in linear time when G is a cycle. In the original variable-dimension it may be solved in $O(n^2)$.

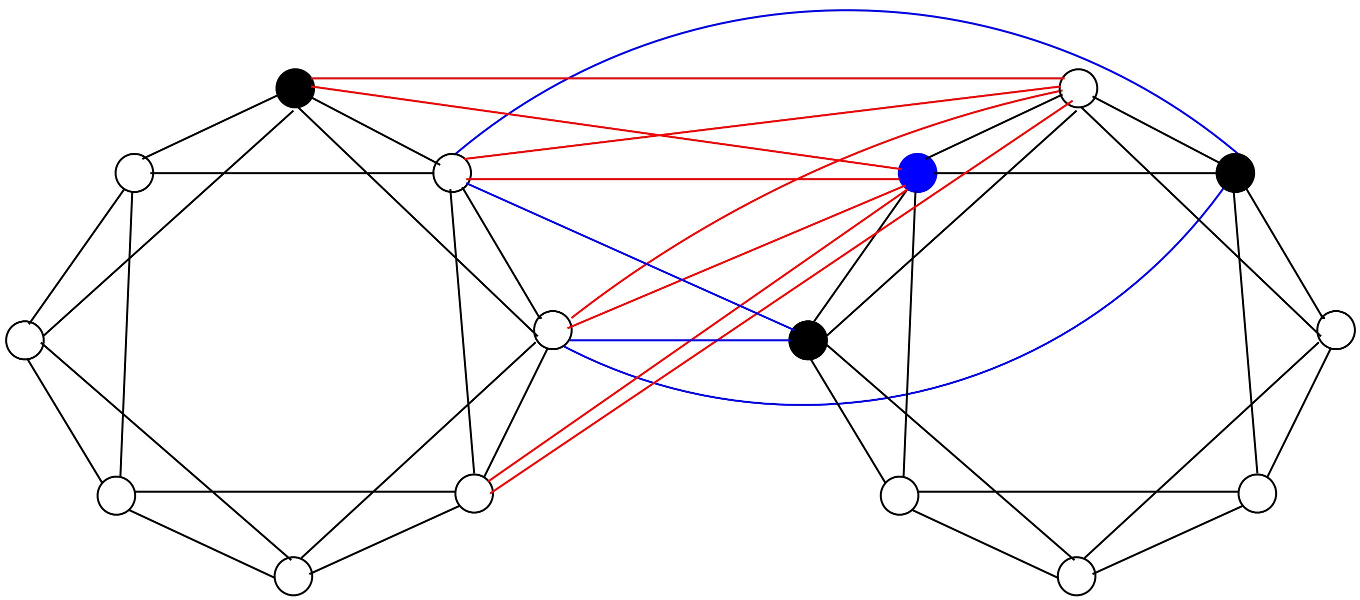
- If we add $\sum_{v \in V} y_v = p$ to $UFLP(\overleftrightarrow{G})$ when G is a cycle, then he bicycle with the lifted g-odd inequalities are redundant.

MWDSP



UFLP





MWSSP