Revenue maximization with heterogeneous discounting: Auctions and pricing*

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Abstract

We characterize the revenue maximizing mechanism in an environment with private valuations and asymmetric discount factors. The optimal mechanism combines auctions to encourage competition and dynamic pricing to screen of buyers' valuations. When buyers are ex-ante symmetric and the seller is more patient than the buyers, the optimal mechanism takes a remarkably simple form. The seller runs a *modified* second price auction and allocates the item to the highest bid buyer if and only if the second highest bid exceeds the reserve price. The winning buyer pays the second highest bid. If the item is not sold in the auction, the seller posts a price path that depends on the second highest bid. The item is then allocated to the highest bid buyer at a strictly positive time. Our results imply that, for a patient seller, auctions and pricing schemes are complements and caution against the presumption that it is ex-ante optimal to commit not to trade when an auction fails.

KEYWORDS: dynamic mechanism design, auctions, pricing, discount rates

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1 Introduction

Most of the mechanism design literature assumes that contracting parties discount time at the same rates. Yet, this assumption may fail in several important applications. For example, agents may borrow at different interest rates (imperfect capital markets), or they may (agree to) disagree on the probability with which the negotiation continues, and this translates into heterogeneous time discounts.[1] While the role of differences in time preferences has been explored in models ranging from bargaining theory and finance to repeated games and revenue management (Rubinstein, 1982; Harrison and Kreps, 1978; Lehrer and Pauzner, 1999; Talluri and Van Ryzin, 2006), little is known about how different discount rates shape outcomes in a multi-agent mechanism design framework. The goal of this paper is to close this important gap.

We consider the following model. There is one seller holding a single object and $N \geq 1$ buyers. Types are independent and valuations are private. The seller and buyers discount rates may be different, but the model is otherwise standard (Myerson, 1981). Transfers and assignments occur simultaneously so neither party can lend to the other. A mechanism maps reports to a decision of who receives the item, when the item is delivered, and how much the buyer pays when the transaction takes place.

Our main contribution is to characterize the revenue maximizing mechanism. When the seller is less patient than the buyers, and buyers are ex-ante symmetric, the optimal mechanism is a standard second price auction with a reserve price. In contrast, when the seller is more patient than the buyers, the optimal mechanisms consists of two steps. At time 0, the seller runs a *modified* second price auction with a reserve price. In this auction, the seller assigns the good to the bidder with the highest bid and charges the second highest bid, whenever the latter exceeds the reserve price. If the second highest bid falls short of the reserve price, the auction fails to allocate the object. In this case, the seller assigns the good to the bidder with the highest valuation who pays a price determined by the second highest bid and receives the good at a strictly positive time. Figure 1 illustrates the optimal mechanism. We show that the optimal mechanism generates more revenue than any static mechanism and cannot be implemented through a simple posted price path. We also extend our analysis to the case in which buyers are ex-ante asymmetric.

A key economic insight emerging from our analysis is that auctions and pricing schemes are complements for a patient seller. Indeed, in our model the auction stage allows the seller to extract rents through competition, whereas the dynamic pricing stage is used to screen buyers through time. Our results therefore caution against the presumption that for a revenue maximizing seller, it is ex-ante optimal to commit not to trade after an auction fails.[2]

Since seller and buyers have different discount rates, our mechanism design problem is not

[1] Prior research also suggests that costumers's discount rates are substantially low in some industries. Yao et al. (2012) show that consumers' discount factors for telecommunication services are about 0.9 per week, which is probably lower than the discount rates of companies.

[2] Previous literature argues that for a seller with full commitment power, the revenue maximizing policy is a static second price auction (Stokey, 1979; Bulow, 1982).

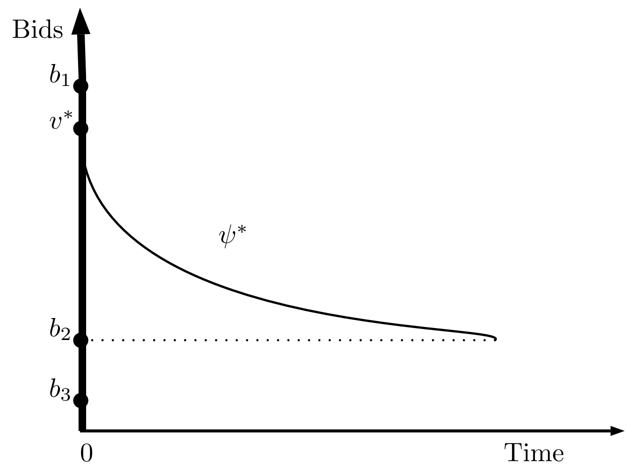

Figure 1: Implementation of the optimal mechanism for ex-ante symmetric buyers. The seller runs a modified second price auction with reserve price v^* at time 0. If the second highest bid is lower than v^*, the seller preannounces a contingent pricing scheme ψ^* which depends on the second highest bid. The item is allocated at time 0 if and only if the highest bid b_1 is above v^*.

quasilinear. In particular, the approach by Myerson (1981) or any of its multiple extensions are not applicable. One of the key innovations in the paper is the concept of *dynamic marginal revenue*. Similar to its static counterpart (Bulow and Roberts, 1989), the dynamic marginal revenue measures the rents that a seller can extract in any incentive compatible mechanism. In contrast, the dynamic marginal revenue depends on the allocation rule that determines the time at which the item is allocated as a function of types.[3]

When the seller is more impatient, the dynamic marginal revenue is always below the standard static marginal revenue and, as a result, the best that the seller can do is to run a static optimal mechanism. Thus, for ex-ante symmetric buyers, the optimal mechanism can be implemented by a second price auction with reserve price. When the seller is more patient than the buyers, delaying trade involves nontrivial tradeoffs. Delaying trade for a type means that the seller charges less and receives the payment later. On the other hand, delaying trade changes the dynamic marginal revenue of higher types, thus allowing the seller to extract more rents (the incentive compatibility constraint is binding for high types pretending to be low types). We formally capture this tradeoff by deriving necessary and sufficient conditions for optimality.

From a mathematical standpoint, our analysis is novel. Standard results from calculus of variations and optimal control impose strong convexity restrictions that are unlikely to hold in nontrivial applications of our mechanism design framework. Instead, we address the problem by discretizing the typespace and taking the limit as the discretization gets finer. We expect that this technical contribution may be useful in other mechanism design

[3]The static marginal revenue only depends on the distribution of types but not on the allocation rule of the mechanism.

problems in which the usual quasilinearity assumption does not hold.

The mechanism design formulation of our problem generates several comparative statics results. We prove that as the seller becomes arbitrarily patient, the optimal mechanism extracts virtually the full surplus. Intuitively, an arbitrarily patient seller can slowly screen buyers who must give up substantial surplus to accelerate trade. We also illustrate our results with a numerical example and explore the relative relevance of the auction stage for the optimal revenues. Our simulations show that when buyers and seller have similar discount rates, the auction stage plays a major role generating revenues. Yet, even when discount rates are similar, the dynamic pricing scheme generates a significant share of revenues. When buyers are much more impatient than the seller, the item is rarely allocated in the auction phase and most of the seller's rent comes from dynamically screening high buyers using a posted price.

Since utility is nontransferable in our model, our paper connects to the auction literature that relaxes the quasilinearity assumption (Maskin and Riley, 1984; Laffont and Robert, 1996; Pai and Vohra, 2014; Baisa, 2017). In those papers, randomizations play a key role in the optimal mechanisms. We add to this literature by expositing a setup with closed form solutions and, more importantly, by bringing the role of time (instead of randomizations) to the design of optimal mechanisms.

Recent work has also highlighted the role of dynamics in mechanism design. As opposed to our work, in which the dynamics are founded on the difference in discount rates, researchers have suggested other reasons to explain trade over time. The most common assumptions in this literature are that consumers arrive stochastically and goods may perish or there is a deadline for trade (Aviv and Pazgal, 2008; Elmaghraby et al., 2009; Osadchiy and Vulcano, 2010; Correa et al., 2016; Board and Skrzypacz, 2016; Gershkov et al., 2017; Briceño-Arias et al., 2017). Other assumptions to explain this phenomenon are that valuations may change over time (Pavan et al., 2014; Garrett, 2016) or that buyers may be short-lived (Pai and Vohra, 2013). On the other hand, the literature on durable goods with heterogeneous discounting has observed that delay on trade occurs in specific environments (Stokey, 1979; Landsberger and Meilijson, 1985; Shneyerov, 2014). Our paper extends these results by deriving the optimal mechanism in a general multi-agent framework .

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 introduces the dynamic marginal revenue and solves the one-buyer case. Section 4 solves the N-buyer case allowing for heterogeneous buyers. Section 5 illustrates the solutions under parametric restrictions. Section 6 concludes. The appendix contains omitted proofs.

2 Model

We consider a model with $N \geq 1$ buyers and one seller. The seller owns an indivisible object and places no value for it. Buyer i has a private valuation for the asset $v_i \in [0, 1]$. We consider independent private values where the valuation for buyer i is drawn independently according to a continuous distribution F_i on $[0,1]$.[4] We denote the density of F_i by f_i and assume the following regularity conditions.

Assumption 1 *For all $v_i \in [0,1]$, we have*

a. $f_i(v_i) > 0$ *(full support).*

b. $f_i(v_i)$ *is continuously differentiable.*

c. $MR_i(v_i) := v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ *is non-decreasing in* v_i.

Parts a. and c. are standard in the literature whereas part b. is a technical restriction.

All players are infinitely lived and can trade at any time $t \geq 0$. If there is trade at period t between the seller and buyer i at price $p \in \mathbb{R}$, then the game ends and the payoffs are given by

$$\begin{aligned} U_S(p,t) &= \exp(-\lambda t)p, \\ U_{B_i}(p,t,v_i) &= \exp(-\mu_i t)(v_i - p), \\ U_{B_j}(p,t,v_j) &= 0 \qquad \text{for } j \neq i, \end{aligned}$$

where $\mu_i, \lambda \in \mathbb{R}_{++}$ are discount rates. While symmetric discount rates is a frequent simplifying assumption in the literature, we allow for heterogeneous discount rates. This additional generality can be motivated by the following examples:

I. Imperfect capital markets. Financial markets are imperfect, and the interest rates at which seller and buyers may borrow need not coincide. In this type of model, λ and μ_i can be interpreted as the interest rates at which seller and buyer i may borrow.

II. Heterogeneous beliefs. Seller and buyers may know that the execution of the contract may become unfeasible at some time t. For example, the seller may leave the market, in which case the mechanism is not executed, and the buyers obtain zero payoffs. Such event occurs at an exponentially distributed random time. However, the seller and buyers may have different estimates of the intensity of the exponential distribution. In other words, players agree to disagree. In this case, the exponential rates λ and μ_i can capture the seller's and buyers's beliefs that the game will no longer be executed.

We restrict attention to mechanisms where transfers and allocations occur simultaneously. This restriction is crucial for our results as, otherwise, a patient seller could generate unbounded revenues by using a Ponzy scheme. The assumption that transfers and allocations are simultaneous implies the idea that buyers do not borrow from sellers and it is likely to hold in environments in which contracts that deliver the good before payments cannot be enforced. Our restriction is motivated by the observation that in practice sellers rarely lend money to buyers.[5]

[4] The only non-trivial assumption regarding the support of F_i is that it intersects the seller's valuation for the object (non-gap case).

[5] Most auction platforms do not lend money to buyers presumably because it is hard for them to enforce payments. Note that, particularly in developing countries, retail companies do provide credit to clients.

When $\lambda \neq \mu_i$ for some i, utility is nontransferable. To see this, suppose there is only one buyer with discount rate μ, and compute the value $H(u)$ that maximizes the seller's payoff subject to the constraint that the buyer receives utility at least u:

$$H(u) := \max_{(t,p)\in\mathbb{R}_+\times\mathbb{R}} e^{-\lambda t}p$$

subject to

$$e^{-\mu t}(v-p) \geq u.$$

By varying u, we can characterize the whole Pareto frontier as the set

$$\Big\{(u, H(u)) \mid u \in \mathbb{R}\Big\}.$$

It is simple to see that when $\lambda \neq \mu$, the function $H(u)$ is equal to

$$H(u) = \Big(\frac{\lambda}{\lambda-\mu}\frac{v}{u}\Big)^{-\lambda/\mu}\Big(v - \big(\frac{\lambda}{\lambda-\mu}\frac{v}{u}\big)\Big).$$

Thus, the slope of the Pareto frontier is not -1 and therefore utility is nontransferable.

The mechanism design problem

We solve the revenue maximizing problem using a mechanism design approach. We restrict attention to deterministic mechanisms and show this is without loss of generality. A *deterministic* mechanism is a family of functions $(p_i, x_i, \tau_i) : [0,1]^N \to \mathbb{R}\times\{0,1\}\times\mathbb{R}\cup\{\infty\}$, for $i = 1,\dots,N$, with $\sum_{i=1}^N x_i(v) \leq 1$, such that when players report $v \in [0,1]^N$, the good is assigned to player i whenever $x_i(v) = 1$, at time $\tau_i(v)$ and at the price $p_i(v)$.

Since our model is not quasilinear, the definition of incentive compatible mechanisms is subtle. In particular, the well known results establishing equivalence between Bayesian and ex-post incentive compatibility in quasilinear environments do not hold (Gershkov et al., 2013). We restrict attention to mechanisms that are ex-post incentive compatible: if buyers's valuations are common knowledge, it is in the best interest of each buyer to report their type. The restriction to mechanisms satisfying ex-post constraints is of interest because buyers often do not know the distributions of valuations of other buyers. We also assume that any buyer can always refuse to participate in the mechanism and gets zero payoffs. We therefore also restrict attention to mechanisms in which buyers get nonnegative payoffs.[6] The following definition introduces the family of mechanisms we focus on.

Definition 1 *A deterministic mechanism* $(p_i, x_i, \tau_i)_{i=1}^N$ *is* Ex-Post Incentive Compatible *(E.P.I.C.) if*

Thus our model seems sensible for most auction platforms but not necessarily for retailers. Exploring the industry conditions under which sellers can lend money to buyers is interesting, but beyond the scope of this paper.

[6] This restriction is also formulated ex-post. Our results would not change if the participation constraint were formulated in an interim sense.

a. For all $i = 1, \ldots, N$ *and all* $v \in [0,1]^N$,

$$v_i \in \arg\max_{v' \in [0,1]} \exp(-\mu_i \tau_i(v', v_{-i})) \big[v_i x_i(v', v_{-i}) - p_i(v', v_{-i}) \big] \, .$$

b. For all $i = 1, \ldots, N$ *and all* $v \in [0,1]^N$,

$$\exp(-\mu_i \tau_i(v_i, v_{-i})) \big[v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \big] \geq 0.$$

A randomized mechanism is a probability distribution over deterministic mechanisms. We say that a randomized mechanism is E.P.I.C. if each player finds it optimal to report his type and to participate even when the whole type profile and the mechanism randomization is common knowledge.

The seller maximizes the expected revenue

$$\mathbb{E}\big[\sum_{i=1}^{N} \exp(-\lambda \tau_i(v_i, v_{-i})) p_i(v_i, v_{-i}) \big] \tag{1}$$

over all randomized E.P.I.C. mechanisms.

The following result shows the restriction to deterministic E.P.I.C. mechanisms is without loss of generality.

Lemma 1 *To find a solution to Problem (1), it is without loss to restrict attention to deterministic E.P.I.C. mechanisms.*

In what follows, we restrict attention to deterministic E.P.I.C. mechanisms. To economize on notation, we call a deterministic E.P.I.C. mechanisms simply E.P.I.C. mechanism.

The envelope theorem and a reformulation

An attractive feature of E.P.I.C. mechanisms is that the standard envelope approach (Milgrom and Segal, 2002) holds and thus transfers are completely determined by the allocation rule (and the utility level of the lowest type).

Lemma 2 *A mechanism* $(p_i, x_i, \tau_i)_{i=1}^N$ *satisfies Definition 1 part a. if and only if*

a. For every $v_{-i} \in [0,1]^{N-1}$, $\exp(-\mu_i \tau_i(v_i, v_{-i})) x_i(v_i, v_{-i})$ *is non-decreasing on* v_i.

b. For all $v \in [0,1]^N$,

$$\begin{aligned} &\exp(-\mu_i \tau_i(v_i, v_{-i}))(v_i x_i(v_i, v_{-i}) - p(v)) \\ &= U_i(0, v_{-i}) + \int_0^{v_i} \exp(-\mu_i \tau_i(s, v_{-i})) x_i(s, v_{-i}) ds \end{aligned}$$

where $U_i(0, v_{-i})$ *is the utility attained by the lowest type* $v_i = 0$.

Using Lemma 2, Fubini's theorem, and the fact that $U_i(0, v_{-i}) = 0$ for any optimal mechanism, we can reformulate the problem of optimal mechanism design as

$$\max \Phi(x, \tau) \tag{2}$$

subject to

$$\exp(-\mu_i \tau_i(\cdot, v_{-i}))x_i(\cdot, v_{-i}) \text{ is non-decreasing and } \sum_{i=1}^{N} x_i(v) \leq 1$$

where

$$\Phi(x, \tau) := \mathbb{E}\left[\sum_{i=1}^{N} x_i(v) \exp(-\lambda \tau_i(v)) \left(v_i - \frac{\int_{v_i}^{1} \exp((\mu_i - \lambda)\tau_i(s, v_{-i})) f_i(s) ds}{\exp((\mu_i - \lambda)\tau_i(v)) f_i(v_i)}\right)\right].$$

The rest of the paper characterizes the solution to this problem.

3 Solving the one buyer case: the dynamic marginal revenue

To solve Problem (2), we first analyze the problem for the one buyer case. We omit the index i (and, for example, write μ for the discount rate of the buyer and x for the allocation).

For the one buyer case, E.P.I.C. mechanisms can be fully characterized by the temporal allocation rule.

Lemma 3 *For any E.P.I.C. mechanism (x, τ, p) there is a (weakly) revenue-improving E.P.I.C. mechanism $(\tilde{x}, T, p)$, with $\tilde{x}(v) = 1$ for all $v \in [0, 1]$.*

Using Lemma 3, the seller's problem can be restated as

$$\max \Phi(T) := \mathbb{E}\left[\exp(-\lambda T(v)) \left(v - \frac{\int_{v}^{1} \exp((\mu - \lambda)T(s)) f(s) ds}{\exp((\mu - \lambda)T(v)) f(v)}\right)\right] \tag{3}$$

subject to

$$T \colon [0, 1] \to \mathbb{R}_+ \cup \{\infty\}, \quad T \text{ is non-increasing.}$$

For any $T \colon [0, 1] \to \mathbb{R}_+$, we define the *dynamic marginal revenue* of type v as

$$MR(v|T) = v - \frac{\int_{v}^{1} \exp((\mu - \lambda)T(s)) f(s) ds}{\exp((\mu - \lambda)T(v)) f(v)}.$$

Then, $\Phi(T) = \mathbb{E}[\exp(-\lambda T(v)) MR(v|T)]$.

The dynamic marginal revenue summarizes the information rents that can be extracted from type v given an allocation rule T. This concept naturally extends the static one: for $\lambda = \mu$, $MR(v|T) = v - \frac{1-F(v)}{f(v)}$ as in Myerson (1981), and Bulow and Roberts (1989).

The following preliminary result shows that when the seller is more impatient than the buyer, all trade occurs at $t = 0$.

Lemma 4 *If $\lambda \geq \mu$, in the optimal mechanism all trade occurs at $t = 0$. Types $v < \bar{v}$ never trade, with $\bar{v} = \frac{1-F(\bar{v})}{f(\bar{v})}$.*

Put another way, when the buyer is more patient than the seller, the seller does not benefit from inter-temporal price discrimination. The proof consists on bounding the dynamic marginal revenue by its static counterpart. Note that since $\mu \leq \lambda$ and T is non-increasing

$$MR(v|T) \leq v - \frac{1 - F(v)}{f(v)}$$

This means that for any trading mechanism, the seller's expected payoff is at most

$$\Phi(T) \leq \mathbb{E}\Big[\exp(-\lambda T(v))MR(v)\Big]$$

The term on the right side is maximized by setting

$$T(v) = \begin{cases} 0 & \text{if } v < \bar{v} \\ \infty & \text{if not.} \end{cases}$$

Thus, when $\lambda \geq \mu$, the optimal trading mechanism is a simple take-it-or-leave it offer as in Myerson (1981).

The optimal mechanism changes radically when the seller is more patient than the buyer. By delaying trade, the seller can extract virtually all the surplus from any type v buyer. The optimal timing of trading solves a non-trivial trade-off. To see this, suppose that F is the uniform distribution and consider the non-increasing policy $T(v) = (1 - v)\kappa$, where $\kappa > 0$. Then,

$$MR(v|T) = v - \frac{1}{\kappa(\lambda - \mu)}\Big(1 - \exp(-\kappa(\lambda - \mu)v)\Big).$$

When $\lambda < \mu$, by taking $\kappa \to \infty$, the seller delays trade and the dynamic marginal revenue from type v goes to v. Yet, delaying trade is costly as the seller discount profits using a rate $\lambda > 0$. The optimal trading policy must balance these two forces.

We now characterize some properties of the optimal trading policy T^*.

Proposition 1 *Let T^* be a solution to (3) and define $\underline{v} = \inf\{v \mid T^*(v) < \infty\}$. If $\mu > \lambda$, then $\underline{v} = 0$ and $\lim_{v \to 0} T^*(v) = \infty$.*

Proposition 1 shows that when the seller is more patient than the buyer, in the optimal mechanism all types trade. This proposition deviates from the standard results from static mechanism design problems in which the seller commits not to trade to maximize revenue. When $\mu > \lambda$, the seller extracts higher surplus by distorting the timing of trade for all types. Intuitively, because the seller is relatively more patient, the cost for delaying a specific type is lower than the benefit of extracting higher rents from higher types (because consumers are more impatient, it is less appealing for them to mimic types that are receiving the object further in the future).

3.1 Optimal trading dynamics

We now characterize the trading dynamics in the optimal mechanism for the one buyer case. To find this optimal mechanism, we first ignore the monotonicity constraint in Problem (3) and solve the relaxed problem first

$$\max_{T\colon [0,1]\to\mathbb{R}} \int_0^1 \exp(-\lambda T(v))\Big(v - \frac{\int_v^1 \exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)T(v))f(v)}\Big)f(v)dv. \tag{4}$$

To derive a necessary optimality condition, define $v^* = \inf\{v \mid T^*(v) = 0\}$. Take $v < v^*$ and slightly perturb $T^*(v) > 0$ to $T^*(v) + \epsilon$, with $\epsilon > 0$. Approximating to the first order, this perturbation has three effects on the seller expected revenue.

First, by changing the trading period, payoffs occur later and the expected revenue is reduced by the quantity

$$-\lambda \exp(-\lambda T^*(v))MR(v|T^*)(v)f(v)\epsilon.$$

Second, by delaying trade with agent v, higher types have less incentive to mimic type v. Thus, from the seller's perspective the benefit from relaxing the incentive constraint of types $s > v$ equals to

$$\exp(-\lambda T^*(v))(\mu-\lambda)\frac{\int_v^1 \exp((\mu-\lambda)T^*(s))f(s)ds}{\exp((\mu-\lambda)T^*(v))}\epsilon.$$

Finally, notice that in an E.P.I.C. mechanism, agent v's surplus is pin-down by the assignment for lower types $s < v$ (Lemma 2). Hence, to encourage type v to wait ϵ more units of time, the seller has to decrease the price $p(v)$, and this has a negative impact on expected revenues:

$$-(\mu-\lambda)\exp((\mu-\lambda)T^*(v))f(v)\int_0^v \exp(-\mu T^*(s))ds\epsilon.$$

Combining these three effects and simplifying some terms, we deduce that any candidate solution T^* must satisfy that for all $v \in [0, v^*]$

$$v - \frac{\int_v^1 \exp((\mu-\lambda)T^*(s))f(s)ds}{\exp((\mu-\lambda)T^*(v))f(v)} = (1-\frac{\lambda}{\mu})\Big(v - \exp(\mu T^*(v))\int_0^v \exp(-\mu T^*(s))ds\Big). \tag{5}$$

The left hand side of the above equation equals the optimal dynamic marginal revenue $MR(v|T^*)$. Using Lemma 2, the term on the right hand side equals $(1-\frac{\lambda}{\mu})$ times the optimal pricing $p^*(v)$. Therefore, in the optimal mechanism, we have that transfers and information rents are connected by the equation:

$$MR(v|T^*) = (1-\frac{\lambda}{\mu})p^*(v) \quad \text{ for } v \le v^* .$$

To solve the integral Equation (5) we need to compute $v^* \in [0,1]$ and $T^*\colon [0,1] \to \mathbb{R}$, with the constraint $T^*(v) = 0$ for $v > v^*$. While the derivation of Equation (5) is simple and intuitive, the proof that is a valid necessary and sufficient condition for our problem is more involved.

Theorem 1 *For $\mu > \lambda$ there exists a unique solution to (v^*, T^*) to Equation* (5)*. Moreover, T^* is non-increasing, twice-continuously differentiable in $]0, v^*[$, and solves Problem* (3)*.*

The proof of this theorem is technical. We now sketch the key steps in the proof; all details are in the appendix. First, we discretize the space of types and consider a discrete version of Problem (3). The discrete problem always has a solution, which satisfies a first order condition. We show that the family of solutions of the discrete problems has a pointwise converging limit, and that the limit is a continuous function that satisfies Equation (5) and solves Problem (3). Second, using results from ordinary differential equations, we deduce that the integral system given by Equation (5) and the constraint $T(v) = 0$ for $v \leq v^*$ has a unique solution (v^*, T^*).

The proof strategy for Theorem 1 is novel. Standard existence theorems do not apply to our problem. Results from calculus of variations impose strong convexity restrictions that are unlikely to be met in interesting applications of our model (Ekeland and Temam, 1999). We therefore need to discretize the space of types and take the limit as the grid grows large to derive our necessary and sufficient optimality conditions.

A direct consequence of Theorem 1 is that the monotonicity constraint in Problem (2) is not binding. Hence, any solution to the seller's problem satisfies Equation (5).

The necessary and sufficient integral in Equation (5) is not easy to work with. It is an integral equation in which the boundary condition is left open. We show that, under mild conditions, the optimal transfer scheme can be found by solving a standard ordinary differential equation.

Assumption 2 *The distribution F is such that the limit $L := \lim_{v \to 0} \frac{f'(v)v}{f(v)}$ exists.*

The following result provides a simple way to find solutions to our mechanism design problem.

Proposition 2 *Suppose that Assumption 2 holds, and define l as the unique solution, in the unit interval, of the equation*

$$2 + L(1 - (1 - \frac{\lambda}{\mu})x) - (1 - \frac{\lambda}{\mu})(2x + \frac{\lambda}{\mu}\frac{x^2}{1-x}) = 0 \, . \tag{6}$$

Then, the following hold:

1. *The differential equation*

$$\begin{cases} 2 + \frac{f'(v)}{f(v)}\Big(v - (1 - \frac{\lambda}{\mu})p(v)\Big) - (1 - \frac{\lambda}{\mu})\Big(2p'(v) + \frac{\lambda}{\mu}\frac{p(v)p'(v)}{v - p(v)}\Big) = 0 \\ p(0) = 0, \ p'(0) = l \end{cases} \tag{7}$$

   has a unique solution $\hat{p} \colon [0, 1] \to \mathbb{R}$.

2. *The optimal cutoff v^* can be computed as the unique solution to*

$$v - \frac{1 - F(v)}{f(v)} = (1 - \frac{\lambda}{\mu})\hat{p}(v).$$

3. The optimal transfer rule solves

$$p^*(v) = \min\{\hat{p}(v^*), \hat{p}(v)\}$$

4. The optimal trading time satisfies

$$T^*(v) = \int_v^{v^*} \frac{\hat{p}'(s)}{\mu(s - \hat{p}(s))} ds.$$

To complement the above proposition, we present a simple formula that characterizes the seller's profit $\pi^* := \Phi(T^*)$. From incentive compatibility, we have that for $v \geq v^*$ the transfer rule has to be constant, thus,

$$\pi^* = (1 - F(v^*))p^*(v^*) + \mathbb{E}[\exp(-\lambda T^*(v))p^*(v)\mathbb{1}_{\{v \leq v^*\}}].$$

Using Equation (5), we obtain that

$$\pi^* = \left(1 - \frac{\lambda}{\mu}\right)^{-1} \big((1 - F(v^*))MR(v^*) + \mathbb{E}[\exp(-\lambda T^*(v))MR(v|T^*)\mathbb{1}_{\{v \leq v^*\}}]\big) .$$

Taking a different route, we can recompute the seller's payoff from the envelope approach derived in Equation (2), obtaining

$$\begin{aligned}\pi^* &= \mathbb{E}[e^{-\lambda T^*(v)}MR(v|T^*)(v)\mathbb{1}_{\{v \leq v^*\}}] + \mathbb{E}[\exp(-\lambda T^*(v))MR(v|T^*)(v)\mathbb{1}_{\{v > v^*\}}] \\ &= \mathbb{E}[\exp(-\lambda T^*(v))MR(v|T^*)(v)\mathbb{1}_{\{v \leq v^*\}}] + \mathbb{E}[MR(v)\mathbb{1}_{\{v > v^*\}}] .\end{aligned}$$

Combining these two equations we get the following result.

Corollary 1 *In the optimal mechanism, the seller's revenue is given by*

$$\pi^* = \frac{\mu}{\lambda}\frac{(1 - F(v^*))^2}{f(v^*)}.$$

Proof. Take the two equations above for π^*. Notice that $\mathbb{E}[MR(v)\mathbb{1}_{\{v > v^*\}}] = (1 - F(v^*))v^*$ and combine the equations to obtain π^* as function of v^*. ■

As discussed earlier, the key drivers of our results are that the buyer is more impatient than the seller and that the seller capitalizes this temporal-advantage by making a slow screening of the demand through time. Notice that the seller's advantage (the degree of impatience) only depends on the ratio $\frac{\mu}{\lambda}$ and not on the absolute values of the discount factors.[7] This result is natural: the unit of account in which the payoffs are computed (e.g., dollars) should be independent of the unit of time (e.g., months).

[7] From Proposition 2 we get that the distribution v^* is only a function of the $\frac{\mu}{\lambda}$ and the original distribution. Therefore, Corollary 1 implies that the revenue that the seller obtains only depends on this ratio.

A remarkable feature of the optimal mechanism is that the threshold v^* entirely determines the optimal mechanism. Thus, it is natural to ask how v^* and $\bar{v}$ compare, where $\bar{v}$ is the optimal reserve price in the classical Myerson setup. We assert that $v^* > \bar{v}$. Indeed, from Equation (7) we get that p^* is strictly positive for $v > 0$, implying that $MR(v^*) > 0$. Hence, by Assumption 1 c., the inequality holds. This inequality has a simple explanation: the dynamic marginal revenue differs from the static benchmark only after time 0, and moreover, it is constructed in such a way that is positive for all types $v > 0$. Consequently, the only way to transform the dynamic marginal revenue of $\bar{v}$ in a strictly positive one is by allocating it after time 0.

Finally, in order to get a sense about how the solution changes as the ratio λ/μ varies, we analyze the limit case $\lambda/\mu \to 0$. Because profits are bounded, Corollary 1 implies that $v^* \to 1$ as $\lambda/\mu \to 0$. Thus, as the buyer becomes more impatient relative to the seller, trade is likely to occur after time 0. We assert that when $\lambda/\mu \to 0$, $p^*(v)$ converges to v for all $v \in [0, 1]$. To show this, notice that the seller's payoff only depends on the ratio λ/μ. Thus, without loss of generality, we can fix λ and take $\mu \to \infty$. The characterization of T^* in Proposition 2, part 4, and the fact trade happens after time 0 with probability one, imply that $p^*(v)$ converges to v for all v.[8] Therefore, the seller extracts all the rents in the economy.

4 The optimal mechanism

In this section, we use previous results to find the optimal mechanism for the N-buyer case. The following lemma shows that the rents that the seller can extract from the buyers are computed by treating each buyer individually. In other words, conditional on giving the item to buyer i, the seller's profit is the present value of the single buyer optimal dynamic marginal revenue.

Lemma 5 *The seller's problem is equivalent to solving*

$$\max_{x_i(\cdot)} \sum_{i=1}^{N} \mathbb{E}\left[x_i(v)\exp(-\lambda T_i^*(v_i)) MR_i(v_i|T_i^*)\right] .$$

subject to

$$\sum_{i=1}^{N} x_i(v) \leq 1 \text{ and } x_i(v) \geq 0,$$

where T_i^* is the solution for the single buyer case with distribution F_i.

Lemma 5 states that, in the optimal mechanism, the reports of all players only influence who gets the item but not when the item is allocated. The dynamic decision only depends

[8] From Proposition 2, part 4, p^* converges for almost every $v \in [0, 1]$. Since p^* is continuous, we extend the conclusion to every element in the interval.

on each buyer's type. Fixing T_i^* as the optimal trading time for the one buyer case with distribution F_i (as in Section 3), we can solve for the optimal mechanism.

Theorem 2 *The solution* (x^*, τ^*, p^*) *to Problem* (2) *is given by*

a. $\tau_i^*(v) = T_i^*(v_i)$.

b. $x_i^*(v) = \begin{cases} \frac{1}{|W|} & \text{if } i \in W := \arg\max_j \exp(-\lambda T_j^*(v_j)) MR_j(v_j|T_j^*) \\ 0 & \text{otherwise} \end{cases}$.

c. $p_i^*(v) = v_i x_i^*(v) - \exp(\mu_i T_i^*(v_i) \int_0^{v_i} \exp(-\mu_i T_i^*(s)) x_i^*(s, v_{-i}) ds$.

Proof. Immediate from Lemma 5. ■

When $\lambda > \mu_i$, the seller always allocates the item. In this case, when the winner's valuation v_i is lower than v_i^* then trade is delayed.

Implementation for the symmetric case

We now characterize the optimal mechanisms for the ex-ante symmetric case.

Proposition 3 *Suppose that valuations are drawn from the same distribution* F *and that players discount rates coincide* $\mu_i = \mu > \lambda$, *for* $i = 1, \dots, N$. *Then the following protocol is a revenue maximizing mechanism:*

1. *At* $t = 0$, *run a modified second price auction. Where the highest bid* $b^{(1)}$ *receives the item and pays the second highest reported bid* $b^{(2)}$ *if* $b^{(2)} \geq v^*$.

2. *If* $b^{(2)} < v^*$, *preannounce a pricing scheme*

$$\psi^*(t; b^{(2)}) = T^{*-1}(t) - \exp(\mu t) \int_{b^{(2)}}^{T^{*-1}(t)} \exp(-\mu T^*(s)) ds \ , \quad t > 0.$$

The highest-valuation bidder trades at time $T^*(b^{(1)})$ *and pays* $\psi^*(T^*(b^{(1)}); b^{(2)})$.

Proof. Immediate from Lemma 5. ■

The optimal mechanisms runs a modified second price auction with reserve price v^* at $t = 0$. The difference with a standard second price auction comes from the entry decision generated by the reserve price. In a standard second price auction, the auction allocates if at least one buyer bids above the reserve price. In our modified second price auction, it is also important whether the second highest bid is above the reserve price. Thus, if two or more bids are above v^*, the auction allocates the good to the bidder with the highest bid, who pays the second highest bid $b^{(2)}$. If not, the auction does not allocate the object, and the seller negotiates a price and a time at which the transaction occurs.[9] The highest valuation

[9]Note that the seller commits to the protocol and so she honors her terms of trade.

bidder pays a price determined by the second highest valuation $b^{(2)}$. The negotiation results in a transaction at time $T^*(b^{(1)})$.

The main message from Proposition 3 is that a patient seller should use an auction to extract rents through competition and a decreasing price path to screen valuations over time. This result casts doubt on the presumption that sellers should commit not to sell after an auction fails. Our results therefore may help rationalize why sellers sometimes commit to sell after an auction.[10]

5 Illustrative example

In this section, we characterize the optimal mechanism for two buyers whose valuations are uniformly distributed on $[0,1]$ and $\mu > \lambda$.

Dynamic Marginal Revenue

We first start by computing the dynamic marginal revenue for each buyer. Using Proposition 2, we compute the optimal temporal distortion and the respective dynamic marginal revenue. For the uniform case, we have that the optimal temporal distortion is

$$T^*(v) = \begin{cases} 0 & \text{if } v \geq v^* \\ \frac{2v^*-1}{\mu\left(1-v^*(1+\frac{\lambda}{\mu})\right)} \log(\frac{v^*}{v}) & \text{if } v_1 \leq v^*, \end{cases}$$

where

$$v^* = \frac{1}{1+\sqrt{\frac{\lambda}{\mu}(2-\frac{\lambda}{\mu})^{-1}}}\,.$$

The dynamic marginal revenue is

$$MR(v|T^*) = \begin{cases} 2v-1 & \text{if } v \geq v^* \\ \left(1-\sqrt{\frac{\lambda}{\mu}(2-\frac{\lambda}{\mu})^{-1}}\right)v & \text{if } v \leq v^*. \end{cases}\,.$$

Optimal allocation

The surplus that could be extracted from player i is $MR(v_i|T^*)$. Because both players have valuations drawn from the same distribution, (and the dynamic marginal revenue is monotone) we have that $x_i^*(v) = 1$ if and only if $v_i > v_{-i}$. When buyer i gets the item, the item will be delivered at time $T^*(v_i)$.

[10]One reason sellers may sell after an auction fails is lack of commitment. Yet, in some marketplaces sellers *commit to sell* if no buyer bids above a reserve price. For example, the platform mercadominero.cl sells second hand mining machinery and announces a dynamic pricing scheme at the beginning of each auction. For details see http://www.mercadominero.cl/sitio/procedimiento.pdf.

Dynamic implementation

Since players are ex-ante symmetric, the optimal mechanism can be implemented as follows. At time 0, run a modified second price auction: if $b^{(2)} \geq v^*$, the player with the highest bid gets the item and pays the second highest bid $b^{(2)}$; if not, post the dynamic pricing scheme

$$\psi^*(t; b^{(2)}) = v^* \frac{c}{1+c} \exp\left(-\frac{\mu t}{c}\right) + b^{(2)} \frac{1}{1+c} \left(\frac{b^{(2)}}{v^*}\right)^c \exp(-\mu t) \ , t \geq 0$$

where $c = \frac{2v^*-1}{1-v^*(1+\frac{\lambda}{\mu})}$.

Comparative statics

To illustrate the solution and gain some intuition, we plot figures containing the dynamic marginal revenues, the reserve price, the optimal price paths for different parameter settings, and the seller's profit as a function of the parameters.

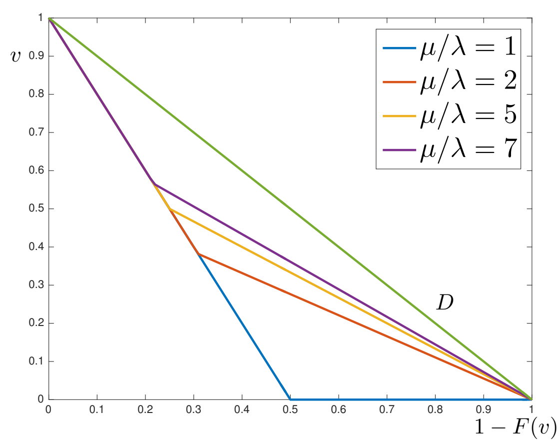

Figure 2: Dynamic Marginal Revenue for the uniform case, and $\lambda = 1$. The green line is the standard demand curve (complete information rents), and the blue line is the marginal revenue in the static case.

Figure 2 shows the optimal dynamic marginal revenue as the buyer's discount rate μ increases: the complete information rents correspond to the demand curve D depicted in green, whereas the static marginal revenue is depicted in blue. As μ/λ increases, the dynamic marginal revenue approaches to the demand curve D. On the other hand, when μ approaches λ, the dynamic marginal revenue approaches the static marginal revenue.

Figure 3 shows that the reserve price is increasing as a function of μ/λ. From the formula described at the beginning of this section, we observe that the reserve price only depends on this ratio. Moreover, when the discount rates are very different, the posted price part of the optimal mechanism becomes more relevant (and likely to end up being used). On

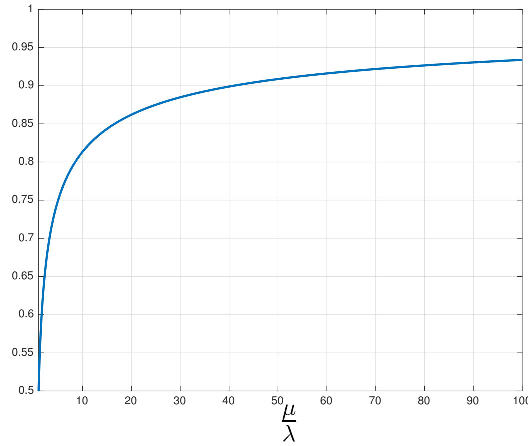

Figure 3: Reserve value v^* as a function of μ/λ.

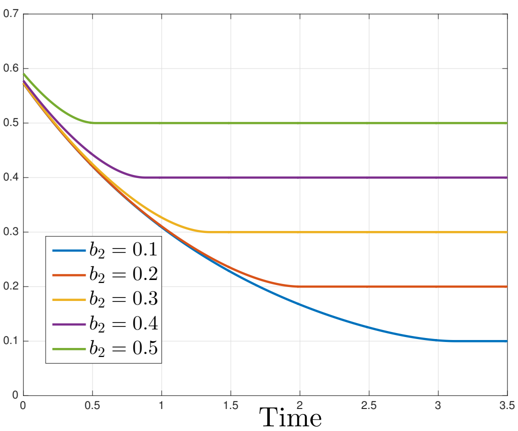

Figure 4: Optimal pricing ψ^* as function of the second highest bid b_2, when $b_2 < v^*$ and $\mu = 3$, $\lambda = 1$.

the contrary, as $\mu/\lambda \to 1$, the optimal dynamic mechanism approaches to the optimal static mechanism.

Regarding the optimal pricing ψ^*, Figure 4 shows the impact of the second highest bid b_2 in ψ^* (conditional that $b_2 < v^*$). We observe that as b_2 increases, so do trading prices.

Figure 5 shows the impact on the pricing scheme as a function of the disagreement in the discount rates. As μ/λ increases, the pricing scheme gets steeper and with higher values. On the other hand, as μ and λ get closer the pricing gets flatter, recovering the optimal static mechanism.

Figure 6 illustrates the relevance of the auction stage in the optimal revenues. The plot shows the percentage of the seller's revenue coming from the auction. We observe that for similar discount factors the auction is an important source of revenues for the seller. Yet,

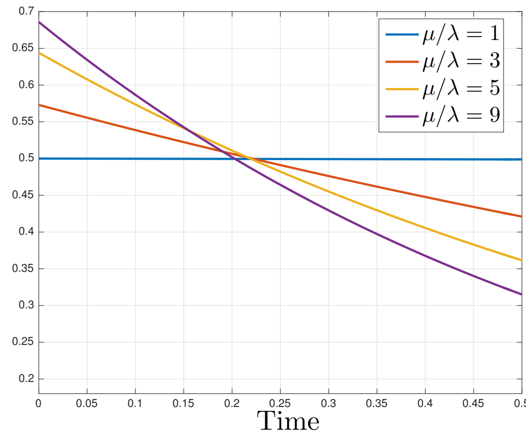

Figure 5: Optimal pricing ψ^* as function of the ratio $\frac{\mu}{\lambda}$, for $b^{(2)} = 0.2$ and $\lambda = 1$.

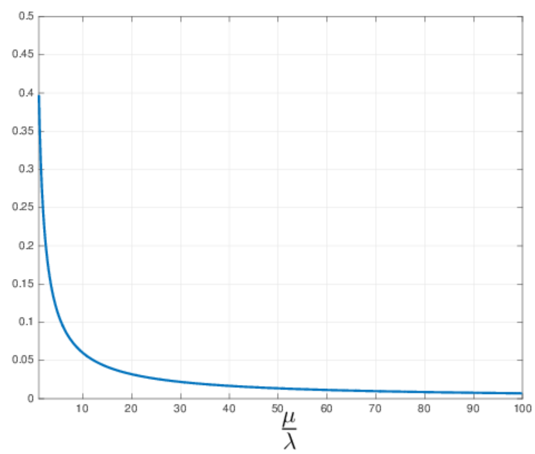

Figure 6: Fraction of the seller's revenue coming from the auction stage, as a function of μ/λ.

even when the discount factors are similar, the pricing stage brings a significant fraction of revenues. When buyers are significantly more impatient than the seller, the auction stage does not play an important role: the pricing stage provides most of the seller's rents. Thus, our results suggest that in markets where the seller and buyers have similar access to capital, auctions should play an important role; by contrast, in markets where buyers are significantly more impatient, posted prices should prevail.

6 Concluding remarks

We derive the revenue maximizing mechanism when a seller and buyers have different discount rates. The optimal policy crucially depends on the comparison between discount rates. When the seller is more impatient than the buyers, the optimal mechanism is a Myerson's optimal auction run at time zero (and trade is not delayed). When the buyers are more impatient than the seller, time is used as a screening device. The optimal mechanism allocates the item to the bidder with the highest discounted (optimal) dynamic marginal revenue. When buyers are ex-ante symmetric the mechanism takes a simple form: An auction with reserve price assigns at time 0 and, when the item is not sold, a dynamic pricing scheme awards the item to the highest-type buyer. Our results show that auctions and pricing schemes are complements for a patient seller and we introduce new tools to solve our mechanism design problem.

An interesting open question concerns the role of time in situations when both parties, seller and buyers, have private valuations about an asset. Consider for instance the setting described by Myerson and Satterthwaite (1983), where a negotiation between a seller and a buyer is mediated by a broker. Both parties have private and independent valuations about the underlying asset. The broker's objective is to maximize the expected profit she can extract from the transaction.[11] To illustrate the relevance of the timing decision in this context, we show that if the broker is more patient than the other agents, time is a useful tool for the broker. On the one hand, if the broker restricts attention to static mechanisms, his profit would be given by

$$\pi_{\text{Static}} = \mathbb{E}\left[(MR(v_b) - MC(v_s))\,\mathbb{1}_{\{MR(v_b)\geq MC(v_s)\}}\right],$$

where $MC(v_s) = v_s + \frac{F(v_s)}{f(v_s)}$ (Myerson and Satterthwaite, 1983, Section 5). On the other hand, if the broker restricts attention to static allocations for the seller, $\tau_s \equiv 0$, but screens the buyer using $\tau_b(v_s, v_b) = T^*(v_b)$ (where T^* is the temporal allocation described in Section 3), the broker's profit is given by

$$\mathbb{E}\left[\left(e^{-\lambda_{\text{Broker}} T^*(v_b)} MR(v_b|T^*) - MC(v_s)\right)\mathbb{1}_{\{e^{-\lambda_{\text{Broker}} T^*(v_b)} MR(v|T^*)\geq MC(v_s)\}}\right] \leq \pi_{\text{Dynamic}}.$$

Noticing that $e^{-\lambda_{\text{Broker}} T^*(v_b)} MR(v_b|T^*) \geq MR(v_b)$, with strict inequality for a nonnegligible set of types, we conclude that $\pi_{\text{Static}} < \pi_{\text{Dynamic}}$. In particular, when valuations are uniformly distributed in $[0,1]$ and discount rates $\frac{\mu_{\text{Buyer}}}{\lambda_{\text{Broker}}} = 5$, it is easy to show that $\pi_{\text{Static}} = 0.0419$ whereas $\pi_{\text{Dynamic}} \geq 0.0461$.

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[11] An important challenge that makes this problem even harder, is that in this framework a feasible E.P.I.C. mechanism has an extra constraint: the seller's temporal allocation has to be always earlier than the buyer's allocation (i.e. $\tau_s(v_s, v_b) \leq \tau_b(v_s, v_b)$).

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A Appendix A: Proofs

This appendix contains proofs for all results but Theorem 1.

Proof of Lemma 1. Consider any randomized E.P.I.C. mechanism. Note that all randomness can be generated using a random variable $\omega \in [-1, 0]$ which is uniform (independent of valuations). Construct a new mechanism design problem, with $N+1$ buyers, where the new buyer has valuation ω. Any general E.P.IC. mechanism in the original problem can be seen as a deterministic E.P.I.C. mechanism in the $N+1$-buyer problem. Theorem 2 shows that in the optimal mechanism for the $N+1$ game, the temporal allocation for each player is deterministic and independent of the others players valuation (in particular, it does not depend on ω). Noting that the buyer with valuation ω never gets the item and never sets the price paid by the winning buyers, we conclude that the optimal mechanism for the $N+1$ problem does not condition on ω. As a result, the restriction to deterministic mechanism for the N-buyer problem is without loss. ■

Proof of Lemma 2. For the proof fix v_{-i} and consider $u_i(v_i', v_i) := \exp(-\mu_i \tau_i(v_i'))(v_i x_i(v_i') - p_i(v_i'))$, and $U(v_i) = u(v_i, v_i)$.[12]

Consider an E.P.I.C. mechanism (p, x, τ), then for every v_i, v_i' we have that $u(v_i, v_i) \geq u(v_i', v_i)$ and $u'(v_i', v_i') \geq u(v_i, v_i')$. This is equivalent to

$$\begin{aligned}\exp(-\mu_i\tau_i(v_i))v_i x_i(v_i) - &\exp(-\mu_i\tau_i(v_i'))v_i' x_i(v_i') \\ &\geq \exp(-\mu_i\tau_i(v_i))p_i(v_i) - \exp(-\mu_i\tau(v_i'))p_i(v_i') \\ &\geq \exp(-\mu_i\tau_i(v_i'))v_i x_i(v_i') - \exp(-\mu_i\tau_i(v_i))v_i' x_i(v_i)\ .\end{aligned}$$

Thus,

$$v_i(\exp(-\mu_i\tau_i(v_i))x_i(v_i) - \exp(-\mu_i\tau_i(v_i'))x_i(v_i')) \geq v_i'(\exp(-\mu_i\tau_i(v_i))x_i(v_i) - \exp(-\mu_i\tau_i(v_i'))x_i(v_i')).$$

This implies that $\exp(-\mu_i\tau_i(v_i))x_i(v_i)$ is non-decreasing.

To show part $b.$, notice that

$$u(v_i', v_1) - u(v_i', v_2) = \exp(-\mu_i\tau_i(v'))x_i(v_i')(v_1 - v_2) \leq |v_1 - v_2|\ ,$$

which implies that $u(v_i', \cdot)$ is absolutely continuous. Because $u(v_i', \cdot)$ is also differentiable, with $u_2(v_i', v_i) = \exp(-\mu_i\tau_i(v_i'))x_i(v_i') \leq 1$, from Theorem 2 in Milgrom and Segal (2002), we get that $U_i(v_i) = U_i(0) + \int_0^{v_i} \exp(-\mu_i\tau_i(s))x_i(s)ds$.

To tackle the converse, consider an E.P.I.C. mechanism (p, x, τ) satisfying $a.$ and $b.$. Because $U_i(v_1) - U_i(v_2) = \int_{v_1}^{v_2} \exp(-\mu_i\tau_i(s))x_i(s)ds \leq |v_1 - v_2|$ we have that $U_i(v_i)$ is Lipschitz and hence $\phi(v_i', v_i) = u(v_i', v_i) - U_i(v_i)$ is absolutely continuous $a.e.$ differentiable on v_i. Thus, for $a.e.$ $v_i \in [0, 1]$ we have that

$$\phi_2(v_i', v_i) = u(v_i', v_i) = U_i'(v_i) = \exp(-\mu_i\tau_i(v_i'))x_i(v_i') - \exp(-\mu_i\tau_i(v_i))x_i(v_i)\ .$$

[12]To simplify notation we omit v_{-i} on x_i and τ_i.

By the monotonicity assumption over (x_i, τ_i), we have that $\phi_2(v_i', v_i) \geq 0$ for a.e. $v_i < v_i'$ and $\phi_2(v_i', v_i) \leq 0$ for a.e. $v_i > v_i'$. The continuity of $\phi(v_i', \cdot)$ implies that $0 = \phi(v_i', v_i') \geq \phi(v_i', v_i)$ for every v_i. Thus, $U_i(v_i) \geq u(v_i', v_i)$ for (v_i', v_i) which correspond to the incentive compatibility definition. ■

Proof of Lemma 3. Given an E.P.I.C. mechanism (x, τ), consider the mechanism $(\tilde{x}, T)$ where $\tilde{x}(v) = 1$ and $T(v) = -\frac{\log(x(v))}{\mu} + \tau(v)$ for $v \in [0,1]$.

Using Lemma 2, we have that $(\tilde{x}, T)$ is also an E.P.I.C. mechanism and the transfers generated under both mechanisms are exactly the same.

Rewriting the seller's problem, we get that

$$\begin{aligned}\Phi(x,\tau) = \mathbb{E}[\exp(-\lambda\tau(v))p(v)] &= \mathbb{E}[\exp(-\lambda T(v))x(v)^{\frac{\lambda}{\mu}}p(v)] \\ &\leq \mathbb{E}[\exp(-\lambda T(v))p(v)] = \Phi(\tilde{x}, T)\ .\end{aligned}$$

Where the inequality holds because $x(v) \in [0,1]$. Hence, we conclude that without loss of generality we can restrict our attention to mechanism where the non-temporal allocation rule is always equals to one. ■

Proof of Proposition 1. Take $T\colon [0,1] \to \mathbb{R}$ non-increasing such that $\underline{v} > 0$, with $\underline{v} < 1$. For $\epsilon, \eta \in \mathbb{R}$ and $\bar{T} = T(\bar{v} + \epsilon)$, define the alternative allocation rule

$$\hat{T}(v) = \begin{cases} T(v) & v \notin [\underline{v} - \eta, \underline{v} + \epsilon] \\ \bar{T} & v \in [\underline{v} - \eta, \underline{v} + \epsilon]. \end{cases}$$

Note that $\hat{T}$ is non-increasing. We claim that we can choose ϵ and η such that $\Phi(\hat{T}) - \Phi(T) > 0$. Now,

$$\begin{aligned}&\Phi(\hat{T}) - \Phi(T) = \int_{\underline{v}-\eta}^{\underline{v}+\epsilon} \exp(-\lambda\bar{T})\Big\{v - \frac{F(\underline{v}+\epsilon) - F(v)}{f(v)} - \frac{\int_{\underline{v}+\epsilon}^{1} \exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)\bar{T})f(v)}\Big\}f(v)dv \\ &- \int_{\underline{v}}^{\underline{v}+\epsilon} \exp(-\lambda T(v))\Big\{v - \frac{\int_{v}^{1} \exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)T(v))f(v)}\Big\}f(v)dv \\ &= \exp(-\lambda\bar{T})\Big\{(\underline{v}-\eta)\big(F(\underline{v}+\epsilon) - F(\underline{v}-\eta)\big)\Big\} - \exp(-\lambda\bar{T})\int_{\underline{v}-\eta}^{\underline{v}+\epsilon} \frac{\int_{\underline{v}+\epsilon}^{1} \exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)\bar{T})f(v)}f(v)dv \\ &- \int_{\underline{v}}^{\underline{v}+\epsilon} \exp(-\lambda T(v))\Big\{v - \frac{\int_{v}^{1} \exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)T(v))f(v)}\Big\}f(v)dv\end{aligned}$$

It follows that

$$
\begin{aligned}
&\exp(\lambda\bar{T})\Big(\Phi(\hat{T})-\Phi(T)\Big)\\
&=(\underline{v}-\eta)\big(F(\underline{v}+\epsilon)-F(\underline{v}-\eta)\big)-\int_{\underline{v}-\eta}^{\underline{v}+\epsilon}\frac{\int_{\underline{v}+\epsilon}^{1}\exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)\bar{T})f(v)}f(v)dv\\
&-\int_{\underline{v}}^{\underline{v}+\epsilon}\exp(-\lambda(T(v)-\bar{T}))\Big\{v-\frac{\int_{v}^{1}\exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)T(v))f(v)}\Big\}f(v)dv\\
\geq&(\underline{v}-\eta)\big(F(\underline{v}+\epsilon)-F(\underline{v}-\eta)\big)-\int_{\underline{v}-\eta}^{\bar{v}+\epsilon}\frac{\int_{\underline{v}+\epsilon}^{1}\exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)\bar{T})f(v)}f(v)dv-\epsilon
\end{aligned}
$$

where the inequality follows since $\exp(-\lambda(T(v)-\bar{T}))\leq 1$ for all $v\in[\bar{v},\bar{v}+\epsilon]$. Take $\eta\in]0,\bar{v}[$ such that $(\bar{v}-\eta)\big(F(\bar{v})-F(\bar{v}-\eta)\big)>0$, define $\delta=(\bar{v}-\eta)\big(F(\bar{v})-F(\bar{v}-\eta)\big)$, and take ϵ and $\bar{T}=T(\bar{v}+\epsilon)$ such that

$$
\epsilon<\delta/2 \text{ and } \int_{\bar{v}-\eta}^{\bar{v}+\epsilon}\frac{\int_{\bar{v}+\epsilon}^{1}\exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)\bar{T})f(v)}f(v)dv<\delta/2
$$

where we use the assumption that $\mu-\lambda>0$ and the fact that $\bar{T}=T(\bar{v}+\epsilon)\to\infty$ as $\epsilon\to 0$. It follows that

$$
\exp(\lambda\bar{T})\Big(\Phi(\hat{T})-\Phi(T)\Big)>(\bar{v}-\eta)(F(\bar{v}+\epsilon)-F(\bar{v}))+\delta-\frac{\delta}{2}-\frac{\delta}{2}>0
$$

and therefore $\Phi(\hat{T})-\Phi(T)>0$. ■

Proof of Proposition 2. We start by showing that l exists. If $f(0)>0$, then since $f'(0)$ exists we have that $L=0$. If instead $f(0)=0$, because F is regular in the myersonian sense, we get that $f'(0)>0$. Thus, independently of the value $f(0)$, we have that $L\geq 0$. Next, noticing that the left hand side of Equation (6) is strictly decreasing in x, that at $x=0$ the value is $2+L>0$, and that as x approaches to one the value goes to $-\infty$; we conclude that l is well defined. More on, $l\in(0,1)$. The uniqueness of l comes from the monotonicity of the function $g(x):=2+L(1-(1-\frac{\lambda}{\mu})x)-(1-\frac{\lambda}{\mu})(2x+\frac{\lambda}{\mu}\frac{x^2}{1-x})$ over the domain $[0,1]$.[13]

I. Existence of Solution of the O.D.E. Consider the optimal allocation T^*. From Theorem (1), we have that T^* is differentiable on $(0,v^*)$. Thus, differentiating Equation (5) we obtain:

$$
T^{*\prime}(v)=-\frac{2+\frac{f'(v)}{f(v)}\frac{\int_v^1\exp((\mu-\lambda)T^*(s))f(s)ds}{\exp((\mu-\lambda)T^*(v))f(v)}}{\mu(1-\frac{\lambda}{\mu})\left[\frac{\int_v^1\exp((\mu-\lambda)T^*(s))f(s)ds}{\exp((\mu-\lambda)T^*(v))f(v)}+\exp(\mu T^*(v))\int_0^v\exp(-\mu T^*(s))ds\right]}\,. \tag{8}
$$

[13]Taking derivative, we have that $g'(x)=-(1-\frac{\lambda}{\mu})(2+L+\frac{2x(1-x)+x^3}{(1-x)^2}<0$.

Using Lemma 2, we get that $p^{*\prime}(v) = -\mu(v - p(v))T^{*\prime}(v)$. Plugging this result and Equation (5) into Equation (8), we obtain that the optimal pricing policy p^* satisfies

$$2 + \frac{f'(v)}{f(v)}(v - (1 - \frac{\lambda}{\mu})p^*(v)) - (1 - \frac{\lambda}{\mu})(2p^{*\prime}(v) + \frac{\lambda}{\mu}\frac{p^*(v)p^{*\prime}(v)}{v - p^*(v)}) = 0 \text{ for } v \le v^* .$$

Clearly, the optimal mechanism satisfies $p^*(0) = 0$. Moreover taking the limit $v \to 0$ in the last equation, and using L'Hopital's rule, we conclude that

$$2 + L(1 - (1 - \frac{\lambda}{\mu})p^{*\prime}(0)) - (1 - \frac{\lambda}{\mu})(2p^{*\prime}(0) + \frac{\lambda}{\mu}\frac{p^{*\prime}(0)^2}{1 - p^{*\prime}(0)}) = 0.$$

Using Lemma 2 part a., we have that p^* is increasing. Hence, since with $p^*(v) \le v$ and $p^*(0) = 0$, we have that $p^*(v) \in (0, 1)$. We conclude that p^* solves the differential equation.

II. Uniqueness of Solution of the O.D.E. We claim that p^* is the unique solution of the Cauchy Problem. Take a general solution p, we assert that $p(v) \le v$ for all $v \in [0, 1]$. Because $p'(0) < 1$, it means that for v in a neighborhood of 0, we have that $p(v) < v$. Now, suppose that there is $\tilde{v}$ such that $p(\tilde{v}) = \tilde{v}$, from the differential equation at $\tilde{v}$ we obtain that $p'(\tilde{v}) = 0$. Consequently, for v in a neighborhood of $\tilde{v}$ we get $p'(v) < 1$, equivalently, $p(v) < v$. Thus, $p(v) < v$ for almost every $v \in [0, 1]$.

Because $(1 - \frac{\lambda}{\mu})p(1) < 1 = MR(1)$ and the inequality is reversed at $v = 0$, i.e. $(1 - \frac{\lambda}{\mu})p(0) > MR(0))$, we obtain that the set $V^* = \{v \mid (1 - \frac{\lambda}{\mu})p(v) = MR(v)\}$ is not empty. For $\breve{v} \in V^*$, we implicitly define $\hat{T}$ by $\exp(\mu\hat{T}(v)) \int_0^v \exp(-\mu\hat{T}(s))ds = (v - p(v))$ for $v \le \breve{v}$ and $T(\breve{v}) = 0$. Writing $u(v) = \int_0^v \exp(-\mu\hat{T}(s))ds$, we get the following differential equation

$$\frac{u(v)}{u'(v)} = (v - p(v)) .$$

Using that $u(\breve{v}) = \breve{v} - p(\breve{v})$, we can easily solve the above differential equation.[14] From the solution we get that

$$\exp(\mu\hat{T}(v)) = \frac{1}{u'(v)} = \frac{v - p(v)}{\breve{v} - p(\breve{v})} \exp\left(- \int_{\breve{v}}^v \frac{ds}{s - p(s)} \right) .$$

Taking to the power of $(1 - \lambda/\mu)$ in both side of the expression, we get that

$$\exp((\mu - \lambda)\hat{T}(v)) = \left[\frac{v - p(v)}{\breve{v} - p(\breve{v})} \exp\left(- \int_{\breve{v}}^v \frac{ds}{s - p(s)} \right) \right]^{\left(1 - \frac{\lambda}{\mu}\right)} . \tag{9}$$

Similarly, consider $\tilde{T}$ such that $\frac{1 - F(\breve{v}) + \int_v^{\breve{v}} \exp((\mu-\lambda)\tilde{T}(s))f(s)ds}{\exp((\mu-\lambda)\tilde{T}(v))f(v)} = v - \left(1 - \frac{\lambda}{\mu}\right)p(v)$ for $v \le \breve{v}$. Since $\breve{v} \in V^*$, we have that $\tilde{T}(\breve{v}) = 0$. Doing the same procedure as we did for $\hat{T}$, we get

$$\exp((\mu - \lambda)\tilde{T}(v)) = \frac{1}{f(v)} \frac{\breve{v} - \left(1 - \frac{\lambda}{\mu}\right)p(\breve{v})}{v - \left(1 - \frac{\lambda}{\mu}\right)p(v)} \exp\left(- \int_{\breve{v}}^v \frac{ds}{s - \left(1 - \frac{\lambda}{\mu}\right)p(s)} \right) . \tag{10}$$

[14] The solution is $\log\left(\frac{u(\breve{v})}{u(v)}\right) = -\int_{\breve{v}}^v \frac{ds}{s - p(s)}$, for $v \le \breve{v}$.

We claim that $\hat{T}(v) = \tilde{T}(v)$ for $v \le \breve{v}$. From Equations (9) and (10), we have that $\hat{T}, \tilde{T}$ are differentiable (since p is differentiable), and moreover, they satisfy

$$(\mu - \lambda)\hat{T}'(v) = -\big(1 - \frac{\lambda}{\mu}\big)\frac{p'(v)}{v - p(v)} \tag{11}$$

$$(\mu - \lambda)\tilde{T}'(v) = -\frac{2 + \frac{f'(v)}{f(v)}\big(v - \big(1 - \frac{\lambda}{\mu}\big)p(v)\big) - \big(1 - \frac{\lambda}{\mu}\big)p'(v)}{\big(v - \big(1 - \frac{\lambda}{\mu}\big)p(v)\big)}\ ; \tag{12}$$

combining the above and using that p solves Equation (2), we conclude $\hat{T}'(v) = \tilde{T}'(v)$ for $v \le \breve{v}$. Finally, since $\hat{T}(\breve{v}) = \tilde{T}(\breve{v}) = 0$ we conclude that $\hat{T} = \tilde{T}$. Hence, the bundle $(\hat{T}, \breve{v})$ solves the equation presented in Theorem 1. The uniqueness result implies that $\hat{T} = \tilde{T} = T^*$ and $\breve{v} = v^*$. Thus, $p(v) = p^*(v)$ is the optimal pricing mechanism and v^* is the unique solution to the equation $(1 - \frac{\lambda}{\mu})p(v^*) = MR(v^*)$. ■

Proof of Lemma 5. Using the same technique as for the single buyer case (see Appendix B), we deduce, via the discretization approach, the first order conditions for the temporal allocation variable for Problem (2).

Given a fixed report $v_{-i} \in [0,1]^{N-1}$, we denote by $v^*(v_{-i})$ the smallest type such that $\tau_i^*(v^*(v_{-i})) = 0$. Then, for $v_i \le v^*(v_{-i})$ there is an interior solution on the temporal allocation, satisfying:

$$x_i(v)\Bigg[-\lambda v_i + \mu_i \frac{\int_{v_i}^1 \exp((\mu_i - \lambda)\tau_i(s, v_{-i}))f_i(s)ds}{\exp((\mu_i - \lambda)\tau_i(v))f_i(v_i)} \\ -(\mu_i - \lambda)\exp(\mu_i \tau_i(v))\int_0^{v_i}\exp(-\mu_i\tau_i(s, v_{-i}))ds\Bigg] = 0. \tag{13}$$

By the monotonicity condition over E.P.I.C. mechanisms (see Lemma 2), we have that whenever $x_i(v_i, v_{-i}) > 0$, then $x_i(v_i', v_{-i}) > 0$ for $v_i' > v_i$. Thus, for $x_i(v^*(v_{-i}), v_{-i}) > 0$ we have that Equation (13) has a not trivial solution. When $x_i(v^*(v_{-i}), v_{-i}) = 0$, the temporal allocation rule is irrelevant for the seller's payoff. In particular, $T_i^*(v_i)$, the optimal allocation for the single case, is a solution to the problem.

Thus, defining $\underline{v}_i := \inf_{v_i}\{v_i \mid x_i(v_i, v_{-i}) > 0\}$ we have that for any type $v_i \in [\underline{v}_i, v^*(v_{-i})]$ the following conditions holds

$$-\lambda v_i + \mu_i \frac{\int_{v_i}^1 \exp((\mu_i - \lambda)\tau_i(s, v_{-i}))f_i(s)ds}{\exp((\mu_i - \lambda)\tau_i(v))f_i(v_i)} - (\mu_i - \lambda)\exp(\mu_i\tau_i(v))\int_0^{v_i}\exp(-\mu_i\tau_i(s, v_{-i}))ds = 0.$$

From Theorem 1 we conclude that there is a unique solution of this integral equation. Moreover, the solution is $\tau_i(v_i, v_{-i}) = T_i^*(v_i)$, where T_i^* is the solution of the single agent problem for a buyer whose distribution is drawn according to F_i and have a discount factor μ_i. ■

B Appendix B: Proof of Theorem 1

This appendix provides a proof for Theorem 1. First, we show the existence of solution for Problem (2). We also show that the optimal solution consists of a pair (T^*, v^*) satisfying Equation (5). The second part of this section shows the uniqueness of the solution. Section B.3, provides proofs for technical steps used in Section B.1 and Section B.2.

B.1 Existence of Solution: A Discretization Approach

For $n \in \mathbb{N}$, consider the partition of the unit interval given by $\mathcal{P}_n = \{\frac{i}{n} \mid i = 1, \ldots, n\}$. Define the problem

$$(P_n) \max_{T:\mathcal{P}_n \to \mathbb{R}_+ \cup \{\infty\}} \sum_{i=1}^{n} \exp(-\lambda T_i) \Big\{ \frac{i}{n} - \frac{\sum_{k=i}^{n} \exp((\mu-\lambda)T_k) f(\frac{k}{n})\frac{1}{n}}{\exp((\mu-\lambda)T_i) f(\frac{i}{n})} \Big\} f(\frac{i}{n}) \frac{1}{n} ,$$

the discretization of Problem (4).

Because $\mathbb{R}_+ \cup \{\infty\}$ is compact (under the extended topology) and the problem is continuous on T, we have that Problem (P_n) has a solution $T^n \in [\mathbb{R}_+ \cup \{\infty\}]^n$.

Even though T_i^n could be infinity for some $i \in \mathcal{P}_n$, we assert that this cannot be true. Take j the largest element in $\mathcal{P}_n$ such that $T_j^n = \infty$. By definition of j we have that $\sum_{k=j+1}^{n} \exp((\mu-\lambda)T_k^n) f(\frac{k}{n})\frac{1}{n} < \infty$. Hence, we can find $\tilde{T}_j < \infty$ sufficiently large such that

$$\frac{j}{n} - \frac{\sum_{k=j+1}^{n} \exp((\mu-\lambda)T_k^n) f(\frac{k}{n})\frac{1}{n} + \exp((\mu-\lambda)\tilde{T}_j) f(\frac{j}{n})\frac{1}{n}}{\exp((\mu-\lambda)\tilde{T}_j) f(\frac{j}{n})} > 0 .$$

Replacing T_j by $\tilde{T}_j$ the seller's payoff strictly increases, violating the optimality of T^n. Therefore, the optimal solution never allocates at time ∞.

Noticing that the optimization problem is smooth we have that for $T_i^n > 0$, first order conditions holds. Thus, after some algebraic manipulation, we obtain

$$-\lambda \frac{i}{n} + \mu \frac{\sum_{k=i}^{n} \exp((\mu-\lambda)T_k^n) f(\frac{k}{n})\frac{1}{n}}{\exp((\mu-\lambda)T_i^n) f(\frac{i}{n})} - (\mu-\lambda)\exp(\mu T_i^n) \sum_{k=1}^{i} \exp(-\mu T_k^n) \frac{1}{n} = 0 \text{ for } T_i^n > 0 . \tag{14}$$

We are now in position to take the limit over the mesh. For this extent, we project T^n over $[\mathbb{R}_+ \cup \{\infty\}]^{[0,1]}$, by defining for $v \in [0,1]$ $T^n(v) := T_i^n$ for $v \in (\frac{i-1}{n}, \frac{i}{n}]$. Similarly, we define $f^n(v) := f^n(\frac{i}{n})$ as the discretization of the density distribution. Using that $\mathbb{R}_+ \cup \{\infty\}$ is a compact set, Tychonoff's Theorem states that we can take a subsequence such that $T_n \to T^*$ (point-wise), and $f_n \to f$, where f is the original density of our problem (Willard, 2004, Chapter 6).

Two intermediate claims are derived from the limit.

Claim 1 T^* *is a continuous function satisfying*

$$-\lambda v + \mu \frac{\int_v^1 \exp((\mu-\lambda)T^*(s)) f(s) ds}{\exp((\mu-\lambda)T^*(v)) f(v)} - (\mu-\lambda)\exp(\mu T^*(v)) \int_0^v \exp(-\mu T^*(s)) ds = 0 \ \textit{for}\ T^*(v) > 0 .$$

Proof. See Apprendix B.3. ■

Claim 2 *T is smooth function. Moreover, there is $v^* \in (0,1]$ such that $T(v) = 0$ if and only if $v \geq v^*$ and T is decreasing on $(0, v^*]$.*

Proof. See Apprendix B.3. ■

Using the above results, we are now in position to show that T solves Problem (2). First, denote by $\Phi(T^n, f^n)$ the value of Problem (4) when is evaluated in T^n and the buyers's distribution is given by f^n. Then, by continuity of T, we have that $\Phi(T^n, f^n) < \Phi(T, f) + \epsilon$ for n sufficiently large.

Consider a feasible continuous solution G of Problem (4). Then, by Riemman-integrability of G, we have that $\Phi(G^n, f^n) > \Phi(G, f) - \epsilon$ where G^n is G's projection onto $\mathcal{P}_n$. Next, from the optimality of T^n in Problem (P_n), we get that $\Phi(G^n, f^n) < \Phi(T^n, f^n)$. Taking $\epsilon \to 0$, we have that for every feasible continuous solution G, $\Phi(G, f) \leq \Phi(T^*, f)$. By a density argument, we can extend this conclusion to a general feasible solution G. Finally, the monotonicity of T^*, permits to conclude that T^* solves Problem (3).

B.2 Uniqueness of the bundle (T^*, v^*)

The objective of this section is to show that there is a unique function $T^* : [0,1] \to \mathbb{R}_+$ and $v^* \in [0,1]$ such that $T^*(v) = 0$ for $v \leq v^*$ and T^* solves Equation (5) for $v \leq v^*$. First, consider v^* as given, we claim that T^* as a function of v^* is uniquely pin-down.

Claim 3 *For $u \in [0,1]$, there is at most one function $T_u : [0,u] \to \mathbb{R}$ such that $T_u(u) = 0$ and solves*

$$v - \frac{\int_v^1 \exp((\mu-\lambda)T_u(s))f(s)ds}{\exp((\mu-\lambda)T_u(v))f(v)} = (1 - \frac{\lambda}{\mu})(v - \exp(\mu T_u(v))\int_0^v \exp(-\mu T_u(s))ds) \quad \forall v \leq u \ , \tag{15}$$

Proof. See Apprendix B.3. ■

To show uniqueness of the bundle (T^*, v^*), suppose for the sake of a contradiction that there are $v_1^* > v_2^*$ and $T_{v_1^*}$ and $T_{v_2^*}$ both solving (15). Define $\tilde{T}_{v_2^*}(v) := T_{v_1^*}(v) - T_{v_1^*}(v_2^*)$. By simple inspection we get that $\tilde{T}_{v_2^*}$ also solves Equation (5) and satisfies $\tilde{T}_{v_2^*}(v_2^*) = 0$. Invoking Claim 3, we conclude that $\tilde{T}_{v_2^*}(v) = T_{v_2^*}(v)$ for $v \geq v_2^*$.

Using the monotonicity of $T_{v_1^*}$, we get

$$v_2^* - \frac{1 - F(v_2^*)}{f(v_2^*)} < v_2^* - \frac{\int_{v_2^*}^1 \exp((\mu-\lambda)T_{v_1^*}(s))f(s)ds}{\exp((\mu-\lambda)T_{v_1^*}(v))f(v)} = v_2^* - \frac{\int_{v_2^*}^1 \exp((\mu-\lambda)\tilde{T}_{v_2^*}(s))f(s)ds}{\exp((\mu-\lambda)\tilde{T}_{v_2^*}(v))f(v)} \tag{16}$$

$$= (1 - \frac{\lambda}{\mu})(v_2^* - \exp(\mu\tilde{T}_{v_2^*}(v_2^*))\int_0^{v_2^*} \exp(-\mu\tilde{T}_{v_2^*}(s))ds) \ . \tag{17}$$

On the other hand, using that $\tilde{T}_{v_2^*}(v) = T_{v_2^*}(v)$ for $v \leq v_2^*$, we get

$$v_2^* - \frac{1 - F(v_2^*)}{f(v_2^*)} = (1 - \frac{\lambda}{\mu})(v_2^* - \exp(\mu T_{v_2^*}(v_2^*))\int_0^{v_2^*} \exp(-\mu T_{v_2^*}(s))ds) \tag{18}$$

$$= (1 - \frac{\lambda}{\mu})(v_2^* - \exp(\mu\tilde{T}_{v_2^*}(v_2^*))\int_0^{v_2^*} \exp(-\mu\tilde{T}_{v_2^*}(s))ds) \ . \tag{19}$$

Equations (17) and (19) are mutually exclusive. Therefore, we conclude that there is a unique $v^* \in [0,1]$ and T^* solving Equation (5).

B.3 Proofs of Claims

Proof of Claim 1. We assert that T^* is continuous if the sequence $(T^n)_{\{n\geq 0\}}$ satisfies the following property at every $v \in [0,1]$:

$$\text{(Property 1) } \forall \epsilon > 0\ \exists N \in \mathbb{N}\ \exists \delta > 0 \text{ s.t } \forall n \geq N \text{ if } |\frac{i}{n} - v| \leq \delta \text{ then } |T^n(\frac{i}{n}) - T^n(\frac{i+1}{n})| < \epsilon\ .$$

Indeed, take $v \in [0,1]$ and a sequence $v_k \to v$. Because $T^n \to T^*$ (pointwise), we have that for n, k sufficiently large, $|T^*(v) - T^*(v_k)| \leq \epsilon + |T^n(\frac{i}{n}) - T^n(\frac{i+1}{n})|$ with $v \in (\frac{i}{n}, \frac{i+1}{n}]$. Hence, so long as $|T^n(\frac{i}{n}) - T^n(\frac{i+1}{n})| < \epsilon$, T^* is continuous at v.

We now proceed to show that Property 1 holds. Consider first the case where $T^n(\frac{i}{n}), T^n(\frac{i+1}{n})$ are both positive. Subtracting Equation (14), when is evaluated at $T^n(\frac{i}{n})$ and when is evaluated at $T^n(\frac{i+1}{n})$, we obtain that

$$\begin{aligned} C_i^n\left(\frac{f(\frac{i}{n})}{f(\frac{i+1}{n})}\exp(-(\mu-\lambda)T^n(\frac{i+1}{n})) - \exp(-(\mu-\lambda)T^n(\frac{i}{n}))\right) \\ = W_i^n\left(\exp(\mu T^n(\frac{i+1}{n})) - \exp(\mu T^n(\frac{i}{n}))\right)\ , \end{aligned} \tag{20}$$

where $C_i^n = \mu\frac{\sum_{k=i}^n \exp((\mu-\lambda)T_k^n)f(\frac{k}{n})\frac{1}{n}}{f(\frac{i}{n})}$ and $W_i^n = (\mu-\lambda)\sum_{k=1}^i \exp(-\mu T_k^n)\frac{1}{n}$.

Suppose for the sake of a contradiction that Property 1 does not hold. Then, there is $\epsilon > 0$, and a sequence of i_n approaching v such that $|T^n(\frac{i_n}{n}) - T^n(\frac{i_n+1}{n})| > \epsilon$. Without loss of generality, consider the case that $T^n(\frac{i_n}{n}) > T^n(\frac{i_n+1}{n}) + \epsilon$ (the other case is analogous). Then, by the monotonicity of the exponential function, we have that

$$\exp(-(\mu-\lambda)T^n(\frac{i_n+1}{n})) - \exp(-(\mu-\lambda)T^n(\frac{i_n}{n})) > \exp(-(\mu-\lambda)T^n(\frac{i_n}{n}))(\exp(-(\mu-\lambda)\epsilon))\ ,$$

$$\exp(\mu T^n(\frac{i_n+1}{n})) - \exp(\mu T^n(\frac{i_n}{n})) < -\exp(\mu T^n(\frac{i_n}{n}))(\exp(\mu\epsilon))$$

These two inequalities have different signs, and are bounded away from zero. On the other hand, the density f is continuous and the tems C_i^n and W_i^n are alway positive. This two facts make impossible to hold Equation (20), which is a contradiction. We conclude that Property 1 holds for the case where $T^n(\frac{i}{n}), T^n(\frac{i+1}{n})$ are both positive.

The second case is when $T^n(\frac{i}{n})$ is zero and $T^n(\frac{i+1}{n})$ is positive. The proof is similar with the subtlety that for $\frac{i}{n}$ the first order conditions are given by

$$-\lambda\frac{i}{n} + \mu\frac{\sum_{k=i}^n \exp((\mu-\lambda)T_k^n)f(\frac{k}{n})\frac{1}{n}}{f(\frac{i}{n})} - (\mu-\lambda)\sum_{k=1}^i \exp(-\mu T_k^n)\frac{1}{n} \leq 0\ .$$

Hence, subtracting Equation (14) at $\frac{i+1}{n}$, in the above expression we obtain

$$C_i^n\left(\frac{f(\frac{i}{n})}{f(\frac{i+1}{n})}\exp(-(\mu-\lambda)T^n(\frac{i+1}{n}))-1\right)\geq W_i^n\left(\exp(\mu T^n(\frac{i+1}{n}))-1\right)-\frac{\mu-\lambda}{n}\ .$$

Noticing $\lim_{\mathcal{P}_n}\frac{f(\frac{i}{n})}{f(\frac{i+1}{n})}\exp(-(\mu-\lambda)T^n(\frac{i+1}{n}))\leq 1$, the above inequality can only be sustained for every n if $T^n(\frac{i+1}{n})\to 0$, i.e., $|T^n(\frac{i+1}{n})-T^n(\frac{i}{n})|\to 0$.

The case where $T^n(\frac{i}{n})$ is positive and $T^n(\frac{i+1}{n})$ is zero is analogous to the last paragraph and is, therefore, omitted.

We conclude that T^* is a continuous function.

To finalize the proof of the claim we take the limit in Equation (14). Since T^* is continuous, it is Riemman-Integrable, therefore by taking the limit over Equation (14) we get

$$-\lambda v+\mu\frac{\int_v^1\exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)T(v))f(v)}-(\mu-\lambda)\exp(\mu T(v))\int_0^v\exp(-\mu T(s))ds=0\ \text{ for } T(v)>0\ .$$

■

Proof of Claim 2. Take v such that $T(v)>0$. We define the real function

$$h_v(x)=-\lambda v+\mu\frac{\int_v^1\exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)x)f(v))}-(\mu-\lambda)\exp(\mu x)f(v)\int_0^v\exp(-\mu T(s))ds\ .$$

We assert that h_v is strictly decreasing. Indeed, note that

$$h_v'(x)=-\mu(\mu-\lambda)\left(\frac{\int_v^1\exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)x)f(v))}+\exp(\mu x)f(v)\int_0^v\exp(-\mu T(s))ds\right)<0 \text{ for } x\in\mathbb{R}.$$

This implies that $T(v)=h_v^{-1}(0)$.

Because $T(v)$ is continuous we have that $h_v(x)$ is differentiable on v. Therefore, by the Implicit Function Theorem we conclude that T is continuously differentiable on v.

Hence, differentiating Equation (14) at v, we get

$$T'(v)=-\frac{2+\frac{f'(v)}{f(v)}\frac{\int_v^1\exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)T(v))f(v)}}{\mu(1-\frac{\lambda}{\mu})\left[\frac{\int_v^1\exp((\mu-\lambda)T(s))f(s)ds}{\exp((\mu-\lambda)T(v))f(v)}+\exp(\mu T(v))\int_0^v\exp(-\mu T(s))ds\right]}\ . \tag{21}$$

Define $v^*:=\min\{v\mid T(v)=0\}$, then the numerator of $T'(v^*)$ is $-[2+\frac{f'(v^*)}{f^2(v^*)}(1-F(v^*))]$ which is strictly negative.[15] Thus, because the denominator of $T'(v^*)$ is positive, T is decreasing on $[v^*-\epsilon,v^*]$. We claim that T is strictly decreasing on $(0,v^*]$. If not, $\tilde{v}=\max\{v\mid v\leq v^*$ and $T'(v)=0\}$ would be well-defined. Then, $T(\tilde{v})>0$ and from

[15] The Myerson-regularity condition on F is equivalent, in terms of the density f, to have $2+\frac{f'(v)}{f^2(v)}(1-F(v))>0$, for every $v>0$.

Equation (21) we would get that the numerator of $T'(\tilde{v})$ is strictly less than $-[2+\frac{f'(\tilde{v})}{f^2(\tilde{v})}(1-F(\tilde{v}))]$. Thus $T'(\tilde{v}) < 0$, contradicting the definition of $\tilde{v}$. Therefore, T is decreasing on $(0, v^*]$ and equals to zero for $v \geq v^*$. ■

Proof of Claim 3. From Equation (15) evaluated at $v = u$, we obtain that $c_u := \int_0^u \exp(-\mu T_u(s))ds$ equals $u-(1-\frac{\lambda}{\mu})^{-1}\left[u-\frac{1-F(u)}{f(u)}\right]$. Also we have that $\int_0^v \exp(-\mu T_u(s))ds = \int_0^u \exp(-\mu T_u(s))ds - \int_v^u \exp(-\mu T_u(s))ds$. These two expression allow us to alternatively restructure Equation (15) as

$$v-\frac{\int_v^1 \exp((\mu-\lambda)T_u(s))f(s)ds}{\exp((\mu-\lambda)T_u(v))f(v)} = (1-\frac{\lambda}{\mu})(v-\exp(\mu T_u(v))\big(c_u-\int_v^u \exp(-\mu T_u(s))ds\big)) \quad (22)$$

By a similar argument as in Claim 2, we have that T_u is differentiable. Hence, differentiating the above equality with respect to v, we get

$$T_u'(v) = -\frac{2+\frac{f'(v)}{f(v)}\frac{\int_v^1 \exp((\mu-\lambda)T_u(s))f(s)ds}{\exp((\mu-\lambda)T_u(v))f(v)}}{\mu(1-\frac{\lambda}{\mu})\left[\frac{\int_v^1 \exp((\mu-\lambda)T_u(s))f(s)ds}{\exp((\mu-\lambda)T_u(v))f(v)}+\exp(\mu T_u(v))\big(c_u-\int_v^u \exp(-\mu T_u(s))ds\big)\right]}.$$

This differential equation can be reformulated as a standard ordinary differential equation of the form

$$T_u'(v) = -\frac{2+\frac{f'(v)}{f(v)}\frac{H(v)}{\exp((\mu-\lambda)T_u(v))f(v)}}{\mu(1-\frac{\lambda}{\mu})\left[\frac{H(v)}{\exp((\mu-\lambda)T_u(v))f(v)}+\exp(\mu T_u(v))(c_u+G(v))\right]} \quad (23)$$

$$H'(v) = -\exp((\mu-\lambda)T_u(v))f(v) \quad (24)$$

$$G'(v) = \exp(-\mu T_u(v)) \quad (25)$$

With this representation, the uniqueness results known in ODE allow us to conclude the claim. Suppose that there are two solutions T_1 and T_2 of Equation (15). Because both function are continuous, denote by $u^* = \min\{v \in [0,u] \mid T_1(v) = T_2(v)\}$. We assert that u^* cannot be positive. If not, define the Cauchy Problem

$$\begin{cases} (T_u'(v), H'(v), G'(v)) & = F(v, T_u(v), H(v), G(v)) \\ (T_u(u^*), H(u^*), G(u^*)) & = (T_1(u^*), \int_{u^*}^1 \exp((\mu-\lambda)T_1(s))ds, 0) \end{cases},$$

where F is defined by equations (23) to (25). By construction, both T_1 and T_2 solve this Problem. Since F is continuously differentiable at u^*, and therefore locally Lipschitz, the Picardi-Lindelof Theorem guarantees uniqueness of solution for $[u^*, u^*+\epsilon]$ for some $\epsilon > 0$ (Coddington and Levinson, 1955, Chapter 1). This contradicts the definition of u^*. Therefore, there is a unique solution to Equation (15). ■