Efficiency Bounds for Stable Matchings and the Design of Priority Rules*

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Abstract

In many school choice programs, students are assigned to schools using the deferred acceptance algorithm. The resulting stable matching need not be efficient and may leave many students assigned to unattractive schools. We derive bounds for the fraction of students assigned to their top schools and the fraction of students that can be Pareto improved in a large market model. Our bounds facilitate the comparison of a variety of different priority rules. We apply our bounds to examine random tie breakings. Distance-based priorities may assign many students to top schools, but can also cause significant efficiency losses. Simulations confirm our theoretical findings.

Keywords: market design, school choice, priorities

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1 Introduction

In many school choice programs worldwide, students are centrally assigned to schools using the deferred acceptance (DA) algorithm proposed by Gale and Shapley (1962). This algorithm results in a stable matching that need not be Pareto efficient for students (Roth and Sotomayor, 1990). Moreover, as Kesten (2010) shows, for any fixed supply of seats, it is possible to construct the demand so that the stable matching assigns each student to the worst or second-worst school. This naturally leads to the question of which policy decisions can mitigate the inefficiencies of the DA algorithm.

The priority rule used by schools to rank students is an important implementation decision that impacts the efficiency of the assignment. In real-world applications, policymakers use a variety of priority criteria (Cantillon et al., 2022). In cities like Boston and Copenhagen, a student is prioritized according to proximity to a school. In New Haven, students get higher priority in schools in which they have siblings. In many cities, schools use random priorities. While in Amsterdam a student gets a random score that applies to all schools, in Chile each student gets a different random score for each school. As Abdulkadiroğlu et al. (2009) and Leshno and Lo (2021) emphasize, the priority structure is critical in school choice implementations and can be even more important than the specific algorithm used to assign students to schools.

This paper provides results to theoretically evaluate the impact of different priority rules on some efficiency measures in school choice programs that employ the DA algorithm. A direct theoretical comparison of priority rules is often unfeasible, as stable matching models are typically too complex to yield closed-form solutions. We thus develop and apply a methodology to overcome this analytical challenge. We derive tight upper and lower bounds for the fraction of students assigned to their top schools in a large market school choice model. This enables a systematic evaluation of the different priority criteria typically employed in practical implementations.

Our analysis focuses on two key performance measures: the fraction of students assigned to their top schools and the fraction that can be Pareto-improved. These measures are motivated by practical implementations. Indeed, authorities in the US and Europe have made explicit decisions to maximize the number of students assigned to their top schools or to minimize the number of students in Pareto improving pairs (Abdulkadiroğlu et al., 2009; Ashlagi and Nikzad, 2020). While our main theoretical results derive bounds for the number of top-school assignments, we apply this analysis to evaluate different priority designs across both efficiency dimensions.

We study a large market model in which a continuum of students applies to a finite number of schools (Azevedo and Leshno, 2016; Abdulkadiroğlu et al., 2015). Students have preferences over schools, while schools have priorities over students. To capture the interplay between students' preferences and schools' priorities, each student has a type. These types govern students' preferences and also determine their scores within each school, subsequently influencing the schools' priorities over students. Introducing types into our matching model is a flexible way to allow for correlation between students' preferences and scores. Our setup encompasses a variety of priority criteria used in applied school choice, including multiple tie breaking–in which each student receives a different random number for each school–and single tie breaking–in which a student obtains a unique random score that determines her priorities in all schools. It also accommodates models in which students and schools are geographically differentiated, and a student's preference and scores are partly determined by the student location.

Our first main result, Theorem 1, provides tight upper and lower bounds for the fraction of students assigned to their top schools in a stable matching. Behind these bounds is the idea that the performance of a stable matching depends on how students congest and get admission to schools they do not consider top choices. We show that the bounds are easy to apply to examples and models that would otherwise pose significant challenges in analysis.

To establish Theorem 1, we follow Azevedo and Leshno (2016) and Abdulkadiroğlu et al. (2015) and observe that stable matchings correspond to solutions to market-clearing conditions. These solutions are hard to find in closed form. We thus introduce some simple but novel relaxed market-clearing conditions, that have solutions that are easier to analyze. More importantly, as we argue in Subsection 3.1, comparing the solutions from the original market-clearing conditions to the solutions from the relaxed conditions provides critical information about the efficiency properties of the stable matching.

Equipped with Theorem 1, we characterize the impact of various priority protocols used in school choice. An important body of literature has studied the role of different random tie breaking rules on the effectiveness of the DA algorithm (Abdulkadiroğlu et al., 2009; Ashlagi and Nikzad, 2020; Arnosti, 2022; Allman et al., 2022). Proposition 1 shows conditions under which more students are assigned to a given school as their top choice under single tie breaking than multiple tie breaking. In contrast to previous results, Proposition 1 applies even when no parametric restriction is imposed on the demand for schools.

We also evaluate distance-based priorities in a general spatial model of school choice. Stable matchings under distance-based priorities are determined by how students value proximity and by the geographical distribution of students and schools. Under distance-based scores, when students significantly value proximity, students' preferences and schools' priorities are compatible: a student who likes a school also has a high score in the school. As Proposition 2 shows, in this case, the resulting stable matching will be Pareto efficient.

In contrast, distance-based priorities may result in important efficiency losses when students' preferences for proximity are not strong. Indeed, Theorem 2 shows that multiple tie breaking may result in more students assigned to their top schools and fewer students that can be Pareto improved than distance-based priorities. This happens even when students value proximity and, as a result, there is a positive correlation between preferences and priorities. Naturally, in markets where students care not only about proximity to schools but also about other aspects –such as the outcomes of standardized tests, extracurricular activities, etc.– the consistency between preferences and priorities is positive but weak. In these markets, under distance-based priorities, a student may be assigned to a school that is not the top choice but the student just happens to live next to. This force leaves relatively few students assigned to top schools under distance-based priorities.

Theorem 2 applies to a market where students are spatially located and have arbitrary preferences over schools. As intuitive as it may seem, Theorem 2 is not prominent in the literature and is not easily proven. Theorem 1 can be used to provide an otherwise difficult characterization of a stable matching in a general model of spatial differentiation.

Theorem 2 also offers a counterpoint to the argument that proximity reduces the inefficiencies of the DA algorithm by inducing a positive correlation between preferences and priorities (Pathak, 2017; Cantillon et al., 2022). While Proposition 2 confirms this observation when preferences for proximity are strong, Theorem 2 shows that distance-based priorities result in significant efficiency losses when preferences for proximity are positive but weak. More broadly, an important lesson that our work brings to market design is that to evaluate priorities, attention to the correlation between scores and preferences is as crucial as the established focus on inter-school score correlation (Abdulkadiroğlu et al., 2009; Ashlagi and Nikzad, 2020; Arnosti, 2022; Allman et al., 2022).

Our theoretical results show that introducing proximity as a priority criterion has ambiguous effects on the fraction of students assigned to their top schools. We confirm our findings by performing several simulations in a model with a finite but large number of students.

Our work connects to several lines of research in market design.

Efficiency and stable matching. The school choice literature has shown that even the student optimal stable matching need not be Pareto efficient (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Several papers derive conditions under which a stable matching is Pareto efficient.[1] Notably, Ergin (2002) introduces a class of school priorities such that, regardless of students' preferences, the stable

[1] There is also an important set of papers proposing alternative algorithms and solutions, including Shapley and Scarf (1974), Kesten (2010), Che and Tercieux (2018), Ehlers and Morrill (2020), Cantillon et al. (2022), and Reny (2022).

matching is efficient.[2] In many practical applications of the deferred acceptance algorithm, efficiency will not be achieved, and therefore, understanding the magnitude of inefficiencies is useful when designing priorities in matching markets. In this regard, our bounds provide crucial information to compare the efficiency losses of different priority rules.

Priority design. Several authors have observed that school priorities can be designed to impact the performance of the DA algorithm. Abdulkadiroğlu et al. (2009), Ashlagi and Nikzad (2020), Arnosti (2022), Shi (2022), Allman et al. (2022) notice that when schools solve indifferences by using lotteries, the correlation between the scores of a student in different schools is important for efficiency. We contribute to this literature by noting that, as Proposition 1 shows, multiple tie breaking can be compared to single tie breaking in each popular school, imposing no functional form on the demand.

More generally, work on priority design emphasizes that the correlation of scores affects the efficiency properties of the assignment. We show how the correlation between students' preferences and scores is also important to evaluate the efficiency of a stable matching. This observation is important for priority design. Consider the comparison between multiple tie breaking and distance-based priorities in a linear city having two schools, with each school located at one of the extremes of the interval. Scores have no correlation under multiple tie breaking, while under distance-based priorities scores have negative correlation.[3] Theorem 2 shows that when students' preferences and scores are uncorrelated (that is, students do not value proximity to schools), multiple tie breaking is more efficient than distance-based priorities. In sharp contrast, when students value proximity and thus preferences and distance-

[2] Ergin (2002) introduces acyclical priorities. In school choice applications, priorities derived under single tie breaking are acyclical. See also Ehlers and Erdil (2010), Salonen and Salonen (2018), Reny (2021), Pakzad-Hurson (2023).

[3] We consider a linear city model to simplify exposition. Our conclusions hold in a general model of spatial differentiation, where the correlation of scores under distance-based priorities need not be negative. For example, if two schools are close enough, a student that gets closer to one of the schools may also get closer to the other school.

based scores are positively related, Proposition 2 shows that multiple tie breaking is less efficient than distance-based priorities. Thus, our results demonstrate that for a comprehensive evaluation of priorities, paying attention to the correlation of scores is as important as focusing on the correlation between scores and preferences.

Distance-based priorities. Our paper also connects to research about distance-based priorities in school choice. Dur et al. (2018) explore how different precedence orders implementing walk-zone reserves impact the fraction of reserve-group students assigned to each school. More closely related, Çelebi and Flynn (2021) show that in a large market model, the optimal coarsening of scores is attained by splitting agents into at most three indifference classes. They also explore a model in which scores are determined by distance and show that the the optimal number of zones depends on the diversity goals of the planner. Our focus is different in that we explore alternative performance measures, and our insights highlight how the correlation between preferences and priorities determine the effectiveness of the deferred acceptance algorithm. We thus see our analysis as complementary to Çelebi and Flynn's (2021).

Large market models. Finally, our work connects to the literature employing large market models to analyze market design questions (Azevedo and Leshno, 2016; Abdulkadiroğlu et al., 2015; Leshno and Lo, 2021; Allman et al., 2022). We provide a method to bound cutoffs for stable matchings in large market models, and derive new insights for priority design in school choice.

Organization of the paper. Section 2 introduces the model. Section 3 presents our bounds for the fraction of students assigned to their top schools. Section 4 applies our bounds to random priorities and distance-based priorities. Section 5 shows simulations. Section 6 presents concluding remarks. All proofs are in the Appendix.

2 Model

2.1 Environment

There is a finite set of schools C with $|C| = N \geq 2$. There is a continuum S of students to be matched to schools. Each student s has a strict preference ordering $\succ^s$ over $C \cup \{\emptyset\}$, where $\emptyset$ is the outcome if s is unassigned. A student s has a score vector $e^s = (e^s_c)_{c=1}^N$. School c has capacity k_c. A school c prefers student s to student s' if and only if $e^s_c > e^{s'}_c$. We simplify exposition and assume that all schools and all students are acceptable.

Students have types $i \in I$. We endow $I \subseteq \mathbb{R}^L$ with a measure ν so that $\int \nu(di) = 1$ and assume that ν is absolutely continuous. Preferences and scores are determined by types. Concretely, for each i there is a distribution F_i over the finite set of preferences over schools, with $\sum_{\succ} F_i(\succ) = 1$ and $F_i(\succ) \geq 0$, so $F_i(\succ)$ is the fraction of type i students having preference $\succ$. Additionally, a type i student has a score $e^s_c = e_c(i) \in [0,1]$. We assume that the probability of a tie in a school is 0 so that for all c and all $x \in [0,1]$, $\nu(\{i \in I \mid e_c(i) = x\}) = 0$. Implicit in our model is the assumption that any correlation between the preferences of a student s and the scores in schools is determined by the type i of the student $s \in S$. Since a student's type determines scores, a student s can be characterized by a type i and preferences $\succ$, $s = (i, \succ)$. We denote by $\bar{\nu}$ the measure induced by ν and $(F_i)_{i \in I}$ over the set of students.[4]

Let $F^k_i(c)$ be the fraction of type i students that rank school c in the k-th position:

$$F^k_i(c) = \sum_{\succ \text{ such that } c \text{ is ranked } k} F_i(\succ)$$

and $\bar{F}_i(c)$ be the fraction of type i students listing school c: $\bar{F}_i(c) = \sum_{k=1}^N F^k_i(c)$. Since we assume that students list all schools, $\bar{F}_i(c) = 1$ for all i and all c.[5]

[4] Given any subset of students $S' \subseteq S$, $\bar{\nu}(S') = \int \sum_{\succ \text{ such that } (i,\succ) \in S'} F_i(\succ)\nu(di)$.

[5] This assumption simplifies exposition. Our results can be easily accommodated to the case in which $\bar{F}_i(c) < 1$ for some i and c. In the Appendix, we provide results and proofs that apply even when students do not apply to all schools.

We assume that all the schools are popular in the sense that for all c,

$$F^1(c) := \int F_i^1(c)\nu(di) > k_c.$$

Our analysis can be extended to the case in which this inequality holds for some but not all schools, but we simplify exposition by imposing the inequality in all schools. We also assume that $\bar{F}(c) = 1 > F^1(c)$ for all c so that each school has a nontrivial mass of students that demand it but not in the top position.

2.2 Priorities

In this paper, we evaluate different priority criteria. We now discuss how prominent priority rules used in school choice models can be cast as special cases of our general model.

Several papers compare single to multiple tie breaking in school choice problems (Abdulkadiroğlu et al., 2009; Ashlagi and Nikzad, 2020; Arnosti, 2022; Allman et al., 2022). Our model also accommodates these priorities.

Example 1 (Random tie breakings)**.** *Take a school choice model in which students have no types and students' preferences are given by a distribution* $F(\succ)$*. Given the set of schools, scores at each school are randomly determined in* $[0,1]$*. Our general model can accommodate these random priorities as follows.*

Let $I = [0,1]^N$ *be the set of types and* ν *be* N *independent uniform distributions over* $[0,1]$*. The c-component of a student type* $i \in [0,1]^N$ *determines the score that student* i *has in school* c*, that is,* $e_c(i) = i_c$*. In this case, our model becomes a school choice problem in which students are ranked according to* <u>*multiple tie breaking*</u> *(MTB) (Abdulkadiroğlu et al., 2009).*

The model can also accommodate the case of <u>*single tie breaking*</u> *(STB). When* $I = [0,1]$*, and* ν *is the uniform distribution on* $[0,1]$*. A type* i *student has score* i *at each school. Our model becomes a school choice problem in which students are ranked according to a single lottery (Abdulkadiroğlu et al., 2009).*

The setup introduced in subsection 2.1 can also be used to model a school choice environment with geographical differentiation and distance-based priorities (Dur et al., 2018; Çelebi and Flynn, 2021).

Example 2 (Horizontal differentiation and distance-based priorities). *$I \subset \mathbb{R}^2_+$ models a city and a student's type is the location $i \in I$ in the city. Schools are located and spread across the city. Let $d(i, c) \in [0, 1]$ be a distance between a student located in i and school c.*[6] *Similar to Abdulkadiroğlu et al. (2017), the utility that a student located in i derives from attending school c is in part determined by $d(i, c)$. For example, one can generate the utility that a type i student derives from school c as*

$$u_{s,c} = \alpha(1 - d(i, c)) + (1 - \alpha)\epsilon_{i,c}$$

where $\epsilon_i = (\epsilon_{i,c})_{c=1}^N$ is a shock vector and has a distribution H_i and α measures the relevance of proximity to rank schools.[7] *In this case, we can construct the distribution over the finite set of preferences as:*

$$F_i(\succ) = Prob[u_{s,c1} \geq u_{s,c_2} \geq \cdots \geq u_{s,c_N}]$$

where $c_1 \succ \cdots \succ c_N$.

Schools can rank students using a variety of criteria (including random tie breaking, discussed above). Under distance-based priorities, the score that a student type i has in school c is given by $e^s_c = 1 - d(i, c)$.

2.3 Stable matchings

A matching is a function $\mu : S \cup C \to (C \cup \{\emptyset\}) \cup 2^S$ such that:

i. For all $s \in S$, $\mu(s) \in C \cup \{\emptyset\}$;

[6]The distance function can be arbitrary. The only property relevant for our analysis is that the distance function $d(i, c)$ satisfies the triangle inequality.

[7]Using this formulation, we can model fixed effects and also interaction effects other than distance (Abdulkadiroğlu et al., 2017).

ii. For all $c \in C$, $\mu(c) \subseteq S$ with $\bar{\nu}\big(\{s|\mu(s) = c\}\big) \leq k_c$;

iii. For all $c \in C$ and all $s \in S$, $\mu(s) = c$ if and only if $s \in \mu(c)$.

iv. For all c, $\{s \in S \mid c \succ_s \mu(s)\}$ is open.

The first condition says that each student is assigned to a school. The second condition says that each school is assigned to a measure of students that does not exceed its capacity. The third condition says that a student is assigned to a school if and only if the school is assigned to that student. The fourth condition is technical and eliminates redundant matchings that differ in a measure 0 of students (Azevedo and Leshno, 2016).

A matching μ is stable if for all $s \in S$ and all $c \in C$ such that $c \succ_s \mu(s)$, the following conditions hold: (i) $\bar{\nu}\big(\{s|\mu(s) = c\}\big) = k_c$; and (ii) $e_c^s < e_c^{s'}$ for all s' with $\mu(s') = c$. Intuitively, a matching is stable if there is no pair (s, c) that can block the matching (Gale and Shapley, 1962). Stability is an important desideratum in matching theory and its many applications (Roth, 1982; Abdulkadiroğlu et al., 2009).

To characterize stability, we follow Abdulkadiroğlu et al. (2015) and Azevedo and Leshno (2016) and find stable matchings as solutions to a supply and demand system of equations. Given cutoffs $p = (p_c)_{c=1}^N$, a student s can get admission to c if $e_c(i) \geq p_c$. A student demands the favorite school among those where can get admission given p. We thus define $D_c(p)$ as the measure of students that demand school c as a function of cutoffs p.[8] A stable matching can be found by means of market-clearing cutoffs $p = (p_c)_{c=1}^N$ that solve

$$D_c(p) = k_c \quad \forall c \tag{2.1}$$

Given market-clearing cutoffs, a stable matching is built by assigning each student to the most preferred school among those where the score exceeds the cutoff.[9]

[8] We provide a formula for $D_c(p)$ in the Appendix; see equation (A.1).

[9] In general, the market-clearing equations can be written as $D_c(p) \leq k_c \; \forall c$, with equality whenever $p_c > 0$. In the paper, the assumptions over scores and demand will be such that $p_c > 0$ holds for all c.

While the system of equations (2.1) is neat and simple to interpret, it can be solved in closed-form solutions only for special cases. When we can find a closed-form solution to (2.1), it is simple to calculate statistics for the resulting stable matching. However, solving the model in closed-form is unfeasible even for relative simple models.[10]

3 Students assigned to their top schools

This section states and discusses our bounds for the measure of students assigned to their top schools. We then provide some examples and sketch some of the arguments in the proof.

For a given matching, let $R^1(c)$ be the mass of students assigned to school c that put c as their top school. Obviously, $0 \leq R^1(c) \leq k_c$. $R^1(c)$ is an important metric usually employed by policy makers to evaluate the effectiveness of a matching (Abdulkadiroğlu et al., 2009). In the next section, we discuss other performance measures.

For each school c, we compute the demands

$$\Lambda_c^1(x) = \int_{e_c(i) \geq x} F_i^1(c)\nu(di) \quad \text{and} \quad \bar{\Lambda}_c(x) = \int_{e_c(i) \geq x} \nu(di)$$

for all $x \in [0,1]$. Let $\phi_c \in [0,1]$ and $\Phi_c \in [0,1]$ be defined by the equations

$$\phi_c = \max\left\{x \in [0,1] \mid \Lambda_c^1(x) = k_c\right\} \tag{3.1}$$

$$\Phi_c = \min\left\{x \in [0,1] \mid \bar{\Lambda}_c(x) = k_c\right\}. \tag{3.2}$$

In contrast to the cutoff p_c that clears the market for school c in a stable matching, cutoffs ϕ_c and Φ_c are entirely determined by the local demand for school c: while ϕ_c

[10] The system of equations (2.1) is non-linear in p. Under multiple tie breaking, each equation in (2.1) is polynomial of degree N.

is determined by the mass of students that demand c first $(F_i^1(c))_i$, Φ_c is determined by the mass of students that list c in any position.

The following result provides estimates for $R^1(c)$ in a stable matching.

Theorem 1. *For any stable matching and all $c = 1, \ldots, N$:*

$$R^1(c) \geq k_c - \bar{\eta}_c \Big(\int_{e_c(i) \geq \phi_c, e_{\hat{c}}(i) < \Phi_{\hat{c}} \text{ for some } \hat{c} \neq c} (1 - F_i^1(c)) \nu(di) \Big) \tag{3.3}$$

and

$$R^1(c) \leq \int_{e_c(i) \geq \Phi_c} F_i^1(c) \nu(di) + \eta_c \Big(\int_{e_c(i) \geq \phi_c, e_{\hat{c}}(i) \geq \phi_{\hat{c}} \text{ for some } \hat{c} \neq c} (1 - F_i^1(c)) \nu(di) \Big) \tag{3.4}$$

where

$$\bar{\eta}_c = \min \Big\{ 1, \frac{\inf_{x \in [\phi_c, \Phi_c]} \frac{d}{dx} \Lambda_c^1(x)}{\sup_{x \in [\phi_c, \Phi_c]} \frac{d}{dx} \big(\Lambda_c^1(x) + \int_{e_c(i) \geq x, e_{\hat{c}}(i) < \Phi_{\hat{c}} \text{ some } \hat{c} \neq c} (1 - F_i^1(c)) \nu(di) \big)} \Big\}$$

and

$$\eta_c = \frac{\inf_{x \in [\phi_c, \Phi_c]} \frac{d}{dx} \Lambda_c^1(x)}{\sup_{x \in [\phi_c, \Phi_c]} \frac{d}{dx} \bar{\Lambda}_c(x)}.$$

Both bounds are tight.

Theorem 1 provides estimates for the measure of students assigned to their top schools. To apply the bounds, we compute cutoffs ϕ_c and Φ_c that are entirely determined by each school c supply and demand. The real numbers η_c and $\bar{\eta}_c$ adjust for the fact that we do not employ stable matching cutoffs p but approximated cutoffs ϕ_c and Φ_c. As we show in Section 4, getting simple expressions for η_c, $\bar{\eta}_c$, ϕ_c and Φ_c for specific models is straightforward. Thus, Theorem 1 can be easily applied to obtain substantive insights for several examples and models.

The idea behind bounds (3.3) and (3.4) is that the measure of students assigned to their top schools depends on how students can congest schools they do not rank top. A student that does not rank a school c at the top may still congest it depending on the score in c and, critically, the scores in schools $\hat{c} \neq c$. Thus, the measure of students assigned to their top schools depends on how types determine preferences for each school and scores across schools. We now discuss each of the bounds.

A. Discussion of lower bound. The first bound in the Theorem, inequality (3.3), provides a condition under which a high fraction of students assigned to school c will rank it as their top school. Most students will be assigned to their top school in c when (i) most students rank c first, that is, $1 \approx F_i^1(c)$; or (ii) students that rank c second, third, etc. have a low score in c, that is, $\int_{e_c(i)\geq\phi_c}(1-F_i^1(c))\nu(di) \approx 0$; or, more generally, (iii) most students that rank c second, third, etc. and have a high score in c are also likely to get admission in some other school, that is, $\int_{e_c(i)\geq\phi_c}(1-F_i^1(c))\nu(di) \approx \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)\geq\Phi_{\hat{c}} \text{ all } \hat{c}\neq c}(1-F_i^1(c))\nu(di)$.[11] In general, to evaluate inequality (3.3), we compute the measure of the set of students that rank c second, third, etc., and have a high score in c and a low score in some other school $\hat{c}$. When this measure is low, most students that get admission to c will naturally rank c top.

Inequality (3.3) can be used to derive conditions under which all students assigned to a school rank the school top.

Corollary 1. *Suppose that for all i and c such that $F_i^1(c) < 1$ and $e_c(i) \geq \phi_c$, we have that $e_{\hat{c}}(i) \geq \Phi_{\hat{c}}$ for all $\hat{c} \neq c$. Then, $R^1(c) = k_c$ for all c and a stable matching is Pareto efficient.*

It is seldom the case that a stable matching will be efficient. The novelty in Corollary 1 is that one can attain efficiency even if priorities and preferences do not conform. The following example illustrates Corollary 1. In the example, efficiency is attained even when some students rank a school top, but that school does not rank

[11] Note that

$$\int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ some } \hat{c}\neq c}(1-F_i^1(c))\nu(di)$$
$$= \int_{e_c(i)\geq\phi_c}(1-F_i^1(c))\nu(di) - \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)\geq\Phi_{\hat{c}} \text{ all } \hat{c}\neq c}(1-F_i^1(c))\nu(di)).$$

those students highly.[12] The example below also shows that the lower bound (3.3) is tight.

Example 3. *Suppose that $N = 2$ and $I = [0,1]$. Each school has capacity $k = \frac{1}{4}$. Students $i \leq 1/4$ are elite students, with outstanding academic performance. For $i \leq 1/4$, scores are given by $e_{c_1}(i) = e_{c_2}(i) = 1 - i$. School c_1 (resp. c_2) is located in 0 (resp. 1) and students $i > 1/4$ are ranked according to distance. Concretely, for $i > 1/4$, $e_{c_1}(i) = 1 - i$ while $e_{c_2}(i) = i - 1/4$. For each i, a fraction $\alpha(i)$ (resp. $1 - \alpha(i)$) of students rank school c_1 first (resp. school c_2 first). Assume that $\alpha(i) = 1$ for $i \leq 1/2$ and $\alpha(i) = 0$ for $i > 1/2$.*

It is simple to see that $\phi_{c_1} = 3/4$, $\Phi_{c_1} = 3/4$, $\phi_{c_2} = 1/2$, $\Phi_{c_2} = 3/4$. Note that all students that rank school c_1 second have scores $e_{c_1} < 1/2 < 3/4$. All students that rank c_2 second and have scores $e_{c_2}(i) > \phi_{c_2} = 1/2$ also have score $e_{c_1}(i) > \Phi_{c_1} = 3/4$. Using Corollary 1, $R^1(c_1) = R^1(c_2) = k$.

B. Discussion of upper bound. The second bound in Theorem 1, inequality (3.4), provides a condition under which a low fraction of students assigned to school c will rank it as their top school. The Theorem shows that in a stable matching, few students assigned to school c will rank it as their top school when (i) most students that rank c top are unlikely to have sufficiently high scores, that is, $\sup\{F_i^1(c) \mid e_c(i) \geq \Phi_c\} \approx 0$, and (ii) most students that rank c second, third, etc. and have a high score in c are unlikely to get admission in some other school, that is, $\int_{e_c(i)\geq\phi_c}(1 - F_i^1(c))\nu(di) \approx \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)\leq\phi_{\hat{c}} \text{ all } \hat{c}\neq c}(1 - F_i^1(c))\nu(di)$. If (i) were not satisfied, then we could find a non-negligible mass of students for whom c is the top choice and are sure to be assigned. Condition (ii) ensures that students for whom c is listed but is not top are admitted to c in a stable matching.

[12] Erdil and Ergin (2008) show simulations in which the preferences of both sides of the market conform and, as a result, the stable matching is efficient. In those simulations, priorities are given by multiple tie breaking and walk zones. As distance becomes more important for students (in their model, that is captured by $\beta \to 1$), the efficiency loss in the stable matching goes to 0 since in the limit both sides of the market have perfectly conforming preferences. See also Salonen and Salonen (2018) for theoretical results. See also Proposition 2.

We now illustrate the upper bound. The next example shows that the upper bound (3.4) is tight.

Example 4. *Suppose that $N = 2$ and $I = [0,1]$. Each school has capacity $k = \frac{1}{4}$. Students i live in position i with preferences given by $F_i^1(c) = 1/2$ for each c. Schools c_1 and c_2 are located at the extremes of the interval. Priorities are distance-based so the scores of agent i are given by $e_{c_1}(i) = 1 - i$ and $e_{c_2}(i) = i$. It is simple to see that $\phi_c = 1/2$ and $\Phi_c = 3/4$. Thus, inequality (3.4) becomes*

$$R^1(c) \leq \int_{1-i \geq 1-k} \frac{1}{2} di + \frac{1}{2} \int_{1-i \geq 1/2, i \geq 1/2} \frac{1}{2} di = \frac{k}{2}.$$

In the unique stable matching, the cutoff equals $p_c = 3/4$, and thus in each school only half of the students assigned to the school rank the school first: $R^1(c) = k/2$. It thus follows that inequality (3.4) holds with equality.

3.1 Proof sketch for Theorem 1

We close this section by discussing the main ideas behind the proof of Theorem 1. Since ϕ_c solves a market-clearing condition for a demand Λ_c^1 that is below the total demand D_c, we deduce that $\phi_c \leq p_c$ for any cutoff vector p from a stable matching. Analogously, $p_c \leq \Phi_c$. See Lemma 1 in the Appendix for details.

Cutoffs ϕ_c and Φ_c provide bounds for cutoffs p characterizing stable matchings. More importantly, ϕ_c and Φ_c are informative about the measure of students assigned to their top schools. Indeed, when $p_c = \phi_c$, then the number of students assigned to their top schools in c equals k_c:[13]

$$R^1(c) = \int_{e_c(i) \geq p_c} F_i^1(c) \nu(di) = \int_{e_c(i) \geq \phi_c} F_i^1(c) = k_c.$$

[13]Moreover, $R^1(c) = k_c$ if and only if $\phi_c = p_c$.

Similarly, we note that

$$R^1(c) = \int_{e_c(i) \geq p_c} F_i^1(c)\nu(di) = \int_{e_c(i) \geq \Phi_c} F_i^1(c)\nu(di) + \int_{p_c \leq e_c(i) \leq \Phi_c} F_i^1(c)\nu(di).$$

It follows that $R^1(c) = 0$ when $\Phi_c = p_c$ and $F_i^1(c) = 0$ for all $e_c(i) \geq \Phi_c$.

In general, however, $\phi_c < p_c < \Phi_c$. The key technical observation that enables us to prove Theorem 1 is that we can bound $p_c - \phi_c$ and $\Phi_c - p_c$. Indeed, we bound $p_c - \phi_c$ and $\Phi_c - p_c$ by using several relaxed market-clearing conditions. More technically, the proof bounds the distance between the solutions to different non-linear market-clearing equations to derive estimates for the measure of students assigned to their top schools. See the Appendix for details.

4 Priorities in school choice

We now explore the impact of different priority structures on the fraction of students assigned to their top schools and efficiency. The setup for this section is the model of horizontal differentiation presented in Example 2. We fix the demand and the capacity of each school and compute the bounds from Theorem 1 for different priorities. Subsection 4.1 explores random multiple and single tie breaking priorities. Subsection 4.2 explores distance-based priorities. Subsection 4.3 compares multiple tie breaking to distance-based priorities. We denote the fraction of students assigned to their top schools under distance-based priorities, multiple tie breaking, and single tie breaking by R^1_{DB}, R^1_{MTB} and R^1_{STB}, respectively.

4.1 Random priorities

This subsection applies our bounds to the widely studied model of school choice with random priorities (Abdulkadiroğlu et al., 2009; Ashlagi and Nikzad, 2020; Arnosti, 2022; Allman et al., 2022).

It is simple to see that under single or multiple tie breaking, cutoff bounds are identical and given by

$$\phi_c^{RP} = 1 - \frac{k_c}{F^1(c)} \quad \text{and} \quad \Phi_c^{RP} = 1 - k_c \tag{4.1}$$

For multiple tie breaking, we can also compute

$$\bar{\eta}_c = \frac{F^1(c)}{F^1(c) + (1 - F^1(c))\left[1 - \prod_{\hat{c} \neq c} k_{\hat{c}}\right]} \quad \text{and} \quad \eta_c = F^1(c). \tag{4.2}$$

Using Theorem 1, in Appendix B.1.1 we deduce that under multiple tie breaking, for each school c[14]

$$\begin{aligned} & k_c \left(\frac{F^1(c)}{F^1(c) + (1 - F^1(c))\big(1 - \prod_{\hat{c} \neq c} k_{\hat{c}}\big)} \right) \\ \leq \quad & R^1_{MTB}(c) \leq \quad k_c \left(1 - (1 - F^1(c)) \prod_{\hat{c} \neq c} \left(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})} \right) \right). \end{aligned} \tag{4.3}$$

Under multiple tie breaking, some students will be assigned to school c even when c is not their top school, but there will always be some students assigned to their top schools.

We can also apply Theorem 1 to a model with single tie breaking (see Appendix B.1.2). We can then compare multiple and single tie breaking.

Proposition 1. *Suppose that*

$$\frac{k_c}{F^1(c)} \leq \frac{\min\{k_{\hat{c}} \mid \hat{c} \neq c\}}{1 - F^1(c) \prod_{\hat{c} \neq c}(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})})}. \tag{4.4}$$

Then,

$$R^1_{STB}(c) \geq R^1_{MTB}(c) \tag{4.5}$$

In a symmetric model, under single tie breaking all assigned students get admission

[14] For multiple tie breaking, it is possible to derive bounds that do not use Theorem 1. By definition,

$$R^1_{MTB}(c) = F^1(c)(1 - p_c) \in [k_c(1 - \Phi_c), k_c(1 - \phi_c)] = [k_c F^1(c), k_c].$$

The bounds given in (4.3) are strictly sharper than these simple bounds.

to their top schools.[15] In this case, single tie breaking obviously dominates multiple tie breaking. Proposition 1 applies to cases in which schools are asymmetric and a school c is sufficiently popular. Concretely, when $\frac{k_c}{F^1(c)}$ is relatively small compared to the right hand side of inequality (4.4) (which does not depend on k_c), more students are assigned to c in the top position under single tie breaking than under multiple tie breaking.

Proposition 1 contributes to the literature that compares the fraction of students assigned to their top schools under multiple and single tie breaking, including Ashlagi and Nikzad (2020), Arnosti (2022), and Allman et al. (2022). Proposition 1 is different in two important aspects. First, it imposes no specific functional form or restrictions on the demand for schools, while the above-mentioned papers assume uniform or multinomial logit models. The added generality in Proposition 1 is useful since empirical work in school choice oftentimes uses flexible models to estimate preferences (Abdulkadiroğlu et al., 2017). Second, Proposition 1 compares single and multiple tie breaking for each school c that is popular enough.[16] Other results compare the total number of students assigned to their top schools in multiple and single tie breaking.

4.2 Distance-based priorities

Under distance-based priorities, school c ranks students according to $e^s_c = 1 - d(i, c)$. In this subsection, we argue that the fraction of students assigned to their top schools depends on several factors, including how much students value proximity, and the capacity and geographical dispersion of schools.

[15]By symmetric model, we mean a model in which all schools have the same number of seats and the same demand profiles. In a symmetric model, under single tie breaking all schools have the same admission cutoff. As a result, a student can get admission to some school iff the student can get admission to the top school.

[16]To be clear, other results impose popularity, that is, $k_c/F^1(c) < 1$. We are imposing that $k_c/F^1(c)$ is relatively small.

High fraction of students assigned to their top schools. We derive a lower bound for $R^1_{DB}(c)$. It is useful to consider the set of all students that can get admission to c given cutoff ϕ^{DB}_c but are rejected by some school $\hat{c}$ given $\Phi^{DB}_{\hat{c}}$:

$$H(c) = \Big\{ i \mid d(i,c) \leq 1 - \phi^{DB}_c \text{ and } d(i,\hat{c}) > 1 - \Phi^{DB}_{\hat{c}} \text{ some } \hat{c} \neq c \Big\}.$$

$H(c)$ estimates the set of all the students that could get admission to c but would be rejected by some school $\hat{c}$. In Appendix B.1.3, we employ Theorem 1 to deduce:

$$R^1_{DB}(c) \geq k_c - \nu\big(H(c)\big) \sup_{d(i,c) \leq 1 - \phi^{DB}_c} (1 - F^1_i(c)) \tag{4.6}$$

This bound shows two forces that make $R^1_{DB}(c)$ close to k_c.

The first force that makes $R^1_{DB}(c)$ close to k_c follows from a well known observation. When all students living within distance $1 - \phi^{DB}_c$ of school c list c at the top,[17] then $R^1_{DB}(c) = k_c$. In this case, preferences and priorities are consistent in the sense that students that have a high score in c (in other words, who live close to c) also rank school c at the top. When preferences and priorities are consistent, all students will be assigned to their top school, and the matching will be Pareto efficient. The observation that consistent preferences and priorities favor efficiency is not new and is discussed by Salonen and Salonen (2018), Echenique et al. (2020), Leshno and Lo (2021), and Cantillon et al. (2022).

The second force that makes $R^1_{DB}(c)$ close to k_c follows by noting that when $\nu(H(c))$ is small, then $R^1_{DB}(c)$ is close to k_c. Intuitively, when $\nu(H(c))$ is close to 0, most students with a score high enough for school c also score high enough in other schools $\hat{c}$. In this case, schools are clustered and distance-based priorities result in a subset of students who are close to all schools and can get admission anywhere. As a result, many of those students get accepted to the school they like the most.[18] The following example illustrates this idea.

[17] That is, when for all i such that $d(i,c) \leq 1 - \phi^{DB}_c$, $1 = F^1_i(c)$.

[18] This intuition is similar to the idea that under single-tie breaking, many students are assigned to their top schools (Allman et al., 2022).

Example 5 (Clustered schools)**.** *Suppose that $N = 2$ and $I = [0, 1]$. Each school has capacity $k < 1/2$. A fraction $\alpha(i)$ (resp. $1 - \alpha(i)$) of students rank school c_1 first (resp. school c_2 first) and we assume that $\alpha(i) = 1 - \alpha(1 - i)$ for all $i < 1/2$. Both schools are located at $1/2$. In the unique stable matching, $p_{c_1} = p_{c_2} = k$. Students in $\tilde{I} = [\frac{1}{2} - p, \frac{1}{2} + p]$ could get accepted to both schools and thus*

$$R^1_{DB}(c_1) = R^1_{DB}(c_2) = k.$$

The example shows that when $\nu(H(c))$ is small for all schools, students close to the schools have high scores and thus get assigned to their top schools. This is similar to how DA works under single tie breaking.

Strong competition and weak preferences for location. We now use Thereom 1 to derive an upper bound for $R^1_{DB}(c)$. Consider the set of all students that could get admission to c and $\hat{c}$ given cutoffs ϕ^{DB}_c and $\phi^{DB}_{\hat{c}}$:

$$A(c, \hat{c}) = \Big\{ i \mid d(i, c) \leq 1 - \phi^{DB}_c \Big\} \cap \Big\{ i \mid d(i, \hat{c}) \leq 1 - \phi^{DB}_{\hat{c}} \Big\}. \tag{4.7}$$

Note that if $d(i, c) \leq 1 - \phi^{DB}_c$, by the triangle inequality, $d(i, \hat{c}) \geq d(c, \hat{c}) - d(i, c) \geq d(c, \hat{c}) - 1 + \phi_c$. So, the set $A(c, \hat{c})$ in equation (4.7) is empty whenever

$$2 \leq d(c, \hat{c}) + \phi^{DB}_c + \phi^{DB}_{\hat{c}}. \tag{4.8}$$

The triangle inequality used to derive this condition captures an important intuition about congestion under distance-based priorities: When cutoffs in schools are high and schools are not clustered, having a score high enough for some school implies that the scores in other schools are below the cutoffs. This means that under distance-based priorities, students located near a school will have limited chances to attend other schools. As we show below, this force makes efficiency under distance-based priorities harder to achieve.

Condition (4.8) holds for all schools provided that for all c

1. The function $x \in [0,1] \mapsto \int_{d(i,c)<x} F_i^1(c)\nu(di)$ has strictly positive derivative at $x = 0$;

2. $d(c, \hat{c}) > 0$ for all $\hat{c} \neq c$; and

3. k_c is sufficiently small.

The first condition says that each school has some demand that is arbitrarily close to it. It is relatively simple to show that under the first condition, $\phi_c^{DB} \to 1$ as $k_c \to 0$. Since $d(c, \hat{c}) > 0$, it follows that (4.8) holds when all capacities $(k_c)_{c=1}^N$ are relatively small.

Under (4.8), it is simple to apply Theorem 1 to obtain:

$$R_{DB}^1(c) \leq k_c \sup_{d(i,c) \leq 1 - \Phi_c^{DB}} F_i^1(c). \tag{4.9}$$

When capacities are low, under distance-based priorities some students get assigned to a nearby school that is not their top choice. This puts an upper bound on the fraction of students assigned to their most preferred schools.[19]

The next example shows that under (4.8), it is entirely possible that an arbitrarily small fraction of students are assigned to their top schools.

Example 6. *Suppose that $N = 2$ and $I = [0,1]$. Each school has capacity $k < 1/2$. A fraction $\alpha(i)$ (resp. $1 - \alpha(i)$) of students rank school c_1 first (resp. school c_2 first) and we assume that $\alpha(i) = 1 - \alpha(1-i)$ for all $i < 1/2$. Assume that $\alpha(i)$ is increasing in i with $\alpha(i) > 0$ for all $i \in [0,1]$. This means that students tend to value schools that are farther away. Schools rank using distance-based priorities. Under*

$$\int_{i \leq 1/2} \alpha(i) > k. \tag{4.10}$$

it follows that $\phi_{DB}(c) > 1/2$ and (4.8) holds. Since $\bar{\Lambda}(x) = 1 - x$, it is simple to see

[19] Clearly, the bound is non-trivial only when some of the students living close to c list the school not in the top.

that $\Phi = 1 - k$. *Clearly,*

$$R^1_{DB}(c) \leq k \sup_{i \leq k} \alpha(i) = k\alpha(k)$$

It follows that for any $\epsilon > 0$, *there exists an increasing function* α *and* $k < 1/2$ *such that (4.10) holds and* $R^1_{DB}(x) < \epsilon$ *for all c.*[20]

4.3 Comparing priorities

We now compare distance-based priorities and multiple tie breaking. We evaluate these priority criteria using the fraction of students assigned to their top schools and the fraction of students that can be Pareto-improved. Given any matching μ, a positive measure set of students $S' \subseteq S$ can be Pareto-improved if there exists a matching $\bar{\mu}$ such that for almost all $s \in S$, $\bar{\mu}(s) \succeq_s \mu(s)$ with strict preferences for $s \in S'$. When the matching $\bar{\mu}$ is such that $\bar{\mu}(c) = \mu(c)$ for all $c \in C \setminus \{c', c''\}$, with $c' \neq c''$, we say that S' is part of Pareto-improving pairs. Define

$$P = \bar{\nu}\Big(\bigcup_{S' \text{ can be Pareto-improved}} S'\Big) \quad \text{and} \quad P^2 = \bar{\nu}\Big(\bigcup_{S' \text{ is part of Pareto-improving pairs}} S'\Big)$$

When $P = 0$, the measure of students that can be Pareto-improved is 0 and thus the matching is Pareto-efficient. More generally, P provides the measure of all students who could envision a Pareto-improvement of the proposed matching μ and thus P is a metric of the efficiency of the matching.[21]

Proposition 2. *Suppose that for all c and all i such that* $d(i, c) \leq 1 - \phi^{DB}_c$, $F^1_i(c) = 1$. *Then, for all c*

$$R^1_{DB}(c) = k_c \quad \textit{and} \quad \textit{and } P^2_{DB} = P_{DB} = 0.$$

In particular, no alternative priority criterion can result in more students assigned to top schools than distance-based priorities.

[20] Take $\alpha(i) \geq \epsilon i$ for all $i \in [0, 1/2]$ and $k < \epsilon/2$. Then, $R^1 \leq k\alpha(k) < \epsilon/2 < \epsilon$.

[21] If S' and S'' can be Pareto-improved, it does not follow that $S' \cup S''$ can be Pareto-improved.

This result shows that when students value distance strongly, then all students are assigned to their top schools under distance-based priorities. To see how the sufficient conditions can be satisfied, fix I, ν, the set of schools C, the distance function $d(i, c)$, and the capacities k_c. For $c \in C$, compute Φ_c^{DB} and assume that capacities are low enough so that $\{i \mid e_c(i) \geq \Phi_c\} \cap \{i \mid e_{\hat{c}}(i) \geq \Phi_{\hat{c}}\} = \emptyset$ for all $c \neq \hat{c}$. Now, construct F_i such that for all i with $e_{\hat{c}}(i) \geq \Phi_{\hat{c}}$, $F_i^1(c) = 1$. This implies that $F_i^1(c) = 1$ for all $i \in I_c$ and therefore $\phi_c^{DB} = \Phi_c^{DB}$ and $\sup_{d(i,c) \leq 1-\phi_c^{DB}} 1 - F_i^1(c) = 0$. Proposition 2 applies.

The following result contrasts with the previous proposition. We now show that when preferences for distance are weak, multiple tie breaking may result in more students assigned to their top schools than distance-based priorities.

Theorem 2. *Assume condition (4.8) and that for all c,*

$$\sup_{d(i,c) \leq 1-\Phi_c^{DB}} \Big(F_i^1(c) - F^1(c)\Big) \leq \frac{\big(1 - F^1(c)\big)\big(\prod_{\hat{c} \neq c} k_{\hat{c}}\big)}{1 + \big(\frac{1}{F^1(c)} - 1\big)\big(1 - \prod_{\hat{c} \neq c} k_{\hat{c}}\big)} \tag{4.11}$$

Then, for all c

$$R_{DB}^1(c) < R_{MTB}^1(c).$$

If we additionally assume that $\mathbb{P}[c \succ c' \mid i] > 0$ for all $c \neq c'$ and all $i \in I$, then

$$P_{MTB}^2 = P_{MTB} < P_{DB} = P_{DB}^2.$$

This result provides conditions under which multiple tie breaking assigns more students to their top schools than distance-based priorities. Note that when types do not determine preferences, that is $F_i(\succ) = F(\succ)$ for all $i \in I$, then the left-hand side of (4.11) equals 0 and thus condition (4.11) holds. More generally, condition (4.11) captures situations in which location has a limited impact on preferences so that $F_i^1(c)$ stays relatively flat as a function of i and close to its average $F^1(c)$.[22] When preferences for nearby schools are weak and competition is strong, distance-

[22]Note that both conditions (4.11) and (4.8) restrict F and k. They simultaneously hold when

based priorities assign some students to schools just because they live nearby even when those schools are not ranked top by them while under multiple tie breaking those students still have a chance to get accepted to their top schools.

We now discuss the second part of Theorem 2 and compare the fraction of students that can be Pareto improved. It is relatively simple to prove that for any matching μ, $P^2 \leq P$ and

$$\sum_{c=1}^{N} R^1(c) + P \leq \sum_{c=1}^{N} k_c. \tag{4.12}$$

We then prove that, under some conditions, these inequalities bind and therefore the fraction of students assigned to top schools and the fraction of students that can be Pareto improved add up to the total capacity of schools. See Appendix B.2.[23]

Heterogeneous school effects. Changing the priority rule can also have opposite effects for different schools in a market. The following two-school example shows that switching from multiple tie breaking to distance-based priorities may result in more students assigned top in one school, but fewer students assigned top in the other school. The example also illustrates the usefulness of our school-specific bounds.

Example 7. *Suppose that $N = 2$ and $I = [0, 1]$. Each school has capacity $k = \frac{1}{4}$. Student i lives in position i. Students located at $i \leq \frac{1}{4}$ and $i \geq \frac{3}{4}$ prefer school c_1 over school c_2, while students located at $i \in (\frac{1}{4}, \frac{3}{4})$ prefer school c_2 over school c_1. School c_1 is located at 0 while school c_2 is located at 1.*

It is simple to see that when using multiple tie breaking, $\phi_1 = \phi_2 = \frac{1}{2}$ and $\Phi_1 = \Phi_2 = \frac{3}{4}$. Using equation (4.1), it follows that $\frac{1}{7} \leq R^1(c_1) = R^1(c_2) \leq \frac{3}{16}$. In contrast,

types have a limited impact on preferences so that for all c

$$\sup_i F_i^1(c) \leq \frac{F^1(c)}{F^1(c) + (1 - F^1(c))\big(1 - \prod_{\hat{c} \neq c} \frac{k_{\hat{c}}}{F^1(\hat{c})}\big)}$$

and, given preferences, k_c is not too big so that (4.8).

[23] Inequality (4.12) can be strict in some important cases. For example, under single tie breaking the matching is Pareto-efficient, but it is possible that not all students are assigned to their top schools. Example 8 in Appendix B.2 shows that under distance-based priorities, inequality (4.12) can be strict when condition (4.8) does not hold.

when using distance based priorities, $\phi_1 = \Phi_1 = \frac{3}{4}$ *and* $\phi_2 = \frac{1}{2}$, $\Phi_2 = \frac{3}{4}$. *Using equations (4.6) and (4.9), we derive that* $\frac{1}{4} \leq R^1(c_1) \leq \frac{1}{4}$ *and* $0 \leq R^1(c_2) \leq 0$. *It follows that*

$$R^1_{MTB}(c_1) \leq R^1_{DB}(c_1) \quad \textit{and} \quad R^1_{MTB}(c_2) \geq R^1_{DB}(c_2).$$

5 Simulations

We now verify our results in a simulated economy with a finite number of students and a finite number of schools. We construct an economy with $10,000$ students and 100 schools with 50 seats each. Schools are located at the integer points of a 10×10 grid and students are located uniformly at random within this grid. Each student is surrounded by four schools, and those are the only schools a student applies to. The utility that a student s assigned to one of the schools c that surrounds the student is

$$u(s, c) = \alpha(1 - d(s, c)) + (1 - \alpha)\epsilon_{s,c}$$

where $d(s, c)$ is the Euclidean distance between student s and school c, and $\epsilon_{s,c} \sim U[0, 1]$ are i.i.d idiosyncratic noise terms. Parameter α captures the relevance of school proximity in students' preferences. When α is close to 1, proximity to schools fully determines a student's preferences. Conversely, when α is close to 0, students' preferences are random and proximity to schools plays no role.

We compare distance-based priorities, multiple and single tie breaking in terms of the fraction of students assigned to their top schools. As suggested by Proposition 2 and Theorem 2, these comparisons will crucially depend on α. When proximity is important (α close to 1), Proposition 2 shows that distance-based priorities place a large fraction of students to their top schools. More subtly, when proximity is moderately important (α small), Theorem 2 shows that distanced-based priorities assign fewer students to their top schools than multiple tie braking. Figure 1 shows the expected fraction of students assigned to top schools for different priority criteria and for different values of $\alpha \in [0, 1]$. This exercise confirms our insights in the finite economy model.[24]

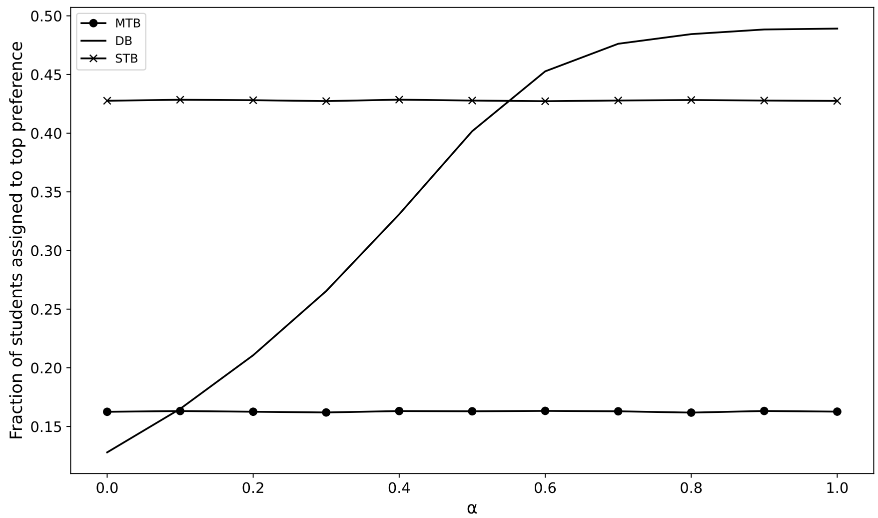

Figure 1: Comparison of priority criteria for different preferences.

6 Concluding remarks

This paper provides tight lower and upper bounds for the measure of students assigned to top schools in a stable matching. These bounds apply to a general large-market matching model and can be easily applied to different environments. Our framework bridges otherwise disparate research lines, connecting the literature on the design of randomization (Abdulkadiroğlu et al., 2009; Ashlagi and Nikzad, 2020) with work on the correlation between preferences and priorities (Erdil and Ergin, 2008; Salonen and Salonen, 2018; Leshno and Lo, 2021; Cantillon et al., 2022).

More importantly, we also derive new insights for market design. While several school districts employ proximity as a priority criterion, little is known about how this policy decision affects the outputs of the DA algorithm. We show that when

[24]Figure 1 also shows that single tie breaking results in more students assigned to top schools than MTB priorities, as in Abdulkadiroğlu et al. (2009).

students highly value proximity, efficient outcomes are achievable. However, under weaker proximity preferences, even multiple tie breaking may assign more students to their top choices than distance-based priorities. Thus, including proximity as a priority criterion is appropriate only if students significantly value proximity. Our results resonate well with the precept that market design is context-specific.

Stable matchings are hard to analyze because comparative statics results and closed-form formulas are typically unavailable. We have overcome these analytical difficulties by introducing sharp bounds on some key performance measures. Future research could sharpen our bounds and extend our methods. Bounding other performance measures, including different diversity metrics or the fraction of students assigned to top l schools, would also be interesting. We leave these research questions for future work.

Appendix

In this Appendix, we provide proofs and supporting material for the main results in the text. In the main text, we simplified exposition and assumed that all students ranked all schools. We relax this assumption. We introduce some notation. Denote the set of schools listed by type i students by $\text{supp}(i) = \{c \mid \bar{F}_i(c) > 0\}$. We abuse notation and for $c \in C$ we denote

$$\text{supp}(c) = \Big\{\hat{c} \in C \setminus \{c\} \mid \exists i \in I \colon c \in \text{supp}(i), \hat{c} \in \text{supp}(i)\Big\}$$

the set of schools that are listed by types that also list c. Note that $\text{supp}(i)$ may not coincide with C. If that is the case, there is some c such that $\bar{F}(c) < 1$.

A Proof of Theorem 1

Define $P_{k,c}$ as the set of all orderings $\succ$ such that school c is listed in position k. Given cutoffs $p \in [0,1]^N$, the demand for school c can be written as

$$D_c(p) = \int_{e_c(i) \geq p_c} F_i^1(c)\nu(di) + \sum_{k=2}^{N} \sum_{\succ \in P_{k,c}} \int_{e_c(i) \geq p_c, e_{\hat{c}}(i) < p_{\hat{c}} \forall \hat{c} \succ c} F_i(\succ)\nu(di). \tag{A.1}$$

The demand is built as follows. Fix a student type i that has school c as its k-th preference. For each one of these student types, a mass $F_i(\succ)$ reveals preference ordering $\succ \in P_{k,c}$. However, only a fraction of $F_i(\succ)$ effectively demands school c. These are students rejected at all $k-1$ schools preferred over c according to $\succ$ ($e_{\hat{c}}(i) < p_{\hat{c}} \forall \hat{c} \succ c$) and accepted at school c ($e_c(i) \geq p_c$). Then, adding up over all possible ranking positions k, all preference orderings $\succ \in P_{k,c}$ with positive measure ($F_i(\succ) > 0$), and aggregating over all student types $i \in I$, we get the total demand for school c.

The following result is useful to derive our efficiency bounds.

Lemma 1. *Let p be a market-clearing cutoff vector characterizing a stable matching. Then, for all c, $\phi_c \leq p_c \leq \Phi_c$.*

Proof. For any $x \in [0,1]^N$, $\Lambda^1_c(x_c) \leq D_c(x) \leq \bar{\Lambda}(x_c)$ which are all decreasing in x_c. Then, fix any p_{-c} and let ϕ_c, p_c, Φ_c be solutions to $\Lambda^1_c(\phi_c) = k_c$, $D_c(p_c, p_{-c}) = k_c$ and $\bar{\Lambda}_c(\Phi_c) = k_c$ respectively, it is true that $\phi_c \leq p_c \leq \Phi_c$. □

A.1 Upper bound

Let p be a cutoff vector for a stable matching. Then

$$
\begin{aligned}
k_c &= D_c(p) \\
&= \int_{e_c(i)\geq p_c} F^1_i(c)\nu(di) + \sum_{k=2}^{N} \sum_{\succ \in P_{k,c}} \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}} \forall \hat{c}\succ c} F_i(\succ)\nu(di) \\
&\geq \int_{e_c(i)\geq p_c} F^1_i(c)\nu(di) + \sum_{k=2}^{N} \sum_{\succ \in P_{k,c}} \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}} \forall \hat{c}\in \mathrm{supp}(i)\setminus\{c\}} F_i(\succ)\nu(di) \\
&= \int_{e_c(i)\geq p_c} F^1_i(c)\nu(di) + \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}} \forall \hat{c}\in \mathrm{supp}(i)\setminus\{c\}} \sum_{k=2}^{N} \sum_{\succ \in P_{k,c}} F_i(\succ)\nu(di) \\
&= \int_{e_c(i)\geq p_c} F^1_i(c)\nu(di) + \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}} \forall \hat{c}\in \mathrm{supp}(i)\setminus\{c\}} \sum_{k=2}^{N} F^k_i(c)\nu(di) \\
&= \int_{e_c(i)\geq p_c} F^1_i(c)\nu(di) + \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}} \forall \hat{c}\in \mathrm{supp}(i)\setminus\{c\}} \big(\bar{F}_i(c) - F^1_i(c)\big)\nu(di) \\
&:= \Lambda(p_c).
\end{aligned}
$$

To see the inequality above, note that for any $k = 2, \dots, N$, $\succ \in P_{k,c}$, and $\hat{c} \succ c$, it follows that $\hat{c} \in \mathrm{supp}(i) \setminus \{c\}$. Thus, for any $k = 2, \dots, N$ and $\succ \in P_{k,c}$

$$
\Big\{ i \in I \mid e_c(i) \geq p_c, e_{\hat{c}}(i) < p_{\hat{c}} \forall \hat{c} \succ c \Big\} \supseteq \Big\{ i \in I \mid e_c(i) \geq p_c, e_{\hat{c}}(i) < p_{\hat{c}} \forall \hat{c} \in \mathrm{supp}(i)\setminus\{c\} \Big\}
$$

and therefore

$$\int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}} \forall \hat{c}\succ c} F_i(\succ)\nu(di) \geq \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}} \forall \hat{c}\in \text{supp}(i)\setminus\{c\}} F_i(\succ)\nu(di).$$

Since $\bar{\Lambda}(\Phi_c) = k_c \geq \Lambda(p_c)$

$$0 \leq \bar{\Lambda}(\Phi_c) - \Lambda(p_c) = \bar{\Lambda}(p_c) + \int_{p_c}^{\Phi_c} \bar{\Lambda}'(s)ds - \Lambda(p_c) \leq (\Phi_c - p_c) \sup_{x\in[\phi_c,\Phi_c]} \bar{\Lambda}' + \bar{\Lambda}(p_c) - \Lambda(p_c)$$

therefore

$$\begin{aligned}
&(\Phi_c - p_c)\Big(- \sup \bar{\Lambda}'\Big) \\
&\qquad \leq \bar{\Lambda}(p_c) - \Lambda(p_c) \\
&\qquad = \int_{e_c(i)\geq p_c} (\bar{F}_i(c) - F_i^1(c))\nu(di) - \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)<p_{\hat{c}} \forall \hat{c}\in \text{supp}(i)\setminus\{c\}} \big(\bar{F}_i(c) - F_i^1(c)\big)\nu(di) \\
&\qquad = \int_{e_c(i)\geq p_c, e_{\hat{c}}(i)\geq p_{\hat{c}} \text{ for some } \hat{c}\in \text{supp}(i)\setminus\{c\}} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&\qquad \leq \int_{e_c(i)\geq \phi_c, e_{\hat{c}}(i)\geq \phi_{\hat{c}} \text{ for some } \hat{c}\in \text{supp}(i)\setminus\{c\}} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&\qquad = \int_{e_c(i)\geq \phi_c} (\bar{F}_i(c) - F_i^1(c))\nu(di) - \int_{e_c(i)\geq \phi_c, e_{\hat{c}}(i)<\phi_{\hat{c}} \forall \hat{c}\in \text{supp}(i)\setminus\{c\}} \big(\bar{F}_i(c) - F_i^1(c)\big)\nu(di).
\end{aligned}$$

Since $\sup \bar{\Lambda}' < 0$, it follows that

$$\begin{aligned}
&(\Phi_c - p_c) \\
&\leq \frac{1}{\big(- \sup \bar{\Lambda}'\big)} \Big(\int_{e_c(i)\geq \phi_c} (\bar{F}_i(c) - F_i^1(c))\nu(di) - \int_{e_c(i)\geq \phi_c, e_{\hat{c}}(i)<\phi_{\hat{c}} \forall \hat{c}\in \text{supp}(i)\setminus\{c\}} \big(\bar{F}_i(c) - F_i^1(c)\big)\nu(di)\Big).
\end{aligned}$$

Finally,

$$
\begin{aligned}
R^1(c) &= \int_{e_c(i)\geq p_c} F_i^1(c)\nu(di) \\
&= \int_{e_c(i)\geq \Phi_c} F_i^1(c)\nu(di) \quad + \int_{p_c\leq e_c(i)\leq \Phi_c} F_i^1(c)\nu(di) \\
&\leq \int_{e_c(i)\geq \Phi_c} F_i^1(c)\nu(di) \quad + \frac{\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(-\int_{e_c(i)\geq x} F_i^1(c)\nu(di))}{\big(-\sup \bar{\Lambda}'\big)} \Bigg(\int_{e_c(i)\geq \phi_c} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&\qquad - \int_{e_c(i)\geq \phi_c, e_{\hat{c}}(i)<\phi_{\hat{c}}, \forall \hat{c}\in \mathrm{supp}(i)\setminus\{c\}} \big(\bar{F}_i(c) - F_i^1(c)\big)\nu(di) \Bigg)
\end{aligned}
$$

where the inequality follows since

$$
\int_{p_c\leq e_c(i)\leq \Phi_c} F_i^1(c)\nu(di) = \Lambda^1(p_c) - \Lambda^1(\Phi_c) \leq (\Phi_c - p_c) \sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(-\Lambda^1(x)).
$$

It follows that

$$
\begin{aligned}
R^1(c) &\leq \int_{e_c(i)\geq \Phi_c} F_i^1(c)\nu(di) + \eta_c \int_{e_c(i)\geq \phi_c, e_{\hat{c}}(i)>\phi_{\hat{c}} \text{ some } \hat{c}\in \mathrm{supp}(i)\setminus\{c\}} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&\leq \int_{e_c(i)\geq \Phi_c} F_i^1(c)\nu(di) + \eta_c \int_{e_c(i)\geq \phi_c, e_{\hat{c}}(i)>\phi_{\hat{c}} \text{ some } \hat{c}\in \mathrm{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di).
\end{aligned}
$$

A.2 Lower bound

Define

$$
\hat{\Lambda}_c(x) = \Lambda_c^1(x) + \int_{e_c(i)\geq x, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ some } \hat{c}\in \mathrm{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di)
$$

and note that

$$
\hat{\Lambda}_c(p_c) \geq D_c(p) = k_c.
$$

Since $\Lambda_c^1(\phi_c) = k_c$,

$$
\Lambda_c^1(\phi_c) \leq \hat{\Lambda}_c(p_c) \leq \hat{\Lambda}_c(\phi_c) + (p_c - \phi_c) \sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}\hat{\Lambda}_c(x).
$$

Rearranging terms,

$$p_c - \phi_c \leq \frac{-1}{\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}\hat{\Lambda}_c(x)} \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ some } \hat{c}\in\text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di).$$

Now,

$$\begin{aligned}
R^1(c) &= \int_{e_c(i)\geq p_c} F_i^1(c)\nu(di) \\
&= \int_{e_c(i)\geq\phi_c} F_i^1(c)\nu(di) - \int_{\phi_c\leq e_c(i)\leq p_c} F_i^1(c)\nu(di) \\
&\geq k_c - (p_c - \phi_c) \sup_{x\in[\phi_c,\Phi_c]} -\frac{d}{dx}\Lambda_c^1(x) \\
&\geq k_c - \frac{\sup_{x\in[\phi_c,\Phi_c]} -\frac{d}{dx}\Lambda_c^1(x)}{-\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}\hat{\Lambda}_c(x)} \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ some } \hat{c}\in\text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di).
\end{aligned}$$

Note that

$$\begin{aligned}
R^1(c) &\geq k_c - \sum_{k=2}^{N} \sum_{\succ\in P_{k,c}} \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)<\Phi_c \ \ \forall\hat{c}\succ c} F_i(\succ)\nu(di) \\
&\geq k_c - \sum_{k=2}^{N} \sum_{\succ\in P_{k,c}} \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)<\Phi_c \ \ \text{ some } \hat{c}\succ c} F_i(\succ)\nu(di) \\
&= k_c - \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)<\Phi_c \ \ \text{ some } \hat{c}\succ c} (\bar{F}_i(c) - F_i^1(c))\nu(di).
\end{aligned}$$

Setting

$$\bar{\eta}_c = \min\left\{1, \frac{\sup_{x\in[\phi_c,\Phi_c]} -\frac{d}{dx}\Lambda_c^1(x)}{-\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}\hat{\Lambda}_c(x)}\right\}$$

it follows that

$$R^1(c) \geq k_c - \bar{\eta}_c \int_{e_c(i)\geq\phi_c, e_{\hat{c}}(i)<\Phi_c \ \ \text{ some } \hat{c}\in\text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di).$$

B Proofs for Section 4

B.1 Applying Theorem 1 to random and distance-based priorities

We apply Theorem 1 for each priority rule.

B.1.1 Multiple tie breaking

Recall that in this setting $I = [0,1]^N$ and ν are N independent uniform distributions. First, we can specify $\Lambda_c^1(x)$ and $\bar{\Lambda}_c(x)$:

$$\Lambda_c^1(x) = \int_{e_c(i)\geq x} F_i^1(c)\nu(di) = \int \left[\int_x^1 F_i^1(c)du\right]\nu(di) = \int F_i^1(c)\nu(di)\int_x^1 du = F^1(c)(1-x),$$

$$\bar{\Lambda}_c(x) = \int_{e_c(i)\geq x} \bar{F}_i(c)\nu(di) = \int \left[\int_x^1 \bar{F}_i(c)du\right]\nu(di) = \int \bar{F}_i(c)\nu(di)\int_x^1 du = \bar{F}_i(c)(1-x)$$

where the first equality obviates the $N-1$ integrals of measure 1. Therefore

$$\phi_c = 1 - \frac{k_c}{F^1(c)} \qquad \Phi_c = 1 - \frac{k_c}{\bar{F}(c)}.$$

Similarly,

$$\begin{aligned}
\Lambda_c(x) &= \Lambda_c^1(x) + \int_{e_c(i)\geq x, e_{\hat{c}}(i)<\Phi_{\hat{c}} \text{ for some } \hat{c}\in \text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&= F^1(c)(1-x) + \int_{e_c(i)\geq x} (\bar{F}_i(c) - F_i^1(c))\nu(di) - \int_{e_c(i)\geq x, e_{\hat{c}}(i)\geq \Phi_{\hat{c}} \forall \hat{c}\in\text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di) \\
&= F^1(c)(1-x) + \int \left[\int_x^1 (\bar{F}_i(c) - F_i^1(c))du\right]\nu(di) - \int \left[\int_x^1 \int_{\Phi_{\hat{c}}, \forall \hat{c}\neq c}^1 (\bar{F}_i(c) - F_i^1(c))du\right]\nu(di) \\
&= F^1(c)(1-x) + (\bar{F}(c) - F^1(c))(1-x) - (\bar{F}(c) - F^1(c))(1-x)\prod_{\hat{c}\neq c}(1-\Phi_{\hat{c}}) \\
&= F^1(c)(1-x) + (\bar{F}(c) - F^1(c))(1-x)\left[1 - \prod_{\hat{c}\neq c}\frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right].
\end{aligned}$$

Having calculated, $\Lambda_c^1(x), \bar{\Lambda}_c(x), \Lambda_c(x)$, we have that

$$\begin{aligned}
\bar{\eta}_c &= \min\left\{1, \frac{\inf_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(\Lambda_c^1(x))}{\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(\Lambda_c(x))}\right\} \\
&= \min\left\{1, \frac{\inf_{x\in[\phi_c,\Phi_c]} -F^1(c)}{\sup_{x\in[\phi_c,\Phi_c]} -F^1(c) - (\bar{F}(c) - F^1(c))\left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}\right\} \\
&= \frac{F^1(c)}{F^1(c) + (\bar{F}(c) - F^1(c))\left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}.
\end{aligned}$$

and

$$\begin{aligned}
\eta_c &= \min \frac{\inf_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(\Lambda_c^1(x))}{\sup_{x\in[\phi_c,\Phi_c]} \frac{d}{dx}(\bar{\Lambda}_c(x))} = \frac{\inf_{x\in[\phi_c,\Phi_c]} -F^1(c)}{\sup_{x\in[\phi_c,\Phi_c]} -\bar{F}(c)} \\
&= \frac{F^1(c)}{\bar{F}(c)}.
\end{aligned}$$

We now calculate our bounds

$$\begin{aligned}
R^1(c) &\geq k_c - \frac{F^1(c)}{F^1(c) + (\bar{F}(c) - F^1(c))\left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]} \frac{k_c}{F^1(c)} (\bar{F}(c) - F^1(c))\left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right] \\
&= k_c \left(\frac{F^1(c)}{F^1(c) + (\bar{F}(c) - F^1(c))\left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}\right)
\end{aligned}$$

and

$$\begin{aligned}
R^1(c) &\leq F^1(c)\frac{k_c}{\bar{F}(c)} + \frac{F^1(c)}{\bar{F}(c)}(\bar{F}(c) - F^1(c))\frac{k_c}{F^1(c)}\left[1 - \prod_{\hat{c}\neq c}\left(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})}\right)\right] \\
&= \frac{k_c}{\bar{F}(c)}\left[F^1(c) + \frac{\bar{F}(c) - F^1(c)}{\bar{F}(c)}\left[1 - \prod_{\hat{c}\neq c}\left(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})}\right)\right]\right] \\
&= k_c\left[1 - \frac{\bar{F}(c) - F^1(c)}{\bar{F}(c)}\prod_{\hat{c}\neq c}\left(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})}\right)\right].
\end{aligned}$$

B.1.2 Single tie breaking

Recall that in this case $I = [0,1]$ and ν is given by the uniform distribution. We first note that for single tie breaking, $\phi_c = 1 - \frac{k_c}{F^1(c)}$ and $\Phi_c = 1 - \frac{k_c}{\bar{F}(c)}$. When $\frac{k_c}{F^1(c)} < \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}$ for all $\hat{c} \neq c$, then $\phi_c \geq \Phi_{\hat{c}}$ for all $\hat{c} \neq c$ and therefore

$$\Big(\int_{e_c(i) \geq \phi_c^{STB}, e_{\hat{c}}(i) < \Phi_{\hat{c}}^{STB} \text{ for some } \hat{c} \neq c} (1 - F_i^1(c))\nu(di)\Big) = 0.$$

Thus, Theorem 1 implies that when $\frac{k_c}{F^1(c)} < \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}$ for all $\hat{c} \neq c$,

$$R^1(c) = k_c.$$

Consider now the case in which $\frac{k_c}{F^1(c)} > \min\{\frac{k_{\hat{c}}}{\bar{F}(\hat{c})} \mid \hat{c} \neq c\}$. This condition implies that $\phi_c < \max\{\Phi_{\hat{c}} \mid \hat{c} \neq c\}$. Since $\bar{\eta}_c \leq 1$, we deduce that

$$\begin{aligned}
R^1(c) &\geq k_c - \bar{\eta}_c\Big(\int_{e_c(i) \geq \phi_c^{STB}, e_{\hat{c}}(i) < \Phi_{\hat{c}}^{STB} \text{ for some } \hat{c} \neq c} (\bar{F}(c) - F_i^1(c))\nu(di)\Big) \\
&= k_c - \left[\int_{e_c(i) \geq \phi_c^{STB}} (\bar{F}_i(c) - F_i^1(c))\nu(di) - \int_{e_c(i) \geq \phi_c^{STB}, e_{\hat{c}}(i) > \Phi_{\hat{c}}^{STB} \forall \hat{c} \in \text{supp}(c)} (\bar{F}_i(c) - F_i^1(c))\nu(di)\right] \\
&= k_c - \left[(\bar{F}(c) - F^1(c))(1 - \phi_c^{STB}) - (\bar{F}(c) - F^1(c))(1 - \max\{\phi_c^{STB}, (\max_{\hat{c} \neq c} \Phi_{\hat{c}}^{STB})\})\right] \\
&= k_c - (\bar{F}(c) - F^1(c))\left[(\max_{\hat{c} \neq c} \Phi_{\hat{c}}^{STB}) - \phi_c^{STB}\right].
\end{aligned}$$

B.1.3 Distance-based priorities

Since $\bar{\eta} \leq 1$, Theorem 1 implies that

$$\begin{aligned}
R^1(c) &\geq k_c - \bar{\eta}_c\Big(\int_{e_c(i) \geq \phi_c, e_{\hat{c}}(i) < \Phi_{\hat{c}} \text{ for some } \hat{c} \neq c} (1 - F_i^1(c))\nu(di)\Big) \\
&= k_c - \bar{\eta}_c \int_{d(i,c) \leq 1 - \phi_c^{DB}, d(i,\hat{c}) > 1 - \Phi_{\hat{c}}^{DB} \text{ some } \hat{c} \neq c} (1 - F_i^1(c))\nu(di) \\
&\geq k_c - \nu(H(c)) \sup_{d(i,c) \leq 1 - \phi_c^{DB}} (1 - F_i^1(c)).
\end{aligned}$$

Under condition (4.8), Theorem 1 implies that

$$\begin{aligned}
R^1(c) &\le \int_{e_c(i)\ge\Phi_c^{DB}} F_i^1(c)\nu(di) + 0 \\
&= \int_{e_c(i)\ge\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} \bar{F}_i(c)\nu(di) \\
&\le k_c \sup_{i:e_c(i)\ge\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} \\
&= k_c \sup_{i:d(i,c)\le 1-\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)}.
\end{aligned}$$

B.2 Students assigned to top schools and Pareto efficiency

Lemma 2. *a. For any matching μ, $P^2 \le P$ and*

$$\sum_{c=1}^{N} R^1(c) + P \le \sum_{c=1}^{N} k_c. \tag{B.1}$$

b. Assume that $\mathbb{P}[c \succ c' \mid i] > 0$ for all $c \neq c'$ and all $i \in I$. When priorities are built from multiple tie breaking and μ is stable, then $P^2 = P$ and (B.1) holds with equality.

c. Assume that $\mathbb{P}[c \succ c' \mid i] > 0$ for all $c \neq c'$ and all $i \in I$. Assume priorities are distance-based, and that conditions (4.11) and (4.8) hold. Then $P^2 = P$ and (B.1) holds with equality.

Proof of Lemma 2. Let μ be any matching and take S^1 as the set of all students assigned to their top schools. Clearly, no subset of S^1 can be Pareto improved. Let

$$S^P = \bigcup_{S'' \text{ can be Pareto improved}} S''$$

and note that $S^P \subseteq \{s \mid s \text{ is assigned by } \mu\}$. It follows that

$$S^1 \cup S^P \subseteq \{s \mid s \text{ is assigned}\} \text{ and } S^1 \cap S^P = \emptyset.$$

As a result,

$$\bar{\nu}(S^1) + \bar{\nu}(S^P) \leq \bar{\nu}(\{s \text{ is assigned}\}).$$

Since $\bar{\nu}(S^1) = \sum_c R^1(c)$, $\bar{\nu}(S^P) = P$ and $\bar{\nu}(\{s \text{ is assigned }\}) \leq \sum_c k_c$, we deduce that

$$\sum_c R^1(c) + P \leq \sum_c k_c.$$

Take now a stable matching under multiple tie breaking. Consider any student s who is assigned to a school that is not her top choice. We will argue that there is a positive measure set S', that contains s, such that S' is part of Pareto-improving pairs. Let $c = \mu(s)$ and consider a school $\hat{c}$ and a set of students $\hat{S}$ assigned to c such that $\hat{S}$ has positive measure and contains s, and all students in $\hat{S}$ prefer $\hat{c}$ over c. Consider the set of all students who prefer c over $\hat{s}$ but only have scores to get admission to $\hat{c}$:

$$\bar{S} = \{s \in S \mid c \succ_s \hat{c}, \quad i_{\hat{c}} \geq p_{\hat{c}}, \quad p_{c'} > i_{c'} \forall c' \neq \hat{c}\}.$$

Clearly,

$$\bar{\nu}(\bar{S}) = \Big(\int \mathbb{P}[c \succ \hat{c} \mid i] \nu(di) \Big) (1 - p_{\hat{c}}) \prod_{c' \neq \hat{c}} p_{c'} > 0.$$

Without loss, assume that $\bar{\nu}(\bar{S}) = \bar{\nu}(\hat{S})$.[25] Construct the matching $\bar{\mu}$ by $\bar{\mu}(c') = \mu(c')$ for all $c' \in C \setminus \{c, \hat{c}\}$ and

$$\bar{\mu}(c) = (\mu(c) \cup \bar{S} \setminus \hat{S} \quad \text{and } \bar{\mu}(\hat{c}) = (\mu(\hat{c}) \cup \hat{S} \setminus \bar{S}.$$

It follows that $\bar{\mu}$ is a matching and $S' = \bar{S} \cup \hat{S}$ is part of Pareto-improving pairs. As a result,

$$\{s \text{ is assigned to a school that is not her top}) \leq \bigcup_{\tilde{S} \text{ is part of Pareto-improving pairs}} \tilde{S}$$

[25]If not, scale down the set with the largest measure so that the measures coincide.

and since

$$\bar{\nu}(\{s \text{ is assigned to a school that is not her top}) = \sum_c (k_c - R^1(c))$$

it follows that

$$\sum_{c=1}^{N}(k_c - R^1(c)) \leq \bar{\nu}(\bigcup_{\tilde{S} \text{ is part of Pareto-improving pairs}} \tilde{S}) = P^2.$$

We deduce that under multiple-tie breaking, $P = P^2$ and $\sum_{c=1}^{N}(k_c - R^1(c)) = P^2$. The proof for distance-based priorities is analogous. □

The following example shows that under distance-based priorities, inequality (4.12) can be strict when condition (4.8) does not hold.

Example 8. *Suppose that $N = 2$ and $I = [0,1]$. School c_1 has capacity k_1 and is located in $3/4$, while school c_2 has capacity k_2 and is located in 1. Students find both schools acceptable and for each i, a fraction 1/2 of students prefer c_1 over c_2.*

Under distance-based priorities, we characterize a stable matching such that all students with score above the cutoff p_1 also have score above p_2 for school c_2:

$$\int_{|i-3/4|<1-p_1} \frac{1}{2} di = k_1 \quad \text{and} \quad \int_{i>p_2} di - \int_{|i-3/4|<1-p_1} \frac{1}{2} di = k_2.$$

The first condition is the market clearing condition for school c_1: the demand for school 1 is given by half of the student living within distance p_1 of the schools. The second condition is the market clearing condition for school c_2: the demand for school 2 is given by all students living with distance p_2 of schools 2 minus the fraction of students that get admission to school 1. We can solve for the cutoffs:

$$(1 - p_1) = k_1 \quad 1 - p_2 = k_1 + k_2$$

with $3/4 - k_1 > 1 - (k_1 + k_2)$ (so that students that can be accepted to c_1 can also be *accepted to c_2). For $k_2 > 1/4$ and $k_1 < k_2$,*

$$R^1_{DB}(c_1) = k_1 \quad \text{and} \quad R^1_{DB}(c_2) = \frac{(k_2 - k_1)}{2}.$$

The matching is Pareto-efficient and $P = 0$, but

$$\sum_c R^1(c) + P < \sum_c k_c.$$

B.3 Proofs for Section 4

Proof of Proposition 1. If $\frac{k_c}{F^1(c)} \leq \min_{\hat{c} \neq c} k_{\hat{c}}$, then $R^1_{STB}(c) = k_c$ and thus obviously $R^1_{STB}(c) > R^1_{MTB}(c)$. Consider now the case $\frac{k_c}{F^1(c)} > \min_{\hat{c} \neq c} k_{\hat{c}}$. In this case, single tie breaking results in more students assigned to their top schools provided:

$$k_c \left[1 - (1 - F^1(c)) \prod_{\hat{c} \neq c} \left(1 - \frac{k_{\hat{c}}}{F^1(\hat{c})}\right)\right] \leq k_c - (1 - F^1(c)) \left[\frac{k_c}{F^1(c)} - \min_{\hat{c} \neq c} \{k_{\hat{c}}\}\right].$$

This condition is equivalent to

$$\frac{k_c}{F^1(c)} \leq \frac{\min_{\hat{c} \neq c} k_{\hat{c}}}{1 - F^1(c) \prod_{\hat{c} \neq c} (1 - \frac{k_{\hat{c}}}{F^1(\hat{c})})}.$$

□

Proof of Proposition 2. From (4.6), we deduce that $R^1_{DB}(c) = k_c$ under the conditions in the statement. □

Proof of Theorem 2 . We provide a condition such that the lower bound for multiple tie breaking is larger than the upper bound for distance-based priorities:

$$\sup_{d(i,c)\leq 1-\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} \leq k_c \left(\frac{F^1(c)}{F^1(c) + (\bar{F}(c) - F^1(c)) \left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]} \right)$$

$$\sup_{d(i,c)\leq 1-\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} - \frac{F^1(c)}{\bar{F}(c)} \leq \frac{F^1(c)}{F^1(c) + (\bar{F}(c) - F^1(c)) \left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]} - \frac{F^1(c)}{\bar{F}(c)}$$

$$\sup_{d(i,c)\leq 1-\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} - \frac{F^1(c)}{\bar{F}(c)} \leq \frac{F^1(c)\bar{F}(c) - F^1(c)F^1(c) - F^1(c)(\bar{F}(c) - F^1(c)) \left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}{\bar{F}(c)F^1(c) + \bar{F}(c)(\bar{F}(c) - F^1(c)) \left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}$$

$$\sup_{d(i,c)\leq 1-\Phi_c^{DB}} \frac{F_i^1(c)}{\bar{F}_i(c)} - \frac{F^1(c)}{\bar{F}(c)} \leq \frac{\left(1 - \frac{F^1(c)}{\bar{F}(c)}\right) \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}}{1 + \left(\frac{\bar{F}(c)}{F^1(c)} - 1\right) \left[1 - \prod_{\hat{c}\neq c} \frac{k_{\hat{c}}}{\bar{F}(\hat{c})}\right]}.$$

□

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