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TSP TOURS IN CUBIC GRAPHS: BEYOND 4/3*

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Abstract. After a sequence of improvements Boyd et al. [*TSP on cubic and subcubic graphs*, Integer Programming and Combinatorial Optimization, Lecture Notes in Comput. Sci. 6655, Springer, Heidelberg, 2011, pp. 65–77] proved that any 2-connected graph whose n vertices have degree 3, i.e., a cubic 2-connected graph, has a Hamiltonian tour of length at most $(4/3)n$, establishing in particular that the integrality gap of the subtour LP is at most 4/3 for cubic 2-connected graphs and matching the conjectured value of the famous 4/3 conjecture. In this paper we improve upon this result by designing an algorithm that finds a tour of length $(4/3 - 1/61236)n$, implying that cubic 2-connected graphs are among the few interesting classes of graphs for which the integrality gap of the subtour LP is strictly less than 4/3. With the previous result, and by considering an even smaller ϵ, we show that the integrality gap of the TSP relaxation is at most $4/3 - \epsilon$ even if the graph is not 2-connected (i.e., for cubic connected graphs), implying that the approximability threshold of the TSP in cubic graphs is strictly below 4/3. Finally, using similar techniques we show, as an additional result, that every Barnette graph admits a tour of length at most $(4/3 - 1/18)n$.

Key words. traveling salesman problem, cubic graphs, integrality gap

AMS subject classifications. 05C85, 05C45, 68W25, 68W40

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1. Introduction.

The traveling salesman problem (TSP) in metric graphs is a landmark problem in combinatorial optimization and theoretical computer science. Given a graph in which edge-distances form a metric the goal is to find a tour of minimum length visiting each vertex exactly once. Alternatively, one can allow arbitrary nonnegative distances but the goal is to find a minimum length tour that visits each vertex at least once. Understanding the approximability of the TSP has attracted much attention since Christofides [7] designed a 3/2-approximation algorithm for the problem. Despite great efforts, Christofides' algorithm continues to be the current champion, while the best known lower bound, recently obtained by Lampis [13] states that the problem is NP-hard to approximate within a factor 185/184, which improved upon the work of Papadimitriou and Vempala [17]. Very recently Karpinski and Schmied [11] obtained explicit inapproximability bounds for the cases of cubic and subcubic graphs. A key lower bound to study the approximability of the problem is the so-called subtour elimination linear program which has long been known to have an integrality gap of at most 3/2 [26]. A long-standing conjecture (see, e.g., Goemans [9]) states that the integrality gap of the subtour elimination LP is precisely 4/3.

There have been several improvements for important special cases of the metric TSP in the last couple of years. Oveis Gharan, Saberi, and Singh [16] designed a $(3/2 - \epsilon)$-approximation algorithm for the case of graph metrics, while Mömke and

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Svensson [14] improved that to 1.461, using a different approach. Mucha [15] then showed that the approximation guarantee of the Mömke and Svensson algorithm is 13/9. Finally, still in the shortest path metric case, Sebö and Vygen [21] found an algorithm with a guarantee of 7/5. These results in particular show that the integrality gap of the subtour LP is below 3/2 in case the metric comes from an unweighted graph. Another notable direction of recent improvements concerns the s-t path version of the TSP on arbitrary metrics, where the natural extension of Christofides' heuristic guarantees a solution within a factor of 5/3 of optimum. An, Shmoys, and Kleinberg [2], found a $(1+\sqrt{5})/2$-approximation for this version of the TSP, while Sebö [20] further improved this result obtaining a 8/5-approximation algorithm.

This renewed interest in designing algorithms for the TSP in graph metrics has also reached the case when we further restrict ourselves to graph metrics induced by special classes of graphs. Gamarnik, Lewenstein, and Sviridenko [8] showed that in a 3-connected cubic graph on n vertices there is always a TSP tour—visiting each vertex at least once—of length at most $(3/2-5/389)n$, improving upon Christofides' algorithm for this graph class. A few years later, Aggarwal, Garg, and Gupta [1] improved the result obtaining a bound of $(4/3)n$ while Boyd et al. [5] showed that the $(4/3)n$ bound holds even if only 2-connectivity assumed. Finally, Mömke and Svensson's [14] approach allows us to prove that the $(4/3)n$ bound holds for subcubic 2-connected graphs. Interestingly, the latter bound happens to be tight and thus it may be tempting to conjecture that there are cubic graphs on n vertices for which no TSP tour shorter than $(4/3)n$ exists. The main result in this paper, proved in sections 3 and 4, shows that this is not the case. Namely, we prove that any 2-connected cubic graph on n vertices has a TSP tour of length $(4/3-\epsilon)n$, for $\epsilon = 1/61236 > 0.000016$. In section 5, we use this result and establish that for cubic graphs, not necessarily 2-connected, there exists a $4/3-\epsilon'$ approximation algorithm for the TSP, where $\epsilon' = \epsilon/(3+3\epsilon)$.

Qian et al. [18] showed that the integrality gap of the subtour LP is strictly less than 4/3 for metrics where all the distances are either 1 or 2. Their result, based on the work of Schalekamp, Williamson, and van Zuylen [19], constitutes the first relevant special case of the TSP for which the integrality gap of the subtour LP is strictly less than 4/3. Our result implies that the integrality gap of the subtour LP is also strictly less than 4/3 in connected cubic graphs.

From a graph theoretic viewpoint, our result can also be viewed as a step towards resolving Barnette's [4] conjecture, stating that every bipartite, planar, 3-connected, cubic graph is Hamiltonian (a similar conjecture was first formulated by Tait [22], then refuted by Tutte [24], then reformulated by Tutte and refuted by Horton, see, e.g., [6], and finally reformulated by Barnette more than 40 years ago). It is worth mentioning that for Barnette's graphs (i.e., those with the previous properties) on n vertices it is straightforward to construct TSP tours of length at most $(4/3)n$; however, no better bound was known. Our result improves upon this, and furthermore as we show in section 2, in this class of graphs our bound improves to $(4/3-1/18)n < 1.28n$. In a very recent work, Karp and Ravi [10] proved that for cubic bipartite connected graphs there exists a TSP tour of length at most $(9/7)n$. This is slightly worse than our $(4/3-1/18)n$ bound but applies to a larger class of graphs, on the other hand the Karp and Ravi result improves upon the bound of $(4/3-1/108)n$ of Larré [12], which applies to cubic bipartite 2-connected graphs.

1.1. Our approach. An *Eulerian subgraph cover* (or simply a *cover*) is a collection $\Gamma = \{\gamma_1, \ldots, \gamma_j\}$ of connected multisubgraphs of G, called *components*, satisfying that (i) every vertex of G is covered by exactly one component, (ii) each component is Eulerian, and (iii) no edge appears more than twice in the same component. Every cover Γ can be transformed into a TSP tour $T(\Gamma)$ of the entire graph by contracting each component, adding a doubled spanning tree in the contracted graph (which is connected) and then uncontracting the components. Boyd et al [5] defined the *contribution* of a vertex v in a cover Γ, as $z_\Gamma(v) = \frac{(\ell+2)}{h}$, where ℓ and h are the number of edges and vertices, respectively, of the component of Γ in which v lies. The extra 2 in the numerator is added for the cost of the double edge used to connect the component to the other in the spanning tree mentioned above, so that $\sum_{v\in V} z_\Gamma(v)$ equals the number of edges in the final TSP tour $T(\Gamma)$ plus 2. Let $\mathcal{D} = \{(\Gamma_i, \lambda_i)\}_{i=1}^k$ be a distribution over covers of a graph. This is, each Γ_i is a cover of G and each λ_i is a positive number so that $\sum_{i=1}^k \lambda_i = 1$. The *average contribution* of a vertex v with respect to distribution $\mathcal{D}$ is defined as $z_\mathcal{D}(v) = \sum_{i=1}^k \lambda_i z_{\Gamma_i}(v)$.

As the starting point of our paper, we study short TSP tours on Barnette graphs (bipartite, planar, cubic, and 3-connected). These types of graphs are very special as it is possible to partition their set of faces into three cycle covers, i.e., Eulerian subgraph covers in which every component is a cycle. By a simple counting argument, one of these cycle covers is composed of less than $n/6$ cycles. By using the transformation described on the previous paragraph, we can obtain a TSP tour of length at most $4n/3$. To get shorter TSP tours we describe a simple procedure that performs local operations to reduce the number of cycles in each of the three initial cycle covers. In these local operations we replace the current cycle cover $\mathcal{C}$ by the symmetric difference between $\mathcal{C}$ and the edges of a given face, provided that the resulting graph is a cycle cover with fewer cycles. By repeating this process on the three initial cycle covers, it is possible to reach a cover with fewer than $5n/36$ cycles, which, in turn, implies the existence of a TSP tour of length at most $(4/3 - 1/18)n$. This result is described in section 2.

The idea of applying local operations to decrease the number of components of an Eulerian subgraph cover can still be applied on more general graph classes. Given a 2-connected cubic graph G, Boyd et al. [5] found a TSP tour T of G with at most $\frac{4}{3}|V(G)| - 2$ edges. Their approach has two phases. In the first phase, they transform G into a simpler cubic 2-connected graph H not containing certain ill-behaved structures (called p-rainbows, for $p \geq 1$). In the second phase, they use a linear programming approach to find a polynomial collection of perfect matchings for H such that a convex combination of them gives every edge a weight of $1/3$. Their complements induce a distribution over cycle covers of H. By performing certain local operations on each cover, they get a distribution of Eulerian subgraph covers having average vertex contribution bounded above by $4/3$. They use this to find a TSP tour for H with at most $\frac{4}{3}|V(H)| - 2$ edges, which can be easily transformed into a TSP tour of G having the desired guarantee. The local operations used by Boyd et al. [5] consist of iterative alternation of 4-cycles and a special type of alternation of 5-cycles (in which some edges get doubled).

Our main result is an improvement on Boyd et al.'s technique that allows us to show that every 2-connected cubic graph G admits a TSP tour with at most $(4/3 - \epsilon)|V(G)| - 2$ edges. The first difference between the approaches, described in section 3, is that our simplification phase is more aggressive. Specifically, we set up a framework to eliminate large families of structures that we use to get rid of all chorded 6-cycles. This clean-up step can very likely be extended to larger families and may ultimately lead to improved results when combined with an appropriate second phase. The second difference, described in section 4, is that we extend the set of local operations of the second phase by allowing the alternation of 6-cycles of a Eulerian subgraph cover. Again, it is likely that one can further exploit this idea to get improved guarantees. Mixing these new ideas appropriately requires significant work but ultimately leads us to find a distribution $\mathcal{D}$ of covers of the simplified graph H for which $\sum_{v\in V(H)} z_{\mathcal{D}}(v) \leq (\frac{4}{3} - \epsilon)n - 2$. From there, we obtain a TSP tour of G with the improved guarantee.

Our analysis allows us to set ϵ as $1/61236 > 0.000016$ for cubic 2-connected graphs. It is worth noting here that by adding extra hypothesis it is possible to improve this constant. For instance, the case of cubic 2-connected bipartite graphs is interesting. This type of graphs is actually 3-edge colorable; therefore, by taking the complements of the three perfect matchings induced by the coloring, we obtain a cycle cover distribution whose support has only 3 cycle covers making the problem much easier to analyze. In fact, by slightly relaxing the simplification phase we can impose that the resulting graph is still bipartite (but allowing a certain type of chorded hexagons) and thus, the operation consisting on the alternation of 5-cycles will never occur. It is possible to show (see Larré's master's thesis [12]) that this modified algorithm yields a tour of length at most $(4/3-1/108)n-2$ for any cubic 2-connected bipartite graph.

2. Barnette graphs. Barnette [4] conjectured that every cubic, bipartite, 3-connected planar graph is Hamiltonian. More than 40 years later and despite considerable effort, Barnette's conjecture is still not settled. This motivates the definition of a *Barnette graph* as a cubic, bipartite, 3-connected planar graph. Even though we are not able to prove or disprove Barnette's conjecture, in this section we show that these graphs admit short tours.

Recall that a tour of a graph G is simply a spanning Eulerian subgraph where every edge appears at most twice. The main idea for obtaining short tours is to find a cycle cover $\mathcal{C}$ of G having small number of cycles. The tour $T(\mathcal{C})$ obtained by taking the union of the edges of $\mathcal{C}$ and a doubled spanning tree of the multigraph obtained by contracting each cycle of $\mathcal{C}$ in G has length $n + 2|\mathcal{C}| - 2$, where $|\mathcal{C}|$ is the number of cycles in $\mathcal{C}$.

Let G be a Barnette graph on n vertices. By 3-connectedness, G has a unique embedding on the sphere up to isomorphism [25]; therefore, its set of faces is well-determined. Furthermore, it has a unique planar dual G^*, which is an Eulerian planar triangulation. As Eulerian planar triangulations are known to be 3-colorable (see, e.g., [23]), Barnette graphs are 3-face colorable. Furthermore, finding such coloring can be done in polynomial time.

We can use this to easily get a tour of G of length at most $4n/3$ as follows. Denote the vertex, edge, and face sets of G as $V = V(G)$, $E = E(G)$, and $F = F(G)$, respectively, and let $c\colon F \to \{1, 2, 3\}$ be a proper 3-face coloring of G. Let $F(i)$ be the set of faces of color i. Since the graph is cubic, $|E| = 3n/2$, and by Euler's formula, $|F| = 2 + |E| - |V| = (n+4)/2$. This means that there is a color i such that $|F(i)| \leq (n + 4)/6$. Since $F(i)$ is a cycle cover, the tour $T(F(i))$ obtained as before has length $n + 2|F(i)| - 2 \leq (4n - 2)/3$.

We will devise an algorithm that finds a short cycle covers of Barnette graphs by performing certain local operations to reduce the number of cycles of the three cycle covers given by $F(1)$, $F(2)$, and $F(3)$. Recall that an even cycle C_0 is *alternating* for a cycle cycle cover $\mathcal{C}$ of G if the edges of C_0 alternate between edges inside $\mathcal{C}$ and edges outside $\mathcal{C}$. If C_0 is an alternating cycle of $\mathcal{C}$, we can define a new cycle cover

$\mathcal{C}\triangle C_0$ whose edge set is the symmetric difference between the edges of $\mathcal{C}$ and those of C_0.

The next procedure constructs cycle covers $\mathcal{C}(i)$, for each $i \in \{1,2,3\}$. Initialize $\mathcal{C}(i)$ as $F(j)$ for some $j \neq i$; for example, $j = i+1 \pmod 3$. As we will see later, each face of $F(i)$ is alternating for $\mathcal{C}(i)$ at every moment of the procedure. Iteratively, check if there is a face f in $F(i)$ such that $\mathcal{C}(i)\triangle f$ has fewer cycles than $\mathcal{C}(i)$. If so, replace $\mathcal{C}(i)$ by the improved cover. Do this step until no improvement is possible to obtain the desired cover. Let $\mathcal{C}$ be the cycle cover of fewer cycles among the three covers found. By returning the tour associated to $\mathcal{C}$ we obtain Algorithm 1 below.

Algorithm 1 To find a tour on a Barnette graph $G = (V, E)$.

Find a 3-face coloring of G with colors in $\{1,2,3\}$.
for each $i \in \{1,2,3\}$ **do**
  $\mathcal{C}(i) \leftarrow F(j)$, for $j = (i+1) \pmod 3$.
  while there is a face $f \in F(i)$ such that $|\mathcal{C}(i)\triangle f| < |\mathcal{C}(i)|$ **do**
    $\mathcal{C}(i) \leftarrow \mathcal{C}(i)\triangle f$.
  end while
end for
Let $\mathcal{C}$ be the cycle cover of smallest cardinality among all $\mathcal{C}(i)$.
Return the tour associated to $\mathcal{C}$.

To analyze the algorithm it will be useful to extend the initial face-coloring to an edge-coloring of G, by assigning to each edge e the color in $\{1,2,3\}$ that is not present in both incident faces of e. Denote as $E(i)$ the set of edges of color i. Then $E(1)$, $E(2)$, and $E(3)$ are disjoint perfect matchings and, furthermore, $E(i) \cup E(j)$ are exactly the edges of $F(k)$, for $\{i,j,k\} = \{1,2,3\}$. Lemma 1 below implies, in particular, that the algorithm is correct.

LEMMA 1. *In every iteration of Algorithm* 1*,* $\mathcal{C}(i)$ *is a cycle cover of* G *containing* $E(i)$ *and every face* $f \in F(i)$ *is alternating for* $\mathcal{C}(i)$*.*

Proof. We proceed by induction. Observe that $\mathcal{C}(i)$ equals $F(j)$ for $j \neq i$ in the beginning so it contains $E(i)$. Suppose this lemma holds at the beginning of an iteration and let f be a face of $F(i)$, so f has no edges of color i. As $\mathcal{C}(i)$ contains $E(i)$, and every vertex v of f has degree 2 in $\mathcal{C}(i)$, we conclude that f is alternating for $\mathcal{C}(i)$ and thus, $\mathcal{C}(i)\triangle f$ is a cover containing $E(i)$. ∎

LEMMA 2. *Let* $\mathcal{C}(i)$ *be the cycle cover obtained at the end of the while-loop in Algorithm* 1 *and let* f *be a face of length* $2k$ *in* $F(i)$*, then* f *intersects at most* $\lfloor \frac{k+1}{2} \rfloor$ *cycles of* $\mathcal{C}(i)$*.*

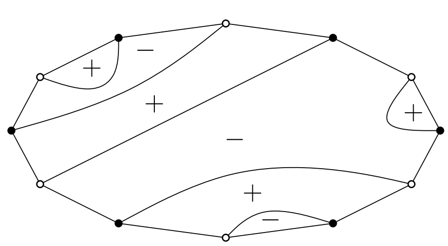

FIG. 1. *The graph* H *in the proof of Lemma* 2.

Proof. Let $\mathcal{C}'$ be the collection of cycles in $\mathcal{C}(i)$ intersecting f and let H be the subgraph of G whose edge set is the union of the edges of $\mathcal{C}'$ and those of f. Using the planar embedding of G having f as the outer face, we can see that H consists of the outer cycle f and a collection of noncrossing paths on the inside connecting vertices of f as in Figure 1.

Label with a plus $(+)$ sign the regions of H bounded by cycles in $\mathcal{C}(i)$ and by a minus $(-)$ sign the rest of the regions except for the outer face, thus $|\mathcal{C}'|$ equals the number of $+$ regions. Furthermore, it is easy to see that the cycle cover $\mathcal{C}(i)\Delta f$ has size equal to $|\mathcal{C}(i)|$ minus the number of $+$ regions plus the number of $-$ regions. By our algorithm's specification, $|\mathcal{C}(i)\Delta f| \geq |\mathcal{C}(i)|$, and therefore the number of $+$ regions must be at most that of $-$ regions. But since f has length $2k$, the total number of regions labeled $+$ and $-$ is $k{+}1$. Since $|\mathcal{C}'|$ is an integer, we get $|\mathcal{C}'| \leq \lfloor (k{+}1)/2 \rfloor$. ∎

With this lemma we can bound the size of the tour returned. For $i \in \{1,2,3\}$ and $k \in \mathbb{N}$ define $F_k(i)$ as the set of faces of color i and length k, and F_k as the total number of faces of length k.

LEMMA 3. *Let $\mathcal{C}(i)$ be the cycle cover obtained at the end of the while-loop in Algorithm* 1. *Then*

$$|\mathcal{C}(i)| \leq 1 + \sum_{k=2}^{\infty} \left\lfloor \frac{k-1}{2} \right\rfloor |F_{2k}(i)| = 1 + |F_6(i)| + |F_8(i)| + 2|F_{10}(i)| + 2|F_{12}(i)| + \cdots .$$

Proof. Let G' be the graph G restricted to the edges of $\mathcal{C}(i)$. Let H be the connected Eulerian multigraph obtained by contracting in G' all the faces in $F(i)$ to vertices. Observe that the edge set of H is exactly $E(i)$. One by one, uncontract each face f in H. In each step we obtain an Eulerian graph which may have more connected components than before. We can estimate the number of cycles in $\mathcal{C}(i)$ as 1 (the original component in H) plus the increase in the number of connected components on each step of this procedure.

Consider the graph H immediately before expanding a face f. Let H_f be the connected component of H containing the vertex associated to f. If f has length $2k$, then, after expanding f, H_f splits into at most $\lfloor \frac{k+1}{2} \rfloor$ connected components. This follows since after expanding all the cycles of $F(i)$, H_f is split into at most that many components by Lemma 2. But then, expanding f increases the number of connected components of H by at most $\lfloor \frac{k+1}{2} \rfloor - 1 = \lfloor \frac{k-1}{2} \rfloor$. ∎

The previous bound is not enough to conclude the analysis. Fortunately, we can find another bound that will be useful.

LEMMA 4. *Let $\mathcal{C}(i)$ be the cycle cover obtained at the end of the while-loop in Algorithm* 1. *If $|C(i)| \neq 1$, then*

$$\begin{aligned} 3|\mathcal{C}(i)| &\leq |\mathcal{C}(i) \cap F_4| + \sum_{k=2}^{\infty} \left\lfloor \frac{k+1}{2} \right\rfloor |F_{2k}(i)| \\ &= |\mathcal{C}(i) \cap F_4| + |F_4(i)| + 2|F_6(i)| + 2|F_8(i)| + 3|F_{10}(i)| + \cdots . \end{aligned}$$

Proof. Since $\mathcal{C}(i)$ contain $E(i)$, every cycle $C \in \mathcal{C}(i)$ must intersect at least two faces in $\mathcal{F}(i)$. In particular, every 4-cycle in $\mathcal{C}(i)$ intersects exactly two faces in $\mathcal{F}(i)$. Consider now a cycle $C \in C(i)$ of length at least 6.

We claim that C must intersect three of more faces of $\mathcal{F}(i)$. If this was not the case, then C intersects exactly two faces f_1 and f_2. Then the edges of C must alternate as one edge of f_1, then one edge in $E(i)$ crossing between the faces, then one edge in f_2, and one edge in $E(i)$ crossing between the faces. In particular, the length of C must be divisible by 4. Let $v_0, v_1, \ldots, v_{4k-1}$ be the vertices of C. By

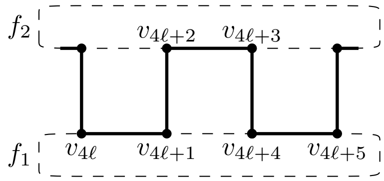

FIG. 2. *A cycle C intersecting two faces f_1 and f_2.*

the previous observation we can assume that for every $1 \le \ell \le k$, $v_{4\ell}v_{4\ell+1}$ is an edge of f_1, $v_{4\ell+2}v_{4\ell+3}$ is an edge of f_2, and the rest of the edges of C are crossing between faces, as depicted in Figure 2.

Suppose that there exists a vertex in f_1 not contained in C, then there must be an index ℓ such that v is in the path P from $v_{4\ell+1}$ to $v_{4\ell+4}$ in f_1, that is internally disjoint with C. Since f_1 is a face of G, we deduce that removing $v_{4\ell+1}$ and $v_{4\ell+4}$ from G disconnects the graph. But this contradicts the 3-connectedness of G. Therefore, C contains all of the vertices in f_1 and, by an analogous argument, all of the vertices of f_2. Since the graph is cubic and connected, the only possibility is that C contains all of the vertices of G. But then $|\mathcal{C}(i)| = 1$ which contradicts the hypothesis of the lemma. Therefore, we have proved the every cycle in $\mathcal{C}(i)$ of length at least six intersects at least 3 faces.

Define the set

$$J(i) = \{(C, H) \in \mathcal{C}(i) \times \mathcal{F}(i) \colon \text{ cycle } C \text{ intersects face } H\ \}.$$

By the previous discussion we have

$$|J(i)| \ge 2|\mathcal{C}(i) \cap F_4| + 3|\mathcal{C}(i) \setminus F_4| = 3|\mathcal{C}(i)| - |\mathcal{C}(i) \cap F_4|.$$

By Lemma 2 we have

$$|J(i)| \le \sum_{k=2}^{\infty} \left\lfloor \frac{k+1}{2} \right\rfloor |F_{2k}(i)|.$$

Combining the last two inequalities we get the desired result. $\square$

Now we have all of the ingredients to bound the size of the cycle cover returned by the algorithm.

LEMMA 5. *Let $\mathcal{C}$ be the cycle cover computed by Algorithm* 1. *Then*

$$|\mathcal{C}| \le \min\left\{\frac{n+4}{6} - \frac{|F_4|}{6}, \frac{(n+1)}{9} + \frac{|F_4|}{6}\right\}.$$

Proof. First, we need a bound on a quantity related to previous lemmas. Let α be $\sum_{k=2}^{\infty} \left\lfloor \frac{k-1}{2} \right\rfloor |F_{2k}|$. We claim that $\alpha \le \frac{1}{2}\left(n - 2 - |F_4|\right)$.

Since $|F| = (n+4)/2$, the claim above is equivalent to proving that $2|F_4| + 2|F| + 4\alpha \le 3n$. Note that $2|F_4| + 2|F| + 4\alpha$ equals

$$\begin{array}{rllllllll} & 2|F_4| & & & & & & \\ + & 2|F_4| + & 2|F_6| + & 2|F_8| + & 2|F_{10}| + & 2|F_{12}| + & 2|F_{14}| + & \cdots \\ & & + \ 4|F_6| + & 4|F_8| + & 8|F_{10}| + & 8|F_{12}| + & 12|F_{14}| + & \cdots \\ = & 4|F_4| + & 6|F_6| + & 6|F_8| + & 8|F_{10}| + & 8|F_{12}| + & 10|F_{14}| + & \cdots, \end{array}$$

which is upper bounded by $\sum_{k=2}^{\infty} 2k|F_{2k}|$. This quantity is the sum of the length of all the faces in G. As each vertex is in three faces, this quantity is at most $3n$, which proves the claim.

By Lemma 3 we have

$$|\mathcal{C}| \leq \frac{1}{3}\sum_{i=1}^{3}|\mathcal{C}(i)| \leq 1+\frac{\alpha}{3} \leq 1+\frac{1}{6}(n-2-|F_4|) = \frac{n+4}{6} - \frac{|F_4|}{6}.$$

On the other hand, using that $|F| = (n+4)/2$ and Lemma 4, we get

$$9|\mathcal{C}| \leq 3\sum_{i=1}^{3}|\mathcal{C}(i)| \leq \sum_{i=1}^{3}|\mathcal{C}(i)\cap F_4| + \alpha + |F| \leq \sum_{i=1}^{3}|\mathcal{C}(i)\cap F_4| + \frac{1}{2}(2n+2-|F_4|).$$

Note that each cycle in F_4 avoids one color; therefore, it can only appear in two cycle covers $\mathcal{C}(i)$, i.e., $\sum_{i=1}^{3}|\mathcal{C}(i)\cap F_4| \leq 2|F_4|$. From here we get that

$$|\mathcal{C}| \leq \frac{1}{9}\left(2|F_4| + n + 1 - |F_4|/2\right) = \frac{n+1}{9} + \frac{|F_4|}{6}. \qquad \square$$

The previous lemma implies the main result of this section.

Theorem 1. *Let $\mathcal{C}$ be the cycle cover computed by Algorithm* 1 *and let T be the tour returned. Then $|\mathcal{C}| \leq \frac{5n+14}{36}$, and the length of T is at most $\frac{23n-22}{18} = \left(\frac{4}{3}-\frac{1}{18}\right)n - \frac{11}{9}$. In particular, every Barnette graph admits a tour of length at most $\left(\frac{4}{3}-\frac{1}{18}\right)n - \frac{11}{9}$.*

Proof. The expression on the right-hand side of Lemma 5 is maximized when $|F_4| = \frac{n+10}{6}$, for which it attains a value of $\frac{5n+14}{36}$. Therefore, for every value of $|F_4|$, this quantity is an upper bound of $|\mathcal{C}|$. To conclude, we just use that the length of T is $n + 2(|\mathcal{C}|-1)$. $\square$

3. 2-connected cubic graphs: Simplification phase. We now go back to general 2-connected cubic graphs. Our algorithm starts by reducing the input graph G to a simpler 2-connected cubic graph H which does not contain a cycle of length six with one or more chords as subgraph. In addition our reduction satisfies that if H has a TSP tour of length at most $(4/3-\epsilon)|V(H)|-2$, then G has a TSP tour of length at most $(4/3-\epsilon)|V(G)|-2$, where $V(H)$ and $V(G)$ denote the vertex sets of H and G, respectively. We will use the notation $\chi^F \in \{0,1\}^E$ of $F \subset E$ to denote the *incidence vector* of F ($\chi^F_e = 1$ if $e \in F$, and 0 otherwise).

Reduction 1. Let γ be a 6-cycle having two chords and let $G[V(\gamma)]$ be the subgraph induced by the set of vertices contained in γ Also, let v_1 and v_2 be the two vertices connecting γ to the rest of G. Our reduction replaces $G[V(\gamma)]$ by a 4-cycle with a chord (shown in Figure 3), identifying v_1 and v_2 with the vertices of degree 2 in the chorded 4-cycle. This procedure in particular removes the *p-rainbow* structure in Boyd et al. [5].

The second step is to consider 6-cycles having only one chord. Let γ be such a cycle and let $G[V(\gamma)]$ be the subgraph induced by the set of vertices contained in γ. Consider the four edges e_1, e_2, e_3, and e_4 connecting γ to the rest of G. Letting w_i be the endpoint of e_i outside γ, we distinguish the following three reductions according to three different cases.

Reduction 2. If only two of the w_i's are distinct, we proceed as in the previous case (Reduction 1); that is, replacing $G[V(\gamma)]$ by a chorded 4-cycle.

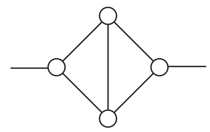

FIG. 3. *A* 4-*cycle with a chord.*

Reduction 3. If three of the w_i's are distinct we replace the 7 vertex structure formed by γ and the w_i adjacent to two vertices in γ by a triangle (3-cycle), identifying the degree two vertices in the structure with those in the triangle. Figure 4 shows an example of this reduction in the specific case that γ has a chord connecting symmetrically opposite vertices.

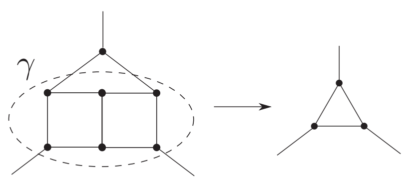

FIG. 4. *Reduction* 3 *in the case that* γ *has a chord connecting symmetrically opposite vertices.*

Reduction 4. The final case is when all four w_i's are distinct. Assume, without loss of generality, that the $w_i's$ are indexed in the cyclic order induced by γ. In this case we replace γ by an edge e and we either connect w_1, w_2 to one endpoint of e and w_3, w_4 to the other, or we connect w_1, w_4 to one endpoint and w_2, w_3 to the other. The previous choice can always be made so that in the reduced graph e is not a bridge, as the following lemma shows.

LEMMA 6. *Let* γ *be a chorded* 6-*cycle considered in Reduction* 4. *Then, the edges in* $G[V(\gamma)]$ *do not simultaneously contain a cut of* G *separating the vertex sets* $\{w_1, w_2\}$ *and* $\{w_3, w_4\}$, *and a second cut separating the vertex sets* $\{w_2, w_3\}$ *and* $\{w_1, w_4\}$.

Proof. By contradiction, suppose that the edge set of $G[V(\gamma)]$ contains two cuts of G such that the first cut separates the vertex sets $\{w_1, w_2\}$ and $\{w_3, w_4\}$, and the second cut separates the vertex sets $\{w_2, w_3\}$ and $\{w_1, w_4\}$. By deleting the vertices and edges of γ, we split G into four connected components, one containing each w_i; thus all paths connecting w_1 and w_3 in G must use the edge e_1, contradicting the 2-connectivity of G. $\square$

Note that all of the above reduction steps strictly decrease the number of vertices in the graph while keeping the graph cubic and 2-connected, and that each step requires only polynomial time. Thus only a linear number of polynomial time steps are needed to obtain a cubic and 2-connected reduced graph H which does not contain 6-cycles with one or more chords. The following result shows that any TSP tour in the reduced graph H of length at most $(4/3-\epsilon)|V(H)|-2$ can be turned into a TSP tour in the original graph G of length at most $(4/3-\epsilon)|V(G)|-2$.

PROPOSITION 1. *Let* T' *a TSP tour in the reduced graph* H *of length at most* $\alpha|V(H)|-2$, *with* $5/4 \leq \alpha \leq 4/3$. *Then, a TSP tour* T *can be constructed in the original graph* G *in polynomial time, such that the length of* T *is at most* $\alpha|V(G)|-2$.

Proof. We distinguish certain cases, depending on what reduction was performed over the graph.

- Case of Reduction 1 or 2: in this case the result is a consequence of the following lemma.

LEMMA 7. *Let $G = (V, E)$ be a graph and let $U \subset V$ be such that the cut $\delta(U)$ has only two elements, say $\delta(U) = \{e_1, e_2\}$. Let $v, w \in U$ be the two end vertices of e_1 and e_2 in U, respectively. Let us suppose that the subgraph $G[U]$ is Hamiltonian and contains a Hamiltonian path connecting v and w. Let H be the graph resulted from replacing the subgraph $G[U]$ by a chorded 4-cycle D and let T' be a TSP tour in H. Then, there exists a TSP tour T in G with length $|T| \leq |T'| + |U| - 4$.*

Proof of Lemma 7. Let χ^P be the incidence vector of some Hamiltonian path P connecting v and w, and let χ^C be the incidence vector of some Hamiltonian cycle C in $G[U]$. Let us denote by x' the vector $\chi^{T'}$ (the incidence vector of T'). We are going to extend the TSP tour T' to the original graph G depending on the value of x' on edges e_1 and e_2. We know that $x'(\{e_1, e_2\}) := x'(e_1) + x'(e_2)$ must be an even number, since T' is a TSP tour. Considering that x' takes values over $\{0, 1, 2\}$, we have that all of the possible cases for the values of $x'(e_1)$ and $x'(e_2)$ are as follows.

- Case $(x'(e_1), x'(e_2)) = (1, 1)$: in this case there is a path in the chorded 4-cycle D and in T' of length 3 connecting v and w. Then, we define the vector x as

$$x(e) = \begin{cases} \chi^P(e) & \text{if } e \in E(G[U]), \\ x'(e) & \text{if } e \in E(G) \setminus E(G[U]]). \end{cases}$$

- Case $(x'(e_1), x'(e_2)) \in \{(2, 0), (0, 2)\}$: in this case there is a 4-cycle in D and in T'. Then, we define the vector x as

$$x(e) = \begin{cases} \chi^C(e) & \text{if } e \in E(G[U]), \\ x'(e) & \text{if } e \in E(G) \setminus E(G[U]). \end{cases}$$

- Case $(x'(e_1), x'(e_2)) = (2, 2)$: in this case we redefine $x'(e_2) = 0$ and then we define the vector x as the previous case.

In any case x is the incidence vector of a TSP tour T in G, of length $|T| \leq |T'| + |U| - 4$. □

It is straightforward to verify that both Reduction 1 and 2 satisfy the hypothesis of Lemma 7. In the case of Reduction 1, the vertex set of the replaced structure has size $|U| = 6$, and in the case of Reduction 2, the vertex set of the replaced structure has size $|U| = 8$. Whatever the case, we have that $|V(G)| = |V(H)| + |U| - 4$, and then

$$\begin{aligned} |T| &\leq |T'| + |U| - 4 \\ &\leq \alpha|V(H)| - 2 + |U| - 4 \\ &= \alpha|V(G)| - 2 - (\alpha - 1)(|V(G)| - |V(H)|) \\ &\leq \alpha|V(G)| - 2, \end{aligned}$$

where the first inequality holds by Lemma 7.

- Case of Reduction 3: in this case the result is a consequence of the following lemma.

LEMMA 8. *Let $G = (V, E)$ a graph and let $U \subset V$ be such that $|U| = 7$ and the cut $\delta(U)$ has only three elements, say $\delta(U) = \{e_1, e_2, e_3\}$. Let $v_1, v_2, v_3 \in U$ be the three end vertices of e_1, e_2, and e_3 in U, respectively. Let us suppose that the subgraph $G[U]$ contains a (not necessarily simple) cycle C of length at most 8 which visits every vertex of U, and for every pair of vertices $v, w \in \{v_1, v_2, v_3\}$ there exists* *a (not necessarily simple) path $P(v,w)$ of length at most 7 which visits every vertex of U. Let H be the graph resulted from replacing the subgraph $G[U]$ by a triangle and let T' be a TSP tour in H. Then, there exists a TSP tour T in G with length $|T| \leq |T'| + 5$.*

Proof of Lemma 8. Let $\chi^P(v,w)$ be the incidence vector of path $P(v,W)$ and χ^C be the incidence vector of C. Let us denote by x' the vector $\chi^{T'}$ (the incidence vector of T'). We are going to extend the TSP tour T' to the original graph G depending on the value of x' on edges e_1, e_2, and e_3. We know that $x'(\{e_1,e_2,e_3\}) := x'(e_1)+x'(e_2)+x'(e_3)$ must be an even number, since T' is a TSP tour. Considering that x' takes values over $\{0,1,2\}$, we have that all of the possible cases for the values of $x'(e_1)$, $x'(e_2)$, and $x'(e_3)$ are as follows.

- Case $(x'(e_1), x'(e_2), x'(e_3)) = (2,0,0)$ (or another possible permutation): in this case we define the vector x as

$$x(e) = \begin{cases} \chi^C(e) & \text{if } e \in E(G[U]), \\ x'(e) & \text{if } e \in E(G) \setminus E(G[U]). \end{cases}$$

- Case $(x'(e_1), x'(e_2), x'(e_3)) \in \{(2,2,0),(2,2,2)\}$ (or another possible permutation): first we redefine $x'(e_2) = x'(e_3) = 0$ and then we define the vector x as the previous case.
- Case $(x'(e_1), x'(e_2), x'(e_3)) = (1,1,0)$ (or another possible permutation): in this case we define the vector x as

$$x(e) = \begin{cases} \chi^{P(v_1,v_2)}(e) & \text{if } e \in E(G[U]), \\ x'(e) & \text{if } e \in E(G) \setminus E(G[U]). \end{cases}$$

- Case $(x'(e_1), x'(e_2), x'(e_3)) = (1,1,2)$ (or another possible permutation): first we redefine $x'(e_3) = 0$ and then we define the vector x as the previous case.

In any case x is the incidence vector of a TSP tour T in G. Note that x was constructed from x', by adding edges from the original graph and removing edges from the triangle. In the first and second case we added at most eight edges and deleted at least 3. In the third and fourth cases we added at most seven edges and deleted at least 2. In any case, we have that $|T| \leq |T'| + 5$. □

To see that all possible structures that are considered in Reduction 3 satisfy the hypothesis of Lemma 8, we need only check by inspection that there exist a cycle C of length at most 8 and paths of length at most 7 connecting every pair of vertices, which visit every vertex of the structures of Figure 5.

Then, applying Reduction 3, we can define a TSP tour T in the original graph with length

$$\begin{aligned} |T| &\leq |T'| + 5 \\ &\leq \alpha|V(H)| - 2 + 5 \\ &= \alpha|V(G)| - 2 + 5 + \alpha(|V(H)| - |V(G)|) \\ &= \alpha|V(G)| - 2 + (5 - \alpha 4) \\ &\leq \alpha|V(G)| - 2, \end{aligned}$$

where the first inequality holds by Lemma 8 and the last one since $5/4 \leq \alpha$.

- Case of Reduction 4: it is easy to construct a TSP tour T in the original graph G with length $|T| \leq |T'| + 4$. The detailed case analysis is omitted since it is very

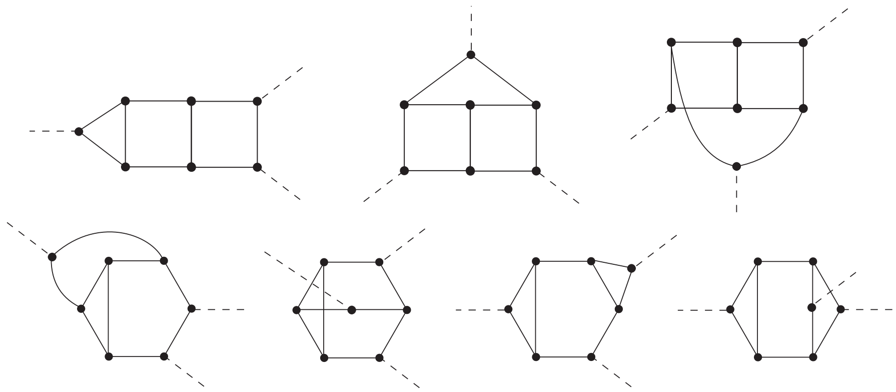

FIG. 5. *All possible structures that are considered in Reduction* 3. *The three figures on the top consider the situations in which* γ *has a chord dividing the cycle into two* 4*-cycles and the extra vertex connects two vertices in* γ *at distance* 1*,* 2*, and* 3*. The next four cases correspond to the situations in which* γ *has a chord dividing the cycle into a* 3*-cycle and a* 5*-cycle and the extra vertex connects two vertices in* γ *at distance* 1 *(middle right), at distance* 2 *(left and right) and at distance* 3 *(middle left). Symmetric cases are displayed only once.*

similar to the previous ones. Then

$$\begin{aligned}|T| &\le |T'| + 4\\ &\le \alpha|V(H)| - 2 + 4\\ &= \alpha|V(G)| - 2 + 4 + \alpha(|V(H)| - |V(G)|)\\ &= \alpha|V(G)| - 2 + (4 - \alpha 4)\\ &\le \alpha|V(G)| - 2,\end{aligned}$$

where the last inequality holds since $\alpha \ge 1$.

Considering this latter case, we finish the proof of Proposition 1. ∎

Finally, note that—as mentioned above—only a linear number of reduction steps are needed, and each step requires only polynomial time, not only to find the desired structure, but also to recover the TSP tour in the original graph. Thus this graph simplification phase runs in polynomial time.

4. 2-connected cubic graphs: Eulerian subgraph cover phase. We say that a matching M is 3*-cut perfect* if M is a perfect matching intersecting every 3-cut in exactly one edge. Boyd et al. [5] have shown the following lemma.

LEMMA 9 (see [5]). *Let* $G = (V, E)$ *be a* 2*-connected cubic graph. Then, the vector* $\frac{1}{3}\chi^E$ *can be expressed as a convex combination of incidence vectors of* 3*-cut perfect matchings of* G*. This is, there are* 3*-cut perfect matchings* $\{M_i\}_{i=1}^k$ *and positive real numbers* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *such that*

$$\sum_{i=1}^{k} \lambda_i = 1 \tag{4.1}$$

and

$$\frac{1}{3}\chi^E = \sum_{i=1}^{k} \lambda_i \chi^{M_i}. \tag{4.2}$$

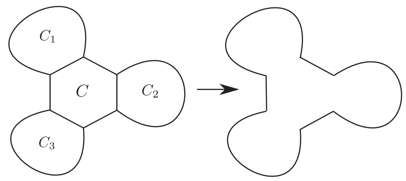

FIG. 6. *Operation (U1).*

Furthermore, Barahona [3] *provides an algorithm to find a convex combination of* $\frac{1}{3}\chi^E$ *having* $k \leq 7n/2 - 1$ *in* $O(n^6)$ *time.*

Consider a graph G that is cubic, 2-connected, and *reduced*. That is, no 6-cycle in G has chords. We also assume that $n \geq 10$ as every cubic 2-connected graph on less than ten vertices is Hamiltonian.

Let $\{M_i\}_{i=1}^k$ and $\{\lambda_i\}_{i=1}^k$ be the 3-cut matchings and coefficients guaranteed by Lemma 9. Let $\{\mathcal{C}_i\}_{i=1}^k$ be the family of *cycle covers* associated to the matchings $\{M_i\}_{i=1}^k$. This is, $\mathcal{C}_i$ is the collection of cycles induced by $E \setminus M_i$. Since each matching M_i is 3-cut perfect, the corresponding cycle cover $\mathcal{C}_i$ does not contain 3-cycles. Furthermore, every 5-cycle in $\mathcal{C}_i$ is induced (i.e., it has no chord in G).

In what follows we define three local operations, (U1), (U2), and (U3), that will be applied iteratively to the current family of covers. Each operation is aimed to reduce the contribution of each component of the family. We stress here that operations (U2) and (U3) are exactly those used by Boyd et al. [5], but for the reader's convenience we explain them here. We start with operation (U1).

(U1) Consider a cycle cover $\mathcal{C}$ of the current family. If C_1, C_2, and C_3 are three disjoint cycles of $\mathcal{C}$ that intersect a fixed 6-cycle C of G (which, because of the graph simplification phase, has no chords), then we merge them into the simple cycle obtained by taking their symmetric difference with C. This is, the new cycle in $V(C_1) \cup V(C_2) \cup V(C_3)$ having edge set $(E(C_1) \cup E(C_2) \cup E(C_3))\Delta E(C)$.

An example of (U1) is depicted in Figure 6. We apply (U1) as many times as possible to get a new cycle cover $\{\mathcal{C}_i^{\mathrm{U1}}\}_{i=1}^k$. Then we apply the next operation.

(U2) Consider a cycle cover $\mathcal{C}$ of the current family. If C_1 and C_2 are two disjoint cycles of $\mathcal{C}$ that intersect a fixed 4-cycle C of G, then we merge them into a simple cycle obtained by taking their symmetric difference with C. This is, the new cycle in $V(C_1) \cup V(C_2)$ having edge set $(E(C_1) \cup E(C_2))\Delta E(C)$.

We apply operation (U2) as many times as possible to obtain a new cycle cover $\{\mathcal{C}_i^{\mathrm{U2}}\}_{i=1}^k$ of G. The next operation we define may transform a cycle cover $\mathcal{C}$ of the current family into a Eulerian subgraph cover Γ, having components that are not necessarily cycles.

(U3) Let Γ be an Eulerian subgraph cover of the current family. If γ_1 and γ_2 are two components of Γ, each one having at least five vertices, whose vertex set intersect a fixed 5-cycle C of G, then combine them into a single component, by adding at most one extra edge.

To explain how we combine the components in operation (U3) we need the following two lemmas.

LEMMA 10 (see [5]). *Let H_1 and H_2 be two connected Eulerian multisubgraphs of a cubic graph G having at least two vertices in common and let H_3 be the sum of H_1 and H_2, i.e., the union of their vertices and the sum of their edges (allowing multiple parallel edges). Then we can remove (at least) two edges from H_3 such that it stays*

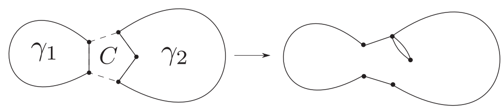

FIG. 7. *Sketch of operation (U3).*

connected and Eulerian.

Proof. Let u and v be in both H_1 and H_2. The edge set of H_3 can be partitioned into edge-disjoint (u, v)-walks P_1, P_2, P_3, and P_4. Since u has degree 3 in G, there must be two parallel edges incident to u that are on different paths, say $e_1 \in P_1$ and $e_2 \in P_2$. If we remove e_1 and e_2, then the graph stays Eulerian. Moreover, it stays connected since u and v are still connected by P_3 and P_4 and every vertex of P_1 and P_2 is still connected to one of u and v. ∎

LEMMA 11 (similar to an observation in [5]). *If v belongs to a component γ of any of the covers Γ considered by the algorithm, then at least two of its three neighbors are in the same component.*

Proof. The above lemma holds trivially when γ is a cycle. In particular, the lemma holds before the application of operation (U3). As the vertex set of a component created by operation (U3) is the union of the vertex set of two previous components, the above lemma also holds after operation (U3). ∎

Observe that if γ is a component of a cover in the current family, and C is an arbitrary cycle of G containing a vertex of γ, then, by the cubicity of G and Lemma 11, C and γ must share at least two vertices. In particular, if γ_1 and γ_2 are the two components intersecting a 5-cycle C considered by operation (U3), then one of them, say γ_1, must contain exactly two vertices of C and the other one must contain the other three vertices (note that they cannot each share two vertices, since then a vertex of C would not be included in the cover). To perform (U3) we first merge γ_1 and C using Lemma 10 removing two edges, and then we merge the resulting component with γ_2, again removing two edges. Altogether, we added the five edges of C and removed four edges. Finally, we remove two edges from each group of triple or quadruple edges that may remain, so that each edge appears at most twice in each component. Figure 7 shows an example of (U3).

Remark 1. Operation (U3) generates components having at least ten vertices. Therefore, any component having nine or fewer vertices must be a cycle. Furthermore, all the cycles generated by (U1) or (U2) contain at least ten vertices (this follows from the fact that G is reduced, and so operation (U2) always involves combining two cycles of length at least 5). From here we observe that any component having nine or fewer vertices must be in the original cycle cover $\{C_i\}_{i=1}^k$.

We say that a 4-cycle C with a chord is isolated if the two edges incident to it are not incident to another chorded 4-cycle. The following is the main result of this section. Before proving it we show it implies the main result of this paper.

PROPOSITION 2 (main proposition). *Let $\{\Gamma_i\}_{i=1}^k$ be the family of Eulerian subgraph covers at the end of the algorithm (that is, after applying all operations), and let $z(v) = z_{\mathcal{D}}(v)$ be the average contribution of vertex v for the distribution $\mathcal{D} = \{(\Gamma_i, \lambda_i)\}_{i=1}^k$. Furthermore, let γ_i be the component containing v in Γ_i and $\Gamma(v) = \{\gamma_i\}_{i=1}^k$. We have the following.*

(P1) *If v is in an isolated chorded 4-cycle, then $z(v) \le 4/3$.*

(P2) *If v is in a nonisolated chorded 4-cycle of G, then $z(v) \le 13/10$.*

(P3) *Else, if there is an induced 4-cycle $\gamma \in \Gamma(v)$, then $z(v) \le 4/3 - 1/60$.*

(P4) *Else, if there is an induced* 5*-cycle* $\gamma \in \Gamma(v)$*, then* $z(v) \leq 4/3 - 1/60$.

(P5) *Else, if there is an induced* 6*-cycle* $\gamma \in \Gamma(v)$*, then we have both* $z(v) \leq 4/3$ *and* $\sum_{w \in V(\gamma)} z(w) \leq 6 \cdot (4/3 - 1/729)$.

(P6) *In any other case* $z(v) \leq 13/10$.

THEOREM 2. *Every* 2*-connected cubic graph* $G = (V, E)$ *admits a TSP tour of length at most* $(4/3 - \epsilon)|V| - 2$*, where* $\epsilon = 1/61236$*. This tour can be computed in polynomial time.*

Proof of Theorem 2. From section 3, we can assume that G is also reduced and so the main proposition holds. Let B be the union of the vertex sets of all isolated chorded 4-cycles of G. We say a vertex is *bad* if it is in B, and *good* otherwise. We claim that the proportion of bad vertices in G is bounded above by 6/7. To see this, construct the auxiliary graph G' from G by replacing every isolated chorded 4-cycle with an edge between its two neighboring vertices. Since G' is cubic, it contains exactly $2|E(G')|/3$ vertices, which are good in G. Hence, for every bad vertex there are at least $(1/4) \cdot (2/3) = 1/6$ good ones, proving the claim.

The Main proposition guarantees that every bad vertex v contributes a quantity $z(v) \leq 4/3$. Now we show that the average contribution of all the good vertices is at most $(4/3 - \delta)$ for some δ to be determined. To do this, define $\mathcal{H} = \{\gamma \in \bigcup_i \Gamma_i \colon |V(\gamma)| = 6\}$ as the collection of all 6-cycles appearing in some cover of the final family, and let $H = \bigcup_{\gamma \in \mathcal{H}} V(\gamma)$ be the vertices included in some 6-cycle of $\mathcal{H}$. It is easy to check that B and H are disjoint. Furthermore, the Main proposition guarantees that if $v \in V \setminus (B \cup H)$, then $z(v) \leq (4/3 - 1/60)$. So we focus on bounding the contribution of the vertices in H.

For every $v \in H$, let $f(v)$ be the number of distinct 6-cycles in $\mathcal{H}$ containing v. Since G is cubic, there is an absolute constant K, such that $f(v) \leq K$. By the Main proposition, $z(v) \leq 4/3$ for $v \in H$ and for every $\gamma \in \mathcal{H}$, $\sum_{v \in V(\gamma)} z(v) \leq 6 \cdot (4/3 - \epsilon')$, where $\epsilon' = 1/729$. Putting this all together we have that

$$
\begin{aligned}
&K \cdot \sum_{v \in H} \left[z(v) - \left(\frac{4}{3} - \frac{\epsilon'}{K} \right) \right] = |H|\epsilon' + K \sum_{v \in H} \left(z(v) - \frac{4}{3} \right) \\
&\leq 6|\mathcal{H}|\epsilon' + \sum_{v \in H} f(v) \left(z(v) - \frac{4}{3} \right) = 6|\mathcal{H}|\epsilon' + \sum_{\gamma \in \mathcal{H}} \sum_{v \in V(\gamma)} \left(z(v) - \frac{4}{3} \right) \\
&\leq 6|\mathcal{H}|\epsilon' - \sum_{\gamma \in \mathcal{H}} 6\epsilon' = 0.
\end{aligned}
$$

It follows that $\frac{1}{|H|} \sum_{v \in H} z(v) \leq (4/3 - \epsilon'/K)$. Since $\epsilon'/K \leq 1/60$, we get

$$
\begin{aligned}
\sum_{v \in V} z(v) &\leq \sum_{v \in B} z(v) + \sum_{v \in H} z(v) + \sum_{v \in V \setminus (B \cup H)} z(v) \\
&\leq \frac{4}{3}|B| + \left(\frac{4}{3} - \frac{\epsilon'}{K} \right) (|V| - |B|) = |V| \left(\frac{4}{3} - \frac{\epsilon'}{7K} \right).
\end{aligned}
$$

We conclude that there is an index i such that $\sum_{v \in V} z_i(v) \leq |V| \left(4/3 - \epsilon'/(7K) \right)$. By adding a double spanning tree of $G/E(\Gamma_i)$ we transform Γ_i into a TSP tour T of length $|V| \left(4/3 - \epsilon'/(7K) \right) - 2$. Noting that $K \leq 12$ and $\epsilon' = 1/729$ we obtain the desired bound.[1] Clearly, all operations can be done in polynomial time. ∎

[1]Consider two edges, e_1 and e_2, adjacent to v. Since there is no chorded 6-cycle, if e_1 and e_2 are contained in a 4-cycle, then v must be contained in at most one 6-cycle. Otherwise, there are at most four 6-cycles which may contain e_1 and e_2. Because there are three possible pairs of edges, we have $K = 12$.

4.1. Proof of the main proposition. We start by a lemma, whose proof is the same as that of [5, Observation 1].

LEMMA 12 ([5]). *For each vertex $v \in V$, and each $i \in \{1, \dots, k\}$, the contribution $z_i(v) := z_{\Gamma_i}(v)$ is*

(a) *at most $\frac{h+2}{h}$, where $h = \min\{t, 10\}$ and v is on a t-cycle belonging to one of the cycle covers $\mathcal{C}_i$, $\mathcal{C}_i^{U1}$, and $\mathcal{C}_i^{U2}$.*

(b) *at most $\frac{13}{10}$ if operation (U3) modified the component containing v.*

We will also use the following notation in our proof. For any subset J of indices in $[k] := \{1, \dots, k\}$, define $\lambda(J) = \sum_{i \in J} \lambda_i$.

The proofs of parts (P1) through (P4) are similar to the arguments used by Boyd et al. [5] to show that $z(v) \leq 4/3$ when v is a 4-cycle or 5-cycle. By using the fact that G is reduced (i.e., it contains no chorded 6-cycles) we obtain a better guarantee in (P2), (P3) and (P4). To prove part (P5) we heavily use the fact that operation (U1) is applied to the initial cycle cover (recall that this operation was not used in [5]).

4.1.1. Proof of part (P1) of the main proposition. Let v be in some isolated chorded 4-cycle C with $V(C) = \{a, b, u_0, u_1\}$ as in Figure 8.

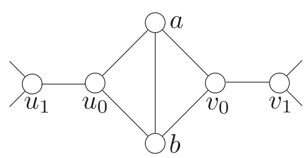

FIG. 8. *A chorded* 4*-cycle.*

For every index i, let C_i be the cycle containing v in the initial cycle cover $\mathcal{C}_i$, and let $\mathcal{C}(v) = \{C_i\}_{i=1}^k$. Consider a cycle $C' \in \mathcal{C}(v)$, and recall that C' cannot be a triangle. If C' does not contain the edge u_0u_1, then $C' = C$. Consider now the case in which C' contains u_0u_1. Then we must also have $ab \in E(C')$ and $v_0v_1 \in E(C')$. Since the graph is reduced, $v_1u_1 \notin E$ as otherwise $u_1 - u_0 - a - b - v_0 - v_1$ would induce a chorded 6-cycle. Hence, the cycle C' cannot be of length 6. It also cannot be of length 7 since then there would be a 3-cut with three matching edges. Therefore, it must be of length at least 8. Using that $\sum_{\{i:\ u_1u_2 \in M_i\}} \lambda_i = \frac{1}{3}$ and applying Lemma 12, we conclude that $z(v) \leq (1/3 \cdot 6/4 + 2/3 \cdot 10/8) = 4/3$.

4.1.2. Proof of part (P2) of the main proposition. Let v be in some non-isolated chorded 4-cycle C with $V(C) = \{a, b, u_0, u_1\}$ as in Figure 8 and recall that $v_1u_1 \notin E$. Without loss of generality, we can assume that u_1 is in a different chorded 4-cycle D. Furthermore, assume that v_1 is not connected by an edge to D, as this would imply the existence of a bridge in G.

Consider, as in the proof of part (P1) a cycle $C' \in \mathcal{C}(v)$. If C' does not contain the edge u_0u_1, then $C' = C$. If on the other hand the edge v_0v_1 is in C', then C' must contain all the vertices of both C and D. It must also contain v_1 and one of its neighbors outside $C \cup D$. In particular, C' has at least ten vertices. By Lemma 12, we have that $z(v) \leq (1/3 \cdot 6/4 + 2/3 \cdot 12/10) = 13/10$.

4.1.3. Proof of part (P3) of the main proposition. Let $\gamma \in \Gamma(v)$ be an induced 4-cycle containing v. By Remark 1, γ is in some initial cycle cover $\mathcal{C}_i$. Since the cycle γ has no chord, then the four edges incident to it (i.e., those sharing one vertex with γ) belong to matching M_i. This observation holds not only for γ but for

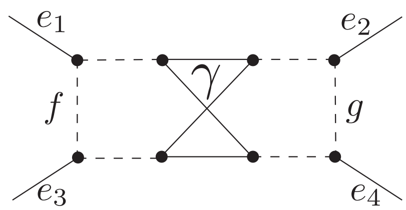

FIG. 9. *4-cycle γ intersecting an 8-cycle.*

any cycle C^* in some initial cycle cover $\mathcal{C}_i$, so we have the following remark.

Remark 2. Let P be a path not sharing edges with a cycle C^* belonging to some initial cycle cover $\mathcal{C}_i$. If P connects any two vertices of C^*, then P has length at least 3.

Furthermore, as the graph is reduced, γ does not share exactly one edge with any other 4-cycle (as this would induce a 6-cycle with a chord). In other words we have the following property.

Remark 3. Let P be a path not sharing edges with γ. If P connects any pair of consecutive edges of γ, then P has length at least 4.

Define the sets $X_p = \{i\colon |C \cap M_i| = p\}$, for $p = 0, 1, 2$ and note that $X_0 \cup X_1 \cup X_2 = [k]$. Define also $x_p = \lambda(X_p)$, for $p = 0, 1, 2$.

By (4.1), we have $x_0 + x_1 + x_2 = 1$. Also, by applying (4.2) to the set of four edges incident to γ we obtain $4x_0 + 2x_1 = 4/3$, which implies that $x_0 = 1/3 - x_1/2$. Finally, by applying (4.2) to the four edges inside γ, we obtain $x_1 + 2x_2 = 4/3$, which implies that $x_2 = 2/3 - x_1/2$.

For every $i \in X_0$, the cycle containing v in $\mathcal{C}_i$ is equal to γ. By Lemma 12 we obtain $z_i(v) \leq 6/4 = 3/2$.

Using Remark 3 we deduce that for every $i \in X_1$, the cycle containing v in $\mathcal{C}_i$ has length at least 7; therefore, by Lemma 12, we have $z_i(v) \leq 9/7$.

Consider now an index $i \in X_2$. Suppose that γ intersects two different cycles of $\mathcal{C}_i$. As each of them has length at least 5 and they both share one edge with a 4-cycle of G we conclude that both cycles are modified by operation (U1) or (U2). Remark 1 implies that v is in a cycle of length at least 10 in $\mathcal{C}_i^{\mathrm{U2}}$. Using Lemma 12 we have $z_i(v) \leq 12/10 = 6/5$.

The only remaining case is if γ is intersected by a single cycle C of $\mathcal{C}_i$. Then, by Remark 3, C has length at least 8. This cycle has length exactly 8 if and only if γ belongs to the structure depicted in Figure 9.

Assume for now that no 8-cycle of an initial cover contains the four vertices of γ. Then, the cycle C in our previous discussion must be of length at least 9, and by Lemma 12, $z_i(v) \leq \max\{11/9, 6/5\} = 11/9$. Putting this all together, we obtain

$$\begin{aligned} z(v) &\leq x_0 3/2 + x_1 9/7 + x_2 11/9 \\ &= (1/3 - x_1/2)3/2 + x_1 9/7 + (2/3 - x_1/2)11/9 \\ &= 71/54 + x_1(9/7 - 3/4 - 11/18) \\ &\leq 71/54 = 4/3 - 1/54. \end{aligned}$$

Now consider the case in which there is an 8-cycle C_j of an initial cover $\mathcal{C}_j$ containing $V(\gamma)$. Then v belongs to the structure depicted in Figure 9, where $e_1 \neq e_3$, $e_2 \neq e_4$, and e_1, e_2, e_3, e_4 are in some matching M_j. As we assumed that $|V(G)| \geq 10$, we cannot simultaneously have $e_1 = e_4$ and $e_2 = e_3$. Let f and g be the leftmost and rightmost edges in the figure. Also, let $Y = \{i\colon f \in M_i\}$ and $Z = \{i\colon g \in M_i\}$. It is easy to check that $Y \cup Z \subseteq X_2$.

Consider an index $i \in X_2$. If $i \in Y \cup Z$ (i.e., if at least one of f and g are in M_i), then the cycle containing v in $\mathcal{C}_i^{\mathrm{U2}}$ has at least ten vertices, and so $z_i(v) \leq 12/10 = 6/5$. If $i \in X_2 \setminus (Y \cup Z)$, then the cycle containing v in $\mathcal{C}_i$ is either the 8-cycle C_j of the structure, or the 8-cycle with edge set $E(C_j)\Delta E(\gamma)$. In any case $z_i(v) \leq 10/8 = 5/4$.

Let $y_1 = \lambda(Y \cup Z)$ and $y_2 = \lambda(X_2 \setminus (Y \cup Z)$, so that $y_1 + y_2 = x_2$. Noting that $y_1 \geq \lambda(Y) = 1/3$, we have

$$\begin{aligned} z(v) &\leq x_0 3/2 + x_1 9/7 + y_1 6/5 + (x_2 - y_1)5/4 \\ &= (1/3 - x_1/2)3/2 + x_1 9/7 + (2/3 - x_1/2)5/4 + y_1(6/5 - 5/4) \\ &= 4/3 - x_1(9/7 - 3/4 - 5/8) - y_1/20 \leq 4/3 - 1/60. \end{aligned}$$

4.1.4. Proof of part (P4) of the main proposition. Let $\gamma \in \Gamma(v)$ be an induced 5-cycle containing v. By Remark 1, γ is in some initial cycle cover $\mathcal{C}_i$. We can assume that no 4-cycle shares exactly one edge with γ, as otherwise operation (U2), or operation (U1) before that, would have modified γ.

The proof for this part is similar to that of part (P3). Define $X_p = \{i\colon |\gamma \cap M_i| = p\}$, for $p = 0, 1, 2$, so that $X_0 \cup X_1 \cup X_2 = [k]$, and let $x_p = \lambda(X_p)$, for $p = 0, 1, 2$.

By (4.1) we have $x_0 + x_1 + x_2 = 1$. Applying (4.2) to the five edges incident to γ, we obtain $5x_0 + 3x_1 + x_2 = 5/3$. This implies that $x_0 = 1/2(1/3 - x_1)$ and $x_2 = 1/2(5/3 - x_1)$.

For every $i \in X_0$, we have $v \in V(\gamma)$ and $\gamma \in \mathcal{C}_i$. By Lemma 12, $z_i(v) \leq 7/5$. For $i \in X_1$, the fact that γ does not share an edge with a 4-cycle implies that v is in a cycle of $\mathcal{C}_i$ having length at least 8, and therefore $z_i(v) \leq 10/8 = 5/4$.

For $i \in X_2$, we have two cases. If γ is intersected by a single cycle C of $\mathcal{C}_i$, then, by Remark 2, C must be of length at least 9, and so, $z_i(v) \leq 11/9$.

The second case is that γ is intersected by two cycles of $\mathcal{C}_i$. One of them, say C', shares exactly one edge with γ (and so, C' cannot be a 4-cycle), and the second one, C'', shares exactly two consecutive edges with γ (by Remark 2, C' cannot be a 4-cycle either). Let $C \in \{C', C''\}$ be the cycle containing vertex v. If C is merged with another cycle during operations (U1) and (U2), then by Remark 1 the resulting cycle containing v in $\mathcal{C}_i^{\mathrm{U2}}$ is of length at least ten, and so $z_i(v) \leq 12/10$. On the other hand, if C is not modified by operations (U1) and (U2), then it must be modified by operation (U3) (this is because C intersects the 5-cycle γ, which in turn intersects two components of $\mathcal{C}_i^{\mathrm{U2}}$ of length at least 5). Lemma 12 guarantees in this case that $z_i(v) \leq 13/10$.

Summarizing, if $i \in X_2$, then $z_i(v) \leq \max\{12/10, 11/9, 13/10\} = 13/10$. Then,

$$\begin{aligned} z(v) &\leq x_0 7/5 + x_1 5/4 + x_2 13/10 \\ &= 1/2(1/3 - x_1) \cdot 7/5 + x_1 5/4 + 1/2(5/3 - x_1) \cdot 13/10 \\ &= 7/30 + 13/12 - x_1/10 \\ &\leq 79/60 = 4/3 - 1/60. \end{aligned}$$

4.1.5. Proof of part (P5) of the main proposition. Let $\gamma \in \Gamma(v)$ be an induced 6-cycle containing v. By Remark 1, γ is in some initial cycle cover $\mathcal{C}_i$. We can assume that no 4-cycle shares exactly one edge with γ, as otherwise operations (U1) or (U2) would have modified γ and so, by the end of the algorithm γ would not be a 6-cycle.

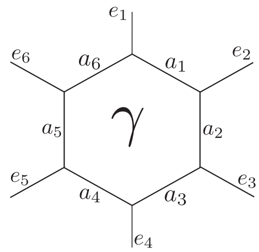

FIG. 10. *Induced* 6-*cycle* γ.

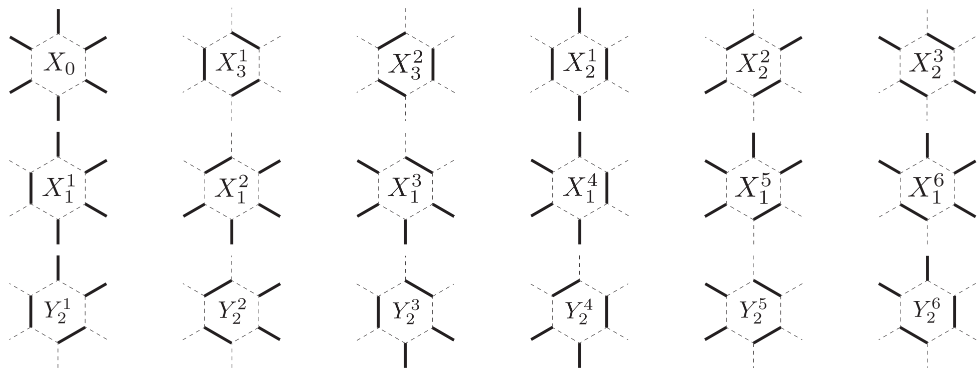

FIG. 11. *The different ways in which a matching can intersect* γ. *We use the same orientation as that of Figure* 10.

We can also assume that γ does not intersect the 5-cycles contained in an initial cycle cover. Indeed, if this was not the case, define $S_5 = \{w \in V(\gamma)\colon\ w \text{ is in some 5-cycle } C \text{ of an initial cycle cover}\}$. If $w \notin S_5$, then in every initial cover, the cycle containing w is of length at least 6; using Lemma 12, part (P4) of the Main proposition, and the fact that $S_5 \neq \emptyset$ implies $|S_5| \geq 2$, we conclude that $\sum_{w\in V(\gamma)} z(w) \leq |S_5|\left(\frac{4}{3} - \frac{1}{60}\right) + |V(C)\setminus S_5|\frac{4}{3} \leq 6\left(\frac{4}{3} - \frac{1}{180}\right)$, and also that $z(w) \leq 4/3$ for all $w \in V$.

Under the assumptions above, all of the components containing v in the final family of covers have length at least 6. Using Lemma 12 we conclude not only that $z(v) \leq \max\{13/10, 8/6\} = 4/3$ (which proves the first statement of P5) but also that $z(w) \leq 4/3$ for the six vertices $w \in V(\gamma)$.

Let us continue with the proof. Denote the edges of γ as $a_1, \ldots, a_6$ and the six edges incident to γ as $e_1, \ldots, e_6$, as in Figure 10.

We now define some sets of indices according on how γ intersects the matchings $M_1, \ldots, M_k$. For every symbol $Z \in \{X_0\} \cup \{X_1^q\}_{q=1}^6 \cup \{X_2^q\}_{q=1}^3 \cup \{Y_2^q\}_{q=1}^6 \cup \{X_3^q\}_{q=1}^2$, we define Z as the set of indices i for which the matching M_i contains the bold edges indicated in Figure 11. For example, $X_0 = \{i\colon \{e_1, \ldots, e_6\} \in M_i\}$, $X_3^1 = \{i\colon \{a_1, a_3, a_5\} \in M_i\}$, and so on. Let also $x_0 = \lambda(X_0)$, $x_i^q = \lambda(X_i^q)$, and $y_2^q = \lambda(Y_i^q)$ for every i and q define

$$x_1 = \sum_{q=1}^{6} x_1^q, \quad x_2 = \sum_{q=1}^{3} x_2^q, \quad y_2 = \sum_{q=1}^{6} y_2^q, \quad x_3 = \sum_{q=1}^{2} x_3^q, \quad \overline{x}_2 = x_2 + y_2.$$

Equation (4.1) implies that $x_0 + x_1 + \overline{x}_2 + x_3 = 1$. Equation (4.2) applied to the set $\{a_1, \ldots, a_6\}$ of edges incident to γ implies that $6x_0 + 4x_1 + 2\overline{x}_2 = 6/3$. Hence, $3x_0 + 2x_1 + \overline{x}_2 = 1$. It follows that

$$2x_0 + x_1 = x_3. \tag{4.3}$$

Recall that there are no 4-cycles in G and no 5-cycles in an initial cycle cover intersecting γ in exactly one edge. Consider $w \in V(\gamma)$ and $i \in [k]$.

If $i \in X_0$ (i.e., M_i shares no edge with γ), then $w \in V(\gamma)$ and $\gamma \in \mathcal{C}_i$. By Lemma 12 we have, $z_i(w) \leq 8/6$. If $i \in X_1 := \cup_{q=1}^{6} X_1^q$ (i.e., M_i contains exactly one edge of γ), then as no 4-cycle shares exactly one edge with γ, w must be in a cycle $C \in \mathcal{C}_i$ of length at least 9; therefore, $z_i(w) \leq 11/9$. If $i \in X_3 := \cup_{q=1}^{2} X_3^q$ (i.e., M_i contains three edges of γ), then we have two cases. The first case is that γ is intersected by one or three cycles of $\mathcal{C}_i$. Then, by the end of operation (U1), w must be in a cycle of $\mathcal{C}_i^{\mathrm{U1}}$ of length at least 9 and so $z_i(w) \leq 11/9$. The second case is that γ is intersected by two cycles of $\mathcal{C}_i$. One of them shares exactly two edges with γ, thence it must be of length at least 8. The other cycle shares exactly one edge with γ and so it must be of length at least 6. Therefore, in this case, four of the vertices w of γ satisfy $z_i(w) \leq 10/8$ and the remaining two satisfy $z_i(w) \leq 8/6$.

We still need to analyze the indices $i \in X_2 := \cup_{q=1}^{3} X_2^q$ and $i \in Y_2 := \cup_{q=1}^{6} Y_2^q$ (i.e., those for which M_i shares two edges with γ). Let $0 < \delta \leq 1$ be a constant to be determined. We divide the rest of the proof in two scenarios.

Scenario 1. If x_3 (which equals $\max\{x_0, x_1, x_3\}$ by (4.3)) is at least δ.

If $i \in X_2 \cup Y_2$, then every vertex $w \in \gamma$ is in a cycle $C \in \mathcal{C}_i$ of length at least 6; therefore $z_i(w) \leq 8/6$ and

$$\begin{aligned} \sum_{w \in V(\gamma)} z(w) &\leq 6 \cdot (x_0 8/6 + x_1 11/9 + \overline{x}_2 8/6) + x_3 \left(2 \cdot \frac{8}{6} + 4 \cdot \frac{10}{8}\right) \\ &\leq 6 \cdot \left((1 - x_3)4/3 + x_3 \left(\frac{4}{3} - \frac{1}{18}\right)\right) \leq 6 \cdot (4/3 - \delta/18). \end{aligned} \tag{4.4}$$

Scenario 2. If x_3 (which equals $\max\{x_0, x_1, x_3\}$ by (4.3)) is at most δ.

We start by stating the following technical lemma.

Lemma 13. *Define $\beta := 1/9 - \delta$. Then at least one of the following cases hold:*

- **Case** 1: $x_2^1, x_2^2, x_2^3 \geq \beta$.
- **Case** 2: $x_2^1, y_2^2, y_2^5 \geq \beta$.
- **Case** 3: $x_2^2, y_2^3, y_2^6 \geq \beta$.
- **Case** 4: $x_2^3, y_2^1, y_2^4 \geq \beta$.
- **Case** 5: $y_2^1, y_2^4, y_2^2, y_2^5 \geq \beta$.
- **Case** 6: $y_2^2, y_2^5, y_2^3, y_2^6 \geq \beta$.
- **Case** 7: $y_2^1, y_2^4, y_2^3, y_2^6 \geq \beta$.

Proof. By applying (4.2) on edges e_1 and a_2, respectively (see Figure 10), we get

$$x_0 + x_1^1 + x_1^4 + x_1^5 + x_1^6 + x_2^1 + y_2^1 + y_2^6 = \frac{1}{3}, \tag{4.5}$$

$$x_1^4 + x_2^1 + y_2^4 + y_2^6 + x_3^2 = \frac{1}{3}. \tag{4.6}$$

Subtracting (4.5) and (4.6), using $\max\{x_0, x_1, x_3\} \leq \delta$ and (4.3), we obtain

$$|y_2^1 - y_2^4| \leq \delta. \tag{4.7}$$

Analogously, we also have

$$|y_2^2 - y_2^5| \leq \delta, \tag{4.8}$$

$$|y_2^3 - y_2^6| \leq \delta. \tag{4.9}$$

Using $\max\{x_0, x_1, x_3\} \leq \delta$, (4.3) and applying (4.2) on edge e_j, for $j \in \{1, \dots, 6\}$, we have

$$x_2^1 + y_2^1 + y_2^6 \geq 1/3 - \delta, \tag{4.10}$$
$$x_2^2 + y_2^2 + y_2^1 \geq 1/3 - \delta, \tag{4.11}$$
$$x_2^3 + y_2^3 + y_2^2 \geq 1/3 - \delta, \tag{4.12}$$
$$x_2^1 + y_2^4 + y_2^3 \geq 1/3 - \delta, \tag{4.13}$$
$$x_2^2 + y_2^5 + y_2^4 \geq 1/3 - \delta, \tag{4.14}$$
$$x_2^3 + y_2^6 + y_2^5 \geq 1/3 - \delta. \tag{4.15}$$

Now we are ready to prove the lemma. Assume, by sake of contradiction, that none of the cases in the lemma holds. As Case 1 does not hold, we can assume, without loss of generality, that one of the following is true:

(i) $x_2^1 < \beta$, $x_2^2, x_2^3 \geq \beta$,
(ii) $x_2^1, x_2^2 < \beta$, $x_2^3 \geq \beta$,
(iii) $x_2^1, x_2^2, x_2^3 < \beta$.

Consider the case that (i) is true. Since Case 3 does not hold and $x_2^2 \geq \beta$, we conclude that $\min\{y_2^3, y_2^6\} < \beta$. Using inequality (4.9) we get $y_2^3, y_2^6 < \beta + \delta$. Analogously, since Case 4 does not hold and $x_2^3 \geq \beta$, we conclude that $\min\{y_2^1, y_2^4\} < \beta$. Using inequality (4.7) we get $y_2^1, y_2^4 < \beta + \delta$. Then we have

$$x_2^1 + y_2^1 + y_2^6 < 3\beta + 2\delta = 1/3 - \delta,$$

which contradicts inequality (4.10).

Consider the case that (ii) is true. Similar as above, since $x_2^3 \geq \beta$ and Case 4 does not hold, we conclude that $y_2^1, y_2^4 < \beta + \delta$. Furthermore, using inequality (4.8) and that Case 6 does not hold, we have at least one of the following inequalities $y_2^2, y_2^5 < \beta + \delta$ or $y_2^3, y_2^6 < \beta + \delta$. If the first one is true, then

$$x_2^2 + y_2^2 + y_2^1 < 3\beta + 2\delta = 1/3 - \delta,$$

which contradicts inequality (4.11). If the second one is true, then

$$x_1^2 + y_2^2 + y_2^6 < 3\beta + 2\delta = 1/3 - \delta,$$

which contradicts inequality (4.10).

Finally, consider the case that (iii) is true. As Cases 5, 6, and 7 do not hold, we have that $\min\{y_2^1, y_2^4, y_2^2, y_2^5\} < \beta$, $\min\{y_2^2, y_2^5, y_2^3, y_2^6\} < \beta$, and $\min\{y_2^1, y_2^4, y_2^3, y_2^6\} < \beta$. Without loss of generality, we can assume that $y_2^1, y_2^2 < \beta$. Using inequalities (4.7) and (4.8) we conclude that $y_2^1, y_2^4 < \beta + \delta$ and $y_2^2, y_2^5 < \beta + \delta$. Therefore,

$$x_2^2 + y_2^2 + y_2^1 < 3\beta + 2\delta = 1/3 - \delta,$$

which contradicts inequality (4.11). $\square$

Denote an index $i \in X_2 \cup Y_2$ as *long* if there are at least two vertices of $V(\gamma)$ contained in a single cycle of $\mathcal{C}_i^{\mathrm{U1}}$ of length at least 7, otherwise denote it as *short*. A set $Z \subseteq [k]$ is called long if Z contains only long indices.

Consider a short index $i \in X_2 \cup Y_2$. Since the matching M_i contains two edges of γ, we must be in the case where γ intersects exactly two cycles of $\mathcal{C}_i^{\mathrm{U1}}$ and both of them are 6-cycles (we assumed at the beginning of the proof of this part that no cycle in $\mathcal{C}_i$ of length at most five intersects γ). The next lemma complements what happens in each of the cases introduced in Lemma 13.

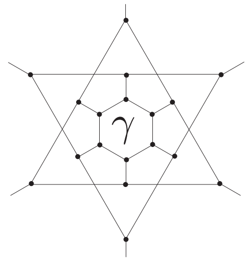

FIG. 12. *6-cycle γ for the case in which X_2^1, X_2^2, and X_2^3 are nonempty and not long.*

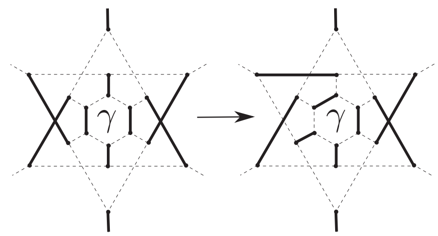

FIG. 13. *Operation (U1) applied to cycles in $\mathcal{C}_{i_1}$, where i_1 is a short index of X_2^1.*

LEMMA 14.

(1) *If X_2^1, X_2^2, and X_2^3 are nonempty, then at least one of them is long.*
(2) *If X_2^1, Y_2^2, and Y_2^5 are nonempty, then at least one of them is long.*
(3) *If X_2^2, Y_2^1, and Y_2^4 are nonempty, then at least one of them is long.*
(4) *If X_2^3, Y_2^3, and Y_2^6 are nonempty, then at least one of them is long.*
(5) *If Y_2^1, Y_2^4, Y_2^2, and Y_2^5 are nonempty, then at least one of them is long.*
(6) *If Y_2^2, Y_2^5, Y_2^3, and Y_2^6 are nonempty, then at least one of them is long.*
(7) *If Y_2^1, Y_2^4, Y_2^3, and Y_2^6 are nonempty, then at least one of them is long.*

Proof. We prove only items 1, 2, and 5, since the proofs for the rest are analogous.

(1) Assume for contradiction that there are short indices $i_1 \in X_2^1$, $i_2 \in X_2^2$, and $i_3 \in X_3^3$. In particular, every vertex of γ is in a 6-cycle of $\mathcal{C}_{i_p}^{\mathrm{U1}}$ (and thus, of $\mathcal{C}_{i_p}$) for $p = 1, 2, 3$. From this, we deduce that the neighborhood of γ in G is exactly as depicted in Figure 12.
Now focus on the short index $i_1 \in X_2^1$. Since G is as in Figure 12, there are three cycles of $\mathcal{C}_{i_1}$ sharing each one edge with a 6-cycle of G. But then, as Figure 13 shows, operation (U1) would merge them into a unique cycle C in $\mathcal{C}_{i_1}^{\mathrm{U1}}$ of length at least 16, contradicting the fact that i_1 is short.
(2) Assume for contradiction that there are short cycles $i_1 \in X_2^i$ $i_2 \in Y_2^2$ and $i_3 \in Y_2^5$. In particular, every vertex of γ is in a 6-cycle of $\mathcal{C}_{i_p}^{\mathrm{U1}}$ (and thus, of $\mathcal{C}_{i_p}$) for $p = 1, 2, 3$. From this, we deduce that the neighborhood of γ in G is exactly as depicted in Figure 14,
Focus on the short index $i_1 \in X_2^1$. Since G is as in Figure 14, there are three cycles of $\mathcal{C}_{i_1}$ that share one edge each with a 6-cycle of G. But in this case, as Figure 15 shows, operation (U1) would merge them into a unique cycle C in $\mathcal{C}_{i_1}^{\mathrm{U1}}$ of length at least 16, contradicting the fact that i_1 is short.
(5) Assume for contradiction that there are short indices $i_1 \in Y_2^1$, $i_2 \in Y_2^4$, $i_3 \in Y_2^2$, and $i_4 \in Y_2^5$. In particular, every vertex of γ is in a 6-cycle of $\mathcal{C}_{i_p}^{\mathrm{U1}}$ (and thus, of $\mathcal{C}_{i_p}$), for $p = 1, 2, 3, 4$. Then, the neighborhood of γ in G is exactly as depicted in Figure 16. But this structure shows a contradiction, as matching M_{i_1} cannot be completed to the entire graph. $\square$

Using Lemmas 13 and 14 we conclude that there is a long set of indices $Z \subseteq X_2 \cup Y_2$ for which $\lambda(Z) \geq \beta$. In particular, using Lemma 12, we conclude that for every $i \in Z$, there are two vertices w in γ with $z_i(w) \leq 9/7$, and for the remaining four vertices of

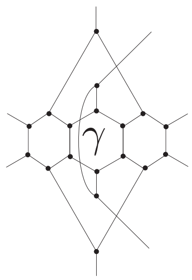

FIG. 14. *6-cycle γ for the cases X_2^1, Y_2^2, Y_2^5 are nonempty and not long.*

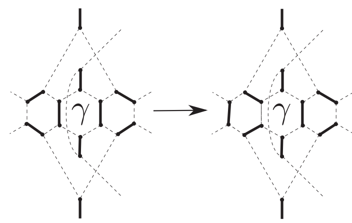

FIG. 15. *Operation (U1) applied to cycles in $\mathcal{C}_{i_1}$, where i_1 is a short index of X_2^1.*

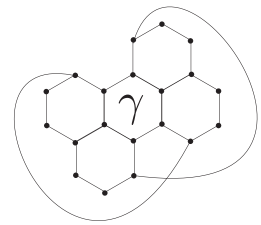

FIG. 16. *6-cycle γ for the cases Y_2^1, Y_2^4, Y_2^2, Y_2^5 are nonempty and not long.*

γ, $z_i(w) \leq 4/3$. Altogether, $\sum_{w \in V(\gamma)} z(w)$ is at most

$$\begin{aligned}
&6 \cdot \left(x_0 \frac{8}{6} + x_1 \frac{11}{9} + (\overline{x}_2 - \beta)\frac{8}{6} \right) + \beta \left(2 \cdot \frac{9}{7} + 4 \cdot \frac{8}{6} \right) + x_3 \left(2 \cdot \frac{8}{6} + 4 \cdot \frac{10}{8} \right) \\
&\leq 6(1-\beta)\frac{4}{3} + \beta \left(2 \cdot \frac{9}{7} + 4 \cdot \frac{8}{6} \right) = 6 \cdot \left(\frac{4}{3} - \frac{1/9 - \delta}{63} \right). \qquad (4.16)
\end{aligned}$$

To end the proof, we set $\delta = 2/81$, so that $(1/9 - \delta)/63 = \delta/18 = 1/729$. From inequalities (4.4) and (4.16) we conclude that in any scenario,

$$\sum_{w \in V(\gamma)} z(w) \leq 6 \cdot (4/3 - 1/729). \qquad (4.17)$$

4.1.6. Proof of part (P6) of the main proposition. If none of the cases indicated by the previous parts hold, then there are no 4, 5, and 6-cycles in $\Gamma(v)$. In other words, all components containing v in the final family of covers have length at least 7. By Lemma 12 we conclude that $z(v) \leq \max\{13/10, 9/7\} = 13/10$.

5. General connected cubic graphs. In this section we give a $4/3 - \epsilon'$ approximation algorithm for the TSP of a connected cubic graph G, where $\epsilon' = 1/183711$. Additionally, our algorithm shows that the integrality gap of the subtour LP for general cubic graphs is also at most $4/3 - \epsilon'$.

Observe that if G is not 2-connected, then the number n of vertices is no longer the optimum value of the subtour LP. In order to get the desired approximation, we need to consider separately the bridges of G, since every feasible tour uses them at twice.

Let F be the set of bridges of G, and let $b = |F|$. Since G is connected, the graph $G \setminus F$ is formed by exactly $(b+1)$ subcubic, 2-edge-connected components. Let $\mathcal{C} = \{G_1, \ldots, G_{b+1}\}$ be the collection of components of $G \setminus F$. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the collection of singleton-components: they are the ones corresponding to cut-vertices of G (if G is 2-connected, then $|\mathcal{C}_0| = 0$). Also, let n_i be the number of vertices in G_i and let $n_0 = |\mathcal{C}_0|$ be the total number of singleton components.

Let SUB be the optimal subtour LP value of G and SUB_i be the optimal subtour LP value of component G_i. Clearly, if e is a bridge, then the corresponding subtour LP variable has to be set to 2 in an optimal solution. Also, for every G_i, $\text{SUB}_i \geq n_i$

and if G_i is a singleton, then $\mathrm{SUB}(i) = 0$. Therefore,

$$\mathrm{OPT} \geq \mathrm{SUB} \geq 2b + \sum_{i=1}^{b+1} \mathrm{SUB}(i) \geq 2b + n - n_0, \tag{5.1}$$

where OPT is the optimal tour value.

The idea of our algorithm is to find a short tour in each G_i and then glue the solutions into a single tour by doubling the bridges. Since each nonsingleton component is bridgeless and has only vertices of degree 2 and 3, we can apply Mömke and Svensson's algorithm [14] to get tour of length at most $(4/3)n_i$ on each of them. Unfortunately, that is not enough to get an overall $(4/3 - \epsilon')$-approximation for G. Instead, on each nonsingleton component we apply algorithms A and B below and choose the solution using the fewer edges. Afterwards, we output the union of all the returned solution together with the doubled bridges.

A: Return the tour given by Mömke and Svensson's algorithm on the component.

B: Replace each vertex of degree 2 by a chorded 4-cycle, so that the resulting graph is cubic. Apply the $(4/3-\epsilon)$-algorithm of Theorem 2 to the expanded cubic 2-connected component to get a tour. Output the tour obtained by contracting the chorded 4-cycles.

THEOREM 3. *The previous algorithm returns a tour of length at most* $(4/3 - \epsilon')\mathrm{SUB}$*, where* $\epsilon' = \epsilon/(3+3\epsilon) = 1/183711$*.*

Proof. Let $A(i)$ and $B(i)$ be the length of the tour returned by algorithms A and B on component G_i, respectively, and let $L(A)$ (respectively, $L(B)$) be the total length of the tour resulting by putting together all tours $A(i)$ (respectively, $B(i)$) and twice the collection of bridges.

Using that $A(i) \leq (4/3)n_i$, for all nonsingleton G_i,

$$L(A) = 2b + \sum_{i=1}^{b+1} A(i) \leq 2b + (4/3)(n - n_0).$$

To analyze the second algorithm we need a little more of work. Let $D(i)$ be the number of vertices of degree 2 in component G_i before expanding it. The expanded components has $n_i + 3D(i)$ vertices. Clearly, the tour of length $B(i)$ is obtained from the tour of length $B^*(i)$ (in the expanded component) by contracting the chorded 4-cycles. Using that $B^*(i)$ contains at least three edges in each chorded 4-cycle, and Theorem 2,

$$\begin{aligned} B(i) &\leq B^*(i) - 3D(i) \\ &\leq \left(\frac{4}{3} - \epsilon\right)(n_i + 3D(i)) - 3D(i) \leq \left(\frac{4}{3} - \epsilon\right)n_i + D(i). \end{aligned}$$

Therefore,

$$L(B) \leq 2b + (4/3 - \epsilon)(n - n_0) + \sum_{i=1}^{b+1} D(i) \leq 4b + (4/3 - \epsilon)(n - n_0).$$

Then, the tour we return has length at most $\min(L(A), L(B))$, which can be checked to be at most

$$\left(1 + \frac{1}{3(1+\epsilon)}\right)(n - n_0 + 2b) \leq \left(1 + \frac{1}{3(1+\epsilon)}\right)\mathrm{SUB},$$

where the last inequality follows from (5.1). $\square$

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