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Dynamic Equilibria in Fluid Queueing Networks

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Flows over time provide a natural and convenient description for the dynamics of a continuous stream of particles traveling from a source to a sink in a network, allowing to track the progress of each infinitesimal particle along time. A basic model for the propagation of flow is the so-called *fluid queue model* in which the time to traverse an edge is composed of a flow-dependent waiting time in a queue at the entrance of the edge plus a constant travel time after leaving the queue. In a dynamic network routing game each infinitesimal particle is interpreted as a player that seeks to complete its journey in the least possible time. Players are forward looking and anticipate the congestion and queuing delays induced by others upon arrival to any edge in the network. Equilibrium occurs when each particle travels along a shortest path.

This paper is concerned with the study of equilibria in the fluid queue model and provides a constructive proof of existence and uniqueness of equilibria in single origin-destination networks with piecewise constant inflow rate. This is done through a detailed analysis of the underlying static flows obtained as derivatives of a dynamic equilibrium. Furthermore, for multicommodity networks, we give a general nonconstructive proof of existence of equilibria when the inflow rates belong to L^p.

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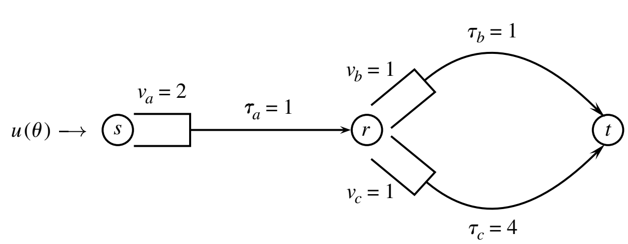
1. Introduction

Understanding time varying flows on networks is relevant in contexts where a steady state is rarely observed such as urban traffic or the Internet. In order to describe the temporal evolution of such systems, one has to consider the propagation of flow across the network by tracking the position of each particle along time. In the most basic model, the so-called *fluid queue model*, a continuous stream of particles is injected at a source s and travels towards a sink t through edges characterized by a *latency* and a *per-time-unit* capacity: flow propagates according to edge dynamics in which particles arriving to an edge e join a queue with service rate ν_e and, after leaving the queue, move along the edge to reach its head after τ_e time units.

Flows over time were initially studied in the framework of optimization. Ford and Fulkerson (1958, 1962) considered a fluid queue model in a discrete time setting and designed an algorithm to compute a flow over time carrying the maximum possible flow from the source s to the sink t in a given timespan. Gale (1959) then showed the existence of a flow pattern that achieves this optimum simultaneously for all time horizons. These results were extended to continuous time by Anderson and Philpott (1994) and Fleischer and Tardos (1998). We refer to Skutella (2009) for an excellent and up-to-date survey.

When network flows suffer from a lack of coordination among the participating agents, it is natural to consider them from a game theoretic perspective. In this setting, each infinitesimal inflow particle is interpreted as a player that seeks to complete its journey in the least possible time, so that equilibrium occurs when each particle travels along an s-t shortest path. The travel time for a particle entering the network at any given time must take into account the queueing delays induced by other particles on the edges along its path. This requires to anticipate the queue lengths by the time when an edge will be reached. The following example provides some intuition on flow propagation and dynamic equilibrium in the fluid queue model (precise definitions will be given in the next section).

Example 1. Consider the network in the figure below with a constant inflow at the source s equal to $u(\theta)=2$ for all times $\theta \geqslant 0$.



The inflow $u(\theta)$ is smaller than the service rate ν_a so that no queue will form on the first edge and a particle departing at time θ reaches the intermediate node r at time $l_r(\theta)=\theta+1$. Thus, a constant flow of 2 arrives at r starting from time $\theta=1$. This flow must be routed along a shortest path

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and, since $\tau_b < \tau_c$, at least initially it goes on edge b. Since the queue rate for this edge is $\nu_b = 1$, a queue starts to grow linearly as $z_b(\theta) = \theta - 1$ for $\theta \geqslant 1$, and consequently the travel time $z_b(\theta)/\nu_b + \tau_b$ (queuing plus latency) will increase until it equalizes τ_c. This occurs exactly at time $\theta = 4$ when the queue length reaches $z_b(4) = 3$. From that point on the flow splits equally between b and c, keeping a constant queue on edge b and an empty queue on c, so that both edges have a constant travel time of 4.

To compute the time $l_t(\theta)$ at which a particle departing at time θ reaches the sink t, we distinguish two cases. For a departure time $\theta \in [0, 3]$ the particle arrives at r at time $\theta + 1$ and then follows the edge b where it faces a queue of length θ and an additional latency $\tau_b = 1$, reaching the sink at time $l_t(\theta) = 2\theta + 2$. For $\theta \geqslant 3$ the particle arrives to r at $\theta + 1$ as before but spends a constant time 4 traversing either b or c to reach the sink at time $l_t(\theta) = \theta + 5$. Both expressions for $l_t(\theta)$ coincide at the breakpoint $\theta = 3$ so this arrival time function is continuous and increasing, and no particle overtakes the flow that entered earlier.

The study of flows over time when particles behave selfishly has mostly been considered in the transportation literature. Probably, the first to consider the fluid queue model as a game was Vickrey (1969), who used it as a tool for evaluating transport investments to mitigate congestion. The seminal paper by Friesz et al. (1993) (see also the book by Ran and Boyce 1996) proposed a general framework to model dynamic equilibrium using an appropriate variational inequality. The model supports very general flow propagation rules and edge dynamics that include the fluid queue model as a particular case. Unfortunately, in this general framework little is known in terms of existence, uniqueness, and characterization of solutions. Under suitable assumptions, an existence result was eventually obtained by Zhu and Marcotte (2000), which however does not apply to the fluid queue model. Also, Meunier and Wagner (2010) established the existence of dynamic equilibria using an alternative specification of the model and exploiting general results for games with a continuum of players. A more detailed discussion of these and related works is postponed until §6.

Recently, Koch and Skutella (2011) obtained a more specific and very useful characterization of dynamic equilibria in the fluid queue model by introducing the concept of *thin flows with resetting*. These thin flows characterize the derivatives of a dynamic equilibrium and can be used to reconstruct equilibria by integration. While the authors did not prove the existence of thin flows, the concept was used to analyze the price-of-anarchy for this class of dynamic routing games. The fluid queue model has also been considered recently by Bhaskar et al. (2014) to investigate the price-of-anarchy in Stackelberg routing games.

Our Contribution. This paper considers flows over time for the fluid queue model as in Koch and Skutella (2011), and is an outgrowth of our previous work in Larré (2010) and Cominetti et al. (2011). We provide a constructive proof for the existence and uniqueness of equilibria, exploiting the key concept of *thin flow with resetting* introduced by Koch and Skutella: a static flow together with an associated labeling that characterize the time derivatives of an equilibrium. We actually consider a slightly more restrictive definition by adding a normalization condition. Using a fixed point formulation we show that normalized thin flows exist, and then we prove that the labeling is unique. As a by-product, this yields an exponential time algorithm to compute a normalized thin flow and shows that this problem belongs to the complexity class Polynomial Parity arguments on Directed graphs (PPAD), though we conjecture that it might be solvable in polynomial time. By integrating these thin flows we deduce the existence of an equilibrium for the case of a piecewise constant inflow rate, and we show that the equilibrium is unique within a natural family of flows over time. Finally, we give a nonconstructive existence proof when the inflow rate belongs to the space of p-integrable functions L^p with $1 < p < \infty$, and we discuss how the result extends to multiple origin-destination pairs.

Organization of the Paper. Section 2 describes the fluid queue model for flows over time. Section 3 characterizes the time derivatives of a dynamic equilibrium using the notion of normalized thin flows with resetting and proves the existence and uniqueness of the latter. In §4 we exploit the previous results to give a constructive proof for the existence of an equilibrium in the case of a piecewise constant inflow rate, and we discuss the uniqueness of this equilibrium. In §5 we present a nonconstructive existence result for more general inflow rates, including the case of multiple origin-destination pairs. Finally, in §6 we compare our findings with previous results in the literature and state some open questions. The appendix at the end summarizes some technical facts used in the paper.

2. A Fluid Queue Model for Dynamic Routing Games

Throughout this paper we consider a network $\mathcal{N} = (G, \nu, \tau, s, t, u)$ consisting of a directed graph G with node set V and edge set E, a vector $\nu = (\nu_e)_{e\in E}$ of positive numbers representing queue service rates, a vector $\tau = (\tau_e)_{e\in E}$ of nonnegative numbers representing edge travel times, a source $s \in V$, a sink $t \in V$, and an inflow rate function u: $\mathbb{R} \to \mathbb{R}_+$ taken from the set $\mathcal{F}_0(\mathbb{R})$ of nonnegative and locally integrable functions which vanish on the negative axis; that is, $u(\theta) = 0$ for a.e. $\theta < 0$. We denote $U(\theta) = \int_0^\theta u(\xi)\, d\xi$ the *cumulative inflow* so that $U \in AC_{\text{loc}}(\mathbb{R})$, the space of locally absolutely continuous functions. For the precise definition of these functional spaces and some of their basic properties, we refer to the appendix as well as Leoni (2009, Chapter 3).

A continuous stream of particles is injected at the source s at a time-dependent rate $u(\theta)$ and flows through the network towards the sink t. Particles arriving to an edge e join

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a queue with service rate ν_e and, after leaving the queue, travel along the edge to reach its head after τ_e time units. Each infinitesimal inflow particle is interpreted as a player that seeks to complete its journey in the least possible time, so that equilibrium occurs when each particle travels along an s-t shortest path. The relevant edge costs for a particle entering the network at time θ must consider the queueing delays induced by other particles along its path by the time when each edge is reached. This introduces intricate spatial and temporal dependencies among the flows that enter the network at different times, possibly at future dates if overtaking occurs. The rest of this section makes these notions more precise.

For simplicity, and without loss of generality, we assume that there is at most one edge between any pair of nodes in G, that there are no loops, and that for each node $v \in V$ there is a path from s to v. An edge $e \in E$ from node v to node w is written vw, while the forward and backward stars of a node $v \in V$ are denoted $\delta^+(v)$ and $\delta^-(v)$. We also suppose that the sum of latencies along any cycle is positive, namely $\sum_{e\in\mathscr{C}} \tau_e > 0$ for every cycle $\mathscr{C}$ in G.

2.1. Flows-Over-Time

The model is formulated in terms of the flow rates on every edge. A *flow-over-time* is a pair $f = (f^+, f^-)$ of arrays of functions $f_e^+, f_e^- \in \mathscr{F}_0(\mathbb{R})$ for each $e \in E$, representing the rate at which flow enters the tail of e and the rate of flow leaving the head of e, respectively. We say that f is *feasible* if the following flow conservation constraints hold at every node $v \neq t$ and for almost all times $\theta \in \mathbb{R}$

$$\sum_{e\in\delta^+(v)} f_e^+(\theta) - \sum_{e\in\delta^-(v)} f_e^-(\theta) = \begin{cases} u(\theta) & \text{for } v = s \\ 0 & \text{for } v \in V\setminus\{s,t\}. \end{cases} \tag{1}$$

The *cumulative inflow* and *cumulative outflow* of an edge e are defined as the $AC_{\text{loc}}(\mathbb{R})$ functions

$$F_e^+(\theta) = \int_0^\theta f_e^+(\xi)\, d\xi,$$

$$F_e^-(\theta) = \int_0^\theta f_e^-(\xi)\, d\xi.$$

2.2. Queue Dynamics and Queuing Delays

An edge e is modeled as a fluid queue with service rate ν_e followed by a link with constant travel time τ_e. The *queue length* $z_e(\theta)$ at any time θ is the net flow that has entered the edge and has not yet left the queue. Accounting for the time τ_e required to reach the head of the edge after leaving the queue, we have

$$z_e(\theta) = F_e^+(\theta) - F_e^-(\theta + \tau_e).$$

Throughout the paper we assume that queues *operate at capacity*. By this we mean that for almost all $\theta \in \mathbb{R}$

$$f_e^-(\theta + \tau_e) = \begin{cases} \nu_e & \text{if } z_e(\theta) > 0, \\ \min\{f_e^+(\theta), \nu_e\} & \text{otherwise.} \end{cases} \tag{2}$$

This condition can be equivalently stated in terms of the queue length dynamics

$$z_e'(\theta) = \begin{cases} f_e^+(\theta) - \nu_e & \text{if } z_e(\theta) > 0 \\ [f_e^+(\theta) - \nu_e]_+ & \text{otherwise,} \end{cases} \tag{3}$$

whose unique solution is given by (see e.g., Prabhu 2002, §1.3)

$$z_e(\theta) = \max_{\eta\in[0,\,\theta]} \int_\eta^\theta [f_e^+(\xi) - \nu_e]\, d\xi. \tag{4}$$

This formula shows that in a queue that operates at capacity, the inflow f_e^+ completely determines the queue length z_e, and therefore the outflow f_e^- is also uniquely determined by (2).

The *queueing delay* experienced by a particle entering e at time θ before it starts traversing the edge is defined as

$$q_e(\theta) = \min\left\{q \geqslant 0\colon\ \int_\theta^{\theta+q} f_e^-(\xi + \tau_e)\, d\xi = z_e(\theta)\right\}. \tag{5}$$

We denote $W_e^\theta = [\theta, \theta + q_e(\theta))$ the interval on which the particle waits in the queue and $Q_e = \{\theta\colon z_e(\theta) > 0\}$ the instants at which the queue is nonempty. Note that for all $\theta' \in W_e^\theta$ the queue remains nonempty since

$$z_e(\theta') = z_e(\theta) + \int_\theta^{\theta'} [f_e^+(\xi) - f_e^-(\xi + \tau_e)]\, d\xi$$
$$\geqslant z_e(\theta) - \int_\theta^{\theta'} f_e^-(\xi + \tau_e)\, d\xi > 0,$$

and therefore $Q_e = \bigcup_\theta W_e^\theta$. The next result shows that in a queue that operates at capacity the queuing delay is exactly $q_e(\theta) = z_e(\theta)/\nu_e$, providing in fact an equivalent characterization of operation at capacity.

Proposition 1. *Let f_e^+, f_e^- be the inflow and outflow on edge e with corresponding queue length z_e.The queue operates at capacity if and only if the next three conditions hold simultaneously*:
(a) *Capacity constraint*: *$f_e^-(\theta) \leqslant \nu_e$ for almost all θ,*
(b) *Nondeficit constraint*: *$z_e(\theta) \geqslant 0$ for all θ,*
(c) *Queueing delay*: *$q_e(\theta) = z_e(\theta)/\nu_e$ for all θ.*

Proof. Suppose the queue operates at capacity. From (2) we clearly have (a) while (4) implies (b). To prove (c) we observe that $\int_\theta^{\theta+q} f_e^-(\xi + \tau_e)\, d\xi \leqslant \nu_e q$ from which it follows that $q_e(\theta) \geqslant z_e(\theta)/\nu_e$. On the other hand, since the queue is nonempty on W_e^θ, condition (2) implies $f_e^-(\xi + \tau_e) = \nu_e$ a.e. $\xi \in W_e^\theta$ and then

$$\int_\theta^{\theta+z_e(\theta)/\nu_e} f_e^-(\xi + \tau_e)\, d\xi = z_e(\theta),$$

which yields $q_e(\theta) = z_e(\theta)/\nu_e$.

Conversely, suppose (a)–(c) hold. From (c) we get $\int_\theta^{\theta+q_e(\theta)} [f_e^-(\xi + \tau_e) - \nu_e]\, d\xi = 0$ so that (a) gives $f_e^-(\xi + \tau_e) = \nu_e$ for almost all $\xi \in W_e^\theta$, and Lemma 8 implies that this equality holds a.e. on $\bigcup_\theta W_e^\theta = Q_e$ proving the first case of (2). For the second case, (b) and Lemma 9: (a) $\Rightarrow$ (c) give that almost everywhere $z_e(\theta) = 0$ implies $0 = z_e'(\theta) = f_e^+(\theta) - f_e^-(\theta + \tau_e)$ and therefore $f_e^-(\theta + \tau_e) = \min\{f_e^+(\theta), \nu_e\}$. $\square$

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2.3. Edge Travel Times

The time at which a particle exits from an edge e can be computed as the sum of the entrance time θ, plus queueing delay, plus latency, namely

$$T_e(\theta) = \theta + \frac{z_e(\theta)}{\nu_e} + \tau_e. \tag{6}$$

For notational convenience, we omit the dependence of T_e on the flow f. Clearly $T_e \in AC_{\text{loc}}(\mathbb{R})$ and using (3) we can compute its derivative almost everywhere as

$$T_e'(\theta) = \begin{cases} \dfrac{1}{\nu_e} f_e^+(\theta) & \text{if } z_e(\theta) > 0, \\ \max\left\{1, \dfrac{1}{\nu_e} f_e^+(\theta)\right\} & \text{otherwise.} \end{cases} \tag{7}$$

Hence $T_e'(\theta) \geqslant 0$ so that T_e is nondecreasing and thus particles traversing e respect FIFO without overtaking. Moreover, all the flow that enters e up to time θ exits by time $T_e(\theta)$. Indeed, since the queue is nonempty over the interval W_e^θ, service at capacity implies $f_e^-(\xi + \tau_e) = \nu_e$ for almost all $\xi \in W_e^\theta$ and then

$$\begin{aligned} F_e^-(T_e(\theta)) &= \int_0^{\theta+\tau_e} f_e^-(\xi)\, d\xi + \int_{\theta+\tau_e}^{T_e(\theta)} \nu_e \, d\xi \\ &= F_e^-(\theta + \tau_e) + z_e(\theta) \\ &= F_e^+(\theta). \end{aligned} \tag{8}$$

2.4. Dynamic Shortest Paths

A flow particle entering a path $P = (e_1, e_2, \ldots, e_k)$ at time θ will reach the endpoint of the path at the time

$$l^P(\theta) = T_{e_k} \circ \cdots \circ T_{e_1}(\theta), \tag{9}$$

Thus, denoting $\mathcal{P}_w$ the set of all s-w paths in G, the earliest time at which a particle starting from s at time θ can reach w is given by

$$l_w(\theta) = \min_{P \in \mathcal{P}_w} l^P(\theta). \tag{10}$$

These functions correspond to shortest paths with edge costs that consider the queueing delays along the path at the appropriate times, taking into account the time it takes to reach every edge. We refer to them as *dynamic shortest paths*.

Since the T_e's are absolutely continuous and nondecreasing, the same holds for their compositions l^P and therefore also for the l_w's (see appendix or Leoni 2009, Chapter 3). Note also that $l_w(\cdot)$ is surjective with $l_w(\theta) \to \pm\infty$ when $\theta \to \pm\infty$. Indeed, for $\theta \to \infty$ this is a consequence of the inequality $l_w(\theta) \geqslant \theta$, while for $\theta \to -\infty$ this follows since all the queues are empty and $l_w(\theta) = \theta + d_{sw}$ with d_{sw} the minimum time from s to w considering only the travel times τ_e and no queuing.

The monotonicity of T_e together with the nondeficit constraints and the fact that the sum of latencies on any cycle is positive, imply that dynamic shortest paths do not contain cycles, and therefore (10) can also be computed by solving

$$l_w(\theta) = \begin{cases} \theta & \text{for } w = s, \\ \displaystyle\min_{e = vw \in \delta^-(w)} T_e(l_v(\theta)) & \text{for } w \neq s. \end{cases} \tag{11}$$

The θ-*shortest-path graph* is defined as the acyclic graph $G_\theta = (V, E_\theta')$ containing all the shortest paths at time θ. An edge $e = vw$ is in E_θ' if and only if $T_e(l_v(\theta)) \leqslant l_w(\theta)$, or equivalently $T_e(l_v(\theta)) = l_w(\theta)$, in which case it is said to be *active*. Note that an inactive edge has $T_e(l_v(\theta)) > l_w(\theta)$, so by continuity it remains inactive nearby. We also denote Θ_e the set of all times θ at which e is active. Note that E_θ' and Θ_e depend on the given flow-over-time f.

2.5. Dynamic Equilibrium

A feasible s-t flow-over-time can be interpreted as a dynamic equilibrium by looking at each infinitesimal inflow particle as a player that travels from the source to the sink along an s-t path that yields the least possible travel time. The following definition makes this notion precise.

Definition 1 (Dynamic Equilibrium). A feasible flow-over-time f is called a *dynamic equilibrium* if for each $e = vw \in E$ we have $f_e^+(\xi) = 0$ for almost all $\xi \in l_v(\Theta_e^c)$.

The next lemma provides an alternative characterization of dynamic equilibrium.

Lemma 1. *A feasible flow-over-time f is a dynamic equilibrium iff for all $e = vw \in E$ and almost all $\xi \in \mathbb{R}$ we have $f_e^+(\xi) > 0 \Rightarrow \xi \in l_v(\Theta_e)$.*

Proof. The condition in the lemma is equivalent to $f_e^+(\xi) = 0$ for almost all $\xi \in l_v(\Theta_e)^c$. Hence, to establish the result it suffices to show that the sets $l_v(\Theta_e)^c$ and $l_v(\Theta_e^c)$ differ on a set of null measure. We note that the first set is included in the second. Indeed, take any $\xi \in l_v(\Theta_e)^c$. Since $l_v(\cdot)$ is surjective we may find $\theta \in \mathbb{R}$ with $\xi = l_v(\theta)$, and since $\xi \notin l_v(\Theta_e)$ it must be the case that $\theta \notin \Theta_e$ so that $\xi \in l_v(\Theta_e^c)$. Now, for each $\xi \in l_v(\Theta_e^c) \setminus l_v(\Theta_e)^c = l_v(\Theta_e^c) \cap l_v(\Theta_e)$ we may find $\theta \in \Theta_e^c$ and $\theta' \in \Theta_e$ such that $\xi = l_v(\theta) = l_v(\theta')$. Since $l_v(\cdot)$ is nondecreasing, it follows that $l_v(\eta) = \xi$ for all η between θ and θ' and since $\theta \neq \theta'$ we may take $\eta \in \mathbb{Q}$ in order to deduce that $l_v(\Theta_e^c) \setminus l_v(\Theta_e)^c \subset l_v(\mathbb{Q})$. This shows that the sets $l_v(\Theta_e)^c$ and $l_v(\Theta_e^c)$ differ on a countable set, hence a set of measure zero. □

Remark. A slightly different notion, which we call *strong dynamic equilibrium*, was considered in Koch and Skutella (2011) requiring that $e \notin E_\theta' \Rightarrow f_e^+(l_v(\theta)) = 0$ for each $e = vw \in E$ and almost all θ. This condition implies dynamic equilibrium (since l_v is absolutely continuous and maps null sets into null sets), and it is in fact strictly stronger as

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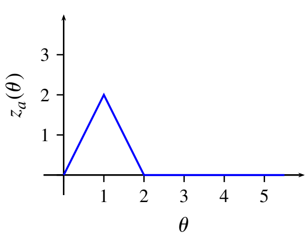
illustrated in the example below. The point is that while the concept of dynamic equilibrium is insensitive to modifications on sets of measure zero, this is not the case for strong equilibrium: the composition $f_e^+(l_v(\theta))$ is not well defined under the standard identification of functions that coincide almost everywhere and depends on the representative function f_e^+ that is chosen. Indeed, since $l_v(\cdot)$ may be constant over a nontrivial interval, a simple modification of f_e^+ at a single point may spoil the *almost everywhere* condition with respect to θ. Definition 1 avoids this difficulty.

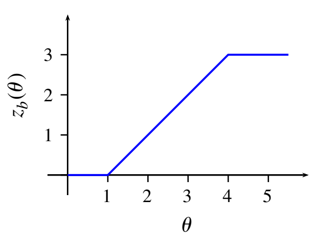
Example 2. Consider the same network as in Example 1 in the introduction with inflow function

$$u(\theta)=\begin{cases}4 & \text{if } 0\leqslant\theta<1,\\ 0 & \text{if } 1\leqslant\theta\leqslant 2,\\ 2 & \text{if } 2<\theta.\end{cases}$$

The inflow of link a is $f_a^+(\theta)=u(\theta)$ so that a queue builds up in the interval $[0,1]$ and is emptied during $[1,2]$, after which it stays empty (see the left plot below). Thus, the queue on edge a has a constant throughput equal to 2 and the outflow, which is also the inflow at the intermediate node r, is given by

$$f_a^-(\theta)=\begin{cases}0 & \text{for } \theta\leqslant 1,\\ 2 & \text{for } \theta\geqslant 1.\end{cases}$$





The outflow of edge a is exactly the same as in Example 1 in the introduction, so that the equilibrium flows in edges b and c are the same as described there. More explicitly, the inflow and outflow functions are

$$f_b^+(\theta)=\begin{cases}0 & \text{for } \theta<1,\\ 2 & \text{for } 1\leqslant\theta<4,\\ 1 & \text{for } \theta\geqslant 4,\end{cases}\qquad f_b^-(\theta)=\begin{cases}0 & \text{for } \theta<2,\\ 1 & \text{for } \theta\geqslant 2,\end{cases}$$

$$f_c^+(\theta)=\begin{cases}0 & \text{for } \theta<4,\\ 1 & \text{for } \theta\geqslant 4,\end{cases}\qquad f_c^-(\theta)=\begin{cases}0 & \text{for } \theta<8,\\ 1 & \text{for } \theta\geqslant 8.\end{cases}$$

For these flows the queue on edge c remains empty at all times and the queue on edge b evolves as in the right plot above. A routine calculation shows that these flows yield a strong dynamic equilibrium with corresponding earliest time functions

$$l_r(\theta)=\begin{cases}1+2\theta & \text{for } 0\leqslant\theta<1,\\ 3 & \text{for } 1\leqslant\theta<2,\\ 1+\theta & \text{for } \theta\geqslant 2,\end{cases}$$

$$l_t(\theta)=\begin{cases}2+4\theta & \text{for } 0\leqslant\theta<1,\\ 6 & \text{for } 1\leqslant\theta<2,\\ 2+2\theta & \text{for } 2\leqslant\theta<3,\\ 5+\theta & \text{for } \theta\geqslant 3.\end{cases}$$

We observe that the edge c is not in E'_θ for any $\theta<3$. If we modify $f_c^+(\theta)$ at just one point by taking $f_c^+(3)>0$, we still have a dynamic equilibrium. However, since $l_r(\theta)=3$ for all $\theta\in[1,2]$ we now have $f_c^+(l_r(\theta))>0$ throughout this interval, and strong equilibrium fails.

2.6. Queues and Cumulative Flows in a Dynamic Equilibrium

It is worth noting that at equilibrium all edges with positive queue must be active. Namely, let E_θ^* denote the set of links with positive queue

$$E_\theta^*=\{e=vw\in E\colon z_e(l_v(\theta))>0\}. \tag{12}$$

Proposition 2. *If f is a dynamic equilibrium, then $E_\theta^*\subseteq E'_\theta$ and we have*

$$E'_\theta=\{e=vw\in E\colon l_w(\theta)\geqslant l_v(\theta)+\tau_e\}, \tag{13}$$

$$E_\theta^*=\{e=vw\in E\colon l_w(\theta)>l_v(\theta)+\tau_e\}. \tag{14}$$

Proof. Let $e=vw\in E_\theta^*$ and consider the largest $\theta'\leqslant\theta$ at which e was active. Equilibrium implies $f_e^+(\xi)=0$ for almost all $\xi\in(l_v(\theta'),l_v(\theta)]$, so the queue must be nonempty throughout this interval and (7) gives $T'_e(\xi)=0$ almost everywhere. Hence T_e is constant in this interval so that

$$T_e(l_v(\theta))=T_e(l_v(\theta'))=l_w(\theta')\leqslant l_w(\theta),$$

which yields $e\in E'_\theta$ proving the inclusion $E_\theta^*\subseteq E'_\theta$.

To show (13), we note that for $e\in E'_\theta$ we have

$$l_w(\theta)=T_e(l_v(\theta))\geqslant l_v(\theta)+\tau_e$$

where the inequality follows from definition of T_e and the nondeficit constraints. Conversely, suppose that $l_w(\theta)\geqslant l_v(\theta)+\tau_e$. If $z_e(l_v(\theta))=0$ this yields $l_w(\theta)\geqslant T_e(l_v(\theta))$ so that $e\in E'_\theta$, whereas in the case $z_e(l_v(\theta))>0$ the same conclusion follows since we already proved that $E_\theta^*\subseteq E'_\theta$.

A similar argument proves (14). For $e\in E_\theta^*$ we have $z_e(l_v(\theta))>0$ and then $e\in E'_\theta$, so that $l_w(\theta)=T_e(l_v(\theta))>l_v(\theta)+\tau_e$. Conversely, if $l_v(\theta)+\tau_e<l_w(\theta)$ the inequality $l_w(\theta)\leqslant T_e(l_v(\theta))$ and the definition of T_e yield $z_e(l_v(\theta))>0$. □

Intuitively, at equilibrium any flow routed through an edge $e=vw$ up to time $l_v(\theta)$ should reach w before the optimal time $l_w(\theta)$. This is, in fact, an equivalent characterization of dynamic equilibrium.

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THEOREM 1. *Let f be a feasible s-t flow-over-time. The following are equivalent*

(a) *f is a dynamic equilibrium*;

(b) *for each $e=vw$ and all θ we have $F_e^+(l_v(\theta)) = F_e^-(l_w(\theta))$*;

(c) *for each $e=vw$ and almost all θ we have $e\notin E'_\theta \Rightarrow f_e^+(l_v(\theta))l'_v(\theta)=0$.*

PROOF. For each θ consider the interval $I_\theta=(\theta',\theta]$ with $\theta'\leqslant\theta$ the largest time such that $T_e(l_v(\theta')) = l_w(\theta)$. Note that θ' is well defined since $l_w(\theta)\leqslant T_e(l_v(\theta))$ and $T_e(l_v(\theta'))\to-\infty$ when $\theta'\to-\infty$. Note also that $I_\theta=\varnothing$ for $\theta\in\Theta_e$ since in this case $\theta'=\theta$. We claim that Θ_e^c coincides with the union of the I_θ's. Indeed, for each $\theta\in\Theta_e^c$ we have $\theta'<\theta$ and therefore $\theta\in I_\theta$ so that $\Theta_e^c\subseteq\bigcup_\theta I_\theta$. Conversely, for $\theta''\in I_\theta$ we have by definition of θ' that $T_e(l_v(\theta''))>l_w(\theta)\geqslant l_w(\theta'')$ so that $\theta''\in\Theta_e^c$ and then $\bigcup_\theta I_\theta\subseteq\Theta_e^c$.

Now, invoking (8), for each θ we have

$$F_e^+(l_v(\theta)) - F_e^-(l_w(\theta)) = \int_{l_v(\theta')}^{l_v(\theta)} f_e^+(\xi)\,d\xi \geqslant 0, \tag{15}$$

with equality iff f_e^+ vanishes almost everywhere on $(l_v(\theta'), l_v(\theta)] = l_v(I_\theta)$. Lemma 8 then shows that (b) holds iff $f_e^+(\xi)=0$ for almost all $\xi\in\bigcup_\theta l_v(I_\theta)=l_v(\Theta_e^c)$, proving (b) $\Leftrightarrow$ (a). Similarly, a change of variables (cf. appendix) allows to rewrite (15) as

$$F_e^+(l_v(\theta)) - F_e^-(l_w(\theta)) = \int_{\theta'}^{\theta} f_e^+(l_v(z))l'_v(z)\,dz \geqslant 0,$$

with equality iff $f_e^+(l_v(z))l'_v(z)=0$ for almost all $z\in I_\theta$. By Lemma 8, (b) holds iff this map vanishes almost everywhere on $\bigcup_\theta I_\theta=\Theta_e^c$, proving (b) $\Leftrightarrow$ (c). □

Theorem 1(b) above provides a way to synchronize the flow over time on the different edges by using the departure time as a common clock. This property motivates the definition of cumulative flow on an edge at a given time, consisting of all flow that departed up to that time and which uses the edge.

DEFINITION 2 (CUMULATIVE FLOW). The *cumulative flow* induced by a dynamic equilibrium f is defined as $x(\theta) = (x_e(\theta))_{e\in E}$ with $x_e(\theta)=F_e^+(l_v(\theta))=F_e^-(l_w(\theta))$ for all $e=vw\in E$ and $\theta\in\mathbb{R}$.

Integrating the flow conservation constraints (1) over the interval $[0, l_v(\theta)]$, it follows that for each $\theta\in\mathbb{R}$ the cumulative flow $x(\theta)$ is a static s-t flow of value $U(\theta)$,

$$\sum_{e\in\delta^+(v)} x_e(\theta) - \sum_{e\in\delta^-(v)} x_e(\theta) = \begin{cases} U(\theta) & \text{for } v=s,\\ 0 & \text{for } v\in V\backslash\{s,t\}.\end{cases} \tag{16}$$

Differentiating, for almost all θ we get that $x'(\theta)$ is a static s-t flow of value $u(\theta)$ with $x'_e(\theta)=0$ for $e\notin E'_\theta$.

2.7. Path Formulation of Dynamic Equilibrium

Since the θ-shortest path graph G_θ is acyclic, $x'(\theta)$ does not route flow on cycles. Hence, denoting $\mathcal{P}$ the set of simple s-t paths we may find a decomposition $u(\theta)=\sum_{P\in\mathcal{P}} h_P(\theta)$ into nonnegative path-flows $h_P(\theta)\geqslant 0$ such that

$$x'_e(\theta) = \sum_{P\ni e} h_P(\theta).$$

Indeed, start with $y=x'(\theta)$ and consider the paths $P\in\mathcal{P}$ in a fixed order setting $h_P(\theta)=\min_{e\in P} y_e$ and updating $y_e\leftarrow y_e-h_P(\theta)$ for $e\in P$. This yields a measurable decomposition $h_P\in\mathcal{F}_0(\mathbb{R})$ such that $h_P(\theta)>0$ only for paths that belong to the θ-shortest-path graph G_θ.

It is appealing to take the latter as the definition of dynamic equilibrium. The difficulty is to properly define *shortest path* since this requires the exit-time functions T_e, which in turn require an appropriate flow-over-time f to be associated with $h=(h_P)_{P\in\mathcal{P}}$. Since f depends on how the flow h propagates along the paths, both f and T_e must be determined simultaneously. This *network loading* process typically requires additional conditions to be well defined, such as an acyclic network structure or when link travel times are bounded away from zero, which is a natural and mild assumption (see e.g., Meunier and Wagner 2010, Xu et al. 1999, Zhu and Marcotte 2000). Since we will not require network loading until §5, we defer its discussion to that section.

3. Derivatives of Dynamic Equilibria: Normalized Thin Flows

The functions x_e and l_w are absolutely continuous, and therefore they can be recovered by integrating their derivatives. In this section we characterize these derivatives, yielding a constructive method to find an equilibrium. Our characterization is closely related to the notion of *thin-flow with resetting* introduced by Koch and Skutella (2011).

Recall that for almost all θ the derivative $x'(\theta)$ is an s-t flow of value $u(\theta)$ with $x'_e(\theta)=0$ for $e\notin E'_\theta$. On the other hand, clearly $l'_s(\theta)=1$ while for $w\neq s$ we may use (11) and the rule of differentiation of a (pointwise) minimum function, which combined with (7) yields almost everywhere

$$l'_w(\theta) = \min_{e=vw\in E'_\theta} T'_e(l_v(\theta))l'_v(\theta) = \min_{e=vw\in E'_\theta} \rho_e(l'_v(\theta), x'_e(\theta)),$$

where for each $e=vw\in E'_\theta$ we set

$$\rho_e(l'_v, x'_e) = \begin{cases} x'_e/\nu_e & \text{if } e\in E^*_\theta,\\ \max\{l'_v, x'_e/\nu_e\} & \text{if } e\notin E^*_\theta.\end{cases}$$

Since E'_θ is acyclic, this allows to compute $l'_w(\theta)$ by scanning the nodes w in topological order.

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This discussion motivates the next definition. Let $u_0 \geqslant 0$ and (E^*, E') be a pair of edge sets such that

$$\text{(H)} \qquad E^* \subseteq E' \subseteq E \quad \text{with } E' \text{ acyclic and for all } v \in V \text{ there is an } s\text{-}v \text{ path in } E'.$$

We denote by $K(E', u_0)$ the nonempty, compact, and convex set of all static s-t flows $x' = (x'_e)_{e\in E} \geqslant 0$ of value u_0 with $x'_e = 0$ for $e \notin E'$. With each $x' \in K(E', u_0)$ we associate the unique labels given as above by $l'_s = 1$ and $l'_w = \min_{e=vw\in E'} \rho_e(l'_v, x'_e)$ for $w \neq s$. Note that the map $x' \mapsto l'$ is continuous.

Definition 3 (Normalized Thin Flow). A flow $x' \in K(E', u_0)$ is called a *normalized thin flow* (*NTF*) *of value* u_0 *with resetting on* $E^* \subseteq E'$ iff $x'_e = 0$ for every edge $e = vw \in E'$ such that $l'_w < \rho_e(l'_v, x'_e)$.

Theorem 2. *Let f be a dynamic equilibrium and $\theta \in \mathbb{R}$ such that the right derivatives $u_0 = (dU/d\theta^+)(\theta)$, $l'_v = (dl_v/d\theta^+)(\theta)$ and $x'_e = (dx_e/d\theta^+)(\theta)$ exist. Then x' is an NTF of value u_0 with resetting on $E^*_\theta \subseteq E'_\theta$, with corresponding labels l'.*

Proof. Differentiating (16), it follows that x' is an s-t flow of value u_0. Moreover, if $e \notin E'_\theta$ then e remains inactive on some interval $[\theta, \theta+\epsilon)$, so the chain rule (see appendix) and equilibrium imply that on this interval $x'_e(\xi) = f^+_e(l_v(\xi))l'_v(\xi) = 0$ a.e., so $x_e(\cdot)$ is constant and $x'_e = 0$. This proves that $x' \in K(E'_\theta, u_0)$.

Let us show that l' are the corresponding labels. Clearly $l'_s = 1$. For the rest of the argument we distinguish two more subsets of E'_θ: E^*_+ contains the links $e = vw$, which have a queue or are about to build one with $z_e(\xi) > 0$ for all ξ on a small interval $(l_v(\theta), l_v(\theta)+\epsilon)$, whereas E'_+ includes the links without queue at time θ but which are active along a strictly decreasing sequence $\theta_n \downarrow \theta$. For $e \in E'_+$ we have $l_w(\theta_n) = T_e(l_v(\theta_n)) \geqslant l_v(\theta_n) + \tau_e$ and $l_w(\theta) = l_v(\theta) + \tau_e$ so that $l_w(\theta_n) - l_w(\theta) \geqslant l_v(\theta_n) - l_v(\theta)$ and dividing by $\theta_n - \theta$ with $n \to \infty$ we get $l'_w \geqslant l'_v$. Similarly, for $e \in E'_\theta \backslash E^*_+$ we may take $\theta_n \downarrow \theta$ with $z_e(l_v(\theta_n)) = 0$ so that $l_w(\theta_n) \leqslant T_e(l_v(\theta_n)) = l_v(\theta_n) + \tau_e$ and we get $l'_w \leqslant l'_v$. Also, for $e = vw \in E'_\theta$ the capacity constraint gives for $\theta' \geqslant \theta$

$$x_e(\theta') - x_e(\theta) = \int_{l_w(\theta)}^{l_w(\theta')} f^-_e(\xi)\, d\xi \leqslant \nu_e(l_w(\theta') - l_w(\theta)), \quad (17)$$

which implies $l'_w \geqslant x'_e/\nu_e$. Finally, when $e \in E^*_+$ we have $z_e(l_v(\theta')) > 0$ for θ' close to θ and as observed after Equation (5) the queue remains nonempty over $[l_v(\theta'), l_v(\theta') + z_e(l_v(\theta'))/\nu_e)$ so that (2) implies that $f^-_e(\xi) = \nu_e$ for almost all $\xi \in [l_v(\theta') + \tau_e, l_w(\theta'))$. This readily gives $f^-_e(\xi) = \nu_e$ almost everywhere on a small interval to the right of $l_w(\theta)$ and then equality holds in (17) for θ' sufficiently close to θ, so that $l'_w = x'_e/\nu_e$ for $e \in E^*_+$. In summary

(a) $l'_w \geqslant l'_v$, for $e = vw \in E'_+$,
(b) $l'_w \leqslant l'_v$, for $e = vw \in E'_\theta \backslash E^*_+$,
(c) $l'_w \geqslant x'_e/\nu_e$, for $e = vw \in E'_\theta$,
(d) $l'_w = x'_e/\nu_e$, for $e = vw \in E^*_+$.

Combining (b) and (d) we get $l'_w \leqslant \min_{e=vw\in E'_\theta} \rho_e(l'_v, x'_e)$ with equality if there is some $e = vw \in E^*_\theta$. To prove the equality when no edge from E^*_θ is incident on w, choose any $\theta_n \downarrow \theta$ and a sequence of active edges $e_n \in E'_{\theta_n}$, and take a subsequence with $e_n = vw$ constant so that $e = vw \in E'_+$. Then (a) and (c) combined give $l'_w \geqslant \rho_e(l'_v, x'_e)$. Altogether this proves $l'_w = \min_{e=vw\in E'_\theta} \rho_e(l'_v, x'_e)$ for $w \neq s$.

Let us finally show that x' is an NTF. Suppose $x'_e > 0$ on some $e = vw \in E'_\theta$ with $l'_w < \rho_e(l'_v, x'_e)$. The latter and (d) imply $e \notin E^*_\theta$, while $x'_e > 0$ gives $x_e(\theta') > x_e(\theta)$ for all $\theta' > \theta$ so e must be active on a sequence $\theta_n \downarrow \theta$ and $e \in E'_+$. Then (a) and (c) yield the contradiction $l'_w \geqslant \rho_e(l'_v, x'_e)$. □

Theorem 2 derives the existence of NTF's from a dynamic equilibrium. To proceed in the other direction, we study the existence of NTF's, and then by integration we reconstruct a dynamic equilibrium.

Theorem 3. *Let $u_0 \geqslant 0$ and (E^*, E') satisfying* (H). *Then there is an NTF of value u_0 with resetting on $E^* \subseteq E'$.*

Proof. Let $K = K(E', u_0)$ and observe that the NTF's are precisely the fixed-points of the set-valued map $\Gamma: K \to 2^K$ with nonempty convex compact values given by

$$\Gamma(x') = \{y' \in K: y'_e = 0 \text{ for all } e \in E' \text{ such that } l'_w < \rho_e(l'_v, x'_e)\}$$

with l' the labels corresponding to x' and E^*. Since $x' \mapsto l'$ is continuous, it follows that Γ is upper-semicontinuous, and a fixed point $x' \in \Gamma(x')$ exists by virtue of Kakutani's fixed point theorem. □

This result shows that finding an NTF belongs to the complexity class PPAD. It also suggests a finite (exponential time) algorithm to compute an NTF: we guess the set E'_0 of links $e \in E'$ that satisfy $l'_w = \rho_e(l'_v, x'_e)$, and then solve

$$\max_{(x', l')} \left\{ \sum_{w\in V} l'_w : x' \in K(E'_0, u_0);\; l'_s = 1;\; l'_w \leqslant \min_{e=vw\in E'} \rho_e(l'_v, x'_e) \right\}.$$

The latter can be restated as a mixed integer linear program and solved in finite time. By considering all possible subsets $E'_0 \subseteq E'$, the method eventually finds an NTF.

In general there may exist different NTF's, each one with its corresponding labels. We show next that the labels in all of them coincide.

Theorem 4. *Let $u_0 \geqslant 0$ and (E^*, E') satisfying* (H). *Then the labels l' are the same for all NTF's of value u_0 with resetting on $E^* \subseteq E'$.*

Proof. Let x' and y' be two NTF's with different labels $l' \neq h'$, and suppose without loss of generality that $S = \{v \in V: l'_v > h'_v\}$ is nonempty. Consider the net flow across the boundary of S: since x' and y' satisfy flow conservation, setting $b_s = u_0$, $b_t = -u_0$ and $b_v = 0$ for $v \in V\backslash\{s, t\}$, we get

$$x'(\delta^+(S)) - x'(\delta^-(S)) = \sum_{v\in S} b_v = y'(\delta^+(S)) - y'(\delta^-(S)). \quad (18)$$

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For $e = vw \in \delta^+(S)$ we have $x'_e \leqslant y'_e$ since otherwise $x'_e > y'_e$ implies $x'_e > 0$ and $l'_w = \rho_e(l'_v, x'_e) > \rho_e(h'_v, y'_e) \geqslant h'_w$ contradicting $w \notin S$. Similarly, $x'_e \geqslant y'_e$ for all $e = vw \in \delta^-(S)$ since $y'_e > x'_e$ implies $y'_e > 0$ and $h'_w = \rho_e(h'_v, y'_e) \geqslant \rho_e(l'_v, x'_e) \geqslant l'_w$ contradicting $w \in S$. These inequalities and (18) imply $x'_e = y'_e$ for all $e \in \delta(S)$, with $y'_e = 0$ for $e \in \delta^-(S)$ since $y'_e > 0$ yields a contradiction as before. Since E' is acyclic, we may find $w \in S$ with all edges $e = vw \in E'$ belonging to $\delta^-(S)$. Now, $l'_w > h'_w \geqslant 0$ and $x'_e = 0$ implies that $e \notin E^*$ for all these edges, and then $\rho_e(l'_v, x'_e) = l'_v$ as well as $\rho_e(h'_v, y'_e) = h'_v$, from which we get the contradiction $h'_w = \min_{vw\in E'} h'_v \geqslant \min_{vw\in E'} l'_v = l'_w$. □

4. Existence and Uniqueness of Dynamic Equilibria

Koch and Skutella (2011) describe a method to extend an equilibrium for the case of a constant inflow rate $u(\theta) \equiv u_0$. Given a feasible flow-over-time f that satisfies the equilibrium conditions in $[0, \theta_k]$, the equilibrium is extended as follows:

(1) Find x' an NTF of value u_0 with resetting on $E^*_{\theta_k} \subseteq E'_{\theta_k}$, and let l' denote the corresponding labels.

(2) Compute $\theta_{k+1} = \theta_k + \alpha$ with $\alpha > 0$ the largest value with

$$l_w(\theta_k)+\alpha l'_w - l_v(\theta_k) - \alpha l'_v \leqslant \tau_e, \quad \text{for all } e = vw \notin E'_{\theta_k}, \tag{19}$$

$$l_w(\theta_k)+\alpha l'_w - l_v(\theta_k) - \alpha l'_v \geqslant \tau_e, \quad \text{for all } e = vw \in E^*_{\theta_k}. \tag{20}$$

(3) Extend the earliest-time functions and the flow-over-time as

$$l_v(\theta) = l_v(\theta_k) + (\theta - \theta_k) l'_v, \quad \text{for } v \in V \text{ and } \theta \in [\theta_k, \theta_{k+1}],$$

$$f^+_e(\xi) = x'_e / l'_v, \quad \text{for } e = vw \in E \text{ and } \xi \in [l_v(\theta_k), l_v(\theta_{k+1})),$$

$$f^-_e(\xi) = x'_e / l'_w, \quad \text{for } e = vw \in E \text{ and } \xi \in [l_w(\theta_k), l_w(\theta_{k+1})).$$

Theorems 3 and 4 imply that x' in step (1) exists and l' is unique. Moreover there are finitely many l', each one corresponding to a different pair (E^*_θ, E'_θ). The α computed in (2) is strictly positive so that each iteration extends the earliest-time functions to a strictly larger interval. The conditions (19) and (20) correspond, respectively, to the maximum ranges on which the inactive edges remain inactive, and the positive queues remain positive. Hence, for $\theta \in [\theta_k, \theta_{k+1})$ the pair (E^*_θ, E'_θ) remains constant, whereas at θ_{k+1} this pair changes and we must recompute the NTF. Note that when $l'_v = 0$ the update of f^+_e does not extend its domain of definition and similarly for f^-_e when $l'_w = 0$. As shown in Koch and Skutella (2011), the extension maintains at all times the conditions for dynamic equilibrium in the strong sense (see Remark after Definition 1).

This extension procedure can be used to establish the existence of a dynamic equilibrium. Starting from the interval $(-\infty, \theta_0]$ with $\theta_0 = 0$ and zero flows, the extension can be iterated as long as required to find a new interval $[\theta_k, \theta_{k+1}]$ with $\theta_{k+1} > \theta_k$ at every step k. Eventually, θ_k may have a finite limit θ_∞: in this case, since the label functions are nondecreasing and have bounded derivatives, we can define the equilibrium at θ_∞ as the limit point of the label functions l, and restart the extension process. A standard argument using Zorn's lemma shows that a maximal solution is defined over all $\mathbb{R}_+$. Note that the f constructed above is right-constant.

Definition 4. A function g: $\mathbb{R} \to \mathbb{R}$ is called *right-constant* if for each $\theta \in \mathbb{R}$ there is an $\epsilon > 0$ such that g is constant on $[\theta, \theta + \epsilon)$. Similarly, g is *right-linear* if for each θ it is affine on $[\theta, \theta + \epsilon)$ for some $\epsilon > 0$.

The extension method works even if the inflow rate function is piecewise constant, so we have the following existence result.

Theorem 5. *Suppose that the inflow u is piecewise constant, i.e., there is an increasing sequence $(\xi_k)_{k\in\mathbb{N}}$ with $\xi_0 = 0$ such that $u(\cdot)$ is constant on each interval $[\xi_k, \xi_{k+1})$. Then there exists a strong dynamic equilibrium f that is right-constant and whose label functions l are right-linear.*

Dynamic equilibria in general are not unique. Consider, for instance, a constant inflow $u(\theta) = \mathbb{1}_{\{\theta \geqslant 0\}}$ in the network in Example 1 but with $\tau_c = \tau_b$. Then all queues remain empty at all times, and any splitting of the outflow $f^-_a(\theta) = \mathbb{1}_{\{\theta \geqslant 1\}}$ among the edges b and c yields a dynamic equilibrium. Nevertheless, using Theorem 4 one can prove that the earliest-time functions in all sufficiently regular dynamic equilibria are the same and coincide with those given by the constructive procedure.

Theorem 6. *Suppose that the inflow u is piecewise constant. Then, the earliest-time functions $(l_v)_{v\in V}$ are the same for all dynamic equilibria f which are right-continuous.*

Proof. When f is right continuous, it follows that the queue lengths $z_e(\theta)$, the exit-time functions $T_e(\cdot)$, and the earliest-time functions $l_v(\theta)$ are right-differentiable everywhere with right-continuous derivatives. Theorem 2 implies that $(dl_v/d\theta^+)(\cdot)$ are an NTF, and Theorem 4 shows that these derivatives are unique. Since they can take only finitely many values, continuity from the right imply that $(dl_v/d\theta^+)(\cdot)$ is right-constant and $l_v(\cdot)$ is right-linear. It follows that any two right-continuous dynamic equilibria must have the same earliest-time functions. Indeed, if these functions coincide up to time θ, their right derivatives at θ coincide, and since they are right-linear they will also coincide on a nontrivial interval $[\theta, \theta + \epsilon]$. This implies that in fact the functions must coincide throughout $\mathbb{R}$. □

5. Existence of Equilibria for Inflow Rates in L^p

The previous sections studied dynamic equilibria for a single origin-destination with piecewise constant inflows.

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We consider next more general inflow rates and then extend the results to multiple origin-destination pairs. We proceed as in Friesz et al. (1993) using a variational inequality for a path-flow formulation of dynamic equilibrium. The analysis is nonconstructive and exploits the following particular case of the existence result (Brézis 1968, Theorem 24). Let $(X, \|\cdot\|)$ be a reflexive Banach space and $\langle\cdot,\cdot\rangle$ the canonical pairing between X and its dual X^*. If $\mathcal{A}\colon K \to X^*$ is a weak-strong continuous map defined on a nonempty, closed, bounded, and convex subset $K \subseteq X$, then the following variational inequality problem has a solution:

$$\mathrm{VI}(K,\mathcal{A}) \quad \text{Find } x \in K \text{ such that } \langle \mathcal{A}x, y-x\rangle \geqslant 0, \quad \text{for all } y \in K.$$

5.1. Variational Inequality Formulation

Let us consider first the case of a single origin-destination pair st and an inflow rate $u \in L^p(0,T)$ where T is a finite horizon and $1 < p < \infty$. We extend $u(\theta) \equiv 0$ outside $[0,T]$ so that u may be seen as a function in $\mathcal{F}_0(\mathbb{R})$. As before, let $\mathcal{P}$ be the set of paths connecting s to t and denote by K the nonempty, bounded, closed, and convex set of *feasible path-flows* given by

$$K = \left\{ h \in L^p(0,T)^{\mathcal{P}}\colon \sum_{P\in\mathcal{P}} h_P = u \text{ and } h_P \geqslant 0 \quad \text{for all } P \in \mathcal{P} \right\}. \tag{21}$$

The space $X = L^p(0,T)^{\mathcal{P}}$ is reflexive with dual $X^* = L^q(0,T)^{\mathcal{P}}$ where $1/p + 1/q = 1$. We will show that a dynamic equilibrium can be obtained by solving the variational inequality $\mathrm{VI}(K,\mathcal{A})$ with $\mathcal{A}\colon K \subseteq X \to X^*$ such that $\mathcal{A}_P(h) \in L^q(0,T)$ is the continuous function $\theta \mapsto l_h^P(\theta) - \theta$ giving the time required to travel from s to t using path P under the path-flow pattern given by h, namely, the problem is to find $h \in K$ as defined by (21) such that

$$\sum_{P\in\mathcal{P}} \int_0^T (l_h^P(\theta) - \theta)(h'_P(\theta) - h_P(\theta))\, d\theta \geqslant 0 \quad \forall h' \in K. \tag{22}$$

To properly define the map $\mathcal{A}$, our first task is to show that every $h \in K$ determines a unique feasible flow-over-time f, which in turn induces link travel times T_e and path travel times l_h^P. This is achieved by the network loading procedure described in the next subsection. In §5.3 we establish the weak-strong continuity of $\mathcal{A}$, and then in §5.4 we conclude the existence of a dynamic equilibrium. Finally, §5.5 extends the existence result to multiple origin-destinations.

5.2. Network Loading

The following network loading procedure requires $\tau_e > 0$ on every link e, which we assume from now on. Let $h = (h_P)_{P\in\mathcal{P}}$ be a given family of path-flows with $h_P \in \mathcal{F}_0(\mathbb{R})$ for all $P \in \mathcal{P}$. A *network loading* is a flow-over-time $f = (f^+, f^-)$ together with nonnegative and measurable link-path decompositions

$$f_e^+(\theta) = \sum_{P\ni e} f_{P,e}^+(\theta), \qquad f_e^-(\theta) = \sum_{P\ni e} f_{P,e}^-(\theta), \tag{23}$$

such that for all links $e = vw$ and almost all $\theta \in \mathbb{R}$ one has

$$f_{P,e}^+(\theta) = \begin{cases} h_P(\theta) & \text{if } e \text{ is the first link of } P, \\ f_{P,e^*}^-(\theta) & \text{if } e^* \text{ is the link in } P \text{ just before } e, \end{cases} \tag{24}$$

together with the link transfer equations

$$\int_0^{T_e(\theta)} f_{P,e}^-(\xi)\, d\xi = \int_0^\theta f_{P,e}^+(\xi)\, d\xi \tag{25}$$

where T_e is the link travel time induced by f_e^+ through Equations (4) and (6). We denote by ω the tuple comprising all the flows f_e^+, $f_{P,e}^+$, f_e^-, $f_{P,e}^-$ for $e \in E$ and $P \in \mathcal{P}$. In order to prove the existence and uniqueness of a network loading, we first establish the following technical lemma.

Lemma 2. *Let a link-path decomposition of the inflow*

$$f_e^+(\theta) = \sum_{P\ni e} f_{P,e}^+(\theta)$$

be given over an initial interval $(-\infty, \bar\theta]$. Then there are unique outflows $f_{P,e}^- \in L^\infty((-\infty, T_e(\bar\theta)])$ satisfying (25), with $0 \leqslant f_{P,e}^-(\xi) \leqslant \nu_e$ for all $\xi \leqslant T_e(\bar\theta)$.

Proof. Since T_e maps $(-\infty, \bar\theta]$ surjectively onto $(-\infty, T_e(\bar\theta)]$, it is clear that there is at most one $f_{P,e}^-$ satisfying (25) (under the usual identification of functions differing on a negligible subset of $\mathbb{R}$). To establish the existence let $A \subseteq (-\infty, \bar\theta]$ be the set of times θ at which the derivative $T'_e(\theta)$ exists and is strictly positive, and set

$$f_{P,e}^-(T_e(\theta)) = \begin{cases} f_{P,e}^+(\theta)/T'_e(\theta) & \text{for } \theta \in A, \\ 0 & \text{otherwise.} \end{cases}$$

This unambiguously defines $f_{P,e}^-(\xi)$ for all $\xi \leqslant T_e(\bar\theta)$ as a nonnegative measurable function. Moreover, for $\theta \in A$ we have

$$\sum_{P\ni e} f_{P,e}^-(T_e(\theta)) = f_e^+(\theta)/T'_e(\theta) = f_e^-(T_e(\theta)) \leqslant \nu_e,$$

which implies $0 \leqslant f_{P,e}^-(\xi) \leqslant \nu_e$ for all $\xi \leqslant T_e(\bar\theta)$ so that the $f_{P,e}^-$'s are essentially bounded. Finally, a change of variables in the integral (see appendix) gives

$$\int_0^{T_e(\theta)} f_{P,e}^-(\xi)\, d\xi = \int_0^\theta f_{P,e}^-(T_e(\xi))T'_e(\xi)\, d\xi = \int_0^\theta f_{P,e}^+(\xi)\, d\xi$$

where we used the equality $f_{P,e}^-(T_e(\xi))T'_e(\xi) = f_{P,e}^+(\xi)$, which follows from the definition of $f_{P,e}^-(\xi)$ when $\xi \in A$ and from the fact that, almost everywhere, (7) implies that if $T'_e(\xi) = 0$ then $f_e^+(\xi) = 0$ and therefore $f_{P,e}^+(\xi) = 0$. □

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Proposition 3. *Suppose that $\tau_e > 0$ on all links e. Then to each path-flow tuple h it corresponds a unique network loading ω.*

Proof. Let $h = (h_P)_{P\in\mathcal{P}}$ be a given family of path-flows and suppose that we have a link-path decomposition satisfying (23)–(25) over an interval $(-\infty, \bar\theta]$. For $\bar\theta = 0$ this is easy since all flows vanish on the negative axis. By Lemma 2, the inflow decompositions $f_e^+(\theta) = \sum_{P\ni e} f_{P,e}^+(\theta)$ over $(-\infty, \bar\theta]$, together with condition (25), determine unique link-path decompositions for the outflows $f_e^-(\theta) = \sum_{P\ni e} f_{P,e}^-(\theta)$ over the interval $(-\infty, T_e(\bar\theta)]$. These intervals include $(-\infty, \bar\theta+\varepsilon]$ with $\varepsilon = \min_e \tau_e > 0$, and then using (24) it follows that the link inflows and their link-path decompositions have unique extensions to $(-\infty, \bar\theta+\varepsilon]$. Proceeding inductively it follows that the inflows and outflows, together with their link-path decompositions, are uniquely defined on all of $\mathbb{R}$. □

5.3. Continuity of Path Travel Times

We prove next that the network loading procedure defines path travel time maps $h\mapsto l_h^P$ that are weak-strong continuous from $K\subset L^p(0,T)^{\mathcal{P}}$ to the space of continuous functions $C([0,T],\mathbb{R})$ endowed with the uniform norm. The proof is split into several lemmas.

Lemma 3. *There exists a constant $M \geqslant 0$ such that all the flows in the network loading corresponding to any $h\in K$ are supported on $[0,M]$.*

Proof. We claim that the queue lengths are bounded by $z_e(\theta) \leqslant \bar z = \int_0^T u(\xi)\,d\xi$. Indeed, an inductive argument based on (24) and (25) shows that for each path P and each link $e\in P$ we have $\int_{\mathbb{R}} f_{P,e}^+(\xi)\,d\xi = \int_{\mathbb{R}} h_P(\xi)\,d\xi$. Since $z_e(\theta)\leqslant F_e^+(\theta)$, using (23) we get

$$z_e(\theta) \leqslant F_e^+(\theta) = \sum_{P\ni e}\int_0^\theta f_{P,e}^+(\xi)\,d\xi \leqslant \sum_P \int_{\mathbb{R}} h_P(\xi)\,d\xi = \int_0^T u(\xi)\,d\xi.$$

This bound implies that the time to traverse a link e is at most $\bar z/\nu_e + \tau_e$. Denoting by δ the maximum of these quantities over all $e\in E$ and setting $M = T + m\delta$ where m is the maximum number of links in all paths $P\in\mathcal{P}$, then $l_h^P(\theta)\leqslant M$ for all $P\in\mathcal{P}$ and $\theta\in[0,T]$. This, together with (24) and (25), implies in turn that all the flows in a network loading are supported on the interval $[0,M]$. □

Lemma 4. *The maps $f_e^+\mapsto z_e$ and $f_e^+\mapsto T_e$ defined by (4) and (6) are weak-strong continuous from $L^p(0,M)$ to $C([0,M],\mathbb{R})$.*

Proof. The continuity of $f_e^+\mapsto T_e$ is immediate from that of $f_e^+\mapsto z_e$. To show the latter, we recall that Arzela-Ascoli's theorem implies that the integration map I: $L^p(0,M)\to C([0,M],\mathbb{R})$ defined by $Ix(\theta) = \int_0^\theta x(\xi)\,d\xi$ is a compact operator, and hence it is weak-strong continuous. It follows that the map $f_e^+\mapsto y_e$ given by $y_e(\theta) = \int_0^\theta [f_e^+(\xi)-\nu_e]\,d\xi$ is weak-strong continuous, and then the same holds for $f_e^+\mapsto z_e$ since (4) gives

$$z_e(\theta) = \max_{\eta\in[0,\theta]}\{y_e(\theta)-y_e(\eta)\} = y_e(\theta) - \min_{\eta\in[0,\theta]} y_e(\eta)$$

and the map $y\mapsto Hy$ operating on $C([0,M],\mathbb{R})$ as $Hy(\theta) = \min_{\eta\in[0,\theta]} y_e(\eta)$ is nonexpansive. □

Lemma 5. *Let Ω denote the set of all the restrictions to $[0,M]$ of the pairs (h,ω) where $h\in K$ and ω is the corresponding network loading. Then Ω is a bounded and weakly closed subset of $L^p(0,M)^k$ where k is the dimension of the tuple (h,ω), namely $k = |\mathcal{P}| + 2|E| + 2|\mathcal{P}||E|$.*

Proof. From Lemma 3 we know that all flows $(h,\omega)\in\Omega$ are supported on $[0,M]$, while (24) and Lemma 2 imply that they are uniformly bounded in $L^p(0,M)$. Let us take a weakly convergent net $(h^\alpha,\omega^\alpha)\rightharpoonup(h,\omega)$ with $(h^\alpha,\omega^\alpha)\in\Omega$. It is clear that conditions (23) and (24) are stable under weak limits so that ω satisfies these equations. In order to show (25) it suffices to pass to the limit in

$$\int_0^{T_e^\alpha(\theta)} f_{P,e}^{\alpha-}(\xi)\,d\xi = \int_0^\theta f_{P,e}^{\alpha+}(\xi)\,d\xi. \tag{26}$$

The right-hand side converges to $\int_0^\theta f_{P,e}^+(\xi)\,d\xi$ while the integral on the left can be written as the sum

$$\int_0^{T_e^\alpha(\theta)} f_{P,e}^{\alpha-}(\xi)\,d\xi = \int_0^{T_e(\theta)} f_{P,e}^{\alpha-}(\xi)\,d\xi + \int_{T_e(\theta)}^{T_e^\alpha(\theta)} f_{P,e}^{\alpha-}(\xi)\,d\xi.$$

The first term on the right converges to $\int_0^{T_e(\theta)} f_{P,e}^-(\xi)\,d\xi$ while the second converges to zero. Indeed, letting $q = p/(p-1)$, by Hölder's inequality we have

$$\left|\int_{T_e(\theta)}^{T_e^\alpha(\theta)} f_{P,e}^{\alpha-}(\xi)\,d\xi\right| \leqslant \|f_{P,e}^{\alpha-}\|_p \sqrt[q]{|T_e^\alpha(\theta)-T_e(\theta)|}$$

so the conclusion follows since $0\leqslant f_{P,e}^{\alpha-}(\xi)\leqslant \nu_e$ implies $\|f_{P,e}^{\alpha-}\|_p \leqslant \nu_e\sqrt[p]{M}$, and according to Lemma 4 we have $T_e^\alpha(\theta)\to T_e(\theta)$. Hence, we may pass to the limit in (26), which proves that w satisfies (25) and therefore $(h,\omega)\in\Omega$ as was to be proved. □

Lemma 6. *The maps $h\mapsto T_e$ defined by the network loading procedure are weak-strong continuous from $K\subset L^p(0,T)^{\mathcal{P}}$ to $C([0,M],\mathbb{R})$.*

Proof. Take a weakly convergent net $h^\alpha\rightharpoonup h$ in K and let ω^α be the corresponding network loading. From Lemma 3 we know that the net ω^α is bounded in $L^p(0,M)$, while Lemma 5 implies that any weak accumulation point of w^α is a network loading for h. Since the latter is unique, it follows that $w^\alpha\rightharpoonup w$. In particular $f_e^{\alpha+}\rightharpoonup f_e^+$ weakly in $L^p(0,M)$ so that the conclusion $T_e^\alpha\to T_e$ strongly in $C([0,M],\mathbb{R})$ is a consequence of Lemma 4. □

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Lemma 7. *For each $P \in \mathcal{P}$ the map $h \mapsto l_h^P$ defined by the network loading procedure is weak-strong continuous from $K \subset L^p(0,T)^{\mathcal{P}}$ to $C([0,T],\mathbb{R})$.*

Proof. Let $P = (e_1, e_2, \ldots, e_k)$. Set $M_i = T + i\delta$ with δ as in the proof of Lemma 3, and consider the restrictions $T_{e_i}\colon [0, M_{i-1}] \to [0, M_i]$ so that for all $\theta \in [0,T]$

$$l_h^P(\theta) = T_{e_k} \circ \cdots \circ T_{e_1}(\theta). \tag{27}$$

By Lemma 6 the maps $h \mapsto T_{e_i}$ are weak-strong continuous, so the conclusion follows by noting that composition is a continuous operation. More precisely, the map $(f,g) \mapsto g \circ f$ defined on the spaces

$$\begin{aligned}\circ\colon\, & C([0,M_{i-1}],[0,M_i]) \times C([0,M_i],[0,M_{i+1}]) \\ & \to C([0,M_{i-1}],[0,M_{i+1}])\end{aligned}$$

is a continuous map (with respect to uniform convergence). Indeed, consider a strongly convergent net $(f^\alpha, g^\alpha) \to (f,g)$. Then for each $\theta \in [0, M_{i-1}]$ we have

$$\begin{aligned}|g^\alpha \circ f^\alpha(\theta) - g \circ f(\theta)| \leqslant{} & |g^\alpha(f^\alpha(\theta)) - g(f^\alpha(\theta))| \\ & + |g(f^\alpha(\theta)) - g(f(\theta))|.\end{aligned}$$

The first term on the right can be bounded by $\|g^\alpha - g\|_\infty$, which tends to 0, while the second term also tends to zero uniformly in θ since g is uniformly continuous and $\|f^\alpha - f\|_\infty$ tends to zero. ☐

5.4. Existence of Dynamic Equilibrium for a Single Origin-Destination

With these preliminary results we may now prove that the variational inequality VI$(K,\mathcal{A})$ has a solution, and the corresponding network loading gives a dynamic equilibrium.

Theorem 7. *Let $u \in L^p(0,T)$ with $1 < p < \infty$ and assume that $\tau_e > 0$ on every link e. Then there exists a dynamic equilibrium.*

Proof. According to Lemma 7 the map $h \mapsto \mathcal{A}(h)$ is weak-strong continuous from K to X^* so that the variational inequality VI$(K,\mathcal{A})$ has a solution $h \in K$. We claim that the corresponding flow-over-time f given by Proposition 3 is a dynamic equilibrium. If not, by Theorem 1 we may find $\theta > 0$ and a link $e = vw \notin E'_\theta$ such that for all $\epsilon > 0$ we have $f_e(l_v(\xi)) l'_v(\xi) > 0$ on a subset of positive measure in $[\theta, \theta+\epsilon]$. Choose ϵ small enough so that E'_ξ decreases on $[\theta,\theta+\epsilon]$ and choose $P \in \mathcal{P}$ with $e \in P$ and $h_P(\xi) > 0$ on a subset $I \subseteq [\theta,\theta+\epsilon]$ with positive measure. Take also $P' \in \mathcal{P}$ with all links in $E'_{\theta+\epsilon}$ so that P' is optimal for each $\xi \in [\theta,\theta+\epsilon]$ (that is, P' is an s-t path in the ξ-shortest-path graph G_ξ), and let $h' \in K$ be identical to h except for $\xi \in I$ where we transfer flow from P to P', that is $h'_P(\xi) = 0$ and $h'_{P'}(\xi) = h_{P'}(\xi) + h_P(\xi)$. A direct calculation then gives

$$\begin{aligned}0 \leqslant \langle \mathcal{A}h, h' - h\rangle &= \int_{[0,T]} \langle \mathcal{A}h(\xi), h'(\xi) - h(\xi)\rangle\, d\xi \\ &= \int_I (l_h^{P'}(\xi) - l_h^P(\xi)) h_P(\xi)\, d\xi.\end{aligned}$$

Since P' is optimal for all $\xi \in I$ while P is not (since $e \notin E'_\xi$), it follows that $l_h^{P'}(\xi) < l_h^P(\xi)$, which yields a contradiction. ☐

5.5. Extension to Multiple Origin-Destination Pairs

The extension to multiple origin-destinations is straightforward. For each pair $st \in N \times N$ let $u_{st} \in L^p(0,T)$ be the corresponding inflow (possibly zero) and let $\mathcal{P}_{st}$ be the set of s-t paths which is assumed nonempty if u_{st} is nonzero. A feasible flow-over-time is now a family of inflows $f_e^+ = \sum_{st} f_{e,st}^+$ and outflows $f_e^- = \sum_{st} f_{e,st}^-$ satisfying flow conservation for each st pair, namely, for all nodes $v \neq t$ and almost all $\theta \in \mathbb{R}$

$$\sum_{e \in \delta^+(v)} f_{e,st}^+(\theta) - \sum_{e \in \delta^-(v)} f_{e,st}^-(\theta) = \begin{cases} u_{st}(\theta) & \text{for } v = s, \\ 0 & \text{for } v \in V \backslash \{s,t\}. \end{cases} \tag{28}$$

The definitions of queue lengths, link travel times, and path travel times remain unchanged, and we only need to introduce the origin-destination optimal times

$$l_{st}(\theta) = \min_{P \in \mathcal{P}_{st}} l^P(\theta).$$

Dynamic equilibrium holds when for each pair st and all $e = vw \in E$ we have $f_{e,st}^+(\xi) = 0$ for almost all $\xi \in l_{sv}(\mathbb{R} \backslash \Theta_e^s)$ where Θ_e^s denotes the set of all times θ at which link $e = vw$ is active for origin s, namely $l_{sw}(\theta) = T_e(l_{sv}(\theta))$.

Denoting $\mathcal{P}$ the union of all the $\mathcal{P}_{st}$'s, the network loading procedure in §5.2 remains unchanged as it did not depend on having a single origin-destination pair. Also the results in §5.3 are easily extended by considering K as the set of path-flows $h = (h_P)_{P \in \mathcal{P}} \in L^p(0,T)^{\mathcal{P}}$, which are nonnegative and that satisfy flow conservation for each pair st, that is

$$\sum_{P \in \mathcal{P}_{st}} h_P = u_{st}.$$

For the bound $\bar{z} = \int_0^T u(\xi)\, d\xi$ of the queue lengths in Lemma 3 it suffices to take u as the sum of all the u_{st}'s. With these preliminaries, the proof of Theorem 7 is readily adapted to establish the existence of a dynamic equilibrium for multiple origin-destinations.

Theorem 8. *Let $u_{st} \in L^p(0,T)$ with $1 < p < \infty$ the inflows for multiple origin-destination pairs $st \in N \times N$, and assume that $\tau_e > 0$ on every link e. Then there exists a dynamic equilibrium.*

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6. Concluding Remarks

Although dynamic traffic assignment has received considerable attention since the seminal work by Merchant and Nemhauser (1978a, b), the existence and characterization of dynamic equilibria still poses many challenging questions. For a review of the literature and open problems we refer to Peeta and Ziliaskopoulos (2001). Several of the previous studies have relied on a strict FIFO condition that requires the exit time functions $T_e(\cdot)$ to be strictly increasing. For instance, Friesz et al. (1993) consider a situation in which users choose simultaneously route and departure time, with link travel times specified as $D_e(y_e) = \alpha_e y_e + \beta_e$ where $y_e = F_e^+(\theta) - F_e^-(\theta)$ is the total flow on link e at time θ and α_e, β_e are strictly positive constants. Strict FIFO was shown to hold for such linear volume-delay functions, which allowed to characterize the equilibrium by a variational inequality, though no existence result was given. Strict FIFO was also used by Xu et al. (1999) to investigate the network loading problem, namely, to determine the link volumes and travel times that result from a given set of path-flow departure rates. Shortly after, the existence of equilibria was established by Zhu and Marcotte (2000) under a strong FIFO condition that holds for linear volume-delay functions (even in the case $\alpha_e = 0$), assuming in addition that inflows are uniformly bounded.

Unfortunately, as illustrated by the example in §2.5, strict FIFO does not hold in our framework and these previous results do not apply. This is somewhat surprising since we also consider linear travel times. The subtle difference is that we consider the queue length z_e instead of the total volume y_e on the link. Note that the fluid queue model could be cast into the linear volume-delay framework by decomposing each link into a pure queueing segment with travel time z_e/ν_e (that is, $\alpha_e = 1/\nu_e$, $\beta_e = 0$), followed by a link with constant travel time τ_e (that is, $\alpha_e = 0$, $\beta_e = \tau_e$). Strict FIFO fails precisely because the queueing segment has $\beta_e = 0$. In this respect it is worth noting that our existence results do not require strict FIFO, as long as $\tau_e > 0$, and Theorem 5 holds even if $\tau_e = 0$.

A general existence result for dynamic network equilibrium beyond strict FIFO was recently presented by Meunier and Wagner (2010). Their model considers both route choice and departure time choice and is based on a weak form of strict FIFO: the travel time $T_e(\cdot)$ strictly increases on any interval on which there is some inflow into the link. This weaker property does hold in our context, and the result applies provided that the inflow $u(\cdot)$ belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R})$.

An interesting feature of the approach in §4, compared with previous existence results, is that it provides a way to construct the equilibrium. In this respect, our work owes much to Koch and Skutella (2011). There are, however, several differences. On the modeling side, we distinguish the notion of dynamic equilibrium from the stronger dynamic equilibrium condition (see Remark 1). Both concepts were used interchangeably in Koch and Skutella (2011), although they might differ as shown in Example 2. In particular, our Theorem 1 precises Koch and Skutella (2011, Theorem 1) which characterizes dynamic equilibria, not strong equilibria. Also, Theorem 2 is an extension of Koch and Skutella (2011, Theorem 2), which applies to the larger class of dynamic equilibria and provides a sharper conclusion by including the normalization condition. The existence and uniqueness for NTF's in Theorems 3 and 4 are new, and so is the subsequent existence and uniqueness of a dynamic equilibrium in Theorems 5 and 6. To the best of our knowledge, the latter uniqueness result has not been observed previously in the literature.

The constructive approach in §4 raises a number of questions. On the one hand, it would be relevant to know if the step sizes computed in step (2) of the extension method are bounded away from 0. In this case the θ_k's would not accumulate, and the equilibrium would be computed in finitely many steps for any given horizon T. A related question is whether a steady state could eventually be attained with $\alpha = \infty$ at some iteration, in which case the algorithm would be finite. A weaker but still difficult conjecture is whether the queue lengths $z_e(\cdot)$ remain bounded as long as the capacity of any s-t cut is large enough, for instance larger than the inflow at any point in time. The difficulty for proving such a claim is that the flow across a cut can be arbitrarily larger than the inflow: the queueing processes might introduce delay offsets in such a way that the flow entering the network at different times reaches the cut simultaneously at a later date, causing a superposition of flows that exceeds the capacity of the cut. On the other hand, while it is easy to give a finite algorithm to compute thin flows with resetting, the computational complexity of the problem remains open. A polynomial time algorithm for this would imply that for piecewise constant inflows one could compute a dynamic equilibrium in polynomial time (in input plus output).

Another interesting question is whether the constructive approach in §4 can be adapted to deal with more general inflows $u(\theta)$. More precisely, let $N(l, u_0)$ denote the unique labels in a normalized thin flow of value u_0 with resetting on the set E^* of all links $e = vw$ with $l_w > l_v + \tau_e$, and E' the set of links with $l_w \geqslant l_v + \tau_e$ (see Proposition 2). Recalling Theorems 2 and 4, an equilibrium could be computed by solving the system of ordinary differential equations

$$l'(\theta) = N(l(\theta), u(\theta))$$

with initial condition $l_v(0)$ equal to the minimum s-v travel time with empty queues. The cumulative flows $x_e(\theta)$ could then be recovered by integrating a measurable selection of the corresponding thin flows. The main difficulty here is that the map N is discontinuous in l so that the standard theory and algorithms for ordinary differential equations do not apply directly. A final open problem is to extend the constructive approach to multiple origin-destinations.

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Appendix. The Spaces $L^p_{\mathrm{loc}}(\mathbb{R})$ and $AC_{\mathrm{loc}}(\mathbb{R})$

We denote $L^p_{\mathrm{loc}}(\mathbb{R})$ the vector space of measurable functions $g\colon \mathbb{R}\to\mathbb{R}$ such that $|g(\cdot)|^p$ is integrable on every bounded interval. Similarly, $AC_{\mathrm{loc}}(\mathbb{R})$ is the vector space of functions $h\colon \mathbb{R}\to\mathbb{R}$ that are absolutely continuous on every bounded interval. For a thorough study of absolutely continuous functions we refer to Leoni (2009, Chapter 3). Here we just summarize a few facts required in our analysis:

- For all $1\leqslant p\leqslant\infty$ we have $L^p_{\mathrm{loc}}(\mathbb{R})\subseteq L^1_{\mathrm{loc}}(\mathbb{R})$.
- The primitive of any $g\in L^1_{\mathrm{loc}}(\mathbb{R})$ belongs to $AC_{\mathrm{loc}}(\mathbb{R})$. Conversely, every $h\in AC_{\mathrm{loc}}(\mathbb{R})$ is differentiable almost everywhere with $h'\in L^1_{\mathrm{loc}}(\mathbb{R})$ and

$$h(\theta)=h(0)+\int_0^\theta h'(\xi)\,d\xi.$$

- If $f,g\in AC_{\mathrm{loc}}(\mathbb{R})$ then their product fg and minimum $\min\{f,g\}$ are also in $AC_{\mathrm{loc}}(\mathbb{R})$.
- If $f,h\in AC_{\mathrm{loc}}(\mathbb{R})$ we do not necessarily have $f\circ h\in AC_{\mathrm{loc}}(\mathbb{R})$, but this holds if either f is Lipschitz or h is monotone. In both cases the following chain rule holds for almost all $y\in\mathbb{R}$:

$$(f\circ h)'(y)=f'(h(y))h'(y).$$

- In particular, if $h\in AC_{\mathrm{loc}}(\mathbb{R})$ is monotone and $g\in L^1_{\mathrm{loc}}(\mathbb{R})$ we have the change of variable formula

$$\int_{h(a)}^{h(b)} g(\xi)\,d\xi=\int_a^b g(h(y))h'(y)\,dy.$$

The following are more specific properties for which we could not find a reference, so we include a proof.

Lemma 8. *Let $g\colon\mathbb{R}\to\mathbb{R}_+$ be a nonnegative function in $L^1_{\mathrm{loc}}(\mathbb{R})$ and $\{(a_i,b_i)\}_{i\in I}$ a possibly uncountable family of intervals. Then g vanishes almost everywhere on each (a_i,b_i) iff it vanishes almost everywhere on $\bigcup_{i\in I}(a_i,b_i)$. The statement also holds for semi-open intervals of the form $[a_i,b_i)$ or $(a_i,b_i]$.*

Proof. Assume with no loss of generality that all intervals are nonempty. Since $\mu(A)=\int_A g(\xi)\,d\xi$ defines a regular measure on the Borel sets $A\subseteq\mathbb{R}$, for $\Theta=\bigcup_{i\in I}(a_i,b_i)$ we have

$$\mu(\Theta)=\sup\{\mu(K)\colon K \text{ compact},\ K\subseteq\Theta\}.$$

Now, each compact $K\subseteq\Theta$ has a finite subcover $K\subseteq\bigcup_{k=1}^n(a_{i_k},b_{i_k})$ so that

$$\mu(K)\leqslant\sum_{k=1}^n\mu((a_{i_k},b_{i_k}))=\sum_{k=1}^n\int_{a_{i_k}}^{b_{i_k}} g(\xi)\,d\xi=0.$$

It follows that $\mu(\Theta)=0$, which implies that $g(\xi)=0$ for almost all $\xi\in\Theta$ and proves the first statement.

The other claims follow, since all three unions differ in countably many elements. Indeed, consider for instance the set $N=\bigcup_{i\in I}[a_i,b_i)\setminus\bigcup_{i\in I}(a_i,b_i)$. Each point $z\in N$ must be an endpoint $z=a_i$ with the corresponding interval (a_i,b_i) disjoint from N. It follows that if $a_j\in N$ is another such point, the corresponding intervals cannot overlap, and therefore there can be at most countably many. A similar argument shows that $\bigcup_{i\in I}(a_i,b_i]\setminus\bigcup_{i\in I}(a_i,b_i)$ is countable. □

Remark. Lemma 8 does not hold for closed intervals $[a_i,b_i]$. In fact, every function g vanishes almost everywhere on each interval $[x,x]$ for $x\in\mathbb{R}$ but clearly not necessarily on $\bigcup_{x\in\mathbb{R}}[x,x]=\mathbb{R}$.

Lemma 9. *Let $z\in AC_{\mathrm{loc}}(\mathbb{R})$ with $z(\theta)=0$ for $\theta<0$. Then the following are equivalent:*
(a) *$z(\theta)\geqslant 0$ for all θ,*
(b) *$z(\theta)\leqslant 0\Rightarrow z'(\theta)\geqslant 0$ for almost all θ,*
(c) *$z(\theta)\leqslant 0\Rightarrow z'(\theta)=0$ for almost all θ.*

Proof. Let N be a null set such that $z'(\theta)$ exists for all $\theta\notin N$.

[(a) ⇔ (b)] Under (a), for all $\theta\notin N$ with $z(\theta)\leqslant 0$ we have $z(\theta)=0$ so that $z'(\theta)\geqslant 0$, which gives (b). Conversely, suppose that (b) holds but $z(\theta)<0$ for some θ, and consider the smallest θ' such that $z(\cdot)$ remains negative on $(\theta',\theta]$. Then $z(\theta')=0$, whereas (b) implies $z'(\xi)\geqslant 0$ for almost all $\xi\in(\theta',\theta)$ from which we get the contradiction $0>z(\theta)=z(\theta')+\int_{\theta'}^{\theta}z'(\xi)\,d\xi\geqslant 0$.

[(c) ⇔ (a)] Clearly (c) implies (b), which in turn implies (a). Conversely, since (a) implies (b), it suffices to show that the set $A=\{\theta\notin N\colon z(\theta)\leqslant 0;\ z'(\theta)>0\}$ is countable, provided that (a) holds. Indeed, for each $\theta\in A$ we have $z(\theta)=0$ and we may find $\epsilon>0$ such that $z(\theta')>0$ for all $\theta'\in I_\theta=(\theta,\theta+\epsilon)$. These intervals I_θ do not meet A so they cannot overlap, and therefore there can be at most countably many. □

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