Protocol Invariance and the Timing of Decisions in Dynamic Games*

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September 26, 2017

Abstract

We characterize a class of dynamic stochastic games that we call separable dynamic games with noisy transitions and establish that these widely used models are protocol invariant provided that periods are sufficiently short. Protocol invariance means that the set of Markov perfect equilibria is nearly the same irrespective of the order in which players are assumed to move within a period. Protocol invariance can facilitate applied work and renders the implications and predictions of a model more robust. Our class of dynamic stochastic games includes investment games, R&D races, models of industry dynamics, dynamic public contribution games, asynchronously repeated games, and many other models from the extant literature.

*We thank Yuichiro Kamada, George Mailath, and Juuso Toikka for very useful comments and suggestions. Doraszelski acknowledges financial support from the Dean's Research Fund at the Wharton School. Escobar acknowledges financial support from the Institute for Research in Market Imperfections and Public Policy, MIPP, ICM IS130002, Ministerio de Economía.

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1 Introduction

The timing of decisions is an essential ingredient into modelling many strategic situations. Asynchronous decisions are a type of commitment, and being able to move first and thus set the stage for rivals can confer a considerable advantage on a player. Synchronous decisions, in contrast, take away the ability to commit as players are neither leaders nor followers. From the basically static models in Cournot (1838) and von Stackelberg (1934) to the genuinely dynamic models in Cyert and DeGroot (1970) and Maskin and Tirole (1987, 1988a, 1988b) and the anti-folk theorems in Rubinstein and Wolinsky (1995) and Lagunoff and Matsui (1997, 2001), a long and distinguished literature has pointed out cases where the protocol of moves matters crucially for equilibrium behavior.

Our paper provides a counterpoint to this literature. We show that a fairly general and widely used class of dynamic models is protocol invariant provided that periods are sufficiently short and moves are therefore sufficiently frequent. Protocol invariance means that the set of equilibria of a model is nearly the same irrespective of the order in which players are assumed to move within a period, including—and extending beyond—simultaneous, alternating, and sequential moves. Protocol invariance can facilitate applied work and renders the implications and predictions of a model more robust.

We focus on infinite-horizon dynamic stochastic games and their stationary Markov perfect equilibria (henceforth Markov perfect equilibria for short). A dynamic stochastic game is a dynamic system that can be in different states in different periods according to a discrete-time Markov process that players can influence through their actions. Dating back to Shapley (1953), dynamic stochastic games have a long tradition in economics and are central to the analysis of strategic interactions among forward-looking players in dynamic environments. The main contribution of this paper is to characterize a class of dynamic stochastic games that are protocol invariant provided that periods are sufficiently short. We call this class *separable dynamic games with noisy transitions*.

We apply dynamic stochastic games to situations in which a player primarily influences his rivals' payoffs by taking action to change the state. While per-period payoffs and state-to-state transitions depend arbitrarily on the state in a separable dynamic game, they are assumed to depend on players' actions in an additive manner: to a first-order approximation, per-period payoffs and state-to-state transitions are built from parts that depend on the actions taken by individual players. To the extent that there are complementarities between players' actions and other non-separabilities in per-period payoffs and state-to-state transitions, they must vanish as periods become short.

Noisy transitions preclude that there is an action that a player can take to guarantee a change in the state. We model the evolution of the state by a discrete-time approximation to a continuous-time Markov process in which the time spent in a state has an exponential distribution with a finite hazard rate. The finite hazard rate implies that transitions are noisy. This assumption reflects the view that models are only an approximation to reality, and so there always is some residual uncertainty associated with taking an action.

While the assumptions of separability and noisy transitions doubtlessly rule out some interesting applications, many dynamic models in the literature are amenable to these assumptions. Examples include investment games (Spence 1979, Fudenberg and Tirole 1983, Hanig 1986, Reynolds 1987, Reynolds 1991, Dockner 1992), R&D races (Reinganum 1982, Lippman and McCardle 1987), models of industry dynamics (Ericson and Pakes 1995), dynamic public contribution games (Marx and Matthews 2000, Compte and Jehiel 2004, Georgiadis 2015), and the recent continuous-time stochastic games with moves at random times (Arcidiacono, Bayer, Blevins, and Ellickson 2016, Ambrus and Lu 2015, Calcagno, Kamada, Lovo, and Sugaya 2014, Kamada and Kandori 2017). We also show that some dynamic models that are not obviously separable, including the asynchronously repeated Bertrand, Cournot, and coordination games in Maskin and Tirole (1988a, 1988b) and Lagunoff and Matsui (1997), can be re-cast to satisfy our assumptions.

Our main result is that separable dynamic games with noisy transitions are protocol invariant provided that periods are sufficiently short. To provide intuition, consider a prototypical investment game between two firms. A firm can undertake a risky investment project to increase its capital stock. A firm's per-period payoff increases in its own capital stock and decreases in its rival's capital stock. The separability assumption is satisfied, as whether its rival invests affects directly neither the firm's per-period payoff nor the probability that the firm succeeds in increasing its capital stock. Moreover, transitions from one state to another are noisy due to the risky nature of the investment project.

Now contrast two protocols of moves. When firms move alternatingly, a forward-looking firm deciding whether to invest understands that its rival's capital stock remains constant for (at least) the period. In contrast, when firms move simultaneously, the firm has to take into account the probability that its rival's capital stock increases over the course of the period. This probability, however, becomes negligible as periods become short because transitions are noisy. The protocol of moves is therefore almost immaterial to the firm's decision.

As intuitive as it may be that our assumptions imply protocol invariance, this intuition neither immediately translates into a proof nor is it always salient in the literature. Consider the large literature on technology adoption and the related debate about the persistence of monopoly (see Reinganum (1989) and the references therein). In a prototypical model with two firms, the state-of-the-art technology evolves over time and a firm decides whether to continue operating its current technology or pay a cost to undertake an upgrade to the state-of-the-art technology. The separability assumption is satisfied as a firm's per-period payoff depends on its rival's current technology but not directly on its rival's adoption decision. Riordan and Salant (1994) approximate a continuous-time game by a discrete-time game with alternating moves and establish that a pattern of increasing dominance arises and all adoptions are by one of the firms. Giovannetti (2001) shows in a discrete-time game with simultaneous moves that firms take turns to adopt in a pattern of perpetual leapfrogging. Most recently, Iskhakov, Rust, and Schjerning (2017) show that a game with alternating moves has a unique Markov perfect equilibrium whereas the game with simultaneous moves has a vast number of equilibria, with some displaying increasing dominance and others leapfrogging.[1] Our main result

[1]In the game with alternating moves uniqueness obtains under certain parameter restrictions; absent these restrictions, the "set of payoffs shrinks dramatically" in the game with alternating moves as compared to the game with simultaneous moves (Iskhakov,

clarifies that these stark differences between simultaneous and alternating moves hinge on the outcome of the adoption decision being certain. Provided that periods are sufficiently short, these differences vanish under the empirically plausible assumption that the process of updating to the state-of-the-art technology involves at least some uncertainty.

The proof of our main result proceeds from the equilibrium conditions for a separable dynamic game with noisy transitions. We fix an arbitrary protocol of moves and take the limit as periods become short and we therefore pass from discrete to continuous time. We observe that the limit conditions are independent of the protocol of moves used to pass to the limit. Protocol invariance follows immediately if the limit conditions admit a unique solution.

However, due to the richness of the class of separable dynamic games with noisy transitions, the limit conditions may admit multiple solutions. In this case, taking the limit of a sequence of equilibria under one protocol of moves may potentially lead to a different solution of the limit conditions than taking the limit under another protocol of moves, thereby causing protocol invariance to fail. To show that this does not happen, we introduce differential topology tools to study the limit conditions. We prove that generically all solutions of the limit conditions can be approximated by the Markov perfect equilibria of a separable dynamic game with noisy transitions and an arbitrary protocol of moves provided that periods are sufficiently short. Our proof draws on and expands ideas in Harsanyi (1973a, 1973b), who establishes robustness to perturbations of payoffs and information in normal-form games. To the best of our knowledge, our paper is the first attempt to use differential topology tools to establish robustness to the timing of decisions.

While we mostly treat the limit conditions as a technical device, they are of interest by themselves. The limit conditions can be interpreted as the equilibrium conditions for a continuous-time stochastic game. We also provide an equivalence result showing that the limit conditions are identical to the equilibrium conditions for a dynamic stochastic game in which in any period one player is randomly selected to make a decision. In this game with random moves, the fact that a player can revise his decision only at random times confers a kind of commitment power on the player similar to that in the games with alternating moves in Maskin and Tirole (1988a, 1988b) and Lagunoff and Matsui (1997). Our equivalence result therefore underscores that the class of separable dynamic games with noisy transitions admits quite rich strategic interactions between players.

Separability, noisy transitions, and Markov perfection appear to be the key properties that underpin protocol invariance. We show that our assumptions are tight in the sense that counterexamples to protocol invariance can be constructed if any one of them is relaxed. In particular, we show that protocol invariance does not extend beyond Markov perfect equilibria to other equilibrium concepts.

Our main result facilitates and informs applied work in a number of ways. First and perhaps most important, determining the protocol of moves that is most realistic and appropriate for the application at hand may be amongst the most difficult choices a modeler has to make. In empirical work, in particular, the

Rust, and Schjerning 2017, p. 47).

timing of decisions and the ability to commit is typically not observable to the researcher. Hence, we may be suspicious of any implication or prediction from a model that is driven by the protocol of moves that the modeler has chosen to impose, a point that has been made forcefully by Rosenthal (1991) and van Damme and Hurkens (1996) for normal-form games and by Kalai (2004) for large Bayesian games. Protocol invariance alleviates this concern and the burden of determining the protocol of moves for the class of separable dynamic games with noisy transitions by ensuring that equilibrium behavior is independent of the timing of decisions provided that periods are sufficiently short. Second, because the timing of decisions and the ability to commit is typically not observable, empirical work sometimes averages over different protocols of moves (Einav 2010). This average depends on the assumed probability distribution over protocols of moves and may be difficult to interpret if it does not correspond to an equilibrium of any game. Protocol invariance renders averaging unnecessary. Third, our main result cautions against the presumption that imposing asynchronous instead of synchronous decisions on a dynamic stochastic game reduces the number of equilibria. Fourth, dynamic stochastic games are often not very tractable analytically and thus call for the use of numerical methods. Doraszelski and Judd (2007) show that the computational burden can vary by orders of magnitude with the protocol of moves. For the class of separable dynamic games with noisy transitions, protocol invariance justifies imposing the protocol of moves that is most convenient from a computational perspective.

We apply and extend our main result in three ways. First, we extend protocol invariance to the limiting case of deterministic transitions and provide a novel dynamic programming characterization of separable dynamic games with noisy transitions as moves become arbitrarily frequent and hazard rates arbitrarily large. We emphasize that this extension relies on a particular way of taking the joint limit. We exposit a discontinuity in the set of Markov perfect equilibria as moves become arbitrarily frequent and hazard rates arbitrarily large and argue that many examples from the extant literature where equilibrium behavior hinges on the protocol of moves can be seen as a manifestation of this discontinuity.

Second, we provide a new rationale for focusing on Markov perfect equilibria. Provided that periods are sufficiently short, we show that if a strict finite-memory equilibrium payoff profile in a separable dynamic game with noisy transitions and simultaneous moves is protocol invariant, then it is arbitrarily close to a Markov perfect equilibrium payoff profile. Thus, the Markovian restriction on equilibrium strategies is not only sufficient but also necessary for protocol invariance. Markov perfect equilibria are therefore the only equilibria that are robust to changes in the protocol of moves. This result adds to the literature providing foundations for Markov perfect equilibria (Maskin and Tirole 2001, Bhaskar and Vega-Redondo 2002, Sannikov and Skrzypacz 2007, Faingold and Sannikov 2011, Bhaskar, Mailath, and Morris 2013, Bohren 2014).

Third, we contribute to the literature on computing Markov perfect equilibria (Pakes and McGuire 1994, Pakes and McGuire 2001, Doraszelski and Judd 2007, Doraszelski and Judd 2012, Weintraub, Benkard, and Van Roy 2008, Ifrach and Weintraub 2017). Doraszelski and Judd (2012) show that the limit conditions that arise as we pass from discrete to continuous time are particularly easy to solve numerically, often reducing the computational burden by orders of magnitude. Our main result provides a justification for solving the limit conditions by establishing that these solutions almost coincide with the Markov perfect equilibria of separable dynamic games with noisy transitions and arbitrary protocols of moves provided that periods are sufficiently short. Moreover, we show that this one-to-one correspondence between the solutions to the limit conditions and the Markov perfect equilibria of dynamic stochastic games obtains more broadly. In particular, we can dispense with the separability assumption if we restrict attention to games with simultaneous moves.

Our paper is related to two strands of literature. First, our notion of protocol invariance builds on and extends the notion of a commitment robust equilibrium in Rosenthal (1991) and van Damme and Hurkens (1996) from two-player normal-form games to N-player dynamic stochastic games. Rosenthal (1991) defines a Nash equilibrium of a two-player normal-form game to be commitment robust if it is also a subgame perfect equilibrium outcome of each of the two extensive-form games in which one of the players moves first, and provides a series of illustrative examples. In contrast to the notion of a commitment robust equilibrium, our notion of protocol invariance pertains to the entire set of equilibria of a fairly general class of dynamic models. Our paper is also related to Kalai (2004), who shows that the Nash equilibria of large anonymous Bayesian games are approximately robust to variations in the extensive-form version of the game. The driving force behind Kalai's (2004) result is the vanishing impact that a player's action has on other players' payoffs as the number of players grows large. In our setting, the impact that a player's action has on other players' payoffs (other than through a change in the state) vanishes as periods become short. From a more technical perspective, Kalai (2004) allows for ϵ-equilibria, while we impose exact equilibrium and prove results for generic payoffs.

Second, previous attempts to exposit dynamic models where the protocol of moves does not matter for equilibrium behavior are few and far between and confined to very specific models. Abreu and Gul (2000) study bilateral bargaining and show that independent of the bargaining protocol the same limit is reached as the time between offers becomes short. Caruana and Einav (2008) study a model in which players repeatedly announce an action but only the final announced action is relevant for payoffs. While players can revise their announcements, they pay a cost each time they do so; in this way, announcements play the role of an imperfect commitment device. Caruana and Einav (2008) show that the order in which players make announcements does not matter as long as the time between announcements is sufficiently short. In contrast to Abreu and Gul (2000) and Caruana and Einav (2008), we do not presuppose that the limit conditions admit a unique solution. Because our class of dynamic stochastic games is much less tightly specified, we require differential topology tools to analyse the limit conditions.

The remainder of this paper is organized as follows. Section 2 introduces separable dynamic games with noisy transitions. Section 3 develops our main result. Section 4 discusses a number of applications and extensions of our main result and Section 5 concludes. Unless noted otherwise, proofs are in the Appendix. An Online Appendix provides further examples and proofs.

2 Separable Dynamic Games with Noisy Transitions

We focus on dynamic stochastic games with finite sets of players, states, and actions. Time $t = 0, \Delta, 2\Delta, \ldots$ is discrete and measured in units of $\Delta > 0$. We refer to Δ as the length of a period; as $\Delta \to 0$, moves become frequent. The time horizon is infinite. Let $\{1, 2, \ldots, N\}$ denote the set of players, Ω the set of states, and $A_i(\omega)$ the set of actions of player i in state ω. Each player strives to maximize the expected net present value of his stream of payoffs and discounts future payoffs using a discount rate $\rho > 0$. Monitoring is perfect.

The protocol of moves determines which players can take an action at time t and which players cannot. We allow for a general protocol of moves that encompasses—and goes beyond—simultaneous, alternating, and sequential moves. To this end, we allow the set of players who have the move to change from one period to the next. The set of players $J^t \subseteq \{1, 2, \ldots, N\}$ who have the move at time t thus becomes part of the state of the system, and we refer to it as the "protocol" state to distinguish it from the familiar "physical" state $\omega^t \in \Omega$. In contrast to the physical state, for simplicity we assume that the protocol state evolves independently of players' actions. In the Online Appendix, we show that our protocol-invariance theorem remains valid without this simplifying assumption.

The game proceeds as follows. It starts at time $t = 0$ from an initial state $(\omega^{t=0}, J^{t=0})$. After observing $(\omega^{t=0}, J^{t=0})$, the players $j \in J^{t=0}$ who have the move choose their actions $a^{t=0}_{J^{t=0}} = \left(a^{t=0}_j\right)_{j \in J^{t=0}}$ simultaneously and independently from each other. Now two things happen, depending on the state $(\omega^{t=0}, J^{t=0})$ and the actions $a^{t=0}_{J^{t=0}}$. First, player i receives a payoff $u^{\Delta}_i(\omega^{t=0}, J^{t=0}, a^{t=0}_{J^{t=0}})$. Second, the system transits from state $(\omega^{t=0}, J^{t=0})$ to state $(\omega^{t=\Delta}, J^{t=\Delta})$. Independent of each another, the transition from $\omega^{t=0}$ to $\omega^{t=\Delta}$ happens with probability $\Pr^{\Delta}\left(\omega^{t=\Delta} | \omega^{t=0}, J^{t=0}, a^{t=0}_{J^{t=0}}\right)$ and that from $J^{t=0}$ to $J^{t=\Delta}$ with probability $\Pr\left(J^{t=\Delta} | J^{t=0}\right)$. While player i receives a payoff irrespective of whether he has the move ($i \in J^{t=0}$) or not ($i \notin J^{t=0}$), the exact amount depends on the actions $a^{t=0}_{J^{t=0}}$ of the players who have the move, as do the state-to-state transitions. In the next round at time $t = \Delta$, after observing $(\omega^{t=\Delta}, J^{t=\Delta})$, the players $j \in J^{t=\Delta}$ who have the move choose their actions $a^{t=\Delta}_{J^{t=\Delta}}$. Then player i receives a payoff $u^{\Delta}_i(\omega^{t=\Delta}, J^{t=\Delta}, a^{t=\Delta}_{J^{t=\Delta}})$ and the state changes again from $(\omega^{t=\Delta}, J^{t=\Delta})$ to $(\omega^{t=2\Delta}, J^{t=2\Delta})$. The game goes on in this way ad infinitum.

To allow for a general protocol of moves, we partition the set of players $\{1, 2, \ldots, N\}$ and assume that the set of players J^t who have the move at time t evolves according to a Markov process that is defined over this partition as follows:

Assumption 1 (Protocol of Moves) *Let $\mathcal{J}$ be a partition of $\{1, 2, \ldots, N\}$ and $\mathcal{P} = \left(\Pr\left(J' | J\right)\right)_{J, J' \in \mathcal{J}}$ a $|\mathcal{J}| \times |\mathcal{J}|$ transition matrix. $\mathcal{P}$ is irreducible and its unique stationary distribution is uniform on $\mathcal{J}$.*

In stating Assumption 1 and throughout the remainder of the paper we omit the time superscript whenever possible and use a prime to distinguish future from current values.

We denote the protocol of moves as $< \mathcal{J}, \mathcal{P} >$ in what follows. Because $\mathcal{J}$ is a partition of $\{1, 2, \ldots, N\}$, Assumption 1 ensures that player i always has the move in conjunction with the same rivals. By requiring the transition matrix $\mathcal{P}$ to have a unique stationary distribution that is uniform on $\mathcal{J}$, Assumption 1 further ensures that all players have the move with the same frequency over a sufficiently large number of periods.

Assumption 1 accommodates synchronous and asynchronous decisions and thus encompasses most dynamic stochastic games in the literature, including games with simultaneous moves (Shapley 1953, Ericson and Pakes 1995), games with alternating moves (Maskin and Tirole 1987, Maskin and Tirole 1988b, Maskin and Tirole 1988a, Lagunoff and Matsui 1997), and games with random moves (Doraszelski and Judd 2007, Iskhakov, Rust, and Schjerning 2017). In games with simultaneous moves, the partition is $\mathcal{J} = \{\{1, \dots, N\}\}$ with the trivial 1×1 transition matrix $\mathcal{P}$; in games with alternating moves the partition is $\mathcal{J} = \{\{1\}, \dots, \{N\}\}$ with the $N \times N$ transition matrix $\mathcal{P}$ with entries $\Pr(\{\mod_N(i+1)\}|\{i\}) = 1$.[2] In games with asynchronous moves, $\mathcal{J} = \{\{1\}, \dots, \{N\}\}$ and the identity of the player who has the move in a given period may follow a deterministic sequence as in games with alternating moves or it may be stochastic. Games with random moves are another special case of games with asynchronous moves. In these games, the probability that a player has the move in a given period is uniform across players and periods. Finally, Assumption 1 accommodates more than one—but less than all—players having the move in a given period and thus settings where decisions are partially synchronous.

In the Online Appendix, we show that Assumption 1 can be relaxed in several ways. First, we show that our protocol-invariance theorem remains valid if the evolution of the protocol state J depends on players' actions a_J and the physical state ω. Second, we show that the uniform stationary distribution in Assumption 1 can be replaced by a non-uniform stationary distribution. Third, we provide an extension of our protocol-invariance theorem that does not require $\mathcal{J}$ to be a partition of the set of players.

We model the evolution of the physical state by a discrete-time approximation to a continuous-time Markov process in order to impose that transitions are noisy:[3]

Assumption 2 (Noisy Transitions) *The transition probability* $\Pr^{\Delta}(\omega'|\omega, J, a_J)$ *is differentiable in* Δ *and can be written as*

$$\Pr^{\Delta}(\omega'|\omega, J, a_J) = \begin{cases} 1 - q_J(\omega, a_J)\Delta + \mathcal{O}(\Delta^2) & \text{if} \quad \omega' = \omega, \\ q_J(\omega, a_J) p_J(\omega'|\omega, a_J)\Delta + \mathcal{O}(\Delta^2) & \text{if} \quad \omega' \neq \omega, \end{cases}$$

where $q_J \colon \{(\omega, (a_j)_{j \in J}) \mid a_j \in A_j(\omega)\} \to \mathbb{R}^+ \cup \{0\}$, $p_J \colon \{(\omega, (a_j)_{j \in J}) \mid a_j \in A_j(\omega)\} \to \mathtt{P}(\Omega)$, *and* $\mathtt{P}(\Omega)$ *is the set of probability distributions over* Ω. *We normalize* $p_J(\omega \mid \omega, a_J) = 0$.

Without loss of generality, we decompose the transition probability $\Pr^{\Delta}(\omega'|\omega, J, a_J)$ into a probability that the state changes in a given period—or that a jump occurs in the lingo of stochastic processes—and a probability distribution over successor states conditional on the state changing. The probability that the state changes is $q_J(\omega, a_J)\Delta$ in proportion to the length of a period Δ and, conditional on the state changing, the probability that it changes from ω to ω' is $p_J(\omega'|\omega, a_J)$. Normalizing $p_J(\omega|\omega, a_J) = 0$ amounts to ignoring

[2]The notation $\mod_N(x)$ refers to the modulo N congruence.

[3]An alternative modelling approach is to start from a discrete-time Markov process and then "rescale" it to shorter periods. This presumes that the discrete-time Markov process is embeddable (Elfving 1937).

a jump from a state to itself and adjusting the hazard rate $q_J(\omega, a_J)$ of a jump occurring accordingly. Importantly, Assumption 2 restricts the model to have a finite hazard rate in a given state so that there is no action that a player can take to guarantee a change in the state.

We finally assume that per-period payoffs and state-to-state transitions have an additively separable structure:

Assumption 3 (Separability) *The per-period payoff $u_i^\Delta(\omega, J, a_J)$ is differentiable in Δ and can be written as*

$$u_i^\Delta(\omega, J, a_J) = |\mathcal{J}| \sum_{j \in J} u_{i,j}(\omega, a_j)\Delta + \mathcal{O}(\Delta^2),$$

where $u_{i,j} \colon \{(\omega, a_j) \mid a_j \in A_j(\omega)\} \to \mathbb{R}$. The hazard rate $q_J(\omega, a_J)$ and transition probability $p_J(\omega' \mid \omega, a_J)$ can be written as

$$q_J(\omega, a_J) = |\mathcal{J}| \sum_{j \in J} q_j(\omega, a_j)$$

and

$$q_J(\omega, a_J) p_J(\omega' \mid \omega, a_J) = |\mathcal{J}| \sum_{j \in J} q_j(\omega, a_j) p_j(\omega' \mid \omega, a_j),$$

where $q_j \colon \{(\omega, a_j) \mid a_j \in A_j(\omega)\} \to \mathbb{R}^+ \cup \{0\}$ and $p_j \colon \{(\omega, a_j) \mid a_j \in A_j(\omega)\} \to \mathtt{P}(\Omega)$.

To a first-order approximation, Assumption 3 builds up the per-period payoff $u_i^\Delta(\omega, J, a_J)$ of player i from the flow payoff $u_{i,j}(\omega, a_j)$ by summing over the players $j \in J$ who have the move. By taking action a_j in state ω, player j "contributes" $|\mathcal{J}| u_{i,j}(\omega, a_j)\Delta$ to the per-period payoff of player i in proportion to the length of a period Δ. This restricts complementarities between players' actions and other non-separabilities to the higher-order term $\mathcal{O}(\Delta^2)$.[4] We discuss below our reason for scaling by the number of elements of the partition $\mathcal{J}$.

While at first glance Assumption 3 may seem to trivialize the strategic interactions between players, it does not.[5] Importantly, Assumption 3 does not restrict how per-period payoffs and state-to-state transitions depend on the state. Because players are forward looking, this allows strategic interactions to be channeled through continuation values. As mentioned in Section 1 and further discussed in Section 2.1, examples of dynamic stochastic games that satisfy Assumption 3 range from investment games over the recent continuous-time stochastic games with moves at random times to asynchronously repeated games.

In conjunction with Assumption 2, Assumption 3 builds up the components of the transition probability $\Pr^\Delta(\omega'|\omega, J, a_J)$ from the player-specific hazard rate $q_j(\omega, a_j)$ and the transition probability $p_j(\omega'|\omega, a_j)$ by

[4] More explicitly, we assume that there exists $\bar{c} > 0$ and $\bar{\Delta} > 0$ such that $\|u_i^\Delta(\omega, J, a_J) - |\mathcal{J}| \sum_{j \in J} u_{i,j}(\omega, a_j)\Delta\| \leq \bar{c}\Delta^2$ for all $\Delta < \bar{\Delta}$.

[5] Assumption 3 has bite in conjunction with Assumption 2 but is a rather mild on its own. Consider the canonical dynamic stochastic game in Shapley (1953). If the state space takes the form $\Omega = \prod_{i=1}^N \Omega_i$ and player i independently controls the evolution of $\omega_i \in \Omega_i$, then the game can be considered separable even if the per-period payoff of player i is $v_i(\omega, a)\Delta$. To see this, redefine the state in period t as $\tilde{\omega}^t = (\omega^{t-1}, a^{t-1}, \omega^t)$ and let the flow payoff of player i be $u_{i,i}(\tilde{\omega}^t, a_i^t) = \frac{1}{\delta} v_i(\omega^{t-1}, a^{t-1})$ and $u_{i,j}(\tilde{\omega}^t, a_j^t) = 0$ if $j \neq i$, where δ is the discount factor.

summing over the players $j \in J$ who have the move. Because it imposes a competing hazards model on the transition probability, a change in the state is with high probability due to the action taken by one of the players having the move.

In what follows, we call the above game a *separable dynamic game with noisy transitions* and denote it by $\Gamma =< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >$. We view the function $u_{i,j} \colon \{(\omega, a_j) \mid a_j \in A_j(\omega)\} \to \mathbb{R}$ as a vector $u_{i,j} \in \mathbb{R}^{\sum_{\omega\in\Omega}|A_j(\omega)|}$ and denote $u_i = (u_{i,j})_{j=1}^N \in \mathbb{R}^{\sum_{j=1}^N \sum_{\omega\in\Omega}|A_j(\omega)|}$ and $u = (u_i)_{i=1}^N \in \mathbb{R}^{N\sum_{j=1}^N \sum_{\omega\in\Omega}|A_j(\omega)|}$. We further denote the collection of hazard rates and transition probabilities $q = (q_j(\omega, a_j))_{\omega\in\Omega, j=1,\ldots,N, a_j\in A_j(\omega)}$ and $p = (p_j(\omega'|\omega, a_j))_{\omega\in\Omega, j=1,\ldots,N, a_j\in A_j(\omega)}$.

A stationary Markovian strategy for player i is a function $\sigma_i : \Omega \to \cup_{\omega\in\Omega}\mathtt{P}(A_i(\omega))$ with $\sigma_i(\omega) \in \mathtt{P}(A_i(\omega))$ for all ω, where $\mathtt{P}(A_i(\omega))$ is the set of probability distributions over $A_i(\omega)$. Because $\mathcal{J}$ is a partition of $\{1, 2, \ldots, N\}$, Assumption 1 ensures that player i always has the move in conjunction with the same rivals. Hence, while the state of the system comprises both the physical state ω and the protocol state J, it suffices to consider Ω as the domain of σ_i. We use $\sigma_i(a_i \mid \omega)$ to denote the probability that action $a_i \in A_i(\omega)$ is played in state ω. We explore more general strategies in Example 8 and Section 4.2.

From hereon, we denote by Σ_i the set of stationary Markovian strategies for player i and $\Sigma = \prod_{i=1}^N \Sigma_i$ the set of strategy profiles. To account for mixed strategies, we extend the flow payoff $u_{i,j}(\omega, \sigma_j(\omega)) = \sum_{a_j\in A_j(\omega)} u_{i,j}(\omega, a_j)\sigma_j(a_j \mid \omega)$ and transition probability

$$\Pr^\Delta(\omega'|\omega, J, \sigma_J(\omega)) = \sum_{a_J\in\prod_{j\in J} A_j(\omega)} \Big(\Pr^\Delta(\omega'|\omega, J, a_J) \prod_{j\in J} \sigma_j(a_j \mid \omega)\Big).$$

A profile of stationary Markovian strategies $\sigma = (\sigma_i)_{i=1}^N$ is a stationary Markov perfect equilibrium if it is a subgame perfect equilibrium of the game Γ. The set of Markov perfect equilibria of the game Γ is denoted $\mathrm{Equil}(\Gamma)$. This set is nonempty (Shapley 1953).[6]

Our main interest is to compare equilibrium behavior under different protocols of moves. Assumption 1 lets us compare two models $\Gamma =< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >$ and $\overline{\Gamma} =< \Delta, \overline{\mathcal{J}}, \overline{\mathcal{P}}, u, p, q, \rho >$ that differ only in the protocol of moves by ensuring that all players move with the same frequency. The scale factor $|\mathcal{J}|$ in Assumption 3 further ensures that a player's action brings about the same payoffs and chances of changing the state in the two models. To see this, contrast a game with simultaneous moves Γ with a game with alternating moves $\overline{\Gamma}$. In the game with simultaneous moves Γ, player j takes an action a_j every Δ units of time, yielding the payoff $u_{i,j}(\omega, a_j)\Delta$ and the hazard rate $q_j(\omega, a_j)\Delta$ (neglecting the higher-order term $\mathcal{O}(\Delta^2)$). Over a stretch of $N\Delta$ units of time, the action a_j thus yields the payoff $u_{i,j}(\omega, a_j)N\Delta$ and the hazard rate $q_j(\omega, a_j)N\Delta$. In the game with alternating moves $\overline{\Gamma}$, in contrast, player j has the move only once every $N\Delta$ units of time. According to Assumption 3, if player j takes an action a_j, then this yields the payoff $|\overline{\mathcal{J}}|u_{i,j}(\omega, a_j)\Delta = Nu_{i,j}(\omega, a_j)\Delta$ and the hazard rate $|\overline{\mathcal{J}}|q_j(\omega, a_j)\Delta = Nq_j(\omega, a_j)\Delta$. Hence, per-period

[6] Shapley (1953) establishes existence for dynamic stochastic games with simultaneous moves. To apply his result, we view the game Γ as a dynamic stochastic game with simultaneous moves in which the players that do not have the move have no impact on per-period payoffs and state-to-state transitions.

payoffs and state-to-state transitions in the game with alternating moves $\overline{\Gamma}$ are comparable to those in the game with simultaneous moves Γ.[7]

2.1 Examples

In the remainder of this section we discuss how prominent examples of dynamic stochastic games from the literature can be cast as special cases of our model.

Example 1 (Entry Games and R&D Races) *Consider $N = 2$ firms that may enter a new market. To enter the market, firm i must complete K steps. For example, to build a cement plant and enter the market, a firm needs to find a location, design the plant, obtain environmental permits, negotiate with contractors, etc. Alternatively, consider an R&D race in which a firm gradually discovers an invention and obtains a patent through a series of intermediate steps (Fudenberg, Gilbert, Stiglitz, and Tirole 1983, Grossman and Shapiro 1987, Harris and Vickers 1987).*

Let $K \geq 1$ be the number of required steps and $\omega_i \in \Omega_i = \{0, 1, \dots, K\}$ the number of steps that firm i has already completed. The state of the game is $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 = \Omega$. To take the next step, firm i can make an investment, denoted by $a_i = 1$, at cost $c_i > 0$. Action $a_i \in A_i(\omega) = \{0, 1\}$ induces the hazard rate

$$q_i(\omega, a_i) = \begin{cases} \lambda a_i & \textit{if} \quad \omega_i \leq K - 1, \\ 0 & \textit{if} \quad \omega_i = K. \end{cases}$$

The transition probability is[8]

$$p_i(\omega' \mid \omega, a_i) = \begin{cases} 1 & \textit{if} \quad \omega_i' = \omega_i + 1, \omega_{-i}' = \omega_{-i}, \omega_i \leq K - 1, \\ 1 & \textit{if} \quad \omega_i' = 0, \omega_{-i}' = \omega_{-i}, \omega_i = K, \\ 0 & \textit{otherwise.} \end{cases}$$

Once firm i has completed all steps it enters the new market (or obtains the patent) and, depending on whether its rival has also completed all steps, obtains the monopoly profit $B_i > 0$ or the duopoly profit $b_i < B_i$. Its flow payoff is

$$u_{i,i}(\omega, a_i) = \begin{cases} B_i - c_i a_i & \textit{if} \quad \omega_i = K, \omega_{-i} \leq K - 1, \\ b_i - c_i a_i & \textit{if} \quad \omega_i = \omega_{-i} = K, \\ -c_i a_i & \textit{otherwise} \end{cases}$$

and $u_{i,j}(\omega, a_j) = 0$ if $j \neq i$. Assumptions 2 and 3 are satisfied.

[7] Instead of scaling by the number of elements of the partition $\mathcal{J}$ in Assumption 3, we can assume that interactions occur at time $t = 0, \Delta/|\mathcal{J}|, 2\Delta/|\mathcal{J}|, \dots$. This alternative formulation ensures that a player has the move on average once every Δ units of time. Our results immediately carry over.

[8] Note that conditional on a jump occurring we specify a transition from $\omega_i = K$ to $\omega_i' = 0$ with probability one. This is immaterial, however, because no jump occurs as $q_i(\omega, a_i) = 0$ if $\omega_i = K$.

Example 2 (Industry Dynamics) *Ericson and Pakes (1995) develop a discrete-time model of industry dynamics. In their model and the large literature following it (see Doraszelski and Pakes (2007) for a survey), incumbent firms decide on investment and exit and compete in the product market; potential entrants decide on entry. Depending on the application, firm i's state variable $\omega_i \in \Omega_i$ encodes its current product quality, production capacity, marginal cost, etc. It further encodes whether firm i is currently an incumbent firm that competes in the product market or a potential entrant. The state of the game is $\omega = (\omega_1, \omega_2, \ldots, \omega_N) \in \prod_{i=1}^{N} \Omega_i = \Omega$.*

Incumbent firm i earns a profit $\pi_i(\omega)$ from competing in the product market (price or quantity competition, depending on the application) that, following the literature, we treat as a reduced-form input into the model. While $\pi_i(\omega)$ depends on the current state of the game ω, it does not depend on the current investment and exit decisions. The cost of investment $c_i(\omega, a_i)$ as well as any cost or benefit pertaining to exit are simply added to $\pi_i(\omega)$. As a result, per-period payoffs are separable in the sense of Assumption 3.

In many applications of the Ericson and Pakes (1995) model, firm i has exclusive control over the evolution of ω_i through its investment, exit, and entry decisions (e.g., Besanko and Doraszelski 2004, Chen 2009, Doraszelski and Markovich 2007). Because the decisions of firm i affect its own state variable but not its rivals' state variables, the transition probabilities are separable in the sense of Assumption 3. In other applications, there is in addition a common shock such as an increase in the quality of the outside good or an industry-wide depreciation shock (e.g., Berry and Pakes 1993, Gowrisankaran 1999, Fershtman and Pakes 2000, de Roos 2004, Markovich 2008). Assumption 3 accommodates a common shock because transitions effected by "nature" can be subsumed into those effected by one of the players.[9]

Because investment may or may not result in a favorable outcome, transitions due to investment decisions are noisy as required by Assumption 2. Transitions due to entry and exit decisions present a difficulty because in the Ericson and Pakes (1995) model, an incumbent firm can exit the industry for sure and a potential entrant can enter the industry for sure. Doraszelski and Judd (2012) show how to formulate exit and entry with finite hazard rates either by way of exit and entry intensities or by way of randomly drawn, privately observed scrap values and setup costs (as in Doraszelski and Satterthwaite 2010). Their formulation satisfies Assumption 2.

Example 3 (Continuous-Time Stochastic Games with Moves at Random Times) *Arcidiacono, Bayer, Blevins, and Ellickson (2016) develop a continuous-time stochastic game in which a player is given the move at random times. Decisions are asynchronous as the probability that more than one player has the move at a given time is zero. Ambrus and Lu (2015), Ambrus and Ishii (2015), Calcagno, Kamada, Lovo, and Sugaya (2014), and Kamada and Kandori (2017) develop closely related continuous-time stochastic games with moves at random times.*

Arcidiacono, Bayer, Blevins, and Ellickson (2016) endow player i with a Poisson process with a constant hazard rate λ. The time between jumps in this process is therefore exponentially distributed. If process i is

[9]One may alternatively represent nature by an extra player 0.

the first of the N processes to jump, then player i is given the move and chooses an action a_i. The state of the game then changes from ω to ω' with probability $l_i(\omega' \mid \omega, a_i)$, with $l_i(\cdot \mid \omega, a_i) \in \mathtt{P}(\Omega)$.

We can formulate this process in our framework by defining the hazard rate

$$q_i(\omega, a_i) = \lambda(1 - l_i(\omega \mid \omega, a_i))$$

and the transition probability[10]

$$p_i(\omega' \mid \omega, a_i) = \begin{cases} \frac{1}{(1-l_i(\omega|\omega,a_i))} l_i(\omega' \mid \omega, a_i) & \textit{if} \quad \omega' \neq \omega, \\ 0 & \textit{if} \quad \omega' = \omega. \end{cases}$$

Finally, the flow payoff of player i is

$$u_{i,j}(\omega, a_j) = \begin{cases} s_i(\omega) + \lambda \pi_i(\omega, a_i) & \textit{if} \quad i = j, \\ 0 & \textit{if} \quad i \neq j, \end{cases}$$

where $s_i(\omega)$ is a baseline payoff and $\pi_i(\omega, a_i)$ an additional payoff that player i receives if he is given the move. To account for the likelihood that player i is given the move, $\pi_i(\omega, a_i)$ is multiplied by λ in the flow payoff. The flow payoff and transition probability in Arcidiacono, Bayer, Blevins, and Ellickson (2016) clearly conform to Assumptions 2 and 3.

Example 4 (Dynamic Public Contribution Games) *Consider N players that contribute towards completing a public project (Marx and Matthews 2000, Compte and Jehiel 2004, Georgiadis 2015). Completing the project requires K steps and $\omega \in \Omega = \{0, 1, \ldots, K\}$ indicates the number of steps that have been completed. Player i's contribution $a_i \in A_i(\omega) \subseteq \mathbb{R}$ induces a hazard rate $q_i(\omega, a_i)$ which is strictly increasing in a_i if $\omega \neq K$, while $q_i(\omega, a_i) = 0$ if $\omega = K$. The transition probability is*

$$p_i(\omega' \mid \omega, a_i) = \begin{cases} 1 & \textit{if} \quad \omega' = \omega + 1, \omega \leq K - 1, \\ 1 & \textit{if} \quad \omega' = 0, \omega = K, \\ 0 & \textit{otherwise.} \end{cases}$$

The public project is completed once state $\omega = K$ is reached and results in flow payoffs B_i for player i. The cost of contribution is $c_i(\omega, a_i)$ for player i. We therefore specify its flow payoff as

$$u_{i,i}(\omega, a_i) = \begin{cases} B_i - c_i(\omega, a_i) & \textit{if} \quad \omega = K, \\ -c_i(\omega, a_i) & \textit{otherwise,} \end{cases}$$

and $u_{i,j}(\omega, a_j) = 0$ if $j \neq i$. Assumptions 2 and 3 are satisfied.

[10] If $l_i(\omega \mid \omega, a_i) = 1$, then $p_i(\cdot \mid \omega, a_i)$ can be defined arbitrarily.

Example 5 (Asynchronously Repeated Games) *Maskin and Tirole (1988a, 1988b) study discrete-time repeated games with asynchronous moves. These games can be re-cast to satisfy our assumptions.*

Restricting Example 3 by setting $N = 2$, $\Omega = \Omega_1 \times \Omega_2$, $A_i(\omega) = \Omega_i$,

$$l_i(\omega' \mid \omega, a_i) = 1 \text{ if and only if } a_i = \omega'_i,$$

and $u_{i,j}(\omega, a_j) = s_i(\omega)$, *we obtain a game in which the state* ω_i *of player* i *is simply a record of the last chosen action. Depending on the application,* $s_i(\omega)$ *represents the payoff from Bertrand competition (Maskin and Tirole 1988a), Cournot competition (Maskin and Tirole 1988b), or a coordination game (Lagunoff and Matsui 1997).*

A separable dynamic game with noisy transitions built from these primitives under a protocol of alternating moves is similar to the asynchronously repeated games in Maskin and Tirole (1988a, 1988b) and Lagunoff and Matsui (1997) in that changes in the payoff-relevant state do not occur at the same time.[11] *The key difference is that in Maskin and Tirole (1988a, 1988b) and Lagunoff and Matsui (1997) the player who has the move can change the state with probability one. We come back to this difference in Section 4.1.*

3 Protocol-Invariance Theorem

Consider the separable dynamic game with noisy transitions $\Gamma = <\Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho>$. We are interested in exploring how the set of Markov perfect equilibria $\text{Equil}(\Gamma)$ of the game Γ changes as we change the protocol of moves $<\mathcal{J}, \mathcal{P}>$.

We endow the set of all flow payoffs $u \in \mathbb{R}^{N \sum_{j=1}^{N} \sum_{\omega \in \Omega} |A_j(\omega)|}$ with the Lebesgue measure and say that a property is *generic* if it does not hold at most on a closed subset of measure zero. In this case we say that the property holds for *almost all* $u \in \mathbb{R}^{N \sum_{j=1}^{N} \sum_{\omega \in \Omega} |A_j(\omega)|}$.

The main result of the paper is a protocol-invariance theorem:

Theorem 1 (Protocol-Invariance Theorem) *Fix* p, q, *and* ρ. *For almost all* u, *all* $<\mathcal{J}, \mathcal{P}>$ *and* $<\overline{\mathcal{J}}, \overline{\mathcal{P}}>$, *and all* $\varepsilon > 0$, *there exists* $\bar{\Delta} > 0$ *such that for all* $\Delta < \bar{\Delta}$ *and* $\sigma \in \text{Equil}(<\Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho>)$, *there exists* $\overline{\sigma} \in \text{Equil}(<\Delta, \overline{\mathcal{J}}, \overline{\mathcal{P}}, u, p, q, \rho>)$ *such that* $\|\sigma - \overline{\sigma}\| < \varepsilon$.

In words, for any Markov perfect equilibrium σ of a game with a protocol of moves $<\mathcal{J}, \mathcal{P}>$, the game with another protocol $<\overline{\mathcal{J}}, \overline{\mathcal{P}}>$ has a Markov perfect equilibrium $\overline{\sigma}$ that is arbitrarily close to σ provided that periods are sufficiently short. Theorem 1 thus shows that the set of Markov perfect equilibria of separable dynamic games with noisy transitions is generically almost independent of the protocol of moves.

To begin establishing Theorem 1, consider a Markov perfect equilibrium $\sigma^{\Delta} = (\sigma_i^{\Delta})_{i=1}^{N}$ of the separable dynamic game with noisy transitions $\Gamma = <\Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho>$. Let $V_i^{\Delta}(\omega, J)$ be the continuation value of

[11]Similar to Maskin and Tirole (1988a, 1988b), we restrict attention to Markov perfect equilibra from the outset. Lagunoff and Matsui (1997) show that there is no loss in doing so in their model, as any outcome that can be sustained by a subgame perfect equilibrium can also be sustained by a Markov perfect equilibrium.

player i if players $J \in \mathcal{J}$ have the move and the state is $\omega \in \Omega$. The discrete-time Bellman equation for a period length of Δ is

$$V_i^\Delta(\omega, J) = u_i^\Delta(\omega, J, \sigma_J^\Delta(\omega)) + \exp(-\rho\Delta) \sum_{\omega' \in \Omega} \sum_{J' \in \mathcal{J}} V_i^\Delta(\omega', J') \Pr(J'|J) \Pr^\Delta\left(\omega'|\omega, J, \sigma_J^\Delta(\omega)\right),$$

where the player discounts payoffs accruing in the subsequent period by $\exp(-\rho\Delta)$ and $\sigma_J^\Delta(\omega) = \left(\sigma_j^\Delta(\omega)\right)_{j\in J}$. Under Assumptions 2 and 3 this becomes

$$\begin{aligned} V_i^\Delta(\omega, J) = |\mathcal{J}| \sum_{j\in J} u_{i,j}(\omega, \sigma_j^\Delta(\omega))\Delta + \exp(-\rho\Delta) \Bigg\{ & \sum_{J'\in\mathcal{J}} V_i^\Delta(\omega, J')\Pr(J'|J) \left(1 - |\mathcal{J}| \sum_{j\in J} q_j(\omega, \sigma_j^\Delta(\omega))\Delta\right) \\ & + \sum_{\omega'\neq\omega} \sum_{J'\in\mathcal{J}} V_i^\Delta(\omega', J')\Pr(J'|J) \left(|\mathcal{J}| \sum_{j\in J} \varphi_j(\omega'|\omega, \sigma_j^\Delta(\omega))\Delta\right) \Bigg\} + \mathcal{O}(\Delta^2), \end{aligned} \tag{3.1}$$

where we use the shorthand notation $\varphi_j(\omega' \mid \omega, a_j) = q_j(\omega, a_j) p_j(\omega'|\omega, a_j)$ and $\varphi_j(\omega' \mid \omega, \sigma_j(\omega)) = \sum_{a_j \in A_j(\omega)} \varphi(\omega' \mid \omega, a_j)\sigma_j(a_j \mid \omega)$.

Let $V^\Delta = \left(V_i^\Delta\right)_{i=1}^N$ be the profile of value functions corresponding to the Markov perfect equilibrium σ^Δ. Consider a sequence $(\sigma^\Delta, V^\Delta)$ indexed by the period length Δ. Assuming that $(\sigma^\Delta, V^\Delta) \to (\sigma^0, V^0)$ (where convergence is possibly through a subsequence Δ_n) and taking the limit of equation (3.1) as $\Delta \to 0$, we deduce that

$$V_i^0(\omega, J) = \sum_{J'\in\mathcal{J}} V_i^0(\omega, J')\Pr(J'|J).$$

Stacking this equation for all $J \in \mathcal{J}$ yields the system of linear equations $\mathcal{P}x = x$, where x is a $|\mathcal{J}|$-dimensional column vector with entries $V_i^0(\omega, J)$. Assumption 1 implies that $V_i^0(\omega, J) = V_i^0(\omega, J')$ for all $J, J' \in \mathcal{J}$.[12] This means that in equilibrium the continuation value of player i is almost independent of the identity of the players who have the move and equals $V_i^0(\omega)$: having the move does not imply a higher or lower payoff. From hereon, let $V_i^0 \colon \Omega \to \mathbb{R}$ be the value function of player i and $V^0 = (V_i^0)_{i=1}^N$ be the profile of value functions in the limit as $\Delta \to 0$.

Equation (3.1) can equivalently be written as

$$\begin{aligned} \frac{1}{\Delta} V_i^\Delta(\omega, J) - \frac{\exp(-\rho\Delta)}{\Delta} \sum_{J'\in\mathcal{J}} V_i^\Delta(\omega, J')\Pr(J'|J) = |\mathcal{J}| \sum_{j\in J} u_{i,j}(\omega, \sigma_j^\Delta(\omega)) \\ + \exp(-\rho\Delta)|\mathcal{J}| \sum_{j\in J} \left(\sum_{\omega'\neq\omega} \sum_{J'\in\mathcal{J}} V_i^\Delta(\omega', J')\Pr(J'|J)\varphi_j(\omega'|\omega, \sigma_j^\Delta(\omega)) - \sum_{J'\in\mathcal{J}} V_i^\Delta(\omega, J')\Pr(J'|J) q_j(\omega, \sigma_j^\Delta(\omega)) \right) + \mathcal{O}(\Delta). \end{aligned}$$

[12]The vector $y = (1, \dots, 1)'$ is always a right eigenvector since $\mathcal{P}$ is stochastic. Since $\mathcal{P}$ is irreducible, the Perron-Frobenious theorem implies that both the left and right eigenvectors associated to the eigenvalue 1 are unique, up to scalar multiplication. It follows that for any solution to the system $\mathcal{P}x = x$, $x_i = x_j$ for all i and j.

Summing this equation for all $J \in \mathcal{J}$ yields

$$\frac{1-\exp(-\rho\Delta)}{\Delta}\sum_{J\in\mathcal{J}} V_i^{\Delta}(\omega,J) = |\mathcal{J}|\sum_{J\in\mathcal{J}}\sum_{j\in J} u_{i,j}(\omega,\sigma_j^{\Delta}(\omega))$$

$$+\exp(-\rho\Delta)|\mathcal{J}|\sum_{J\in\mathcal{J}}\sum_{j\in J}\left(\sum_{\omega'\neq\omega}\sum_{J'\in\mathcal{J}} V_i^{\Delta}(\omega',J')\Pr(J'|J)\varphi_j(\omega'|\omega,\sigma_j^{\Delta}(\omega)) - \sum_{J'\in\mathcal{J}} V_i^{\Delta}(\omega,J')\Pr(J'|J)q_j(\omega,\sigma_j^{\Delta}(\omega))\right) + \mathcal{O}(\Delta^2),$$

where we use the fact that, under Assumption 1, $\sum_{J\in\mathcal{J}}\Pr(J'|J)=1$.[13] Taking the limit as $\Delta\to 0$ yields the continuous-time Bellman equation

$$\begin{aligned}\rho V_i^0(\omega) &= \sum_{J\in\mathcal{J}}\sum_{j\in J} u_{i,j}(\omega,\sigma_j^0(\omega)) + \sum_{J\in\mathcal{J}}\sum_{j\in J}\left(\sum_{\omega'\neq\omega} V_i^0(\omega')\varphi_j(\omega'|\omega,\sigma_j^0(\omega)) - V_i^0(\omega)q_j(\omega,\sigma_j^0(\omega))\right)\\ &= \sum_{j=1}^{N} u_{i,j}(\omega,\sigma_j^0(\omega)) + \sum_{j=1}^{N}\sum_{\omega'\neq\omega}\left(V_i^0(\omega')-V_i^0(\omega)\right)\varphi_j(\omega'|\omega,\sigma_j^0(\omega)),\end{aligned} \tag{3.2}$$

where the last equality uses that, under Assumption 1, there exists a unique $J\in\mathcal{J}$ such that $j\in J$ and the fact that $\sum_{\omega'\neq\omega} p_j(\omega'|\omega,a_i)=1$. Importantly, condition (3.2) is independent of the protocol of moves $<\mathcal{J},\mathcal{P}>$ used to pass from discrete to continuous time.

The discrete-time optimality condition for a period length of Δ is

$$\sigma^{\Delta}(a_i\mid\omega)>0 \Rightarrow$$
$$a_i\in\arg\max_{\tilde{a}_i\in A_i(\omega)} u_i^{\Delta}(\omega,J,\tilde{a}_i,\sigma_{J\setminus\{i\}}^{\Delta}(\omega)) + \exp(-\rho\Delta)\sum_{\omega'\in\Omega}\sum_{J'\in\mathcal{J}} V_i^{\Delta}(\omega',J')\Pr(J'|J)\Pr{}^{\Delta}\left(\omega'|\omega,J,\tilde{a}_i,\sigma_{J\setminus\{i\}}^{\Delta}(\omega)\right).$$

Since $\sigma^{\Delta}\to\sigma^0$, $\sigma_i^0(a_i\mid\omega)>0$ implies $\sigma_i^{\Delta}(a_i\mid\omega)>0$ if the period length Δ is sufficiently small. Dividing by Δ, rearranging terms, and taking the limit as $\Delta\to 0$ (as we did in the previous paragraph) thus yields the continuous-time optimality condition

$$\sigma_i^0(a_i\mid\omega)>0\Rightarrow a_i\in\arg\max_{\tilde{a}_i\in A_i(\omega)} u_{i,i}(\omega,\tilde{a}_i)+\sum_{\omega'\neq\omega}\left(V_i^0(\omega')-V_i^0(\omega)\right)\varphi_i(\omega'\mid\omega,\tilde{a}_i). \tag{3.3}$$

Condition (3.3) is again independent of the protocol of moves. It formalizes that player i faces the same tradeoff between current and future payoffs under any protocol of moves $<\mathcal{J},\mathcal{P}>$ and that this tradeoff is not directly affected by his rivals' actions.

The intuition is best seen by contrasting two protocols of moves in a separable dynamic game with noisy transitions. With alternating moves, if player i has the move, then to choose an action a_i he must consider the contribution $u_{i,i}(\omega,a_i)\Delta$ to his per-period payoff that his action yields and the impact his action

[13] Recall that if the transition matrix $\mathcal{P}$ is irreducible and its unique stationary distribution is uniform on $\mathcal{J}$, then $\mathcal{P}$ is doubly stochastic.

has on state-to-state transitions through $q_i(\omega, a_i)p_i(\omega'|\omega, a_i)\Delta$ (neglecting the higher-order term $\mathcal{O}(\Delta^2)$). With simultaneous moves, two additional considerations arise. First, player i must consider how his rivals' actions change the contribution to his per-period payoff that his action yields and the impact his action has on state-to-state transitions. However, because complementarities between players' actions and other non-separabilities in per-period payoffs and state-to-state transitions are restricted to the higher-order term $\mathcal{O}(\Delta^2)$, player i can neglect his rivals' actions if the period length Δ is sufficiently small. Second, player i must consider the possibility that his rivals' actions further change the state of the game. The probability that two or more players cause the state to change is, however, negligible if the period length Δ is sufficiently small. Assumption 1 finally ensures that irrespective of the protocol of moves all players move with the same frequency over a sufficiently large number of periods. Thus, in the limit as $\Delta \to 0$, player i faces the same tradeoff between current and future payoffs.

Conditions (3.2) and (3.3) are the limit as $\Delta \to 0$ of the equilibrium conditions for the separable dynamic game with noisy transitions Γ. We provide economic interpretations of these conditions in Section 3.1. Here we merely observe that they impose restrictions on the limit strategy and continuation value profiles (σ^0, V^0). Noting that the limit conditions (3.2) and (3.3) may admit multiple solutions and that V^0 is entirely determined by σ^0 using condition (3.2), we denote the set of strategy profiles $\sigma^0 \in \Sigma$ satisfying condition (3.3) as $\text{Equil}^0(< u, p, q, \rho >)$. This set does not depend on the protocol of moves $< \mathcal{J}, \mathcal{P} >$ used to pass to the limit.

We summarize the above discussion in a lemma:

Lemma 1 *Consider a sequence* (σ^Δ) *with* $\sigma^\Delta \in \text{Equil}(< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >)$. *If* $\sigma^\Delta \to \sigma^0$, *then* $\sigma^0 \in \text{Equil}^0(< u, p, q, \rho >)$.

Unfortunately, Lemma 1 does not imply Theorem 1 because conditions (3.2) and (3.3) may admit multiple solutions. In this case, taking the limit of a sequence of equilibria under different protocols of moves may potentially lead to different solutions of the limit conditions. To overcome this difficulty, we would ideally show that all solutions to the limit conditions (3.2) and (3.3) can be approximated by the Markov perfect equilibria of a separable dynamic game with noisy transitions and an arbitrary protocol of moves provided that periods are sufficiently short.

An example makes plain that this cannot always be done. Consider the entry game in Example 1 with $K = 1$ and $(c_i, b_i, B_i) = (c, b, B)$ for all i. We further restrict $\lambda B/\rho = c$ and $b < 0$ so that a duopolist incurs a loss. A firm's only nontrivial decision is whether to invest in state $(0, 0)$. Define the pure strategy profile σ^0 in which both firms invest by $\sigma_i^0(1 \mid (0,0)) = 1$. Let us verify that $\sigma^0 \in \text{Equil}^0(< u, p, q, \rho >)$. The limit conditions (3.2) and (3.3) become

$$\rho V_i^0(0,0) = -c + \lambda(\frac{B}{\rho} - V_i^0(0,0)) + \lambda(0 - V_i^0(0,0))$$

and

$$-c+\lambda(\frac{B}{\rho}-V_i^0(0,0))\geq 0, \tag{3.4}$$

where we use the fact that $V_i^0(\omega)=B/\rho$ when $\omega_i\neq\omega_{-i}=0$ and $V_i^0(\omega)=0$ when $\omega_i\neq\omega_{-i}=1$. We deduce that $V_i^0(0,0)=\frac{1}{\rho+2\lambda}(-c+\lambda\frac{B}{\rho})=0$ and σ^0 is indeed a solution to the limit conditions.

However, this solution cannot be approximated by a discrete-time game with simultaneous moves. In this game, a firm has an incentive to invest if and only if

$$-c+e^{-\rho\Delta}\Delta\lambda^2\frac{b\Delta}{1-e^{-\rho\Delta}}+e^{-\rho\Delta}\lambda(1-\lambda\Delta)\frac{B\Delta}{1-e^{-\rho\Delta}}\geq 0. \tag{3.5}$$

Because $\lambda B/\rho=c$ and $b<0$, this condition holds in the limit as $\Delta\to 0$ but not for $\Delta>0$.[14]

This example illustrates the problem we have to solve in establishing Theorem 1. While the expected net present value of the stream of payoffs in a discrete-time game is virtually independent of the protocol of moves and arbitrarily close to its continuous-time counterpart, equilibrium behavior is governed by differences in continuation values. The mere fact that continuation values converge does not ensure that the sign of these differences coincide in discrete and continuous time. In the example, condition (3.4) shows that in the limit as $\Delta\to 0$ the payoff from investing is greater than or equal to the payoff from not investing. Yet, in the discrete-time game the fact that condition (3.5) does not hold for $\Delta>0$ shows that the payoff from investing is less than the payoff from not investing.

We proceed as follows. To rule out the above example, we first restrict attention to solutions $\sigma^0\in\text{Equil}^0(<u,p,q,\rho>)$ that are *regular*. The formal definition of regularity is in the Appendix and allows for both pure and mixed strategy profiles.[15] We note that a pure strategy profile is regular if it is strict, i.e., if the maximization problem in condition (3.3) admits a unique solution. Our key insight is that for a regular solution σ^0 of the limit conditions (3.2) and (3.3), the differences in continuation values that govern equilibrium behavior have the same sign as their discrete-time counterparts under any protocol of moves. As a result, σ^0 can be approximated by a Markov perfect equilibrium of a separable dynamic game with noisy transitions and an arbitrary protocol of moves. We then use differential topology tools to establish that for almost all flow payoffs u, the restriction to regular solutions is without loss of generality. This yields the following lemma:

Lemma 2 *Fix p, q, and ρ. For almost all u, all $\sigma^0\in\text{Equil}^0(<u,p,q,\rho>)$, all $<\mathcal{J},\mathcal{P}>$, and all $\varepsilon>0$, there exists $\bar{\Delta}>0$ such that for all $\Delta<\bar{\Delta}$, there exists $\sigma\in\text{Equil}(<\Delta,\mathcal{J},\mathcal{P},u,p,q,\rho>)$ such that $||\sigma-\sigma^0||<\varepsilon$.*

Lemmas 1 and 2 finally combine to yield Theorem 1.

[14] This solution also cannot be approximated by a discrete-time game with alternating moves. In the Online Appendix, we complement this example by showing that the solution can be approximated by a slightly modified discrete-time game with alternating moves but not with simultaneous moves.

[15] In the Online Appendix we provide an example showing that the limit conditions (3.2) and (3.3) in general may not admit a solution in pure strategies.

The proof of Lemma 2 draws on and expands ideas in Harsanyi (1973a, 1973b). Existing genericity results for dynamic stochastic games (Haller and Lagunoff 2000, Doraszelski and Escobar 2010) do not apply to our setting because the limit conditions (3.2) and (3.3) are in continuous time and separable dynamic games with noisy transitions restrict per-period payoffs and state-to-state transitions and are therefore a subset of measure zero of the dynamic stochastic games considered in the literature.[16]

3.1 Interpretations of Limit Conditions

We offer two economic interpretations of the limit conditions (3.2) and (3.3). First, they can be interpreted as the equilibrium conditions for a continuous-time stochastic game along the lines of Doraszelski and Judd (2012). In this game, moves are simultaneous and the state follows a continuous-time Markov process that players can influence through their actions. Properly defining mixed strategies in continuous time is, however, subtle because it requires working with a continuum of independent and identically distributed random variables that satisfy a law of large numbers. As in Bolton and Harris (1999), we can use time to "purify" these strategies and avoid the continuum of independent and identically distributed random variables. Beyond this observation, we follow the literature and alert the reader that a rigorous foundation for mixed strategies in continuous time is an open problem (Bolton and Harris 1999, Faingold and Sannikov 2011).

Second, the limit conditions (3.2) and (3.3) can be interpreted as the equilibrium conditions for a dynamic stochastic game with random moves. The following construction, known as uniformization (Serfozo 1979), is adapted from single-agent decision problems. Fix any $B > N \max_{j=1,\dots,N,\omega\in\Omega,a_j\in A_j(\omega)} q_j(\omega, a_j)$. Define the per-period payoff $\widetilde{u}_{i,j}(\omega, a_j) = \frac{N}{\rho+B} u_{i,j}(\omega, a_j)$, the discount factor $\beta = \frac{B}{\rho+B} < 1$, and the transition probability

$$\widetilde{\varphi}_j(\omega' \mid \omega, a_j) = \begin{cases} \frac{N}{B}\varphi_j(\omega' \mid \omega, a_j) & \text{if} \quad \omega' \neq \omega, \\ 1 - \frac{N}{B} q_j(\omega, a_j) & \text{if} \quad \omega' = \omega. \end{cases}$$

Note that $\widetilde{\varphi}_j(\cdot \mid \omega, a_j) \in \mathsf{P}(\Omega)$ by construction of B. Now formulate a dynamic stochastic game with random moves in which in any period one player $j \in \{1, \dots, N\}$ is randomly and uniformly selected to make a decision $a_j \in A_j(\omega)$. Each player strives to maximize the expected net present value of his stream of payoffs and discounts future payoffs using the discount factor β. Denote by $\text{Equil}^R(\tilde{u}, \tilde{\varphi}, \beta)$ the set of Markov perfect equilibria of this game.

The following proposition shows that the Markov perfect equilibria of the dynamic stochastic game with random moves constructed above are the solutions of the limit conditions (3.2) and (3.3):

Proposition 1 $\text{Equil}^R(<\tilde{u}, \widetilde{\varphi}, \beta>) = \text{Equil}^0(<u, p, q, \rho>)$.

The proof of Proposition 1 is simple and illustrative. The equilibrium conditions for $\sigma \in \text{Equil}^R(<$

[16] Our proof also shows that $\text{Equil}^0(<u, p, q, \rho>)$ consists of a finite number of isolated solutions. This generalizes results on the generic finiteness of the set of Markov perfect equilibria in Haller and Lagunoff (2000) and Doraszelski and Escobar (2010) to continuous-time stochastic games.

$\tilde{u}, \widetilde{\varphi}, \beta >$) are

$$V_i(\omega) = \sum_{j=1}^{N} \frac{1}{N} \left(\widetilde{u}_{i,j}(\omega, \sigma_j(\omega)) + \beta \sum_{\omega' \in \Omega} V_i(\omega) \widetilde{\varphi}_j(\omega' \mid \omega, \sigma_j(\omega)) \right) \tag{3.6}$$

and

$$\sigma_i(a_i \mid \omega) > 0 \Rightarrow a_i \in \arg \max_{\tilde{a}_i \in A_i(\omega)} \widetilde{u}_{i,i}(\omega, \tilde{a}_i) + \beta \sum_{\omega' \in \Omega} V_i(\omega') \widetilde{\varphi}_i(\omega' \mid \omega, \tilde{a}_i). \tag{3.7}$$

These conditions can be equivalently written as

$$V_i(\omega) = \sum_{j=1}^{N} \frac{1}{N} \left\{ \frac{N}{\rho + B} u_{i,j}(\omega, \sigma_j(\omega)) + \frac{B}{\rho + B} \left(\sum_{\omega' \neq \omega} V_i(\omega') \frac{N}{B} \varphi_j(\omega' \mid \omega, \sigma_j(\omega)) + (1 - \frac{N}{B} q_j(\omega, \sigma_j(\omega)) V_i(w)) \right) \right\}$$

and

$$\sigma_i(a_i \mid \omega) > 0 \Rightarrow a_i \in \arg \max_{\tilde{a}_i \in A_i(\omega)} \frac{N}{\rho + B} u_{i,i}(\omega, \tilde{a}_i) + \frac{B}{\rho + B} \left(\sum_{\omega' \neq \omega} V_i(\omega') \frac{N}{B} \varphi_i(\omega' \mid \omega, \tilde{a}_i) + (1 - \frac{N}{B} q_i(\omega, \tilde{a}_i) V_i(w)) \right).$$

Rearranging terms, the limit conditions (3.2) and (3.3) are therefore identical to the equilibrium conditions (3.6) and (3.7) for the dynamic stochastic game with random moves constructed above.

While dynamic stochastic games with random moves are sparsely used, several important papers study repeated games with alternating moves. For example, Maskin and Tirole (1988a) explore a repeated Bertrand game with alternating moves and show how Edgeworth cycles can arise. Maskin and Tirole (1988b) show that an analog to limit pricing can arise in a model of repeated Cournot competition with alternating moves and large fixed costs. Lagunoff and Matsui (1997) show how players can coordinate on the efficient outcome in a repeated coordination game with alternating moves. These results are driven by the fact that a player remains committed to his previously chosen action over a stretch of time.[17] The dynamic stochastic game with random moves constructed above shares this feature. Similarly rich dynamic phenomena thus appear in the continuous-time stochastic game that we obtain as we pass to the limit and, by Theorem 1, in separable dynamic games with noisy transitions and arbitrary protocols of moves provided that periods are sufficiently short.

3.2 Discussion of Assumptions

To illustrate the tightness of our assumptions, we provide a series of examples showing that protocol invariance may fail if any one of them is relaxed.

Example 6 (Separability) *The literature provides a number of examples in which complementarities between players' actions and other non-separabilities in per-period payoffs preclude protocol invariance. Our*

[17] As Lagunoff and Matsui (1997) point out, what matters is that moves are asynchronous "rather than the specific structure of asynchronous choice" (p. 1473).

example with non-separable per-period payoffs is inspired by Lagunoff and Matsui (1997) and Wen (2002). In the Online Appendix we present a closely related example with non-separable state-to-state transitions.

Consider a coordination game with the following payoff matrix:

	L	R
T	$2,2$	$0,0$
B	$0,0$	$1,1$

Denote by $b(a) = (b_1(a), b_2(a))$ the payoff profile given the action profile $a = (a_1, a_2) \in \{T, B\} \times \{L, R\}$.

We construct a dynamic stochastic game with a trivial state space $|\Omega| = 1$ (that we omit along with specifying the transition probability) and contrast the set of Markov perfect equilibria under simultaneous and alternating moves. In the game with simultaneous moves, the per-period payoff of player i is $u_i^\Delta(\omega, \{1,2\}, a) = b_i(a)\Delta$. Irrespective of the period length Δ, there are two Markov perfect equilibria, namely $\sigma_1(T) = \sigma_2(L) = 1$ and $\tilde{\sigma}_1(B) = \sigma_2(R) = 1$.

In the game with alternating moves, in violation of Assumption 3 the per-period payoff of player i is $u_i^\Delta(\omega, \{1\}, a_1) = 0$ and $u_i^\Delta(\omega, \{2\}, a) = b_i(a)\Delta$, meaning that payoffs "materialize" after player 2 moves. Since player 1's action a_1 is payoff relevant for player 2, a Markovian strategy for player 2 includes a_1 as a state variable. Irrespective of the period length Δ, the unique Markov perfect equilibrium is $\sigma_1(T) = 1$ and $\sigma_2(T \mid L) = \sigma_2(B \mid R) = 1$.

Example 7 (Noisy Transitions) *Consider the entry game in Example 1 with $K = 1$. We further restrict $b_i < 0$ so that a duopolist incurs a loss.*

We change Example 1 by assuming that if firm i takes action $a_i = 1$, then its state changes for sure from $\omega_i = 0$ to $\omega_i' = 1$. Irrespective of the protocol of moves, given a set of players $J \subseteq \{1,2\}$ who have the move, the transition probability takes the form

$$\Pr^\Delta(\omega' \mid \omega, J, a_J) = \begin{cases} 1 & \textit{if} \quad \omega_J' = a_J, \omega = (0,0), \\ 1 & \textit{if} \quad \omega' = \omega, \omega_i = 1 \textit{ for some } i, \\ 0 & \textit{otherwise}, \end{cases}$$

and does not satisfy Assumption 2. In the game with alternating moves, if Δ is sufficiently small, then the unique Markov perfect equilibrium outcome is that the firm that moves first enters whereas its rival never enters. In contrast, in the game with simultaneous moves, there exists a Markov perfect equilibrium in which both firms enter with positive probability.

Finally, we show that protocol invariance does not extend beyond Markov perfect equilibria to more general equilibrium concepts. For a separable dynamic game with noisy transitions $\Gamma =< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >$, we say that a strategy σ_i^T for player i has *finite memory* $T \geq 0$ if $\sigma_i^T(h) = \sigma_i^T(\tilde{h})$ for any two histories h and $\tilde{h}$ (perhaps of different length) that coincide in the current state and the outcomes of the previous T rounds of interactions between players. If $T = 0$, then we recover the definition of a stationary Markovian strategy in Section 2.

Example 8 (Markov Perfect Equilibrium) *Consider a partnership game and construct a separable dynamic game with noisy transitions and a trivial state space (that we again omit). There are $N = 2$ players. The set of actions of player i is $A_i = \{0, 1\}$, his flow payoff is*

$$u_{i,j}(a_j) = \begin{cases} -a_j & \text{if} \quad j = i, \\ 2a_j & \text{if} \quad j \neq i, \end{cases}$$

and the discount rate is ρ. Irrespective of the protocol of moves $< \mathcal{J}, \mathcal{P} >$ and the period length Δ, the unique Markov perfect equilibrium of this game is $\sigma_1(0) = \sigma_2(0) = 1$ and has players repeating $(0, 0)$.

We show that this not the case for strict subgame perfect equilibria in finite memory strategies. In the game with simultaneous moves, consider a finite memory strategy σ_i^T with $T \geq 1$ for player i such that player i chooses $a_i^t = 1$ in period t if $t = 0$ or if the players have chosen the same action over the last $\min\{T, t\}$ rounds: $a_1^{\tilde{t}} = a_2^{\tilde{t}}$ for all $\tilde{t} \in \{t-1, \ldots, t - \min\{t, T\}\}$. The strategy profile $\sigma^T = (\sigma_1^T, \sigma_2^T)$ is a strict subgame perfect equilibrium if $\frac{1-e^{-\rho\Delta T}}{1-e^{-\rho\Delta}} > e^{\rho\Delta}$. This condition holds if $T \geq 2$ and the period length Δ is sufficiently small. Hence, there exists a strict subgame perfect equilibrium in finite memory strategies in which players repeatedly play $(1, 1)$.

Turning to the game with alternating moves, consider a finite memory strategy σ_i^T with $T \geq 1$ for player i. We argue that for any strategy profile σ^T to be a strict subgame perfect equilibrium it must be a Markov perfect equilibrium. Hence, the unique strict subgame perfect equilibrium in finite memory strategies is the Markov perfect equilibrium in which players repeat $(0, 0)$.

To complete the argument, suppose player i moves in round t. Because player $-i$ moves after player i and conditions his decision on the previous T periods of interactions, the continuation value of player i depends on a_i^t and the previous $T-1$ periods of interactions. The current payoff of player i moreover depends only on a_i^t. Since σ^T is a strict subgame perfect equilibrium, the maximization problem of player i admits a unique solution which depends, at most, on the previous $T-1$ rounds of interactions. This means that σ_i^T actually conditions on the previous $T-1$ periods of interactions. Continuing iteratively, we deduce that the strategy profile σ^T cannot condition on any previous interactions.

4 Applications and Extensions

We apply and extend our main result in three ways. We first provide an extension to arbitrarily large hazard rates. Then we offer a new rationale for focusing on Markov perfect equilibria and discuss computing these equilibria.

4.1 Protocol Invariance with Arbitrarily Large Hazard Rates

We modify the model in Section 2 by assuming that hazard rates are of form $\tilde{q}_i(\omega, a_i) = \lambda q_i(\omega, a_i)$, where $\lambda \geq 1$ is a parameter. As $\lambda \to \infty$, hazard rates become arbitrarily large. We explore how $\text{Equil}^0(< u, p, \tilde{q}, \rho >)$ changes in response.

Consider $\sigma^{0,\lambda} \in \text{Equil}^0(< u, p, \tilde{q}, \rho >)$ and assume that $\sigma^{0,\lambda} \to \sigma^{0,\infty}$ as $\lambda \to \infty$. Taking the limit of condition (3.2) yields the Bellman equation

$$\rho V_i^{0,\lambda}(\omega) = \sum_{j=1}^{N} u_{i,j}(\omega, \sigma_j^{0,\lambda}(\omega)) + \sum_{j=1}^{N} \sum_{\omega' \neq \omega} \left(V_i^{0,\lambda}(\omega') - V_i^{0,\lambda}(\omega) \right) \lambda \varphi_j(\omega' | \omega, \sigma_j^{0,\lambda}(\omega)),$$

where $\varphi_j(\omega'|\omega, a_j) = q_j(\omega, a_j) p_j(\omega'|\omega, a_j)$. Fix a state $\omega^0 \in \Omega$ and define the function $h_i^\lambda(\omega) = \lambda(V_i^{0,\lambda}(\omega) - V_i^{0,\lambda}(\omega_0))$. Assuming that $h_i^\lambda(\omega) \to h_i(\omega)$ for some $h_i \colon \Omega \to \mathbb{R}$, the Bellman equation becomes

$$v_i = \sum_{j=1}^{N} u_{i,j}(\omega, \sigma_j^{0,\infty}(\omega)) + \sum_{j=1}^{N} \sum_{\omega' \neq \omega} \left(h_i(\omega') - h_i(\omega) \right) \varphi_j(\omega' | \omega, \sigma_j^{0,\infty}(\omega)), \tag{4.1}$$

where $v_i \in \mathbb{R}$ does not depend on ω.[18] Analogously, taking the limit of condition (3.3) as $\lambda \to \infty$ yields the optimality condition

$$\sigma_i^{0,\infty}(a_i \mid \omega) > 0 \Rightarrow a_i \in \arg\max_{\tilde{a}_i \in A_i(\omega)} u_{i,i}(\omega, \tilde{a}_i) + \sum_{\omega' \neq \omega} \Big(h_i(\omega') - h_i(\omega) \Big) \varphi_i(\omega' \mid \omega, \tilde{a}_i). \tag{4.2}$$

Conditions (4.1) and (4.2) extend conditions (3.2) and (3.3) to arbitrarily large hazard rates and characterize the solutions to the limit conditions.

The following proposition summarizes the discussion:

Proposition 2 (Protocol Invariance with Arbitrarily Large Hazard Rates) *Consider a sequence $\left(\sigma^{0,\lambda}\right)$ with $\sigma^{0,\lambda} \in \text{Equil}^0(< u, p, \tilde{q}, \rho >)$. Assume $\sigma^{0,\lambda} \to \sigma^{0,\infty}$ as $\lambda \to \infty$. Fix $\omega^0 \in \Omega$ and assume that, for all i, $\lambda(V_i^{0,\lambda}(\omega) - V_i^{0,\lambda}(\omega_0))$ is uniformly bounded.*[19] *Then, for all i, there exists $h_i \colon \Omega \to \mathbb{R}$ and $v_i \in \mathbb{R}$, such that $\sigma^{0,\infty}$ satisfies conditions (4.1) and (4.2).*

Proposition 2 extends protocol invariance to the limiting case of deterministic transitions and provides a novel dynamic programming characterization of separable dynamic games with noisy transitions as moves become arbitrarily frequent and hazard rates arbitrarily large.[20]

[18] To see this, note that for all $\epsilon > 0$, there exists $\bar{\lambda}$ such that for all $\lambda > \bar{\lambda}$, $|V_i^\lambda(\omega) - V_i^\lambda(\omega^0) - h(\omega)/\lambda| < \epsilon/\lambda$. As a result, $\lim_{\lambda\to\infty} V_i^\lambda(\omega) = \lim_{\lambda\to\infty} V_i^\lambda(\omega^0)$.

[19] This type of condition appears in dynamic programming problems without discounting (Arapostathis, Borkar, Fernández-Gaucherand, Ghosh, and Marcus 1993). It is simple to check in applications. To provide a sufficient condition, fix a strategy profile σ and define $\tau = \inf\{t \mid \omega^t = \omega_0\}$. If $\mathbb{E}_\sigma[\tau \mid \omega^{t=0} = \omega]$ is finite for all $\omega \in \Omega$, all $\sigma \in \Sigma$, and all λ sufficiently large, then $\lambda(V_i^{0,\lambda}(\omega) - V_i^{0,\lambda}(\omega_0))$ is uniformly bounded.

[20] This result contributes to the well-known literature drawing often subtle connections between discrete- and continuous-time

Given our previous effort in Example 7 to show that protocol invariance fails if transitions are deterministic, Proposition 2 may seem puzzling. Example 7 and Proposition 2 can be reconciled by noting that there is a discontinuity in the set of Markov perfect equilibria as moves become arbitrarily frequent and hazard rates arbitrarily large.

Many examples from the extant literature where equilibrium behavior hinges on the protocol of moves even if periods are short (as in Maskin and Tirole 1988a, 1988b and Lagunoff and Matsui 1997) can be seen as a manifestation of this discontinuity. To illustrate, consider the asynchronously repeated game in Example 5 with $\Omega_1 = \{T, B\}$, $\Omega_2 = \{L, R\}$, and $s_i\colon \Omega \to \mathbb{R}$ given by the following payoff matrix:

	L	R
T	$2, 2$	$0, 0$
B	$0, 0$	$1, 1$

This is a repeated coordination game in the spirit of Lagunoff and Matsui (1997) and the specific example used by Mailath and Samuelson (2006, Section 5.4.5) to illustrate their results.

In line with Proposition 2 but in stark contrast to Maskin and Tirole (1988a, 1988b) and Lagunoff and Matsui (1997), protocol invariance arises. It is relatively easy to see that

$$\lim_{\lambda\to\infty} \text{Equil}^0(< u, p, \tilde{q}, \rho >) = \lim_{\lambda\to\infty} \lim_{\Delta\to 0} \text{Equil}(< \Delta, \mathcal{J}, \mathcal{P}, u, p, \tilde{q}, \rho >) = \{\sigma^*\}$$

with $\sigma_1^*(\omega) = T$ and $\sigma_2^*(\omega) = L$ for all ω. With arbitrarily frequent moves and arbitrarily large hazard rates, players therefore coordinate on the efficient state (T, L) regardless of the protocol of moves that the game assumes.[21]

To expose the discontinuity in the set of Markov perfect equilibria, consider a protocol of simultaneous moves $< \mathcal{J}^{sim}, \mathcal{P}^{sim} >$. Imposing $\lambda\Delta = 1$, players determine the state with probability one when they move. It is relatively simple to show that

$$\text{Equil}(< \Delta, \mathcal{J}^{sim}, \mathcal{P}^{sim}, u, p, q, \rho >) = \{\sigma^*, \sigma'\},$$

where $\sigma_1'(\omega) = B$ and $\sigma_2'(\omega) = R$ for all ω. Intuitively, if $\lambda\Delta = 1$, then players can coordinate on one of the two Nash equilibria (T, L) and (B, R). We conclude that

$$\lim_{\lambda\to\infty} \lim_{\Delta\to 0} \text{Equil}(< \Delta, \mathcal{J}, \mathcal{P}, u, p, \tilde{q}, \rho >) \subsetneq \liminf_{\lambda\Delta=1, \Delta\to 0} \text{Equil}(< \Delta, \mathcal{J}^{sim}, \mathcal{P}^{sim}, u, p, \tilde{q}, \rho >)$$

stochastic games with infinite hazard rates (Fudenberg and Tirole 1985, Simon and Stinchcombe 1989). Fudenberg and Tirole (1985), in particular, show that passing to the limit is non-trivial in games with infinite hazard rates even if strategies are restricted to be Markovian. Proposition 2, in contrast, provides a quite tractable model of a dynamic game with arbitrarily frequent moves and arbitrarily large hazard rates.

[21] Up to the fact that the limit as $\Delta \to 0$ is a continuous-time stochastic game, the logic of this result follows from Lagunoff and Matsui (1997). Note that if player 1's state is $\omega_1 = T$, then player 2 has an incentive to choose $a_2 = L$ to obtain $s(T, L) = 2$. In a given state ω, player 1 thus knows that if his state changes to T, then his rival will switch to $a_2 = L$ relatively soon as long as λ is sufficiently large.

and thus that there is a discontinuity in the joint limit as moves become arbitrarily frequent and hazard rates arbitrarily large.

Moreover, the limit of the set of Markov perfect equilibria as $\Delta \to 0$ keeping $\lambda\Delta$ constant depends on the protocol of moves. Consider a protocol of alternating moves $< \mathcal{J}^{alt}, \mathcal{P}^{alt} >$. Imposing $\lambda\Delta = 1/2$, a player determines his state with probability one when he moves.[22] From Theorem 1 in Lagunoff and Matsui (1997), keeping $\lambda\Delta = 1/2$ and taking the period length Δ sufficiently small,[23]

$$\text{Equil}(< \Delta, \mathcal{J}^{alt}, \mathcal{P}^{alt}, u, p, \tilde{q}, \rho >) = \{\sigma^*\}.$$

This implies that

$$\lim_{\lambda\Delta=1/2,\Delta\to 0} \text{Equil}(< \Delta, \mathcal{J}^{alt}, \mathcal{P}^{alt}, u, p, \tilde{q}, \rho >) \subsetneq \liminf_{\lambda\Delta=1,\Delta\to\infty} \text{Equil}(< \Delta, \mathcal{J}^{sim}, \mathcal{P}^{sim}, u, p, \tilde{q}, \rho >).$$

In this sense, we can replicate the results in Lagunoff and Matsui (1997) that the protocol matters when moves are frequent by taking the joint limit $\lambda\Delta \to 1$ and $\Delta \to 0$, but this is just one of many ways of taking the joint limit in our setting. Proposition 2, in contrast, shows that protocol invariance arises if we first take moves to be arbitrarily frequent and then take hazard rates to be arbitrarily large.

4.2 Justification of Markov Perfect Equilibria

We apply our main result to provide a new justification for focusing on Markov perfect equilibria in a class of dynamic stochastic games. Provided that periods are sufficiently short and a robustness requirement is imposed, we show that the set of Markov perfect equilibrium payoffs in separable dynamic games with noisy transitions and simultaneous moves almost coincides with the set of payoffs that can be attained under more general equilibrium concepts.

We focus on strict subgame perfect equilibria in finite memory strategies. By definition, a strict equilibrium involves only pure strategies. Strictness is a natural robustness requirement. In repeated public monitoring games only strict subgame perfect equilibria in finite memory strategies are robust to private monitoring (Mailath and Morris 2002, Mailath and Samuelson 2006, Bhaskar, Mailath, and Morris 2013). Equilibria that fail to be strict are also fragile to perturbations of payoffs and information (Harsanyi 1973a, Harsanyi 1973b, Doraszelski and Escobar 2010).

As we change the protocol of moves $< \mathcal{J}, \mathcal{P} >$ of a separable dynamic game with noisy transitions $\Gamma =< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >$, the sets of histories change and are therefore difficult to compare. To circumvent this difficulty, we explore how the set of payoff profiles $\text{Payoffs}^F(\Gamma) \subseteq \mathbb{R}^N$ associated with strict subgame

[22] Recall from Assumption 3 that the hazard rates are scaled by $|\mathcal{J}|$. This is the reason we take $\lambda\Delta = 1$ with simultaneous moves and $\lambda\Delta = 1/2$ with alternating moves.

[23] The restriction to a discount factor close to one in Lagunoff and Matsui (1997) translates into a period length Δ close to zero in our setting with $\lambda\Delta$ fixed.

perfect equilibria in finite memory strategies changes as we change the protocol of moves. We also define the set of payoff profiles $\text{Payoffs}^M(\Gamma) \subseteq \mathbb{R}^N$ corresponding to the set of Markov perfect equilibria $\text{Equil}(\Gamma)$.

Let $\Gamma^{sim} =< \Delta, \mathcal{J}^{sim}, \mathcal{P}^{sim}, u, p, q, \rho >$ denote a separable dynamic game with noisy transitions under a protocol of simultaneous moves $< \mathcal{J}^{sim}, \mathcal{P}^{sim} >$, with $\mathcal{J}^{sim} = \{\{1, \ldots, N\}\}$. We say that the payoff profile $v \in \text{Payoffs}^F(\Gamma^{sim})$ is *approachable* if for all $\epsilon > 0$ there exists some protocol of asynchronous moves $< \mathcal{J}^{asy}, \mathcal{P}^{asy} >$, with $\mathcal{J}^{asy} = \{\{1\}, \{2\}, \ldots, \{N\}\}$, and a payoff profile $w \in \text{Payoffs}^F(< \Delta, \mathcal{J}^{asy}, \mathcal{P}^{asy}, u, p, q, \rho >)$ such that $\|v - w\| < \epsilon$. In words, focusing on strict subgame perfect equilibria in finite memory strategies, an equilibrium payoff profile of the game with simultaneous moves is approachable if there exists a nearby equilibrium payoff profile of the game for some asynchronous protocol of moves.

The following proposition shows that an approachable equilibrium payoff profile of the game with simultaneous moves almost coincides with a payoff profile corresponding to a Markov perfect equilibrium provided that periods are sufficiently short:

Proposition 3 *Fix p, q, and ρ. For almost all u, and all $\epsilon > 0$, there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, if $v \in \text{Payoffs}^F(\Gamma^{sim})$ is approachable, then there exists $w \in \text{Payoffs}^M(\Gamma^{sim})$ such that $\|v - w\| < \epsilon$.*

Proposition 3 implies that there is no loss in restricting attention to Markov perfect equilibria in separable dynamic games with noisy transitions and simultaneous moves and thus provides a rationale for doing so.

To prove Proposition 3, we build on related results for dynamic stochastic games with asynchronous moves by Bhaskar and Vega-Redondo (2002) and Bhaskar, Mailath, and Morris (2009) and combine them with our Theorem 1. The proof of Proposition 3 draws on the insight from Example 8 that although some payoff profiles can be attained with strict subgame perfect equilibria in finite memory strategies when moves are simultaneous, these payoff profiles cannot be attained when moves are alternating.

Proposition 3 complements several arguments in favor of Markov perfect equilibria given for a variety of dynamic models (Maskin and Tirole 2001, Bhaskar and Vega-Redondo 2002, Sannikov and Skrzypacz 2007, Faingold and Sannikov 2011, Bhaskar, Mailath, and Morris 2013, Bohren 2014). Approachability is conceptually similar to purifiability in Bhaskar, Mailath, and Morris (2013) in that both are robustness requirements: approachability says that equilibrium payoffs should survive changes in the protocol of moves, whereas purifiability says that equilibrium strategies should survive the introduction of private information. We show that only Markov perfect equilibria are approachable in our separable dynamic games with noisy transitions and simultaneous moves, whereas Bhaskar, Mailath, and Morris (2013) show that only Markov perfect equilibria are purifiable in dynamic stochastic games with asynchronous moves.

Proposition 3 also limits possible extensions of Theorem 1. By showing that an equilibrium payoff that is robust to alternative specifications of the protocol of moves must be a Markov perfect equilibrium payoff, Proposition 3 implies that the assumption of Markov perfection is not only sufficient (as shown in Theorem 1) but also necessary for protocol invariance.[24]

[24]Note, however, that Proposition 3 applies only when strategies have finite memory. In the tightly specified model in Example 8, under arbitrary protocols of moves there exists a subgame perfect equilibrium with unbounded memory in which $a_i^t = 1$ for all

4.3 Computation of Markov Perfect Equilibria

Dynamic stochastic games are often not very tractable analytically and thus call for the use of numerical methods. Our main result has a number of implications for computing Markov perfect equilibria.

First, Doraszelski and Judd (2007) show that the computational burden can vary by orders of magnitude with the protocol of moves. For the class of separable dynamic games with noisy transitions, Theorem 1 justifies imposing the protocol of moves that is most convenient from a computational perspective.

Second, Doraszelski and Judd (2012) contrast the burden of computing Markov perfect equilibria in discrete- and continuous-time stochastic games with simultaneous moves. They argue that, under widely used laws of motion for the evolution of the state, computing the expectation over successor states ω' in a continuous-time stochastic game does not suffer from the curse of dimensionality that plagues the discrete-time stochastic game, and that this can reduce the computational burden by orders of magnitude. While passing from discrete to continuous time is computationally advantageous, a natural question is if this changes the nature of the strategic interactions among players. Theorem 1 answers this question for the class of separable dynamic games with noisy transitions. Moreover, the techniques we develop allow us to more broadly establish a tight link between discrete- and continuous-time stochastic games even in the absence of Assumption 3.

Consider a dynamic stochastic game with noisy transitions and simultaneous moves. The per-period payoff is $u_i^\Delta(\omega, \{1,\dots,N\}, a) = u_i(\omega, a)\Delta + \mathcal{O}(\Delta^2)$. The probability that the state changes is $q(\omega, a)\Delta$ in proportion to the length of a period Δ and, conditional on the state changing, the probability that it changes from ω to ω' is $p(\omega'|\omega, a)$. Hence, while Assumptions 1 and 2 are satisfied, Assumption 3 is not. Overloading notation, let $\text{Equil}(\langle \Delta, u, p, q, \rho\rangle)$ be the set of Markov perfect equilibria of this game and $\text{Equil}^0(\langle u, p, q, \rho\ \rangle)$ the set of solutions to the analog of the limit conditions (3.2) and (3.3).[25]

Proposition 4 *Fix p, q, and ρ. For almost all u,* $\lim_{\Delta\to 0} \text{Equil}(\langle \Delta, u, p, q, \rho\rangle) = \text{Equil}^0(\langle u, p, q, \rho\rangle)$.

In words, provided that periods are sufficiently short the Markov perfect equilibria of the discrete-time stochastic game with simultaneous moves almost coincide with those of the continuous-time stochastic game, although the latter are much easier to compute than the former. We note that Proposition 4 does not carry over from simultaneous to alternating moves. We also note that with a continuum of actions, a version of Proposition 4 (and of Theorem 1) can be obtained by considering approximate equilibria as in Fudenberg and Levine (1986).

i and after all on-path histories. Without restrictions on strategies, the properties of the set of equilibrium payoffs as $\Delta \to 0$ are generally not well understood in the literature. The existing results consider either the limit $\Delta \to 0$ with simultaneous moves (Peski and Wiseman 2015) or the limit $\rho \to 0$ (Dutta 1995, Yoon 2001, Hörner, Sugaya, Takahashi, and Vieille 2011).

[25] Given a sequence (A_ν) indexed by $\nu \in \mathbb{N}$, with $A_\nu \subseteq \mathbb{R}^n$, we define

$$\lim\inf_{\nu\to\infty} A_\nu = \{x \in \mathbb{R}^n \mid \limsup_{\nu\to\infty} d(x, A_\nu) = 0\} \quad \text{and} \quad \limsup_{\nu\to\infty} A_\nu = \{x \in \mathbb{R}^n \mid \liminf_{\nu\to\infty} d(x, A_\nu) = 0\},$$

where $d(x, A) = \inf\{\|y - x\| \mid y \in A\}$. If both limits coincide, we denote their common value by $\lim_{\nu\to\infty} A_\nu$.

5 Conclusions

The timing of decisions is an essential ingredient into modeling many strategic situations. Yet, determining the protocol of moves that is most realistic and appropriate for the application at hand can be challenging. While the literature abounds with examples where the protocol of moves matters crucially for equilibrium behavior, our paper is a first attempt to show that the implications and predictions of a fairly general and widely used class of dynamic models are independent of the timing of decisions and thus more robust for the purposes of applied work.

We introduce separable dynamic games with noisy transitions and establish that they are protocol invariant provided that periods are sufficiently short and moves are therefore sufficiently frequent. Separable dynamic games with noisy transitions are well-suited for situations in which a player primarily influences his rivals' payoffs by taking action to change the state and there is some residual uncertainty if the taken action brings about such a change. A particular highlight of this class of dynamic stochastic games is that per-period payoffs and state-to-state transitions can depend arbitrarily on the state. We show that investment games, R&D races, models of industry dynamics, dynamic public contribution games, the recent continuous-time stochastic games with moves at random times, asynchronously repeated games, and many other models from the extant literature can be cast as special cases of separable dynamic games with noisy transitions.

In addition to alleviating the burden of determining the most realistic and appropriate protocol of moves, our main result and its extensions have a number of implications for applied work. They provide a new justification for focusing on Markov perfect equilibria in dynamic stochastic games and facilitate computing these equilibria. They further clarify a driving force behind some of the well-known and important examples in the literature where equilibrium behavior hinges on the protocol of moves. Many of these models, including entry games and the asynchronously repeated Bertrand, Cournot, and coordination games in Maskin and Tirole (1988a, 1988b) and Lagunoff and Matsui (1997), are separable but transitions are assumed to be deterministic. If hazard rates are instead finite, the protocol of moves ceases to matter.

Appendix

This Appendix consists of two parts. Appendix A.1 provides the proof of Theorem 1 and Appendix A.2 provides the proofs for Section 4.

A.1 Proof of Theorem 1

A.1.1 Notation and Preliminary Definitions

Enumerate the state space as $\Omega = \{\omega^1, \dots, \omega^{|\Omega|}\}$ and the set of actions for player i as $A_i(\omega) = \left\{a_i^1, \dots, a_i^{|A_i(\omega)|}\right\}$. Given a strategy profile $\sigma = (\sigma_i)_{i=1}^N \in \Sigma$, define the matrix $\mathcal{P}_\sigma \in \mathbb{R}^{|\Omega| \times \sum_{\omega\in\Omega}\sum_{i=1}^N |A_i(\omega)|}$ as

$$\begin{pmatrix} \sigma_1(a_1^1 \mid \omega^1) & \dots & \sigma_1(a_1^{|A_1(\omega)|} \mid \omega^1) & \sigma_2(a_2^1 \mid \omega^1) & \dots & \sigma_N(a_N^{|A_N(\omega)|} \mid \omega^1) & 0 & \dots & 0 & \dots \\ 0 & & \dots & & \dots & & \sigma_1(a_1^1 \mid \omega^2) & \dots & \sigma_N(a_N^{|A_N(\omega)|} \mid \omega^2) & \dots \\ 0 & & \dots & & \dots & & & \dots & & \dots \end{pmatrix}.$$

Define the matrix $Q \in \mathbb{R}^{\sum_{\omega\in\Omega}\sum_{i=1}^N |A_i(\omega)| \times |\Omega|}$ as

$$\begin{pmatrix} q_1(a_1, \omega^1) & 0 & \dots & \\ \vdots & & & \\ q_1(a_1^{|A_1(\omega)|}, \omega^1) & 0 & \dots & \\ \vdots & & & \\ q_N(a_N^{|A_N(\omega)|}, \omega^1) & 0 & & \\ 0 & q_1(a_1, \omega^2) & 0 & \dots \\ \vdots & & \vdots & \end{pmatrix}$$

and the matrix $P \in \mathbb{R}^{\sum_{\omega\in\Omega}\sum_{i=1}^N |A_i(\omega)| \times |\Omega|}$ as

$$P_{(i,a_i,\omega),\omega'} = \begin{cases} \varphi_i(\omega' \mid a_i, \omega) & \text{if} \quad \omega' \neq \omega, \\ 0 & \text{if} \quad \omega' = \omega. \end{cases}$$

Given a player $i \in \{1, \dots, N\}$, limit condition (3.2) can be written as

$$\Big(\rho\mathbb{1} + \mathcal{P}_\sigma(Q - P)\Big)V_i^0 = \mathcal{P}_\sigma u_i,$$

where $\mathbb{1}$ is the identity matrix, $V_i^0 \in \mathbb{R}^{|\Omega|}$, and $u_i \in \mathbb{R}^{\sum_{j=1}^N \sum_{\omega\in\Omega}|A_j(\omega)|}$. The matrix $\rho\mathbb{1} + \mathcal{P}_\sigma(Q-P)$ is strictly dominant diagonal and therefore invertible.[26] We emphasize the dependence of the unique solution to limit condition (3.2) by writing $V_i^0(\cdot) = V_i^0(\cdot, \sigma)$. This solution is

$$V_i^0(\cdot, \sigma) = \Big(\rho\mathbb{1} + \mathcal{P}_\sigma(Q - P)\Big)^{-1}\mathcal{P}_\sigma u_i.$$

[26] A strictly dominant diagonal matrix X is a square matrix with entries X_{ij} such that $|X_{ii}| > \sum_{j\neq i}|X_{ij}|$ for all i.

Given $i \in \{1, \ldots, N\}$ and $u_i \in \mathbb{R}^{\sum_{j=1}^N \sum_{\omega \in \Omega} |A_j(\omega)|}$, consider the vector

$$\begin{aligned}
u_i + (P-Q)V_i^0(\cdot, \sigma) &= u_i + (P-Q)\left(\rho \mathbb{1} + \mathcal{P}_\sigma (Q-P)\right)^{-1} \mathcal{P}_\sigma u_i \\
&= u_i + (P-Q)\frac{1}{\rho}\left(\mathbb{1} - \frac{1}{\rho}\mathcal{P}_\sigma(Q-P) + \frac{1}{\rho^2}(\mathcal{P}_\sigma(Q-P))^2 \ldots\right)\mathcal{P}_\sigma u_i \\
&= \left(\mathbb{1} - \frac{1}{\rho}(Q-P)\mathcal{P}_\sigma + \frac{1}{\rho^2}((Q-P)\mathcal{P}_\sigma)^2 - \frac{1}{\rho^3}((Q-P)\mathcal{P}_\sigma)^3 + \ldots\right) u_i \\
&= \left(\mathbb{1} + \frac{1}{\rho}(P-Q)\mathcal{P}_\sigma\right)^{-1} u_i,
\end{aligned}$$

where the inversion is justified by strict diagonal dominance. The map

$$u_i \in \mathbb{R}^{\sum_{j=1}^N \sum_{\omega \in \Omega} |A_j(\omega)|} \mapsto \left(\mathbb{1} + \frac{1}{\rho}(P-Q)\mathcal{P}_\sigma\right)^{-1} u_i \in \mathbb{R}^{\sum_{j=1}^N \sum_{\omega \in \Omega} |A_j(\omega)|}$$

is invertible.

The above results have been presented for a given strategy profile $\sigma \in \Sigma$. Following Appendix A.1 in Doraszelski and Escobar (2010), we construct an open set $\Sigma^\varepsilon \subset \mathbb{R}^{\sum_{j=1}^N \sum_{\omega \in \Omega} |A_j(\omega)|}$ that strictly contains Σ such that all the preceding operations are valid for any $\sigma \in \Sigma^\varepsilon$.

A.1.2 Regularity

We begin by providing a formal definition of regularity and establishing the key technical point that for almost all flow payoffs $u \in \mathbb{R}^{N \sum_{j=1}^N \sum_{\omega \in \Omega} |A_j(\omega)|}$, the restriction to regular solutions is without loss of generality.

Given $i \in \{1, \ldots, N\}$, $\omega \in \Omega$, $a_i \in A_i(\omega)$, and $\sigma \in \Sigma^\varepsilon$, define the function

$$\mathcal{U}_i(\omega, a_i, \sigma) = u_{i,i}(\omega, a_i) + \sum_{\omega' \neq \omega} \left(V_i^0(\omega', \sigma) - V_i^0(\omega, \sigma)\right) \varphi_i(\omega' \mid \omega, a_i).$$

In light of limit condition (3.3), we interpret $\mathcal{U}_i(a_i, \omega, \sigma)$ as the objective function that player $i \in \{1, \ldots, N\}$ maximizes over $a_i \in A_i(\omega)$ given state $\omega \in \Omega$ and continuation play $\sigma \in \Sigma^\varepsilon$.

Consider $\bar{\sigma} \in \text{Equil}^0(< u, p, q, \rho >)$. Choose a_i^ω such that $\bar{\sigma}_i(a_i^\omega \mid \omega) > 0$ for all $i \in \{1, \ldots, N\}$ and all $\omega \in \Omega$. Given $a_i \neq a_i^\omega$ and $\sigma \in \Sigma^\varepsilon$, define

$$f_{i,a_i,\omega}(\sigma) = \sigma_i(a_i \mid \omega)\Big(\mathcal{U}_i(a_i, \omega, \sigma) - \mathcal{U}_i(a_i^\omega, \omega, \sigma)\Big)$$

while

$$f_{i,a_i^\omega,\omega}(\sigma) = \sum_{a_i \in A_i(\omega)} \sigma_i(a_i \mid \omega) - 1.$$

By definition, $f(\bar{\sigma}) = 0$. In this subsection, we sometimes emphasize the dependence of f on u by writing $f(\sigma, u)$. Note that $f\colon \Sigma^\varepsilon \times \mathbb{R}^{N \sum_{\omega \in \Omega} \sum_{i=1}^N |A_i(\omega)|} \to \mathbb{R}^{\sum_{\omega \in \Omega} \sum_{i=1}^N |A_i(\omega)|}$ is continuously differentiable.

Definition 1 $\bar{\sigma} \in \text{Equil}^0(< u, p, q, \rho >)$ *is* regular *if the Jacobian of f with respect to σ, $\frac{\partial f}{\partial \sigma}(\bar{\sigma})$, has full rank $\sum_{j=1}^{N} \sum_{\omega \in \Omega} |A_j(\omega)|$.*

We present two preliminary lemmas. We say that a strategy profile $\sigma \in \Sigma^{\varepsilon}$ is *completely mixed* if $\sigma_i(a_i \mid \omega) > 0$ for all $i \in \{1, \ldots, N\}$, $\omega \in \Omega$, and all $a_i \in A_i(\omega)$.

Lemma 3 *If $\sigma \in \Sigma^{\varepsilon}$ is completely mixed, then the Jacobian of f with respect to (σ, u), $\frac{\partial f}{\partial(\sigma,u)}(\sigma, u)$, has full rank $\sum_{j=1}^{N} \sum_{\omega \in \Omega} |A_j(\omega)|$.*

Proof. Define the matrix $M(\sigma, i) \in \mathbb{R}^{\sum_{\omega\in\Omega}(|A_i(\omega)|-1) \times \sum_{j=1}^{N}\sum_{\omega\in\Omega}|A_j(\omega)|}$ such that, for all $a_i \neq \bar{a}_i^{\omega}$, its (i, a_i, ω) row equals 0 in all components save for the (i, a_i, ω) column, where we write $\sigma_i(a_i \mid \omega)$, and for the $(i, a_i^{\omega}, \omega)$ column, where we write $-\sigma_i(a_i \mid \omega)$. The function f can be expressed as

$$f_i(\sigma, u) = \begin{cases} \sum_{a_i \in A_i(\omega)} \sigma_i(a_i \mid \omega^1) - 1, \\ \vdots \\ \sum_{a_i \in A_i(\omega)} \sigma_i(a_i \mid \omega^{|\Omega|}) - 1, \\ M(\sigma, i)\left(\mathbb{1} + \frac{1}{\rho}(P - Q)\mathcal{P}_{\sigma}\right)^{-1} u_i. \end{cases}$$

Up to permutation (which are irrelevant to determine the rank of the Jacobian), we can write

$$\frac{\partial f(\sigma, u)}{\partial(\sigma, u)} = \left(\begin{array}{c|c|c|c|c|c|c|c} \sigma_1 & \sigma_2 & \ldots & \sigma_N & u_1 & u_2 & \ldots & u_N \\ \hline X_1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & X_2 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & X_N & 0 & 0 & \ldots & 0 \\ \hline & & & & Z_1 & 0 & \ldots & 0 \\ & & & & 0 & Z_2 & \ldots & 0 \\ Y_1 & Y_2 & \ldots & Y_N & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & 0 & \ldots & Z_N \end{array}\right),$$

where X_i equals

$$\left(\begin{array}{ccc|ccc|c|ccc} & \sigma_i(\cdot \mid \omega^1) & & & \sigma_i(\cdot \mid \omega^2) & & \ldots & & \sigma_i(\cdot \mid \omega^{|\Omega|}) & \\ \hline 1 & \ldots & 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 1 & \ldots & 1 \end{array}\right)$$

and has rank $|\Omega|$, while $Z_i = M(\sigma, i)\left(\mathbb{1} + \frac{1}{\rho}(P - Q)\mathcal{P}_{\sigma}\right)^{-1}$. Since $M(\sigma, i)$ has full rank $\sum_{\omega\in\Omega}(|A_i(\omega)| - 1)$

and $(\mathbb{1} + \frac{1}{\rho}(P-Q)\mathcal{P}_\sigma)^{-1}$ has full rank $\sum_{\omega\in\Omega}\sum_{j=1}^N |A_j(\omega)|$, Z_i has rank $\sum_{\omega\in\Omega}(|A_i(\omega)|-1)$. We deduce that the Jacobian has full rank $\sum_{j=1}^N\sum_{\omega\in\Omega}|A_j(\omega)|$ and the lemma follows. ■

Given $\sigma \in \Sigma$, $i \in \{1,\ldots,N\}$, and $\omega \in \Omega$, define the *best reply* as

$$B_i(\sigma,\omega) = \arg\max_{a_i\in A_i(\omega)} \mathcal{U}_i(\omega, a_i, \sigma)$$

and the *carrier* as

$$C_i(\sigma,\omega) = \Big\{a_i \in A_i(\omega) \mid \sigma_i(a_i \mid \omega) > 0\Big\}.$$

Using this notation, $\sigma \in \text{Equil}^0(< u, p, q, \rho >)$ if and only if $C_i(\sigma,\omega) \subseteq B_i(\sigma,\omega)$ for all $i \in \{1,\ldots,N\}$. We say that $\sigma \in \text{Equil}^0(< u, p, q, \rho >)$ is *quasi-strict* if $C_i(\sigma,\omega) = B_i(\sigma,\omega)$ for all $i \in \{1,\ldots,N\}$.

Lemma 4 *For almost all $u \in \mathbb{R}^{N\sum_{j=1}^N\sum_{\omega\in\Omega}|A_j(\omega)|}$, any $\sigma \in \text{Equil}^0(< u, p, q, \rho >)$ is quasi-strict.*

Proof. Given $i \in \{1,\ldots,N\}$, consider correspondences $B_i^*\colon \Omega \to \cup_{\omega\in\Omega}A_i(\omega)$ and $C_i^*\colon \Omega \to \cup_{\omega\in\Omega}A_i(\omega)$, with $C_i^*(\omega) \subseteq B_i^*(\omega) \subseteq A_i(\omega)$ for all $\omega\in\Omega$. Define $G(B^*,C^*)$ as the set of all u having some $\sigma \in \text{Equil}^0(< u, p, q, \rho >)$ with best replies $B^* = (B_i^*)_{i=1}^N$ and carriers $C^* = (C_i^*)_{i=1}^N$. Formally,

$$G(B^*,C^*) = \Big\{u \mid \text{ there exists } \sigma \in \text{Equil}^0(< u, p, q, \rho >) \text{ with } B_i(\sigma,\cdot) = B_i^* \text{ and } C_i(\sigma,\cdot) = C_i^* \text{ for all } i = 1,\ldots,N\Big\}.$$

Consider first $\bar\sigma \in \text{Equil}^0(< \bar u, p, q, \rho >)$ such that $B_i(\sigma,\omega) = B_i^*(\omega)$ for all $\omega\in\Omega$. Fix a_i^ω such that $\bar\sigma_i(a_i^\omega \mid \omega) > 0$ and note that the indifference condition $\mathcal{U}_i(a_i,\omega,\bar\sigma) - \mathcal{U}_i(a_i^\omega,\omega,\bar\sigma) = 0$ holds for all $i \in \{1,\ldots,N\}$ and all $a_i \in B_i^*(\omega)$. For all $\omega\in\Omega$ and all $i\in\{1,\ldots,N\}$, define the matrix $P_i(\sigma) \in \mathbb{R}^{\sum_{\omega\in\Omega}(|B_i^*(\omega)|-1)\times\sum_{\omega\in\Omega}|A_i(\omega)|}$, such that for all $a_i \in B_i^*(\omega)$, its (ω, a_i) row equals 0 save for the (ω, a_i) component, where it equals 1, and the (ω, a_i^ω) component, where it equals -1. We can therefore stack all the indifference conditions by writing

$$M(\sigma,u) = \begin{pmatrix} P_1(\sigma)\Big(\mathbb{1} + \frac{1}{\rho}(P-Q)\mathcal{P}_\sigma\Big)^{-1}u_1 \\ \vdots \\ P_N(\sigma)\Big(\mathbb{1} + \frac{1}{\rho}(P-Q)\mathcal{P}_\sigma\Big)^{-1}u_N \end{pmatrix}$$

and note that $M(\bar\sigma,\bar u) = 0$. The Jacobian $\frac{\partial M}{\partial u}(\sigma,u)$ can be computed as

$$\frac{\partial M}{\partial u}(\sigma,u) = \begin{pmatrix} P_1(\sigma)\Big(\mathbb{1} + \frac{1}{\rho}(P-Q)\mathcal{P}_\sigma\Big)^{-1} & 0 & \cdots & 0 \\ 0 & P_2(\sigma)\Big(\mathbb{1} + \frac{1}{\rho}(P-Q)\mathcal{P}_\sigma\Big)^{-1} & \cdots & 0 \\ & \vdots & & \\ 0 & \cdots & & P_N(\sigma)\Big(\mathbb{1} + \frac{1}{\rho}(P-Q)\mathcal{P}_\sigma\Big)^{-1} \end{pmatrix}.$$

Since $P_i(\sigma)$ has full rank $\sum_{\omega\in\Omega}(|B_i^*(\omega)|-1)$, the Jacobian $\frac{\partial M}{\partial u}(\sigma,u)$ has rank $\sum_{i=1}^N\sum_{\omega\in\Omega}(|B_i^*(\omega)|-1)$.

In particular, since $M(\bar{\sigma}, \bar{u}) = 0$, we can construct open sets $\mathcal{N}$, $\mathcal{N}_1 \subseteq \mathbb{R}^{\sum_{i=1}^N \sum_{\omega \in \Omega}(N|A_i(\omega)|-|B_i^*(\omega)|+1)}$, $\mathcal{N}_2 \subseteq \mathbb{R}^{\sum_{i=1}^N \sum_{\omega \in \Omega}(|B_i^*(\omega)|-1)}$, with $\bar{\sigma} \in \mathcal{N}$ and $\bar{u} \in \mathcal{N}_1 \times \mathcal{N}_2$, and a continuously differentiable function Φ such that for all $(\sigma, u_1) \in \mathcal{N} \times \mathcal{N}_1$ there exists a unique $u_2 = \Phi(\sigma, u_1) \in \mathcal{N}_2$ which is a solution to $M(\sigma, (u_1, u_2)) = 0$. Without loss, all these open sets are balls with rational centers and radii and we emphasize their dependence on $(\bar{\sigma}, \bar{u})$ by writing $\mathcal{N}_1^{\bar{\sigma},\bar{u}}$, $\mathcal{N}_2^{\bar{\sigma},\bar{u}}$, and $\mathcal{N}^{\bar{\sigma},\bar{u}}$.

Now take C^* such that for some $i \in \{1, \ldots, N\}$ and some $\omega \in \Omega$, $C_i^*(\omega) \subsetneq B_i^*(\omega)$. Consider the set

$$\begin{aligned} R^{\bar{\sigma},\bar{u}}(B^*, C^*) =& \Big\{ u \in \mathcal{N}_1^{\bar{\sigma},\bar{u}} \times \mathcal{N}_2^{\bar{\sigma},\bar{u}} \mid \text{ there exists } (\sigma, u^1) \in (\mathcal{N}^{\bar{\sigma}} \cap A(C^*)) \times \mathcal{N}_1^{\bar{\sigma},\bar{u}} \text{ such that } u_2 = \Phi(\sigma, u_1) \Big\} \\ &\subseteq \mathbb{R}^{N \sum_{\omega \in \Omega} \sum_{j=1}^N |A_j(\omega)|}, \end{aligned}$$

where $A(C^*) = \{\sigma \in \Sigma \mid C_i(\cdot, \omega) = C_i^*(\omega) \text{ for all } i = 1, \ldots, N\}$. Note that the dimension of $(\mathcal{N}^{\bar{\sigma}} \cap A(C^*)) \times \mathcal{N}_1^{\bar{\sigma},\bar{u}}$ equals $N \sum_{\omega \in \Omega} \sum_{j=1}^N |A_j(\omega)| - \sum_{i=1}^N \sum_{\omega \in \Omega} |B_i^*(\omega)| + \sum_{i=1}^N \sum_{\omega \in \Omega} |C_i^*(\omega)| < N \sum_{\omega \in \Omega} \sum_{j=1}^N |A_j(\omega)|$. Therefore, $M^{\bar{\sigma},\bar{u}}(B^*, C^*)$ has measure zero. Since we are choosing the neighborhoods from a countable set, it follows that $G(B^*, C^*) \subseteq \cup_{n \in \mathbb{N}} Q^n$, where $Q^n = R^{\bar{\sigma}_n, \bar{u}_n}(B^*, C^*)$, has measure zero as well. ■

The following is the main result of this subsection.

Proposition 5 *For almost all* $u \in \mathbb{R}^{N \sum_{j=1}^N \sum_{\omega \in \Omega} |A_j(\omega)|}$*, all* $\bar{\sigma} \in \mathrm{Equil}^0(< u, p, q, \rho >)$ *are regular.*

Proof. From Lemma 4, we can rule out games $u \in \mathbb{R}^{N \sum_{j=1}^N \sum_{\omega \in \Omega} |A_j(\omega)|}$ having non quasi-strict solutions and focus on games having only quasi-strict solutions. Since there is a finite number of correspondences $B_i^* : \Omega \to A_i$, it is enough to prove that the set of games having a non-regular equilibrium σ with

$$B_i(\sigma, \omega) = C_i(\sigma, \omega) = B_i^*(\omega)$$

for all $i \in \{1, \ldots, N\}$ and all $\omega \in \Omega$ has measure zero. Considering the submatrix $\bar{J}(\sigma, u)$ obtained from $\frac{\partial f}{\partial \sigma}(\sigma, u)$ by crossing out all rows and columns corresponding to components (a_i, ω) with $a_i \notin B_i^*(\omega)$, it follows that $\bar{J}(\sigma)$ has full rank if and only if so does $\frac{\partial f}{\partial \sigma}(\sigma, u)$. Noting that $\bar{J}(\sigma, u)$ is the Jacobian of a completely mixed solution, without loss of generality we can therefore assume that $B^*(\omega)$ does not depend on ω and restrict attention to completely mixed solutions. Using Lemma 3 and the transversality theorem, we deduce that for almost all games, all completely mixed equilibria are regular. ■

A.1.3 Establishing Lemma 2

Fix a game $u \in \mathbb{R}^{N \sum_{\omega \in \Omega} \sum_{i=1}^N |A_i(\omega)|}$ and a regular solution $\sigma^0 \in \mathrm{Equil}^0(< u, p, q, \rho >)$. Let $< \mathcal{J}, \mathcal{P} >$ be a protocol of moves and $\Delta > 0$ the period length. We establish that the regular solution σ^0 can be approximated by a Markov perfect equilibrium of a separable dynamic game with noisy transitions and an arbitrary protocol of moves if the period length Δ is sufficiently small. To do so, we apply a version of the implicit function theorem to the limit conditions.

Proof of Lemma 2. In the separable dynamic game with noisy transitions $\Gamma =< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >$, write the continuation value of player $i \in \{1, \ldots, N\}$ if players $J \in \mathcal{J}$ have the move and the state is $\omega \in \Omega$ as $V_i^{\Delta}(\omega, J)$. Note that the value function $V_i^{\Delta}\colon \Omega \times \mathcal{J} \to \mathbb{R}$ is uniquely determined by the strategy profile $\sigma \in \Sigma$. We therefore write $V_i^{\Delta}(\cdot, \cdot) = V_i^{\Delta}(\cdot, \cdot, \sigma)$. Note that $V_i^{\Delta}(\cdot, \cdot, \sigma)$ is a continuous function of (σ, Δ) and its differential with respect to σ at $\Delta = 0$ exists. In particular, for all $J \in \mathcal{J}$ and all $\sigma \in \Sigma$, $V_i^{\Delta}(\cdot, J, \sigma) \to V_i^{0}(\cdot, \sigma)$ as $\Delta \to 0$.

A strategy profile σ^{Δ} is a Markov perfect equilibrium of the separable dynamic game with noisy transitions $\Gamma =< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >$ if for all $i = 1, \ldots, N$, $\omega \in \Omega$, and all $a_i \in A_i(\omega)$

$$\sigma_i^{\Delta}(a_i \mid \omega) > 0 \Rightarrow a_i \in \arg\max_{\tilde{a}_i \in A_i(\omega)} \mathcal{U}_i^{\Delta}(\omega, \tilde{a}_i, \sigma^{\Delta})$$

with

$$\mathcal{U}_i^{\Delta}(\omega, a_i, \sigma^{\Delta}) = u_{i,i}(\omega, a_i) + \exp(-\rho\Delta) \sum_{\omega' \neq \omega} \sum_{J' \in \mathcal{J}} \Big(V_i^{\Delta}(\omega', J', \sigma^{\Delta}) - V_i^{\Delta}(\omega, J', \sigma^{\Delta})\Big) \varphi_i(\omega' \mid \omega, a_i) \Pr(J' \mid J) + \mathcal{O}(\Delta)$$

and $J \in \mathcal{J}$ is such that $i \in J$.

Consider the profile $(a_i^{\omega})_{i=1,\ldots,N,\omega\in\Omega}$ that is used in the construction of the function f in Appendix A.1.2 for which σ^0 is regular. Abusing notation, construct the function $f\colon [0,1] \times \Sigma^{\varepsilon} \to \mathbb{R}^{\sum_{\omega\in\Omega}\sum_{i=1}^{N}|A_i(\omega)|}$ such that for all $a_i \neq a_i^{\omega}$

$$f_{i,a_i,\omega}(\Delta, \sigma) = \sigma_i(a_i, \omega)\Big(\mathcal{U}_i^{\Delta}(\omega, a_i, \sigma) - \mathcal{U}_i^{\Delta}(\omega, a_i^{\omega}, \sigma)\Big)$$

while

$$f_{i,a_i^{\omega},\omega}(\Delta, \sigma) = \sum_{a_i \in A_i(\omega)} \sigma_i(a_i \mid \omega) - 1.$$

Observe that $f(\Delta, \sigma)$ is a continuous function, with a well-defined differential with respect to σ, $D_{\sigma} f(0, \sigma)$, at $(0, \sigma)$. Moreover, $f(0, \sigma^0) = 0$ and $D_{\sigma} f(0, \sigma^0)$ has full rank $\sum_{j=1}^{N} \sum_{\omega\in\Omega} |A_j(\omega)|$. A version of the implicit function theorem (see Lemma 5 below) implies that for all $r > 0$ there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, there exists $\sigma^{\Delta} \in \Sigma^{\varepsilon}$ with $\|\sigma^0 - \sigma^{\Delta}\| < r$ such that $f(\Delta, \sigma^{\Delta}) = 0$. Moreover, we can take $\bar{\Delta}$ and r small enough so that (i) $\sigma_i^{\Delta}(a_i, \omega) > 0$ whenever $\sigma_i^{0}(a_i, \omega) > 0$, and (ii) $\mathcal{U}_i^{\Delta}(\omega, a_i, \sigma^{\Delta}) < \mathcal{U}_i^{\Delta}(\omega, a_i^{\omega}, \sigma^{\Delta})$ whenever $\mathcal{U}_i(\omega, a_i, \sigma^0) < \mathcal{U}_i(\omega, a_i^{\omega}, \sigma^0)$.

To prove that σ^{Δ} is a Markov perfect equilibrium of the separable dynamic game with noisy transitions $< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >$, consider first $a_i \in A_i(\omega)$ and $\omega \in \Omega$ such that $\sigma^0(a_i \mid \omega) = 0$. Since σ^0 is regular, it is also quasi-strict and therefore $\mathcal{U}_i(\omega, a_i, \sigma^0) < \mathcal{U}_i(\omega, a_i^{\omega}, \sigma^0)$. From (ii), $\mathcal{U}_i^{\Delta}(\omega, a_i, \sigma^{\Delta}) < \mathcal{U}_i^{\Delta}(\omega, a_i^{\omega}, \sigma^{\Delta})$. Since $f(\Delta, \sigma^{\Delta}) = 0$, it follows that $\sigma^{\Delta}(a_i \mid \omega) = 0$. Next consider $a_i \in A_i(\omega)$ and $\omega \in \Omega$ such that $\sigma^0(a_i \mid \omega) > 0$. We can use (i) to deduce that $\sigma^{\Delta}(a_i \mid \omega) > 0$ and, since $f(\Delta, \sigma^{\Delta}) = 0$, $\mathcal{U}_i^{\Delta}(\omega, a_i, \sigma^{\Delta}) = \mathcal{U}_i^{\Delta}(\omega, a_i^{\omega}, \sigma^{\Delta})$. All of these observations prove that whenever $\sigma^{\Delta}(a_i, \omega) > 0$, a_i solves $\max_{\tilde{a}_i \in A_i(\omega)} \mathcal{U}_i^{\Delta}(\omega, \tilde{a}_i, \sigma^{\Delta})$. Therefore, σ^{Δ} is a Markov perfect equilibrium of the separable dynamic game with noisy transitions $< \Delta, \mathcal{J}, \mathcal{P}, u, , q, \rho >$.

Since for almost all $u \in \mathbb{R}^{N \sum_{j=1}^{N} \sum_{\omega \in \Omega} |A_j(\omega)|}$, all $\sigma^0 \in \text{Equil}^0(< u, p, q, \rho >)$ are regular, Lemma 2 follows.

■

It remains to prove the implicit function theorem we used above. The textbook presentation of the implicit function theorem (Section M.E in Mas-Colell, Whinston, and Green 1995) applies to continuously differentiable functions defined on open sets. In our setup, the set of parameters $\Delta \in [0, 1]$ is closed and, moreover, we are interested in the boundary case $\Delta = 0$. The following result is a modification of Theorem A in Halkin (1974).

Lemma 5 (Implicit Function Theorem) *Assume $f\colon [0,1] \times \Sigma^\varepsilon \to \mathbb{R}^{\sum_{\omega \in \Omega} \sum_{i=1}^{N} |A_i(\omega)|}$ is a continuous function such that its differential with respect to $\sigma \in \Sigma^\varepsilon$ at $\Delta = 0$, $D_\sigma f(0, \sigma)$, exists. Let $\sigma^0 \in \Sigma$ be such that $f(0, \sigma^0) = 0$ and $D_\sigma f(0, \sigma^0)$ has full rank $\sum_{j=1}^{N} \sum_{\omega \in \Omega} |A_j(\omega)|$. Then, for all $r > 0$ there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, there exists σ^Δ such that $\|\sigma^0 - \sigma^\Delta\| < r$ and $f(\Delta, \sigma^\Delta) = 0$.*

Proof. Consider the function $\varphi(\sigma, \Delta) = \sigma - [D_\sigma f(0, \sigma^0)]^{-1} f(\Delta, \sigma)$ and note that the problem of finding σ^Δ such that $f(\Delta, \sigma^\Delta) = 0$ reduces to the problem of finding a fixed point of $\varphi(\cdot, \Delta)$. Note that $D_\sigma \varphi(0, \sigma^0) = 0$ and therefore we can assume, without loss, that $r > 0$ is small enough so that for all $\|\sigma - \sigma^0\| < r$, $\sigma \in \Sigma^\varepsilon$ and

$$\frac{\|\varphi(\sigma, 0) - \varphi(\sigma^0, 0)\|}{\|\sigma - \sigma^0\|} < \frac{1}{2}.$$

Since $\varphi(\sigma^0, 0) = \sigma^0$, we can therefore deduce that for all $\|\sigma - \sigma^0\| \le r$, $\|\varphi(\sigma, 0) - \sigma^0\| \le r/2$.

Define now $m(\Delta) = \max_{\{\sigma | \|\sigma - \sigma^0\| \le r\}} \|\varphi(\sigma, \Delta) - \varphi(\sigma, 0)\|$. Berge's maximum theorem (Theorem 17.31 in Aliprantis and Border 2006) implies that m is continuous in $\Delta \in [0, 1]$. Since $m(0) = 0$, there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, $m(\Delta) < r/2$. We thus deduce that for all σ such that $\|\sigma - \sigma^0\| \le r$ and $\Delta < \bar{\Delta}$

$$\begin{aligned} \|\varphi(\sigma, \Delta) - \sigma^0\| &\le \|\varphi(\sigma, \Delta) - \varphi(\sigma, 0)\| + \|\varphi(\sigma, 0) - \sigma^0\| \\ &\le m(\Delta) + \|\varphi(\sigma, \Delta) - \sigma^0\| \\ &\le r. \end{aligned}$$

It follows that for all $\Delta < \bar{\Delta}$, the continuous function $\varphi(\cdot, \Delta)$ maps the convex and compact set $\{\sigma \mid \|\sigma - \sigma^0\| \le r\}$ into itself. For any such $\Delta < \bar{\Delta}$, Brouwer's fixed point theorem (Theorem M.I.1 in Mas-Colell, Whinston, and Green 1995) implies the existence of σ^Δ within distance r of σ^0 such that $\varphi(\sigma^\Delta, \Delta) = \sigma^\Delta$. ■

A.1.4 Proof of Theorem 1

Proof of Theorem 1. From Lemma 2, take one of the generic flow payoffs $u \in \mathbb{R}^{N \sum_{\omega \in \Omega} \sum_{j=1}^{N} |A_j(\omega)|}$ and any two protocols of moves as in the statement of Theorem 1. From Lemma 1, there exists $\tilde{\Delta} > 0$ such that for all $\Delta < \tilde{\Delta}$ and all $\sigma^\Delta \in \text{Equil}(< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >)$, there exists $\sigma^0 \in \text{Equil}^0(< u, p, q, \rho >)$ such that $\|\sigma^\Delta - \sigma^0\| < \varepsilon/2$. From Lemma 2 we can find $\hat{\Delta} > 0$ such that for all $\Delta < \hat{\Delta}$, there exists

$\hat{\sigma}^{\Delta} \in$ Equil($< \Delta, \bar{\mathcal{J}}, \bar{\mathcal{P}}, u, p, q, \rho >$) such that $\|\bar{\sigma}^{\Delta} - \sigma^0\| < \varepsilon/2$. Taking $\bar{\Delta} = \min\{\tilde{\Delta}, \hat{\Delta}\} > 0$, Theorem 1 follows from the triangle inequality. ■

A.2 Proofs for Section 4

Proof of Proposition 3. Take any u as in Theorem 1. Define Payoffs^0($< u, p, q, \rho >$) $\subseteq \mathbb{R}^N$ to be the set of payoff profiles associated with solutions in pure strategy profiles to the limit conditions (3.2) and (3.3):

$$\text{Payoffs}^0(< u, p, q, \rho >) =$$
$$\Big\{ V^0(\omega^{t=0}) \in \mathbb{R}^N \mid (V^0, \sigma^0) \text{ solves (3.2) and (3.3) for some pure strategy profile } \sigma^0 \Big\},$$

where $\omega^{t=0} \in \Omega$ is the initial state of the game. For $\epsilon > 0$ take $\bar{\Delta}$ such that for all $\Delta < \bar{\Delta}$, and all protocols $< \mathcal{J}, \mathcal{P} >$, the Hausdorff distance between Payoffs^0($< u, p, q, \rho >$) and Payoffs^F($< \Delta, < \mathcal{J}, \mathcal{P} >, u, p, q, \rho >$) is less than $\epsilon/3$.

Take $v \in$ Payoffs^F(Γ^{sim}). Because v is approachable, for all $n \geq 1$, there exists an asynchronous protocol $< \mathcal{J}^n, \mathcal{P}^n >$ and $w^n \in$ Payoffs^F($< \Delta, \mathcal{J}^n, \mathcal{P}^n, u, p, q, \rho >$) such that $\|v - w^n\| < 1/n$. Restrict the sequence such that $1/n < \epsilon/3$. From Bhaskar, Mailath, and Morris (2013), we can actually take $w^n \in$ Payoffs^M($< \Delta, \mathcal{J}^n, \mathcal{P}^n, u, p, q, \rho >$). By construction, for any such w^n we can find $\tilde{w}^n \in$ Payoffs^0($< u, p, q, \rho >$) such that $\|w^n - \tilde{w}^n\| < \epsilon/3$. Since Payoffs^0($< u, p, q, \rho >$) has a finite number of elements, we can assume that $\tilde{w}^n = \tilde{w}$ does not depend on n (perhaps, by taking a subsequence). Now, take $w \in$ Payoffs^M($< \Delta, \mathcal{J}^{sim}, \mathcal{P}^{sim}, u, p, q, \rho >$) such that $\|w - \tilde{w}\| < \epsilon/3$. It follows that

$$\|v - w\| \leq \|v - w^n\| + \|w^n - \tilde{w}\| + \|\tilde{w} - w\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3},$$

which proves the result. ■

Proof Sketch of Proposition 4. The proof follows from the analysis in Appendix A.1 and arguments in Doraszelski and Escobar (2010). Details are available upon request. To provide a sketch, consider the analog to the limit conditions (3.2) and (3.3) that arise without Assumption 3:

$$\rho V_i(\omega) = u_i(\omega, \sigma(\omega)) + \sum_{\omega' \neq \omega} (V_i(\omega') - V_i(\omega)) \varphi(\omega' \mid \omega, \sigma(\omega))$$

and

$$\sigma_i(a_i \mid \omega) > 0 \Rightarrow a_i \in \arg\max_{\tilde{a}_i \in A_i(\omega)} u_i(\omega, \tilde{a}_i, \sigma_{-i}(\omega)) + \sum_{\omega' \neq \omega} (V_i(\omega') - V_i(\omega)) \varphi(\omega' \mid \omega, \tilde{a}_i, \sigma_{-i}(\omega)).$$

From these limit conditions, we can construct a function f (as we did in Appendix A.1) such that all solutions are zeros of f and, moreover, for almost all flow payoffs u, all solutions are regular. We then apply the implicit function theorem to deduce the result. ■

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Online Appendix

This Online Appendix consists of five parts. Appendix OA.1 presents a slightly modified version of the entry game in Section 3, Appendix OA.2 shows that the limit conditions (3.2) and (3.3) may not admit a solution in pure strategies, Appendix OA.3 provides a counterexample complementing Example 6, and Appendices OA.4 and OA.5 generalize our notion of a protocol of moves and provide extensions of Theorem 1.

OA.1 Modified Entry Game

We slightly modify the discrete-time entry game in Section 3 by assuming that the expected net present value of the stream of payoffs to a monopolist is $\frac{B(\Delta+\Delta^2)}{1-e^{-\rho\Delta}}$ instead of $\frac{B\Delta}{1-e^{-\rho\Delta}}$. Clearly, this does not change the limit conditions (3.2) and (3.3) and the fact that σ^0 is a solution to these conditions provided that $\lambda B/\rho = c$ and $b < 0$.

We first show that this solution can be approximated by the discrete-time game with alternating moves provided that $\rho < 2$. A firm has an incentive to invest if and only if

$$-c + e^{-\rho\Delta}\lambda B\frac{\Delta+\Delta^2}{1-e^{-\rho\Delta}} \geq 0.$$

This condition holds for Δ sufficiently small. To see this, note that the inequality holds with equality at $\Delta = 0$ while the derivative of the left-hand side at $\Delta = 0$ equals $-\lambda B + \lambda B(1/2 + 1/\rho) > 0$.

Next we show that this solution cannot be approximated by the discrete-time game with simultaneous moves. A firm has an incentive to invest if and only if

$$-c + e^{-\rho\Delta}\Delta(\lambda)^2 b\frac{\Delta}{1-e^{-\rho\Delta}} + e^{-\rho\Delta}\lambda(1-\lambda\Delta)B\frac{\Delta+\Delta^2}{1-e^{-\rho\Delta}} \geq 0.$$

This condition holds in the limit as $\Delta \to 0$ but not for $\Delta > 0$. To see this, note that the inequality holds with equality at $\Delta = 0$ while the derivative of the left-hand side at $\Delta = 0$ equals $\lambda B(1/\rho - 1/2 - \lambda) + \lambda^2 b/\rho < 0$.

OA.2 Non-Existence of Solution in Pure Strategies

Consider a separable dynamic game with noisy transitions, $N = 2$ players, $\Omega = \{1, 2\}$, $A_i(\omega) = \{1, 2\}$ if $\omega = i$, and $A_i(\omega) = \{1\}$ if $\omega \neq i$. This means that player i makes a nontrivial decision only when $\omega = i$. Flow payoffs are $u_{i,i}(\omega, a_i) = 0$ if $\omega = i$ for all $a_i \in \{1, 2\}$, while

$$u_{1,2}(a_2, 2) = \begin{cases} 10 & \text{if } a_2 = 1, \\ -10 & \text{if } a_2 = 2 \end{cases}$$

and

$$u_{2,1}(a_1, 1) = \begin{cases} -10 & \text{if } a_1 = 1, \\ 10 & \text{if } a_1 = 2. \end{cases}$$

Hence, the flow payoff of player i is 0 when $\omega = i$, but his decision determines whether the flow payoff of player $-i$ is 10 or -10. Transition probabilities are determined as in Example 3 with $\lambda = 1$ and

$$l_1(1 \mid 1, a_1) \begin{cases} 0 & \text{if } a_1 = 1, \\ 1 & \text{if } a_1 = 2 \end{cases}$$

and

$$l_2(1 \mid 2, a_2) = \begin{cases} 1 & \text{if } a_2 = 1, \\ 0 & \text{if } a_2 = 2, \end{cases}$$

while $l_i(i \mid i, a_{-i}) = 1$. This means that in state $\omega = i$, player i (and only player i) determines a probability distribution over the successor state ω'. (Note that given λ, l_1, and l_2, we can construct p and q as we did in Example 3.)

We show that the limit conditions (3.2) and (3.3) do not admit a solution in pure strategies. The intuition is similar to the non-existence of a Nash equilibrium in pure strategies in matching pennies. Consider a solution $\sigma^* \in \text{Equil}^0(< u, p, q, \rho >)$ in pure strategies. If $\sigma_1^*(\cdot \mid 1) = (1, 0)$, then it must be the case that player 2 is choosing $\sigma_2^*(1 \mid 2) = 1$ for otherwise player 1 makes a loss in state $\omega = 2$ while he can secure 0 by playing $a_1 = 2$. But this would mean that player 2 is willing to make a loss in state $\omega = 1$, while he can secure 0 by playing $a_2 = 2$. Similarly, it cannot be that $\sigma_1^*(\cdot \mid 1) = (0, 1)$. Thus, the limit conditions (3.2) and (3.3) do not admit a solution in pure strategies. They do, however, admit a solution in mixed strategies in which player i chooses $\sigma_i(\cdot \mid \omega) = (1/2, 1/2)$ when $\omega = i$.[27]

OA.3 Non-Separable State-to-State Transitions

Consider a dynamic stochastic game with $N = 2$ players, $\Omega = \{0, 1\}$, and $A_i = \{0, 1\}$. The hazard rate in state $\omega = 0$ is $q(0, a_1, a_2) = 1$ if and only if $a_1 = a_2 = 1$, and $q(0, a_1, a_2) = 0$ otherwise, whereas in state $\omega = 1$, $q(1, a) = 0$. The transition probability satisfies $p(1 \mid 0, (a_1, a_2)) = 1$. State $\omega = 1$ is thus absorbing. Flow payoffs do not depend on actions and take the form $u_{i,i}(\omega) = \omega$. In the game with simultaneous moves, it is simple to see that there exist two Markov perfect equilibria in pure strategies. In one of them, the state is stuck in $\omega = 0$. In contrast, in the game with alternating moves and transitions "materializing" only once both players have made a decision (in a violation of Assumption 3 similar to Example 6), the unique Markov perfect equilibrium is $\sigma_1^* = 1$, $\sigma_2^*(a_1) = a_1$, and the state eventually jumps to $\omega' = 1$.

[27] This example can be easily adapted to show that additive-reward, additive-transition dynamic stochastic games may not admit a Markov perfect equilibrium in pure strategies. It is much simpler than other examples in the literature.

OA.4 Generalized Protocol of Moves

We relax Assumption 1 by generalizing our notion a protocol of moves. We allow the evolution of the protocol state J to depend on players' actions a_J and the physical state ω. We maintain that $\mathcal{J}$ is a partition of the set of players, but allow for a non-uniform stationary distribution. We show that Theorem 1 remains valid.

Assumption 4 (Generalized Protocol of Moves) *Let $\mathcal{J}$ be a partition of $\{1, 2, \ldots, N\}$ and $\mathcal{P} = (\Pr(J'|J, \omega, a_J))_{J,J' \in \mathcal{J}}$ a $|\mathcal{J}| \times |\mathcal{J}|$ transition matrix for all $\omega \in \Omega$ and all selections $J \mapsto a_J \in \prod_{j \in J} A_j(\omega) = A_J(\omega)$. Assume that*

$$\mathcal{P}(\cdot \mid \cdot, \omega, \sigma_J) = \sum_{a_J \in \prod_{j \in J} A_j(\omega)} \mathcal{P}(\cdot \mid \cdot, \omega, a_J) \prod_{j \in J} \sigma_j(a_j)$$

is irreducible for all $\omega \in \Omega$ and all selections $J \mapsto \sigma_J \in \prod_{j \in J} \Sigma_j = \Sigma_J$ and its unique stationary distribution $\pi = (\pi(J))_{J \in \mathcal{J}} \in \mathtt{P}(\mathcal{J})$ is independent of ω and σ_J.

We call $< \mathcal{J}, \mathcal{P} >$ a *generalized protocol of moves.* Under a generalized protocol of moves, the current physical state and action profile may make a transition from one protocol state to another more likely, but on average all protocol states are visited with frequencies that are independent of physical states and action profiles. The remaining aspects of the model are unchanged.

Denote the above game by $\Gamma =< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >$ and consider a Markov perfect equilibrium $\sigma^\Delta = (\sigma_i^\Delta)_{i=1}^N$. Under Assumptions 2, 3, and 4, the discrete-time Bellman equation for a period length of Δ is

$$\begin{aligned} V_i^\Delta(\omega, J) &= |\mathcal{J}| \sum_{j \in J} u_{i,j}(\omega, \sigma_j^\Delta(\omega))\Delta \\ &+ \exp(-\rho\Delta) \Bigg\{ \sum_{J' \in \mathcal{J}} V_i^\Delta(\omega, J') \sum_{a_J \in A_J(\omega)} \sigma_J^\Delta(a_J \mid \omega) \Pr(J'|J, \omega, a_J) \left(1 - |\mathcal{J}| \sum_{j \in J} q_j(\omega, a_j)\Delta\right) \\ &+ \sum_{\omega' \neq \omega} \sum_{J' \in \mathcal{J}} V_i^\Delta(\omega', J') \sum_{a_J \in A_J(\omega)} \sigma_J^\Delta(a_J \mid \omega) \Pr(J'|J, \omega, a_J) \left(|\mathcal{J}| \sum_{j \in J} \varphi_j(\omega'|\omega, a_j)\Delta\right) \Bigg\} + \mathcal{O}(\Delta^2). \end{aligned} \tag{OA.1}$$

Taking the limit as $\Delta \to 0$, we deduce that

$$V_i^0(\omega, J) = \sum_{J' \in \mathcal{J}} V_i^0(\omega, J') \Pr(J'|J, \omega, \sigma_J^0(\omega)).$$

Assumption 4 implies that the transition matrix $\Pr(J' \mid J, \omega, \sigma_J^0)$ has a unique (and uniform) right eigenvector so that $V_i^0(\omega, J) = V_i^0(\omega, J')$ for all $J, J' \in \mathcal{J}$. Let $V_i^0 \colon \Omega \to \mathbb{R}$ be the value function of player i and $V^0 = (V_i^0)_{i=1}^N$ be the profile of value functions in the limit as $\Delta \to 0$.

The Bellman equation can equivalently be written as

$$\frac{1}{\Delta}V_i^{\Delta}(\omega,J)-\frac{\exp(-\rho\Delta)}{\Delta}\sum_{J'\in\mathcal{J}}\sum_{a_J\in A_J(\omega)}\sigma_J^{\Delta}(a_J\mid\omega)V_i^{\Delta}(\omega,J')\Pr(J'|J,\omega,a_J)=|\mathcal{J}|\sum_{j\in J}u_{i,j}(\omega,\sigma_j^{\Delta}(\omega))$$

$$+\exp(-\rho\Delta)|\mathcal{J}|\sum_{j\in J}\left\{\sum_{\omega'\neq\omega}\sum_{J'\in\mathcal{J}}\sum_{a_J\in A_J(\omega)}\sigma_J^{\Delta}(a_J\mid\omega)V_i^{\Delta}(\omega',J')\Pr(J'|J,\omega,a_J)\varphi_j(\omega'|\omega,a_j)\right.$$

$$\left.-\sum_{J'\in\mathcal{J}}\sum_{a_J\in A_J(\omega)}\sigma_J^{\Delta}(a_J\mid\omega)V_i^{\Delta}(\omega,J')\Pr(J'|J,\omega,a_J)q_j(\omega,a_j)\right\}+\mathcal{O}(\Delta).$$

Multiplying by $\pi(J)$ and summing over $J\in\mathcal{J}$ yields

$$\frac{1}{\Delta}\sum_{J\in\mathcal{J}}\pi(J)V_i^{\Delta}(\omega,J)-\frac{\exp(-\rho\Delta)}{\Delta}\sum_{J\in\mathcal{J}}\pi(J)\sum_{J'\in\mathcal{J}}\sum_{a_J\in A_J(\omega)}\sigma_J^{\Delta}(a_J\mid\omega)V_i^{\Delta}(\omega,J')\Pr(J'|J,\omega,a_J)$$

$$=|\mathcal{J}|\sum_{J\in\mathcal{J}}\pi(J)\sum_{j\in J}u_{i,j}(\omega,\sigma_j^{\Delta}(\omega))$$

$$+\exp(-\rho\Delta)|\mathcal{J}|\sum_{J\in\mathcal{J}}\pi(J)\sum_{j\in J}\left\{\sum_{\omega'\neq\omega}\sum_{J'\in\mathcal{J}}V_i^{\Delta}(\omega',J')\Pr(J'|J,\omega,\sigma_J^{\Delta}(\omega))\varphi_j(\omega'|\omega,\sigma_j^{\Delta}(\omega))\right.$$

$$\left.-\sum_{J'\in\mathcal{J}}V_i^{\Delta}(\omega,J')\Pr(J'|J,\omega,\sigma_J^{\Delta}(\omega))q_j(\omega,\sigma_j^{\Delta}(\omega))\right\}+\mathcal{O}(\Delta^2).$$

Using the facts that $\sum_{J\in\mathcal{J}}\Pr(J'|J,\omega,\sigma_J(\omega))\pi(J)=\pi(J')$ and $\sum_{J'\in\mathcal{J}}\Pr(J'|J,\omega,\sigma_J(\omega))=1$, and taking the limit as $\Delta\to 0$, we obtain the continuous-time Bellman equation

$$\rho V_i^0(\omega)=|\mathcal{J}|\sum_{J\in\mathcal{J}}\pi(J)\sum_{j\in J}u_{i,j}(\omega,\sigma_j^0(\omega))+|\mathcal{J}|\sum_{J\in\mathcal{J}}\pi(J)\sum_{j\in J}\left(\sum_{\omega'\neq\omega}V_i^0(\omega')\varphi_j(\omega'|\omega,\sigma_j^0(\omega))-V_i^0(\omega)q_j(\omega,\sigma_j^0(\omega))\right).\tag{OA.2}$$

The discrete-time optimality condition for a period length Δ is

$$\sigma^{\Delta}(a_i\mid\omega)>0\Rightarrow a_i\in\arg\max_{\tilde{a}_i\in A_i(\omega)}u_i^{\Delta}(\omega,J,\tilde{a}_i,\sigma_{J\setminus\{i\}}^{\Delta}(\omega))$$

$$+\exp(-\rho\Delta)\sum_{\omega'\in\Omega}\sum_{J'\in\mathcal{J}}\sum_{a_{J\setminus\{i\}}}\sigma_{J\setminus\{i\}}(a_{J\setminus\{i\}}\mid\omega)V_i^{\Delta}(\omega',J')\Pr\left(J'|J,\omega,a_i,a_{J\setminus\{i\}}\right)\Pr^{\Delta}\left(\omega'|\omega,J,\tilde{a}_i,a_{J\setminus\{i\}}\right).$$

Dividing by Δ, rearranging terms, and taking the limit as $\Delta\to 0$, we deduce the continuous-time optimality condition

$$\sigma_i^0(a_i\mid\omega)>0\Rightarrow a_i\in\arg\max_{\tilde{a}_i\in A_i(\omega)}u_{i,i}(\omega,\tilde{a}_i)+\sum_{\omega'\neq\omega}\Big(V_i^0(\omega')-V_i^0(\omega)\Big)\varphi_i(\omega'\mid\omega,\tilde{a}_i).\tag{OA.3}$$

Conditions (OA.2) and (OA.3) are the analogs of conditions (3.2) and (3.3) for a generalized protocol of moves.

Consider the generalized protocols of moves $< \mathcal{J}_1, \mathcal{P}_1 >$ and $< \mathcal{J}_2, \mathcal{P}_2 >$ with stationary distributions π_1 and π_2. For all $j = 1, \ldots, N$, define $J_1(j)$ to be the unique element in $\mathcal{J}_1$ such that $j \in J_1(j)$. Define $J_2(j)$ analogously. We say that the generalized protocols of moves $< \mathcal{J}_1, \mathcal{P}_1 >$ and $< \mathcal{J}_2, \mathcal{P}_2 >$ are *comparable* if $|\mathcal{J}_1|\pi_1(J_1(j)) = |\mathcal{J}_2|\pi_2(J_2(j))$ for all $j = 1, \ldots, N$. Given a protocol $< \mathcal{J}, \mathcal{P} >$l, a player j moves a fraction $\pi(J(j))$ of the time and has an impact on payoffs and transitions which is scaled by $|\mathcal{J}|$. Thus, comparability means that the total impact of a player's strategy on payoffs and transitions does not depend on the particular protocol that we use in the model. All protocols that satisfy Assumption 1 are comparable.

Theorem 1 remains valid for generalized protocols of moves that are comparable.[28]

Theorem 2 (Generalized Protocol-Invariance Theorem) *Fix p, q, and ρ. For almost all u, all generalized protocols of moves $< \mathcal{J}, \mathcal{P} >$ and $< \overline{\mathcal{J}}, \overline{\mathcal{P}} >$ that are comparable, and all $\varepsilon > 0$, there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$ and $\sigma \in \mathrm{Equil}(< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >)$, there exists $\overline{\sigma} \in \mathrm{Equil}(< \Delta, \overline{\mathcal{J}}, \overline{\mathcal{P}}, u, p, q, \rho >)$ such that $\|\sigma - \overline{\sigma}\| < \varepsilon$.*

OA.5 Non-Partition Protocol of Moves

We relax Assumption 1 and assume that $\mathcal{J}$ is not a partition of the set of players but contains subsets $J \subseteq \{1, \ldots, N\}$ such that for all $i = 1, \ldots, N$, there exists $J \in \mathcal{J}$ such that $i \in J$. This allows player i to have the move in conjunction with different sets of rivals. To simplify the exposition, we assume that $|\{J \in \mathcal{J} \mid i \in J\}| = \kappa$ for all $i = 1, \ldots, N$. As before, there is an irreducible Markov chain $\mathcal{P}$ defined on $\mathcal{J}$ that has a unique stationary distribution that is uniform on $\mathcal{J}$. We call $< \mathcal{J}, \mathcal{P} >$ a *non-partition protocol of moves.*

With a non-partition protocol of moves $< \mathcal{J}, \mathcal{P} >$, the per-period payoff $u_i^{\Delta}(\omega, J, a_J)$ is written as

$$u_i^{\Delta}(\omega, J, a_J) = \frac{|\mathcal{J}|}{\kappa} \sum_{j \in J} u_{i,j}(\omega, a_j)\Delta + \mathcal{O}(\Delta^2),$$

and the hazard rate $q_J(\omega, a_J)$ and transition probability $p_J(\omega' \mid \omega, a_J)$ are written as

$$q_J(\omega, a_J) = \frac{|\mathcal{J}|}{\kappa} \sum_{j \in J} q_j(\omega, a_j)$$

and

$$q_J(\omega, a_J) p_J(\omega' \mid \omega, a_J) = \frac{|\mathcal{J}|}{\kappa} \sum_{j \in J} q_j(\omega, a_j) p_j(\omega' \mid \omega, a_j),$$

where $q_j \colon \{(\omega, a_j) \mid a_j \in A_j(\omega)\} \to \mathbb{R}^+ \cup \{0\}$ and $p_j \colon \{(\omega, a_j) \mid a_j \in A_j(\omega)\} \to \mathtt{P}(\Omega)$. The remaining aspects of the model are unchanged.

With a non-partition protocol of moves $< \mathcal{J}, \mathcal{P} >$, the identity of the players that have the move in

[28]Strictly speaking, here we only show that Lemma 1 remains valid. The proof that Lemma 2 remains valid is available upon request.

conjunction with player i is a state variable. Thus, a Markovian strategy for player i is a function $\sigma_i\colon \Omega\times\{J \in \mathcal{J} \mid i \in J\} \to \cup_{\omega\in\Omega}\mathsf{P}(A_i(\omega))$. Overloading notation, we use $\mathrm{Equil}(< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >)$ to denote the set of Markov perfect equilibria. We say that a Markov perfect equilibrium σ is *simple* if $\sigma_i(a_i \mid \omega, J) = \sigma_i(a_i \mid \omega, \tilde{J})$ for all $i = 1, \ldots, N$, $\omega \in \Omega$, $a_i \in A_i$, and all $J, \tilde{J} \in \mathcal{J}$. In this case, we write $\sigma_i(a_i \mid \omega)$.

The following proposition partially extends Theorem 1 to a non-partition protocol of moves:

Proposition 6 *Assume* $\mathrm{Equil}^0(< u, p, q, \rho >)$ *only contains strict solutions. Then there exists* $\bar{\Delta} > 0$ *such that for all* $\Delta < \bar{\Delta}$*,* $\sigma \in \mathrm{Equil}(< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >)$ *is simple and*

$$\mathrm{Equil}(< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >) = \mathrm{Equil}^0(< u, p, q, \rho >).$$

In contrast to Theorem 1, Proposition 6 restricts attention to strict and thus pure solutions. When mixed solutions are considered, the limit conditions may have a continuum of solutions if players use the payoff-irrelevant realization of J to randomize over actions and our differential topology tools therefore cannot be directly applied.

Proof. Consider a sequence $\left(\sigma^\Delta\right)$ with $\sigma^\Delta \in \mathrm{Equil}(< \Delta, \mathcal{J}, \mathcal{P}, u, p, q, \rho >)$ and $\sigma^\Delta \to \sigma^0$ (possibly through a subsequence) as $\Delta \to 0$. Let V^Δ be the profile of value functions corresponding to σ^Δ and assume it converges to V^0. Similar to Section 3, we can deduce that $V^0(\omega, J)$ does not depend on $J \in \mathcal{J}$ and simply write $V^0(\omega)$. We can also follow Section 3 to deduce that

$$\rho V_i^0(\omega) = \frac{1}{\kappa} \sum_{J\in\mathcal{J}} \sum_{j\in J} \left(u_{i,j}(\omega, \sigma_j(\omega, J)) + \sum_{\omega'\neq\omega} \left(V_i^0(\omega') - V_i^0(\omega)\right) \varphi_j(\omega' \mid \omega, \sigma_j(\omega, J)) \right)$$

and

$$\sigma_i^0(a_i \mid \omega, J) > 0 \Rightarrow a_i \in \arg\max_{\tilde{a}_i\in A_i(\omega)} u_{i,i}(\omega, \tilde{a}_i) + \sum_{\omega'\neq\omega} \left(V_i^0(\omega') - V_i^0(\omega)\right) \varphi_i(\omega' \mid \omega, \tilde{a}_i). \tag{OA.4}$$

Define $\tilde{\sigma}_i(\cdot \mid \omega) = \frac{1}{\kappa} \sum_{J\in\mathcal{J}} \sigma_i^0(\cdot \mid \omega, J)$ for all $i = 1, \ldots, N$ and all $\omega \in \Omega$, and note that $(\tilde{\sigma}, V^0)$ is a solution to the limit conditions (3.2) and (3.3). Since $\mathrm{Equil}^0(< u, p, q, \rho >)$ only contains strict solutions, the profile $\tilde{\sigma} = (\tilde{\sigma}_i)_{i=1}^N$ must be a strict solution and thus the maximization problem in equation (OA.4) has a unique solution. Therefore $\sigma_i^0(a_i \mid \omega, J)$ does not depend on J and σ^0 is simple. In particular, $\sigma^0 \in \mathrm{Equil}^0(< u, p, q, \rho >)$ and therefore there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, $\sigma^\Delta \in \mathrm{Equil}^0(< u, p, q, \rho >)$. To see the converse, note that $\mathrm{Equil}^0(< u, p, q, \rho >)$ has a finite number of pure solutions that are all strict. For any solution (σ^0, V^0) to the limit conditions (3.2) and (3.3), σ^0 satisfies the conditions for a (simple) Markov perfect equilibrium of a separable dynamic game with noisy transitions and the non-partition protocol of moves $< \mathcal{J}, \mathcal{P} >$ since the continuation values in such a game converge to V^0 and, as a result, the incentive constraints are satisfied if $\Delta > 0$ is sufficiently small. ■