Existence and Uniqueness of Equilibria for Flows over Time[⋆]

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Abstract. Network flows that vary over time arise naturally when modeling rapidly evolving systems such as the Internet. In this paper, we continue the study of equilibria for flows over time in the single-source single-sink deterministic queuing model proposed by Koch and Skutella. We give a constructive proof for the existence and uniqueness of equilibria for the case of a piecewise constant inflow rate, through a detailed analysis of the static flows obtained as derivatives of a dynamic equilibrium.

Keywords: Flows Over Time, Network Equilibrium.

1 Introduction

Understanding time varying flows in a network is fundamental in contexts where a *steady state* is rarely attained, including traffic networks and the Internet. Furthermore, these systems are usually characterized by the lack of coordination among participating agents and therefore need to be considered from a game theoretic perspective.

Research in flows over time has mainly focused in optimization. The first to consider this model in a discrete time setting were Ford and Fulkerson [6,7] who designed an algorithm to find a flow over time carrying the maximum possible total flow in a given time horizon. Gale [8] then showed the existence of a flow pattern that achieves this optimum simultaneously for all time horizons. These results were extended to a continuous time setting by Fleischer and Tardos [5] and by Anderson and Philpott [1] respectively. We refer the reader to Skutella [12] for an excellent survey.

The study of network flows over time when different flow particles act selfishly has mostly been considered in the transportation science literature. The book of Ran and Boyce [11] describes a very general model of equilibria, for which unfortunately very little is known. More recently, Koch and Skutella [9] considered a simpler model in which there is an inflow rate at a single source node that travels across the network towards a single sink node through edges that are characterized by their travel time (or latency) and their *per-time-unit* capacity. The model can be interpreted as a fluid relaxation of a deterministic queuing model, and is a special case of Ran and Boyce's general framework. Very recently this model was also considered by Macko, Larson, and Steskal [10] to study Braess's type paradoxes, and by Bhaskar, Fleischer, and Anshelevich [2] to design an effective stackleberg routing strategy.

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Our Contribution. Building upon the work of Koch and Skutella, in this paper we give a constructive (algorithmic) proof for the existence and uniqueness of equilibria for flows over time. Our work is based on the key concept of *thin flow with resetting* introduced by Koch and Skutella. This concept defines a static flow together with an associated labeling which coincides with the derivatives of the distance labels of an equilibrium. We actually consider a slightly more restrictive definition by adding an additional technical constraint, and show how to transform any thin flow with resetting into a flow and labeling satisfying this extra property. Then we prove that such a flow always exists using an equivalent nonlinear complementarity problem which is shown to admit a solution by proving that an associated variational inequality has an interior solution. Furthermore, we prove that a thin flow with resetting with our additional condition is also unique. By integrating these thin flows with resetting, the results allow us to prove the existence and uniqueness (among a natural family) of equilibria for flows over time when the inflow rate is piecewise constant. Finally, we give a non-constructive existence proof when the inflow rate function is measurable and its value raised to the p-th power has finite integral, for some $1 < p < \infty$ (i.e., belongs to L^p).

As a by-product of our existence result for thin flows with resetting, we obtain that the problem of finding such a flow and labeling belongs to TFNP (since a solution is guaranteed to exist). Furthermore our proof strategy shows that the problem belongs to PPAD as it reduces to finding a fixed point. In any case, we conjecture that the problem is actually solvable in polynomial time.

Organization of the paper. In §2, we describe the equilibrium model for flows over time introduced by Koch and Skutella [9]: First, we present the dynamic flow model, and then we define the routing game and its equilibria. In §3 we prove our main results concerning the existence and uniqueness of our refined definition of thin flow with resetting. Finally, in §4 we discuss how the latter results apply to equilibria for flows over time. Due to space limitations, several proofs are deferred to the full version.

2 A Model for Dynamic Routing Games

We consider a network (G, u, τ, s, t) consisting of a directed graph $G = (V, E)$ with node set V and edge set E, a vector $u = (u_e)_{e \in E}$ of strictly positive real numbers representing edges capacities, a vector $\tau = (\tau_e)_{e \in E}$ of nonnegative real numbers representing free flow transit times, a source $s \in V$, and a sink $t \in V$. An edge $e \in E$ from node v to node w is denoted (v, w) or just vw. To avoid confusion, we assume without loss of generality that there is at most one edge between any pair of nodes in G and that there are no loops. We also assume that for every cycle C of G, the sum of the free flow transit times along C is not zero, i.e.,

$$\sum_{e \in C} \tau_e > 0, \quad \text{for every cycle } C \text{ of } G. \tag{1}$$

2.1 Flows over Time

A *flow over time* is a pair of arrays $f := (f^+, f^-)$ of Lebesgue-integrable functions whose components $f_e^+ : \mathbb{R}_+ \to \mathbb{R}_+$ and $f_e^- : \mathbb{R}_+ \to \mathbb{R}_+$ for each $e \in E$ represent respectively the rate at which flow enters the tail of e and the rate of flow leaving the head of e at each time θ. For every edge $e \in E$ we define the cumulative inflow $F_e^+(\theta)$, the cumulative outflow $F_e^-(\theta)$, and the queue size $z_e(\theta)$ at time $\theta \geq 0$ as

$$F_e^+(\theta) := \int_0^\theta f_e^+(\xi)d\xi, \quad F_e^-(\theta) := \int_0^\theta f_e^-(\xi)d\xi, \quad z_e(\theta) := F_e^+(\theta) - F_e^-(\theta + \tau_e).$$

Definition 1. *A flow over time* $f = (f^+, f^-)$ *is said to be* feasible *if and only if it satisfies for almost all* $\theta \geq 0$*:*

1. *the* capacity constraints $f_e^-(\theta) \leq u_e$ *for each* $e \in E$
2. *the* non-deficit constraints $z_e(\theta) \geq 0$ *for each* $e \in E$
3. *the* flow conservation constraints

$$\sum_{e \in \delta^-(v)} f_e^-(\theta) = \sum_{e \in \delta^+(v)} f_e^+(\theta) \quad \textit{for each } v \in V \setminus \{s, t\}. \tag{2}$$

We also define the *network inflow rate* (at node s) as

$$u_0(\theta) = \sum_{e \in \delta^+(s)} f_e^+(\theta) - \sum_{e \in \delta^-(s)} f_e^-(\theta),$$

and the *waiting time at edge* e at time θ by

$$q_e(\theta) := \frac{z_e(\theta)}{u_e} = \frac{F_e^+(\theta) - F^-(\theta + \tau_e)}{u_e} \tag{3}$$

so that the mapping $\theta \mapsto \theta + q_e(\theta)$ represents the time at which a flow particle entering e at time θ exits from the queue. Koch and Skutella [9] showed the following.

Proposition 1. *The function* $\theta \mapsto \theta + q_e(\theta)$ *is nondecreasing and continuous.*

2.2 Label Functions for Flows over Time

For each $w \in V$ let $\mathcal{P}_w$ denote the set of all s-w paths (not necessarily simple) in G. Given an s-t flow over time, a node $w \in V$ and a path $P = (e_1, e_2, ..., e_k) \in \mathcal{P}_w$, with a corresponding sequence of nodes $(v_1, v_2, ..., v_{k+1})$ (*i.e.*, $e_i = v_i v_{i+1}$, $v_1 = s$ and $v_{k+1} = w$), we define $\ell_{w,P} : \mathbb{R}_+ \to \mathbb{R}_+$, the *label function of node* w *using path* P, by

$$\begin{aligned} \ell_{s,P}(\theta) &:= \theta, \\ \ell_{v_{i+1},P}(\theta) &:= \ell_{v_i,P}(\theta) + \tau_{e_i} + q_e(\ell_{v_i,P}(\theta)), \text{ for all } i \in \{1, ..., k\}. \end{aligned}$$

Thus $\ell_{w,P}(\theta)$ represents the time a flow particle reaches w traveling through P, starting from s at time θ. Note that these label functions depend on the full flow over time f through the waiting time functions q_e. We may then define the minimum time at which a flow particle can reach a node as follows (as we shall see, this definition is consistent with that of Koch and Skutella [9]).

Definition 2 (Label functions). *Given a feasible s-t flow over time f we define for every node $v \in V$ the label functions $\ell_v : \mathbb{R}_+ \to \mathbb{R}_+$ as*

$$\ell_v(\theta) := \min_{P \in \mathcal{P}_v} \ell_{v,P}(\theta). \tag{4}$$

Naturally, a particle traveling from s to w in the shortest possible time will not use a path containing a cycle. This is indeed a consequence of the non-decreasing monotonicity of the mapping $\theta \mapsto \theta + q_e(\theta)$, $q_e \geq 0$ and $\tau_e \geq 0$, for all $e \in E$, so that the minimum $\ell_{w,P}(\theta)$ for $P \in \mathcal{P}_w$ is attained over simple paths. Also, as a consequence of the non-decreasing monotonicity and continuity of the map $\theta \mapsto \theta + q_e(\theta)$, we have that for each node $v \in V$, the label function ℓ_v is non-decreasing and continuous. In addition, the functions $(\ell_v)_{v \in V}$ satisfy the definition in [9], which states that

$$\ell_w(\theta) = \begin{cases} \theta & \text{for } w = s \\ \min\limits_{e = vw \in \delta^-(w)} \{\ell_v(\theta) + \tau_e + q_e(\ell_v(\theta))\} & \text{for } w \in V \setminus \{s\}. \end{cases}$$

2.3 Equilibria

A feasible s-t flow over time $f = (f^+, f^-)$ on a network (G, u, τ, s, t), can be seen as a *dynamic equilibrium* if we identify each flow particle that travels from s to t at a time $\theta \in \mathbb{R}_+$ as players. In a dynamic equilibrium, all flow particles travel on s-t paths that yield the shortest possible total travel time, taking into account the aggregate congestion generated within the network due to other particles. For a formal definition we refer to [9]. In what follows we rely on an equivalent characterization given in Theorem 1 below.

Definition 3. *We say that edge $e \in E$ is contained in a shortest path at time $\theta \geq 0$ if and only if $\ell_w(\theta) = \ell_v(\theta) + \tau_e + q_e(\ell_v(\theta))$. We denote E_θ the set of all such arcs and we consider the induced acyclic graph $G_\theta = (V, E_\theta)$ which we call the* ***current shortest paths network****. We also say that* ***flow is sent along currently shortest paths*** *if, for each edge $e = vw \in E$, the following holds for almost all $\theta \geq 0$*

$$f_e^+(\ell_v(\theta)) > 0 \Longrightarrow e \in E_\theta.$$

In other words, a flow that is sent along currently shortest paths is such that at time $\ell_v(\theta)$ there is no flow assigned to any edge $e = vw \in E$ not lying in a shortest path at time θ. Note that there are edges in G not contained in a current shortest path network. Also, since (G, u, τ, s, t) satisfies Condition (1), G_θ is acyclic for every $\theta \geq 0$.

Following Koch and Skutella, let us consider a particle at s at time θ, $x_e^+(\theta)$ the amount of flow assigned to $e = vw$ before this particle can reach v, and $x_e^-(\theta)$ the flow leaving $e = vw$ before this particle can reach w, that is:

$$x_e^+(\theta) := F_e^+(\ell_v(\theta)), \quad x_e^-(\theta) := F_e^-(\ell_w(\theta)).$$

Theorem 1 ([9]). *For a feasible s-t flow over time, the following are equivalent:*

(i) The given flow over time is an equilibrium.
(ii) Flow is only sent along currently shortest paths.
(iii) For each edge $e \in E$ and at all times $\theta \geq 0$, it holds that $x_e^+(\theta) = x_e^-(\theta)$.

Whenever any of the three statements above is satisfied, then $x^+ = x^-$. In that case, we define $x_e(\theta) := x_e^+(\theta)$ for all $\theta \geq 0$ and each $e \in E$. From now on, we call $(x_e(\theta))_{e \in E}$ the *underlying static flow at time θ*.

3 Derivative of an Equilibrium Flow: A Special Labeling

In this section we study a key node labeling for flows over time, proving its existence and uniqueness. This labeling arises from the derivative of the distance label functions of a flow over time at equilibrium, and can be used in order to construct such an equilibrium from its derivatives.

Definition 4 (Thin Flow with Resetting). *Let* (G, u, s, t) *be a network,* $E_1 \subset E(G)$ *a subset of edges and* $F \geq 0$. *A labeling* $(\ell'_v)_{v \in V(G)}$ *is a thin flow of value* F *with resetting on* E_1 *if there exists a static* s-t F*-flow* $(x'_e)_{e \in E(G)} \geq 0$ *(i.e., a nonnegative and conservative static flow sending an amount* F *from* s *to* t*) such that:*

$$\ell'_s = 1, \tag{5}$$

$$\ell'_w \leq \ell'_v \qquad \text{for all } e = vw \in E(G) \setminus E_1, x'_e = 0, \tag{6}$$

$$\ell'_w = \max\{\ell'_v, \frac{x'_e}{u_e}\} \qquad \text{for all } e = vw \in E(G) \setminus E_1, x'_e > 0, \tag{7}$$

$$\ell'_w = \frac{x'_e}{u_e} \qquad \text{for all } e = vw \in E_1, \tag{8}$$

$$\ell'_w \geq \min_{vw \in \delta^-_G(w)} \ell'_v \qquad \text{if } \delta^-_G(w) \cap E_1 = \emptyset. \tag{9}$$

The original definition of thin flow with resetting, due to Koch and Skutella [9], does not consider Condition (9). They showed that the derivatives of the label functions of a dynamic equilibrium, restricted to the current shortest paths network, satisfy Conditions (5)-(8). We introduce Condition (9), to guarantee uniqueness of thin flows with resetting, as illustrated in Example 1 below. The following theorem shows that the derivatives of the label functions of a dynamic equilibrium, restricted to the current shortest paths network, are a thin flow with resetting. This result is applicable if the derivatives of the label and the underlying static flow functions exist. Interestingly, both the label functions and the underlying static flow functions are monotonically nondecreasing implying that both families of functions are differentiable almost everywhere.

Theorem 2. *Consider a feasible* s-t *flow over time* f *which is an equilibrium on a network* (G, u, τ, s, t) *with corresponding label functions* $(\ell_v)_{v \in V}$*, network inflow rate* u_0*, edge waiting time functions* $(q_e)_{e \in E}$ *and underlying static flow* $(x_e)_{e \in E}$*. Then for almost all* $\theta \geq 0$, $\frac{dx_e}{d\theta}(\theta)$ *and* $\frac{d\ell_v}{d\theta}(\theta)$ *exist for each* $e \in E$ *and* $v \in V$*. Moreover, for almost all* $\theta \geq 0$*, on the current shortest paths network* G_θ*, the derivatives* $\left(\frac{d\ell_v}{d\theta}(\theta)\right)_{v \in V}$ *form a thin flow of value* $u_0(\theta)$ *with resetting on the waiting edges* $E_1 := \{e = vw \in E \mid q_e(\ell_v(\theta)) > 0\}$*. A corresponding static flow fulfilling* (5) *to* (9) *is given by the derivatives* $\left(\frac{dx_e}{d\theta}(\theta)\right)_{e \in E}$.

The proof, found in the full version of the paper, follows closely that of Koch and Skutella though we need to be careful when dealing with Condition (9).

Example 1. To illustrate the role of Condition (9) in the construction of an equilibrium, consider a network composed of three nodes $\{s, v, t\}$ and three edges: one from s to t, another from s to v, and the last one from v to t. Every edge e has capacity $u_e = 1$, and the travel times are $\tau_{st} = \tau_{vt} = 0$ and $\tau_{sv} = 1$. Suppose that the network inflow rate is $u_0(\theta) = 2$ if $\theta \in [0, 1)$ and $u_0(\theta) = 1/2$ if $\theta \in [1, \infty)$. At equilibrium, in the beginning flow is sent over the edge st only, so that a queue begins to build up on edge st until time $\theta = 1$, when the waiting time is $q_{st}(1) = 1$, the shortest network path becomes the complete network and the network inflow rate changes in value to $u_0 = 1/2$. After time $\theta = 1$ flow is only sent over edge st again. It is easy to check that the label functions from $\theta = 1$ (and at least for a while) are $\ell_v(\theta) = 1 + \theta$ and $\ell_t(\theta) = 3/2 + \theta/2$, so that the derivatives (from the right) at time $\theta = 1$ are $\frac{d\ell_v}{d\theta}(1) = 1$ and $\frac{d\ell_t}{d\theta}(1) = 1/2$. Indeed, for the shortest path network at time $\theta = 1$ (the complete network), the corresponding thin flow of value $1/2$ with resetting on $E_1 = \{st\}$ is $\ell'_v = 1$ and $\ell'_t = 1/2$, but without Condition (9), the label ℓ'_v could take any value between $1/2$ and 1.

3.1 Existence of Thin Flows with Resetting

In this section we prove the existence of thin flows with resetting when the underlying graph is acyclic. This result will then be used to conclude the existence of equilibria for flows over time. To prove the existence of thin flows with resetting we show that this is equivalent to the existence of solutions of a nonlinear complementarity problem, which in turn is equivalent to finding an interior solution of a certain variational inequality.

We start with a basic result that will be needed later. Given a subset K of the Euclidean n-dimensional space $\mathbb{R}^n$ and a mapping $\Phi : K \to \mathbb{R}^n$, the variational inequality, denoted $\mathrm{VI}(K, \Phi)$, is to find a vector $x \in K$ such that

$$(y - x)^t \Phi(x) \geq 0, \quad \forall y \in K.$$

The solution set to this problem is denoted $\mathrm{SOL}(K, \Phi)$. When $K = \mathbb{R}^n_+$, then VI admits an equivalent form known as a *nonlinear complementarity problem* . Given a mapping $\Phi : \mathbb{R}^n_+ \to \mathbb{R}^n$, the nonlinear complementarity problem, denoted $\mathrm{NCP}(\Phi)$, is to find a vector $x \in \mathbb{R}^n_+$ satisfying

$$\Phi(x) \geq 0 \quad \text{and} \quad x^t \Phi(x) = 0.$$

The equivalence between $\mathrm{NCP}(\Phi)$ and $\mathrm{VI}(\mathbb{R}^n_+, \Phi)$ is easy (see *e.g.* [4]). Also, we have the following result, whose proof follows from Brouwer's fixed point theorem [4].

Theorem 3. *Let $K \subset \mathbb{R}^n$ be compact convex and let $\Phi : K \to \mathbb{R}^n$ be continuous. Then the set $\mathrm{SOL}(K, \Phi)$ is nonempty and compact.*

The first step to show existence of a thin flow with resetting is to get rid of Condition (9), which is done in the next lemma.

Lemma 1. *Let (G, u, s, t) be an acyclic network, $E_1 \subset E(G)$ and $F \geq 0$. Suppose that there exists an F-flow x' on G and a nonnegative node labeling ℓ' satisfying conditions* (5) *to* (8)*. Then ℓ' can be modified in at most $O(|V(G)|^2)$ operations so that the new ℓ' is a thin flow of value F with resetting on E_1 and x' is an associated static flow.*

Proof. We only need to modify the labeling ℓ' so that (9) holds. Suppose that (9) fails at a certain $w \in V$ so that $\delta^-_G(w) \cap E_1 = \emptyset$ and $\ell'_w < \min_{vw \in \delta^-_G(w)} \ell'_v$. From (7) it follows that for every edge $e = vw \in \delta_G^-(w)$ we must have $x'_e = 0$. Hence, redefining $\ell'_w := \min_{vw \in \delta_G^-(w)} \ell'_v$, the modified labels ℓ' still satisfy (5)-(8) while (9) now holds at an additional node w. Repeating this procedure with all the nodes w that violate (9), we get an ℓ' satisfying (5)-(9) in $O(|V(G)|^2)$. □

The following lemma relates a thin flow with resetting to the solutions of an appropriate nonlinear complementarity problem.

Lemma 2. *Let* (G, u, s, t) *be a network,* $E_1 \subset E(G)$ *and* $F \geq 0$*. Consider the mapping* $\Phi : \mathbb{R}_+^{E(G) \cup V(G)} \to \mathbb{R}^{E(G) \cup V(G)}$ *defined as*

$$\Phi_i(x', \ell') := \begin{cases} \max\{\ell'_v, x'_e/u_e\} - \ell'_w & \text{if } i = e = vw \in E(G) \backslash E_1 \\ x'_e/u_e - \ell'_w & \text{if } i = e = vw \in E_1 \\ \ell'_s - 1 & \text{if } i = s \\ \sum_{e \in \delta^-(v)} x'_e - \sum_{e \in \delta^+(v)} x'_e & \text{if } i = v \in V(G) \setminus \{s, t\} \\ \sum_{e \in \delta^-(t)} x'_e - \sum_{e \in \delta^+(t)} x'_e - F & \text{if } i = t\,. \end{cases} \tag{10}$$

Then, every solution (x', ℓ') *of* $\mathrm{NCP}(\Phi)$ *satisfies conditions* (5) *to* (8) *and induces a feasible static flow of value* F*. Reciprocally, every thin flow of value* F *with resetting on* E_1 *is a solution of* $\mathrm{NCP}(\Phi)$*.*

Proof. Let (x', ℓ') be a solution of $\mathrm{NCP}(\Phi)$. Let us check that conditions (5)-(8), and the flow constraints are verified. For this we study each of the following cases:

(i) $e \in E(G) \setminus E_1$. If $x'_e > 0$ then $\ell'_w = \max\{\ell'_v, x'_e/u_e\}$. Also, if $x'_e = 0$, since $\max\{\ell'_v, x'_e/u_e\} - \ell'_w \geq 0$ then $\ell'_w \leq \max\{\ell'_v, x'_e/u_e\} = \ell'_v$. Thus, conditions (6) and (7) are satisfied.

(ii) $e \in E_1$. If $x'_e > 0$ then $\ell'_w = x'_e/u_e$, while if $x'_e = 0$, since $x'_e/u_e - \ell'_w \geq 0$ and $\ell'_w \geq 0$ then $\ell'_w = x'_e/u_e = 0$. Thus, (8) is satisfied.

(iii) If $\ell'_s > 0$ then $\ell'_s = 1$. The case $\ell'_s = 0$ is not possible because $\ell'_s - 1 \geq 0$. Thus, (5) is satisfied.

(iv) $v \in V(G) \setminus \{s, t\}$. Assume $\ell'_v = 0$ and $\sum_{e \in \delta^-(v)} x'_e - \sum_{e \in \delta^+(v)} x'_e > 0$, then there exists $e \in \delta^-(v)$ such that $x'_e > 0$, which implies (by (7) or (8)) that $\ell'_v > 0$, a contradiction. If $\ell'_v > 0$ then $\sum_{e \in \delta^-(v)} x'_e - \sum_{e \in \delta^+(v)} x'_e = 0$. Thus, $\sum_{e \in \delta^-(v)} x'_e - \sum_{e \in \delta^+(v)} x'_e = 0$ for all $v \in V(G) \setminus \{s, t\}$. Similarly, for t we have $\left(\sum_{e \in \delta^-(t)} x'_e - \sum_{e \in \delta^+(t)} x'_e\right) - F = 0$. We conclude that $(x'_e)_{e \in E(G)}$ is a conservative F-flow.

The reciprocal assertion is verified directly. □

Theorem 4. *Let* (G, u, s, t) *an acyclic network,* $E_1 \subset E(G)$ *and* $F \geq 0$*. Then, there exists a thin flow of value* F *with resetting on* E_1*.*

Proof. Let Φ as in (10). By Lemmas 1 and 2, and since $\mathrm{VI}(\mathbb{R}_+^{E(G) \cup V(G)}, \Phi)$ is equivalent to $\mathrm{NCP}(\Phi)$, we need to prove that $\mathrm{VI}(\mathbb{R}_+^{E(G) \cup V(G)}, \Phi)$ has a solution.

Let δ and ϵ positive numbers such that $\epsilon > \delta$. Define $u_{min} := \min_{e \in E(G)} u_e$, $M := \max\{1, F/u_{min}\}$ and $K \subset \mathbb{R}_+^{E(G)\cup V(G)}$ as the compact convex set

$$K := \Big\{(x', \ell') \in \mathbb{R}_+^{E(G)\cup V(G)} \ : \forall e \in E(G), 0 \leq x'_e \leq (M+\epsilon)u_e, \\ \forall v \in V(G), 0 \leq \ell'_v \leq M + \delta\Big\}.$$

By Theorem 3 there exist a solution $(x', \ell') \in K$ of $\mathrm{VI}(K, \Phi)$. Unfortunately, this solution may not coincide with a solution to $\mathrm{NCP}(\Phi)$ since it can lie in the frontier of K. However, if there is an interior solution in K of $\mathrm{VI}(K, \Phi)$, it is easily proven that such solution also solves $\mathrm{VI}(\mathbb{R}_+^{E(G)\cup V(G)}, \Phi)$. The rest of the proof is technical and is devoted to prove that there is an interior solution of $\mathrm{VI}(K, \Phi)$, i.e., a solution satisfying:

$$\forall e \in E(G), x'_e < (M+\epsilon)u_e \quad \text{and} \quad \forall v \in V(G), \ell'_v < M + \delta. \tag{P}$$

Suppose for a contradiction that there exists $e = vw \in E(G)$ such that $x'_e = (M + \epsilon)u_e$. Define $(y, h) \in K$ such that $y_e := 0$, $y_a := x'_a$ for every $a \in E(G) \setminus \{e\}$ and $h := \ell'$. Because (x', ℓ') is solution of $\mathrm{VI}(K, \Phi)$ we have that

$$-(M+\epsilon)u_e\,(M + \epsilon - \ell'_w) \geq 0.$$

Therefore $(M + \epsilon - \ell'_w) \leq 0$, so that $M + \epsilon \leq \ell'_w \leq M + \delta < M + \epsilon$, a contradiction. Thus, we have that for every $e \in E(G), x'_e < (M+\epsilon)u_e$,which together with the fact that (x', ℓ') solves $\mathrm{VI}(K, \Phi)$, implies the complementarity condition

$$x'_e[\Phi(x', \ell')]_e = 0 \qquad \text{for every } e = vw \in E(G). \tag{CC}$$

Therefore for every $e = vw \in E_1$, $x'_e = \ell'_w u_e$. Indeed, if $x'_e > 0$ this follows directly form (CC). Otherwise, if $x'_e = 0$, this follows by taking an element in K which equals (x', ℓ') in every coordinate except in $e = vw$, where it takes some positive value. The variational inequality then says that $\ell'_w \leq x'_e/u_e = 0$.

Note that $\ell'_s = 1$. Define $b(v) = 0$ for every $v \in V \setminus \{s, t\}$ and $b(t) = -F$. Then, if $(y, h) \in K$ we have,

$$\begin{aligned}
&\left(\begin{pmatrix} y \\ h \end{pmatrix} - \begin{pmatrix} x' \\ \ell' \end{pmatrix}\right)^t \Phi(x', \ell') \\
&= \sum_{e=vw\in E\setminus E_1} (y_e - x'_e)(\max\{\ell'_v, x'_e/u_e\} - \ell'_w) + \sum_{e=vw\in E_1} (y_e - x'_e)(x'_e/u_e - \ell'_w) \\
&\quad + \sum_{v\in V\setminus\{s\}} (h_v - \ell'_v)\Big(\sum_{e\in\delta^-(v)} x'_e - \sum_{e\in\delta^+(v)} x'_e + b(v)\Big) \\
&= \sum_{e=vw\in E\setminus E_1 : x'_e = 0} y_e(\ell'_v - \ell'_w) + \sum_{v\in V\setminus\{s\}} (h_v - \ell'_v)\Big(\sum_{e\in\delta^-(v)} x'_e - \sum_{e\in\delta^+(v)} x'_e + b(v)\Big),
\end{aligned}$$

where the last equality follows from (CC).

Now we show that for every $v \in V(G)$, $\ell'_v < M + \delta$. Since G is acyclic, consider a topological ordering $v_1, v_2, ..., v_{|V(G)|}$ of $V(G)$ and let $w := v_{|V(G)|}$. By contradiction, suppose that $\ell'_w = M + \delta$, and define the minimal $k \in \{1, ..., |V(G)|\}$ such that every node label in $T := \{v_k, ..., v_{|V(G)|}\}$ equals ℓ'_w. Note that $\ell'_s = 1$, then $T \neq V(G)$, then there exists $a = uv \in \delta^-(T)$ such that $\ell'_u < \ell'_v$. Observe that if $a \notin E_1$ then $x'_a > 0$. Indeed, suppose that $x'_a = 0$ and define $(y, h) \in K$ such that $h_v := \ell'_v$ for every $v \in V(G)$, $y_e = x'_e$ if $e \neq a$, and $y_a > 0$. Then, because $\ell'_u < \ell'_w$, we have that

$$\left(\begin{pmatrix} y \\ h \end{pmatrix} - \begin{pmatrix} x' \\ \ell' \end{pmatrix}\right)^t \Phi(x', \ell') = y_a(\ell'_u - \ell'_w) < 0,$$

a contradiction. Thus, $x'_a = \ell'_v u_a$, since if $a \notin E_1$ we have just shown that $x'_a > 0$ so that from (CC) $\ell'_v = \max\{\ell'_u, x'_a/u_a\} = x'_a/u_a$, and if $a \in E_1$ this was shown above.

Define $(y, h) \in K$ such that $h_v := \ell'_v$ for every $v \in V(G) \setminus T$, $h_v := 0$ for every $v \in T$ and $y = x'$. Then, we obtain a contradiction since

$$\begin{aligned}\left(\begin{pmatrix} y \\ h \end{pmatrix} - \begin{pmatrix} x' \\ \ell' \end{pmatrix}\right)^t \Phi(x', \ell') &= -\ell'_w\Big(\sum_{e\in\delta^-(T)} x'_e - \sum_{e\in\delta^+(T)} x'_e + \sum_{v\in T} b(v)\Big) \\ &= -\ell'_w\Big(\sum_{e\in\delta^-(T)} x'_e + \sum_{v\in T} b(v)\Big) = -\ell'_w\Big(\ell'_w \sum_{e\in\delta^-(T)} u_e - F\Big) \\ < -\Big(\ell'_w \sum_{e\in\delta^-(T)} u_e - F\Big) &< -((\max\{1, F/u_{min}\}u_a + \delta u_a) - F) < 0.\end{aligned}$$

Then, $\ell'_v < M + \delta$ for every $v \in T$, which implies the complementarity condition

$$\ell'_v[\Phi(x', \ell')]_v = 0, \qquad \text{for every } v \in T.$$

Therefore, if $\ell'_w > 0$, for every $v \in T$ we have $\sum_{e\in\delta^-(v)} x'_e - \sum_{e\in\delta^+(v)} x'_e + b(v) = 0$. In other words either x' satisfies flow conservation in every node in T or $\ell'_w = 0$ (and also $\ell'_v = 0$ for all $v \in T$), and in the latter case $x'(\delta^-(T)) = 0$.

If $T \cup \{s\} \neq V(G)$, we consider the graph G' obtained by deleting the node set T and the edge set $\cup_{v\in T}\delta^-(v) \cup \delta^+(v)$, and redefine $b(v) := b(v) - x'(\delta^+_G(v) \cap \delta^-(T))$. Thus $\sum_{v\in V(G)} b(v) = -F$, so that we can repeat the argument until $V(G') = \{s\}$. Because in each repetition at least one node is removed, this procedure finishes, concluding that $\ell'_v < M + \delta$ for every $v \in V(G)$. □

3.2 Uniqueness of Thin Flows with Resetting

We now show that thin flows with resetting are unique on acyclic networks. This means that the labeling $(\ell'_v)_{v\in V}$ is unique, but not necessarily the static flow $(x'_e)_{e\in E}$.

Theorem 5. *Let (G, u, s, t) be an acyclic network, $E_1 \subset E(G)$ and $F \geq 0$. Then, there exists a unique thin flow of value F with resetting on E_1.*

Proof. Let ℓ' and h' two thin flows of value F with resetting on E_1, and let x' and y' be corresponding static flows. We define the node set

$$S := \{v \in V(G) \mid \ell'_v \geq h'_v\}.$$

We first show the following property:

$$e = vw \in \delta_G^-(S) \implies x'_e \geq y'_e. \tag{P_1}$$

In order to prove (P_1), given an edge $e = vw \in \delta_G^-(S)$, we consider three cases:

- Case $e \in E_1$: in this case $x'_e = \ell'_w u_e \geq h'_w u_e = y'_e$.
- Case $e \notin E_1$ and $\ell'_v \geq \ell'_w$: since $v \notin S$ we have $h'_v > \ell'_v \geq \ell'_w \geq h'_w$, then $h'_v > h'_w$ and therefore $y'_e = 0$, which implies $x'_e \geq y'_e$.
- Case $e \notin E_1$ and $\ell'_v < \ell'_w$: in this case $x'_e = \ell'_w u_e \geq h'_w u_e \geq y'_e$.

Similarly, we can show the following:

$$e = vw \in \delta_G^+(S) \implies y'_e \geq x'_e. \tag{P_2}$$

Indeed, given an edge $e = vw \in \delta_G^+(S)$, we consider three cases:

- Case 1, $e \in E_1$: Since $w \notin S$ then $h'_w > \ell'_w$. Since $e \in E_1$, we have $x'_e = \ell'_w u_e$ and $y'_e = h'_w u_e$, thus $y'_e > x'_e$.
- Case 2, $e \notin E_1$ and $h'_v \geq h'_w$: Since $w \notin S$ then $h'_w > \ell'_w$. Since $v \in S$, we have $\ell'_v \geq h'_v$. Thus $\ell'_v > \ell'_w$ and therefore $x'_e = 0$, which implies $y'_e \geq x'_e$.
- Case 3, $e \notin E_1$ and $h'_v < h'_w$: Clearly $y'_e = h'_w u_e > \ell'_w u_e \geq x'_e$. Thus, $y'_e > x'_e$.

Consider $x'(\delta_G(S)) := \sum_{e \in \delta_G^+(S)} x'_e - \sum_{e \in \delta_G^-(S)} x'_e$. It follows from (P_1) and (P_2) that $x'(\delta_G(S)) \leq y'(\delta_G(S))$. On the other hand, we know that $x'(\delta_G(S)) = y'(\delta_G(S))$ because both flows are conservative. Then, for every $e \in \delta_G^+(S)$, Case 1 and Case 3 (where $y'_e > x'_e$) cannot hold, and therefore the only possible scenario is Case 2, so that

$$\text{for all } e \in \delta_G^+(S),\ x'_e = 0 \text{ and } e \notin E_1. \tag{P_3}$$

To finish the proof we study two cases, depending whether $F > 0$ or $F = 0$.
Case $F > 0$: From (P_3) we have

$$x'(\delta_G(S)) = \sum_{e \in \delta_G^+(S)} x'_e - \sum_{e \in \delta_G^-(S)} x'_e = - \sum_{e \in \delta_G^-(S)} x'_e. \tag{11}$$

On the other hand, we know that $s \in S$. This implies that if $t \notin S$, then (since $F > 0$) there exists $e \in \delta_G^+(S)$ such that $x'_e > 0$, contradicting P_3. Thus $s, t \in S$, and therefore $x'(\delta_G(S)) = 0$, which together with equation (11) implies that

$$\text{for all } e \in \delta_G^-(S),\ x'_e = 0. \tag{P_4}$$

From properties (P_1) and (P_4) we conclude that, for all $e \in \delta_G^-(S)$, $y'_e = 0$, and therefore, from the conservation constraints of the static flow y', we have that for all $e \in \delta_G^+(S)$, $y'_e = 0$. We have proved that

$$\text{for all } e \in \delta_G^+(S),\ x'_e = y'_e = 0 \text{ and } e \notin E_1. \tag{P_5}$$

Suppose there exists $w \in V(G) \setminus S$, i.e., $\ell'_w < h'_w$. Since G is acyclic, we can choose w such that $\delta_G^-(w) \subset \delta_G^+(S)$, and therefore, from (P_5), every $e \in \delta_G^-(w)$ satisfies

$x'_e = y'_e = 0$ and $e \notin E_1$. It follows from (9) that $\ell'_w = \min_{vw \in \delta^-_G(w)} \ell'_v$ and $h'_w = \min_{vw \in \delta^-_G(w)} h'_v$. However, since any $e = vw \in \delta^-_G(w)$ satisfies $v \in S$, we have

$$\ell'_w = \min_{vw \in \delta^-_G(w)} \ell'_v \geq \min_{vw \in \delta^-_G(w)} h'_v = h'_w,$$

contradicting that $w \in V(G) \setminus S$. We conclude that $S = V(G)$, i.e., for all $v \in V(G)$, $\ell'_v \geq h'_v$. Because the result is symmetric with respect to ℓ' and h', we have that $\ell' = h'$. Case $F = 0$: We have that $e \in \delta^+_G(S)$, $x'_e = y'_e = 0$ and $e \notin E_1$. Repeating the argument of the case $F > 0$ yields the desired result. □

4 Existence and Uniqueness of Equilibria for Flows over Time

4.1 Existence for Piecewise Constant Inflow Rate

Koch and Skutella [9] showed a method to extend an equilibrium in case the network inflow rate function u_0 is constant, say $u_0 = F$. Given a feasible flow over time f which satisfies the equilibrium conditions in $[0, T]$ (i.e., flow is only sent along currently shortest paths), the method to extend this equilibrium is as follows:

1. Define the edge set $E_1 := \{e = vw \mid q_e(\ell_v(T)) > 0\} \subset E(G_T)$ and find a thin flow of value F with resetting on E_1 on the network G_T, say (ℓ', x').
2. Find the largest $\alpha > 0$ such that:

$$\ell_w(T) + \alpha\ell'_w - \ell_v(T) - \alpha\ell'_v \leq \tau_e \qquad \text{for all } e = vw \in E \setminus E(G_T), \tag{12}$$

$$\ell_w(T) + \alpha\ell'_w - \ell_v(T) - \alpha\ell'_v \geq \tau_e \qquad \text{for all } e = vw \in E_1. \tag{13}$$

3. Update the flow over time parameters in this order for all $\theta \in [T, T + \alpha]$

$$\begin{aligned} \ell_v(\theta) &:= \ell_v(T) + (\theta - T)\ell'_v && \text{for all } v \in V, \\ f^+_e(\ell_v(\theta)) &:= x'_e/\ell'_v && \text{for all } e = vw \in E(G_T), \\ f^-_e(\ell_w(\theta)) &:= x'_e/\ell'_w && \text{for all } e = vw \in E(G_T). \end{aligned}$$

From Theorem 4 we know that there exists a thin flow with resetting on Step 1, thus we can always extend an equilibrium. Because the label functions are non-decreasing and bounded in every set of the form $[0, T]$, that extension can be repeated as many times as wanted. Thus, starting from the interval $[0, 0]$, we can iterate this method to find a new $\alpha_i > 0$ and a new extension at every iteration i. Eventually, $\sum_{i=1}^{n} \alpha_i$ can have a limit point: in this case, since the label functions are non-decreasing and bounded in every set of the form $[0, T]$, we can define the equilibrium at point $\sum_{i=1}^{\infty} \alpha_i$ as the limit point of the label functions ℓ, and continue with the extension method. Note that, with Condition (9) at hand, this extension method works even if the inflow rate function is piecewise constant (by Theorem 2). So we have the following existence result.

Theorem 6. *Suppose that u_0 is piecewise constant, i.e, there exists a sequence $(\xi_k)_{k \in \mathbb{N}}$ such that $\xi_0 = 0$, $\sum_k \xi_k = \infty$ and u_0 restricted to the interval $[\xi_k, \xi_{k+1})$ is constant. Then, there exists a flow over time f which is an equilibrium whose network inflow rate is u_0 and whose label functions ℓ are piecewise linear.*

4.2 Uniqueness for Piecewise Constant Inflow Rate

Theorems 2 and 5, imply that there is a unique equilibria for flows over time (in the sense of labels) that can be constructed by the previous extension method. However, other type of bizarrely behaving equilibria might exist. To rule out this possibility it would be enough to show that at every iteration of the extension method $\alpha_i > \varepsilon > 0$.

Theorem 7. *Suppose that* u_0 *is piecewise constant. Then, there exists a unique array of label function* $(\ell_v)_{v \in V}$ *for every equilibrium in the family of flows over time which satisfy that for every* $v \in V$ *and every* $\theta \in \mathbb{R}_+$*, there exists* $\epsilon > 0$ *such that* ℓ_v *restricted to* $[\theta, \theta + \epsilon]$ *is an affine function.*

4.3 Existence for General Inflow Rate

In the previous sections, we showed existence in the case of the network inflow rate function u_0 is piecewise constant. We can show existence of equilibria for much more general inflow rate functions, namely when $u_0 \in L^p(\mathbb{R}_+)$, with $1 < p < \infty$ (i.e., u_0 is measurable and $\int_0^\infty |u_0(t)|^p dt < \infty$, see e.g. [3]). The proof, found in the full version of the paper, is neither algorithmic nor constructive, it is based on functional analysis techniques and on finding solutions to an appropriate variational problem.

Theorem 8. *Let* p *be such that* $1 < p < \infty$*. Let* $u_0 \in L^p(\mathbb{R}_+)$ *be a network inflow rate function of the network* (G, u, τ, s, t)*. Then, there is an equilibrium* s*-*t *flow over time.*

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References

1. Anderson, E.J., Philpott, A.B.: Optimisation of flows in networks over time. In: Kelly, F.P. (ed.) Probability, Statistics and Optimisation, pp. 369–382. Wiley, New York (1994)
2. Bhaskar, U., Fleischer, L., Anshelevich, E.: A stackelberg strategy for routing flow over time. In: ACM-SIAM Symposium on Discrete Algorithms, pp. 192–201 (2011)
3. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2010)
4. Facchinei, F., Pang, J.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. I. Springer, New York (2003)
5. Fleischer, L., Tardos, E.: Efficient continuous-time dynamic network flow algorithms. Operations Research 23(3-5), 71–80 (1998)
6. Ford, L.R., Fulkerson, D.R.: Constructing maximal dynamic flows from static flows. Operations Research 6, 419–433 (1958)
7. Ford, L.R., Fulkerson, D.R.: Flows in Networks. Princeton University Press, Princeton (1962)
8. Gale, D.: Transient flows in networks. Michigan Mathematical Journal 6, 59–63 (1959)
9. Koch, R., Skutella, M.: Nash equilibria and the price of anarchy for flows over time. Theory of Computing Systems (to appear)
10. Macko, M., Larson, K., Steskal, Ľ.: Braess's Paradox for Flows over Time. In: Kontogiannis, S., Koutsoupias, E., Spirakis, P.G. (eds.) Algorithmic Game Theory. LNCS, vol. 6386, pp. 262–275. Springer, Heidelberg (2010)
11. Ran, B., Boyce, D.: Modeling Dynamic Transportation Networks. Springer, Berlin (1996)
12. Skutella, M.: An introduction to network flows over time. In: Cook, W., Lovasz, L., Vygen, J. (eds.) Research Trends in Combinatorial Optimization, pp. 451–482. Springer, Berlin (2009)