# The Competition Complexity of Dynamic Pricing

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We study the competition complexity of dynamic pricing relative to the optimal auction in the fundamental single-item setting. In prophet inequality terminology, we compare the expected reward  $A_m(F)$  achievable by the optimal online policy on m i.i.d. random variables drawn from F to the expected maximum  $M_n(F)$  of n i.i.d. draws from the same distribution. We ask how big does m have to be to ensure that  $(1+\varepsilon)A_m(F) \ge M_n(F)$  for all F.

We resolve this question and exhibit a stark phase transition: When  $\varepsilon = 0$  the competition complexity is unbounded. That is, for any *n* and any *m* there is a distribution *F* such that  $A_m(F) < M_n(F)$ . In contrast, for any  $\varepsilon > 0$ , it is sufficient and necessary to have  $m = \phi(\varepsilon)n$  where  $\phi(\varepsilon) = \Theta(\log \log 1/\varepsilon)$ . Therefore, the competition complexity not only drops from being unbounded to being linear, it is actually linear with a very small constant.

The technical core of our analysis is a loss-less reduction to an infinite dimensional and non-linear optimization problem that we solve optimally. A corollary of this reduction, which may be of independent interest, is a novel proof of the factor  $\approx 0.745$  i.i.d. prophet inequality, which simultaneously establishes matching upper and lower bounds.

# $\texttt{CCS Concepts:} \bullet \textbf{Theory of computation} \rightarrow \textbf{Algorithmic game theory and mechanism design}.$

Additional Key Words and Phrases: posted pricing, prophet inequality

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# **1** INTRODUCTION

An important line of work at the intersection of Economics and Computation concerns the *competition complexity* of auctions [3, 4, 14, 15]. The basic idea is to examine how many bidders need to be added to a simple, suboptimal auction mechanism so that its performance is guaranteed to match that of the optimal but more complicated auction mechanism.

This competition complexity approach originates in a seminal paper of Bulow and Klemperer [4], who asked this question for the revenue achievable by the simple but suboptimal second-price auction and Myerson's optimal auction. They showed that for i.i.d. bidders whose valuations are drawn from a regular distribution F, the second-price auction with n + 1 bidders is guaranteed to achieve at least the expected revenue of the optimal auction with n bidders. They concluded that rather than going for the more complicated auction mechanism, one could simply attract one more buyer to the simpler auction mechanism.

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Subsequent work has extended this basic result to a variety of more complex auction settings [3, 14, 28], and also introduced the idea of approximate competition complexity where instead of shooting for optimality one aims at 99% or 99.9% of optimal [15].

#### 1.1 Our Question

In this work, we initiate the study of the competition complexity of posted pricing. We focus on the fundamental single-item case and compare optimal dynamic pricing versus optimal auction. We study the social welfare case, but all our results translate to revenue (under the usual regularity assumptions).

Since we are focusing on social welfare, the simplest way to state our question is in prophet inequality terminology. Our goal is to compare the expected reward  $A_m(F)$  achievable by the optimal policy found by backward induction on  $m \ge n$  i.i.d. draws from a distribution F, to the expected maximum  $M_n(F)$  of n i.i.d. draws from F. For fixed  $\varepsilon \ge 0$  and fixed n, we want to find the smallest  $m \ge n$  such that for every F we have

$$(1+\varepsilon) \cdot A_m(F) \ge M_n(F).$$

We refer to the functional dependence of *m* on *n* and  $\varepsilon$  as the *competition complexity of dynamic pricing*. We sometimes refer to the case  $\varepsilon = 0$  as *exact* competition complexity, and to the case  $\varepsilon > 0$  as the *approximate* version.

## 1.2 Warm-Up: The Uniform Case

As a warm-up, consider the case where F = U[0, 1] is a uniform distribution over [0, 1], and convince ourselves that in this case  $A_{2m} \ge M_n$  for all n, so the exact competition complexity is linear. We have that  $M_n$  is just the maximum of n i.i.d. draws from a uniform distribution over [0, 1], and therefore  $M_n = n/(n + 1)$ . On the other hand, we can compute  $A_n$  through the usual backward induction. The recursion is  $A_{n+1} = \mathbb{E}(\max\{X, A_n\})$  for  $n \ge 1$  and  $A_1 = \mathbb{E}(X)$  where  $X \sim U[0, 1]$ . In other words,  $A_{n+1} = 1/2 + A_n^2/2$  for  $n \ge 1$  and  $A_1 = 1/2$ .

Solving this recursion explicitly seems difficult. What we use instead, is that it leads to the following formula for the difference between two consecutive terms. Namely,  $A_{n+1}-A_n = (1-A_n)^2/2$  for  $n \ge 1$ . We proceed by induction. It is easy to verify that the claim holds for n = 1. So we assume  $A_{2n} \ge M_n$ , and we want to show  $A_{2n+2} \ge M_{n+1}$ . Note that if  $A_{2n+1} \ge M_{n+1}$  then also  $A_{2n+2} \ge A_{2n+1} \ge M_{n+1}$ , and there we are done, so we consider the case  $A_{2n+1} < M_{n+1}$ . We have

$$A_{2n+2} = A_{2n} + (A_{2n+2} - A_{2n+1}) + (A_{2n+1} - A_{2n})$$
$$= A_{2n} + \frac{1}{2}(1 - A_{2n+1})^2 + \frac{1}{2}(1 - A_{2n})^2.$$

Since the function  $f(x) = x + \frac{1}{2}(1-x)^2$  is increasing in  $\mathbb{R}_+$ , and given that  $A_{2n} \ge M_n$ , we obtain a lower bound that together with  $A_{2n+1} < (n+1)/(n+2)$  yields

$$A_{2n+2} \ge M_n + \frac{1}{2} \left( \left( \frac{1}{n+1} \right)^2 + \left( \frac{1}{n+2} \right)^2 \right).$$

The argument is completed by observing that what we add to  $M_n$  on the right-hand side is at least  $M_{n+1} - M_n = 1/((n+1)(n+2))$ . We conclude that for the uniform distribution it suffices to choose  $m \ge 2n$ . A closer examination of the asymptotic behavior of  $A_m$  and  $M_n$  shows that this analysis is in fact tight. Indeed for large m and n,  $A_m \approx 1 - 2/(m + \log(m) + 1.76799)$  [17, 29] while  $M_n \approx 1 - 1/n$  which roughly shows that we need m = 2n + o(n).



Fig. 1. Plot of  $\phi(\varepsilon)$  as a function of  $\varepsilon$  on the left, and as a function of  $1/\varepsilon$  on the right. Plotting  $\phi(\varepsilon)$  as a function of  $1/\varepsilon$  serves to illustrate the very slow growth of  $\phi(\varepsilon)$  as  $\Theta(\log \log 1/\varepsilon)$  when  $\varepsilon \to 0$ . The dashed blue line in the left plot is at  $\varepsilon = \phi^{-1}(1) \approx 0.342$  which implies the optimal factor  $1/(1 + \phi^{-1}(1)) \approx 0.745$  for the i.i.d. prophet inequality. In the other plot the two blue dashed lines are at  $1/\varepsilon = 100$  and  $1/\varepsilon = 1000$  which correspond to approximation ratios of 99.9% and 99.99%. The value of  $\phi(\varepsilon)$  at these points is the constant required to obtain these approximation ratios.

## 1.3 Our Contribution

The above analysis of the uniform case already rules out a "plus constant" result as in Bulow and Klemperer [4]. It leaves some hope that the exact competition complexity of dynamic pricing may be linear or, if not, then at least polynomial with a small polynomial.

Our first main result shows that this hope is unfounded. Indeed, the exact competition complexity is not only "large," it is in fact unbounded.

**Main Result 1 (exact competition complexity):** Consider  $\varepsilon = 0$  and any *n*. Then, for any  $m \ge n$  there exist a distribution *F* such that  $A_m(F) < M_n(F)$ .

In light of this strong impossibility, a natural question is whether this impossibility persists if we relax our goals and aim for 99% or 99.99% of optimal. It turns out that things change, and quite drastically so. This is formalized by our second main result, which nails down the approximate competition complexity in terms of function  $\phi : \mathbb{R}_+ \to \mathbb{R}_+$  given by

$$\phi(\varepsilon) = \int_0^1 \frac{1}{y(1 - \log(y)) + \varepsilon} \mathrm{d}y.$$

**Main Result 2 (approximate competition complexity):** Consider  $\varepsilon > 0$  and any *n*. Then, we have  $(1 + \varepsilon)A_m(F) \ge M_n(F)$  for every *F* if  $m \ge \phi(\varepsilon)n$ , and for large *n* this is tight.

This establishes a remarkable phase transition: While our first main result showed that the exact competition complexity of dynamic pricing is unbounded, our second main result shows that if we aim for approximate optimality, then the competition complexity not only drops from being unbounded to being linear, it is actually *linear with a very small constant*.

We illustrate this in Figure 1. In the technical part of the paper, we show that the function  $\phi(\epsilon)$  grows as  $\Theta(\log \log 1/\epsilon)$  as  $\epsilon \to 0$ , with very small constants hidden in the big-O notation. For example, to obtain 99% of optimal it is sufficient to have  $m \ge 2.30 \cdot n$ , and to obtain 99.99% of optimal it is sufficient to have  $m \ge 2.53 \cdot n$ .

An interesting implication of our analysis is that it yields the factor 0.745 i.i.d. prophet inequality [8, 25, 27, 30] and its tightness [21] as a special case. Here is how: Rather than fixing  $\varepsilon$  and

finding  $m(n, \varepsilon)$ , we may fix  $m(n, \varepsilon) = n$  and find  $\varepsilon$ . The equality  $m(n, \varepsilon) = \phi(\varepsilon)n$  corresponds to solving  $\phi(\varepsilon) = 1$ . This yields  $\varepsilon = \phi^{-1}(1)$  and corresponds to an approximation guarantee of  $1/(1 + \phi^{-1}(1)) \approx 0.745$ .

# 1.4 Our Techniques

Our argument for the uniform distribution F = U[0, 1] that we presented above relied on a formula for the differences between two consecutive terms  $A_{n+1}$  and  $A_n$ , and at its core compared  $A_{2(n+1)} - A_{2n}$  to  $M_{n+1} - M_n$ . Intuitively, we explored properties of the rate of growth and curvature of the two sequences  $A_1, A_2, \ldots, A_m$  and  $M_1, M_2, \ldots, M_n$ .

Our general argument builds on this intuition. Our first key observation characterizes the sequences  $A_1, A_2, \ldots, A_m$  that can arise. Namely, we show that for any distribution F, the corresponding infinite sequence  $(A_i(F))_{i \in \mathbb{N}}$  satisfies the following three properties. Moreover, for any infinite sequence  $(A_i)_{i \in \mathbb{N}}$  satisfying these properties there is a distribution F that leads to this sequence. The three properties are:

- (1) The sequence  $(A_i)_{i \in \mathbb{N}}$  is non-decreasing,
- (2) The sequence  $(A_{i+1} A_i)_{i \in \mathbb{N}}$  is non-increasing, and
- (3) The sequence  $((A_{i+2} A_{i+1})/(A_{i+1} A_i))_{i \in \mathbb{N}}$  is non-decreasing.

Our second key observation is that given a *fixed* infinite sequence  $(A_i)_{i \in \mathbb{N}}$  with these properties, we can identify the compatible distribution F that maximizes  $M_n$ . This worst-case distribution is a simple piece-wise constant distribution, and allows us to express the largest possible  $M_n$  as a function of the  $(A_i)_{i \in \mathbb{N}}$ . We thus reduce the problem of checking whether for a fixed n and m,  $(1 + \varepsilon)A_m(F) - M_n(F) \ge 0$  for all F, to an infinite dimensional optimization problem that seeks to minimize  $(1 + \varepsilon)A_m(F) - M_n(F)$  over all infinite sequences satisfying properties (1)–(3): The inequality is satisfied by all F if and only if the objective value of this infinite dimensional optimization problem is non-negative. To show our two main results we then proceed to solve this infinite dimensional optimization problem optimally. This reduces the problem to the analysis of a recursion, which can be pointwise bounded by a differential equation which, by a careful analysis, leads to the function  $\phi(\varepsilon)$ .

#### 1.5 Other Gaps and Future Work

An additional set of questions that fits the wider theme of this paper concerns the competition complexity of *static pricing*. Here—unlike in the case of dynamic pricing—there are two questions we could ask. The first comparison is between static pricing  $A'_m$  and the optimal auction  $M_n$ , the other is between static pricing  $A'_m$  and dynamic pricing  $A_n$ .

For the first comparison between  $A'_m$  and  $M_n$  we observe the following. First, since  $A'_m \leq A_m$  for all *m*, our impossibility (Main Result 1) implies that the exact competition complexity of static pricing is unbounded. Moreover, while the approximate competition complexity of static pricing may be linear (similar to our Main Result 2 for dynamic pricing), the dependence on  $\varepsilon$  certainly has to be worse. This follows from considering the uniform case where for *m* and *n* large,  $A'_m \approx 1 - \log(m)/m$ while  $M_n \approx 1 - 1/n$ . This means that to ensure that  $(1 + \varepsilon)A'_m \geq M_n$ , we approximately need that  $(1 + \varepsilon)(1 - \log(m)/m) \geq 1 - 1/n$ . Then, for  $\varepsilon$  small with respect to *n*, say  $\varepsilon = 1/n^2$ , we can approximate by subtracting  $\varepsilon$  from the left hand side. We get  $1 - (1 + \varepsilon) \log(m)/m \geq 1 - 1/n$ , which happens if and only if  $m/\log(m) \geq n(1 + \varepsilon)$ , which for  $\varepsilon$  of this order implies that we need at least m = cn with  $c = \Omega(\log(1/\varepsilon))$ .

For the other comparison between  $A'_m$  and  $A_n$ , we observe that for the exact version we need to have at least  $m = \Omega(n \log(n))$  even for the uniform distribution. This again follows from the asymptotic formulas for  $A'_m \approx 1 - \log(m)/m$  and  $A_n \approx 1 - 2/(n + \log(n) + 1.76799)$ , which show

that roughly what we need is that  $m/\log(m) \ge n/2$  and therefore  $m = \Omega(n \log(n))$ . We leave the full resolution of these gaps, which will shine additional light on the relative power of static and dynamic pricing, to future work.

# 1.6 Further Related Work

Our work examines the relative power of a simple mechanism (dynamic pricing) to that of an optimal mechanism (the optimal auction) and thus fits under the broader umbrella of *simple vs. optimal mechanisms* (e.g., [19, 20]).

At the technical core of our work, we rely on a connection between posted-price mechanisms and prophet inequalities that was pioneered and explored in the last fifteen years [5, 6, 9, 18]. This line of work motivated work on prophet inequalities more generally. Most relevant for us is the work on the i.i.d. single-item prophet inequality [1, 7, 8, 25, 27, 30], but there is also exciting work on combinatorial extensions such as [11, 13, 16, 26]. A closely related line of work has examined the gap between various simple mechanisms including posted-price mechanisms and the optimal mechanism on the same number of bidders [2, 12, 22–24].

# 2 FORMAL STATEMENT OF OUR RESULTS

For our analysis it will be convenient to consider  $\mathbb{N} = \{0, 1, 2, ...\}$ , the natural numbers including zero. We consider distributions F over the non-negative reals with finite expectation. For a distribution F, we let  $M_0(F) = 0$  and for  $n \ge 1$  we let  $M_n(F) = \mathbb{E}(\max\{X_1, X_2, ..., X_n\})$ , where  $X_1, ..., X_n$  is an i.i.d. sample distributed according to F. We denote by  $A_n(F)$  the value of the optimal policy and the sequence  $(A_n(F))_{n \in \mathbb{N}}$  satisfies the following recurrence:  $A_0(F) = 0$ ,  $A_1(F) = \mathbb{E}(X)$  and  $A_{n+1}(F) = \mathbb{E}(\max\{X, A_n(F)\})$ , where X is a random variable distributed according to F. We now formally state our main results.

THEOREM 2.1. For every positive integer n > 1, and every positive integer  $m \ge n$ , there exists a distribution F such that  $A_m(F) < M_n(F)$ .

THEOREM 2.2. Let  $\varepsilon > 0$  and let n be a positive integer. Then, for every  $m \ge \phi(\varepsilon)n = \Theta(\log \log 1/\varepsilon)n$ , and every distribution F we have  $(1 + \varepsilon)A_m(F) \ge M_n(F)$ . Conversely, for any  $\delta > 0$ , there exists a distribution G such that for n sufficiently large and  $m < (\phi(\varepsilon) - \delta)n$ , we have  $(1 + \varepsilon)A_m(G) < M_n(G)$ .

While Theorem 2.1 shows that the exact competition complexity of dynamic pricing is unbounded, Theorem 2.2 shows that the approximate competition complexity not only drops from being unbounded to being linear, it is actually linear with a very small constant (see Figure 1).

# **3 AN EQUIVALENT OPTIMIZATION PROBLEM**

In this section we develop the main building block of our analysis. The key result of this section, Theorem 3.1, shows that the question of whether for a given  $\varepsilon \ge 0$ ,  $n \ge 1$ , and  $m \ge 1$  it holds that  $(1 + \varepsilon)A_m(F) \ge M_n(F)$  for all F reduces to showing whether the following infinite-dimensional, non-linear optimization problem has a non-negative objective.

minimize 
$$(1+\varepsilon)\sum_{i=0}^{m-1}\delta_i - \sum_{i=0}^{\infty}\left(1 - \left(\frac{\delta_{i+1}}{\delta_i}\right)^n\right)\delta_i$$
 (1)

subject to  $\delta_{j+1} \leq \delta_j$  for every integer  $j \geq 0$ , (2)

 $\delta_j^2 \le \delta_{j-1} \delta_{j+1}$  for every integer  $j \ge 1$ , (3)

 $\delta_0 = 1 \text{ and } \delta_j > 0 \text{ for every integer } j \ge 1.$  (4)

THEOREM 3.1. Let  $\varepsilon \ge 0$ , and let n and m be two positive integers. Then, we have  $(1 + \varepsilon)A_m(F) \ge M_n(F)$  for every distribution F if and only if the optimal value of the optimization problem (1)-(4) is non-negative.

We prove this theorem by characterizing the sequences  $(A_j(F))_{j \in \mathbb{N}}$  that can result from distributions *F*, and by relating the value of  $M_n(F)$  to the values of the sequence  $(A_j(F))_{j \in \mathbb{N}}$ . The characterization uncovers the properties of the sequences that can arise. Given a sequence of non-negative real values  $(S_n)_{n \in \mathbb{N}}$ , we denote by  $(\partial S_n)_{n \in \mathbb{N}}$  the sequence such that  $\partial S_n = S_{n+1} - S_n$ for every non-negative integer *n*. Consider the following properties:

- (a) The sequence  $(S_n)_{n \in \mathbb{N}}$  is strictly increasing.
- (b) The sequence  $(\partial S_n)_{n \in \mathbb{N}}$  is non-increasing.
- (c) The sequence  $(\partial S_{n+1}/\partial S_n)_{n \in \mathbb{N}}$  is non-decreasing.

Observe that the properties (b)-(c) imply that the sequence  $(\partial S_{n+1}/\partial S_n)_{n\in\mathbb{N}}$  is not only nondecreasing, but also bounded with  $\partial S_{n+1}/\partial S_n \leq 1$  for every  $n \in \mathbb{N}$ , and therefore it is convergent to a limit value of at most one. In what follows, given a distribution F, let  $\omega_0(F) = \inf\{y \in \mathbb{R} : F(y) > 0\}$ and  $\omega_1(F) = \sup\{y \in \mathbb{R} : F(y) < 1\}$  be the left and right end points of the support of F.

We need a few lemmas to prove Theorem 3.1. We also use the following proposition about the optimal policy. The proof of this proposition can be found in the full version of the article.

**PROPOSITION 3.2.** For every distribution F the following holds:

(i) 
$$A_{n+1}(F) = A_n(F) + \int_{A_n(F)}^{\infty} (1 - F(y)) dy$$
 for every  $n \in \mathbb{N}$ 

- (*ii*)  $A_{n+2}(F) = A_{n+1}(F) + \int_{A_n(F)}^{A_{n+1}(F)} F(y) dy$  for every  $n \in \mathbb{N}$ .
- (*iii*)  $\lim_{n\to\infty} A_n(F) = \omega_1(F)$ .

(iv) If  $\omega_0(F) < \omega_1(F)$  and F has finite expectation, then  $A_n(F) < A_{n+1}(F)$  for every  $n \in \mathbb{N}$ .

An important implication of Proposition 3.2(iv) is that the sequence  $(A_j(F))_{j \in \mathbb{N}}$  is strictly increasing, unless F is a distribution that puts probability 1 on a single value. For these distributions F, however,  $A_m(F) = M_n(F)$  for all  $m, n \ge 1$ , so they trivially satisfy  $(1 + \varepsilon)A_m(F) \ge M_n(F)$ .

In the remainder, we will consider distributions *F* with  $\omega_0(F) < \omega_1(F)$ . We begin by showing that for such distributions *F* the sequence  $(A_j(F))_{j \in \mathbb{N}}$  satisfies properties (a)-(c).

LEMMA 3.3. For every distribution F with  $\omega_0(F) < \omega_1(F)$ , the sequence  $(A_n(F))_{n \in \mathbb{N}}$  satisfies properties (a)-(c).

PROOF. Consider a distribution F with  $\omega_0(F) < \omega_1(F)$  and a non-negative integer n. Observe that property (a) holds directly for the sequence  $(A_n(F))_{n \in \mathbb{N}}$  from Proposition 3.2(iv). By Proposition 3.2(ii), it holds that

$$A_{n+2}(F) - A_{n+1}(F) = \int_{A_n(F)}^{A_{n+1}(F)} F(y) \mathrm{d}y \le A_{n+1}(F) - A_n(F),$$

where the inequality holds since  $F(y) \le 1$  for every  $y \in \mathbb{R}$ . Therefore, property (b) holds. Observe that thanks to Proposition 3.2(ii) again, we have

$$\frac{A_{n+2}(F) - A_{n+1}(F)}{A_{n+1}(F) - A_n(F)} = \frac{1}{A_{n+1}(F) - A_n(F)} \int_{A_n(F)}^{A_{n+1}(F)} F(y) dy,$$

and since F is monotone non-decreasing we therefore have

$$F(A_n(F)) \le \frac{A_{n+2}(F) - A_{n+1}(F)}{A_{n+1}(F) - A_n(F)} \le F(A_{n+1}(F)),$$

from where we conclude that that  $(A_n(F))_{n \in \mathbb{N}}$  satisfies property (c).

Next we show that for the type for distributions we are interested in, it is possible to prove an upper bound on the value of  $M_n(F)$  in terms of the values of the sequence  $(A_j(F))_{j \in \mathbb{N}}$ .

LEMMA 3.4. For every distribution F with  $\omega_0(F) < \omega_1(F)$ , we have

$$M_n(F) \leq \sum_{j=0}^{\infty} \left( 1 - \left( \frac{\partial A_{j+1}(F)}{\partial A_j(F)} \right)^n \right) \partial A_j(F).$$

PROOF. Consider the concave function  $\varphi : \mathbb{R} \to \mathbb{R}$  given by  $\varphi(x) = 1 - x^n$ , and for every non-negative integer j let  $\mu_j(y) = 1/\partial A_j(F)$  for every  $y \in [A_j(F), A_{j+1}(F))$  and zero elsewhere. In particular,  $\mu_j$  is a probability density function over  $[A_j(F), A_{j+1}(F))$ . Then, by Jensen's inequality, we have

$$\begin{split} \frac{1}{\partial A_j(F)} \int_{A_j(F)}^{A_{j+1}(F)} (1 - F(y)^n) \mathrm{d}y &= \int_{\mathbb{R}} \varphi(F(y)) \mu_j(y) \mathrm{d}y \\ &\leq \varphi \left( \int_{\mathbb{R}} F(y) \mu_j(y) \mathrm{d}y \right) \\ &= 1 - \left( \frac{1}{\partial A_j(F)} \int_{A_j(F)}^{A_{j+1}(F)} F(y) \mathrm{d}y \right)^n = 1 - \left( \frac{\partial A_{j+1}(F)}{\partial A_j(F)} \right)^n, \end{split}$$

where the last equality holds by Proposition 3.2(ii). In particular, for every non-negative integer j we have

$$\int_{A_j(F)}^{A_{j+1}(F)} (1 - F(y)^n) \mathrm{d}y \le \left(1 - \left(\frac{\partial A_{j+1}(F)}{\partial A_j(F)}\right)^n\right) \partial A_j(F).$$
(5)

Therefore, we have

$$M_n(F) = \int_0^\infty (1 - F(y)^n) dy = \sum_{j=0}^\infty \int_{A_j(F)}^{A_{j+1}(F)} (1 - F(y)^n) dy$$
$$\leq \sum_{j=0}^\infty \left( 1 - \left(\frac{\partial A_{j+1}(F)}{\partial A_j(F)}\right)^n \right) \partial A_j(F),$$

where the second equality holds by Proposition 3.2(iii) and the inequality comes from (5).  $\Box$ 

Our final ingredient is a reverse to the previous two lemmas. It shows that for any sequence satisfying properties (a)-(c) we can construct a distribution G for which  $(A_j(G))_{j \in \mathbb{N}}$  matches the values of the sequence and  $M_n(G)$  matches the upper bound on  $M_n(G)$  in terms of the values of the sequence.

LEMMA 3.5. For every  $(B_n)_{n \in \mathbb{N}}$  with  $B_0 = 0$ , and satisfying (a)-(c), there exists a distribution G such that  $A_n(G) = B_n$  for every non-negative integer n. Furthermore, we have

$$M_n(G) = \sum_{j=0}^{\infty} \left( 1 - \left( \frac{\partial B_{j+1}}{\partial B_j} \right)^n \right) \partial B_j.$$

PROOF. We construct explicitly the distribution *G* satisfying the statement of the lemma. Recall that since  $(B_n)_{n \in \mathbb{N}}$  satisfies properties (b)-(c) the sequence  $(\partial B_{n+1}/\partial B_n)_{n \in \mathbb{N}}$  converges to a value  $\rho \in (0, 1]$ . We prove the following claim.

CLAIM 1. Suppose that  $\rho < 1$ . Then, there exists a value  $\mathcal{B} > 0$  such that  $\lim_{n\to\infty} B_n = \mathcal{B}$ .

Since the sequence  $(B_n)_{n \in \mathbb{N}}$  satisfies property (c), we have that  $\partial B_n \leq \rho \partial B_{n-1}$ , and therefore  $\partial B_n \leq \rho^n \partial B_0 = \rho^n B_1$  for every  $n \in \mathbb{N}$ . On the other hand, we have

$$B_n = \sum_{j=0}^{n-1} (B_{j+1} - B_j) = \sum_{j=0}^{n-1} \partial B_j \le B_1 \sum_{j=0}^{n-1} \rho^n \le \frac{B_1}{1 - \rho},$$

which implies that the sequence  $(B_n)_{n \in \mathbb{N}}$  is upper bounded. Since by property (a) the sequence  $(B_n)_{n \in \mathbb{N}}$  is strictly increasing, we conclude that  $(B_n)_{n \in \mathbb{N}}$  is a convergent sequence and we call  $\mathcal{B}$  the value of this limit.

We now construct the distribution *G* satisfying the conditions of the statement. Consider *G* :  $\mathbb{R} \to \mathbb{R}$  defined as follows: G(x) = 0 for every  $x \in (-\infty, 0)$ , for every non-negative integer *j* and every  $x \in [B_j, B_{j+1})$  we have  $G(x) = \partial B_{j+1}/\partial B_j$ , and let G(x) = 1 for every  $x \ge \lim_{n\to\infty} B_n$ . Since the sequence  $(B_n)_{n\in\mathbb{N}}$  satisfies property (a), the function *G* is well defined for every non-negative integer *n*. Furthermore, since the sequence  $(B_n)_{n\in\mathbb{N}}$  satisfies (c), we have that *G* is non-decreasing, and property (b) implies that  $G(x) \le 1$  for every  $x \in \mathbb{R}_+$ . If  $\rho = 1$  then  $\lim_{x\to\infty} G(x) = 1$ . Otherwise, if  $\rho < 1$ , by Claim 1 there exists a value  $\mathcal{B} > 0$  such that  $\lim_{n\to\infty} B_n = \mathcal{B}$ , and therefore G(x) = 1for every  $x \ge \mathcal{B}$ . Therefore, we conclude that *G* is a distribution.

In what follows we show that  $A_n(G) = B_n$  for every non-negative integer *n*. We proceed by induction. By construction we have  $A_0(G) = 0 = B_0$ . Suppose that  $B_i = A_i(G)$  for every  $i \in \{0, 1, ..., n\}$ . By Proposition 3.2, for every positive integer *n* it holds that

$$\int_{A_{n-1}(G)}^{A_n(G)} G(y) \mathrm{d}y = \int_{A_n(G)}^{\infty} (1 - G(y)) \mathrm{d}y = A_{n+1}(G) - A_n(G),$$

and therefore the inductive step implies that

$$\int_{B_{n-1}}^{B_n} G(y) \mathrm{d}y = A_{n+1}(G) - B_n.$$
(6)

On the other hand, by construction of G it holds that

$$\int_{B_{n-1}}^{B_n} G(y) \mathrm{d}y = \frac{B_{n+1} - B_n}{B_n - B_{n-1}} \cdot (B_n - B_{n-1}) = B_{n+1} - B_n = \partial B_n,$$

and therefore together with (6) we conclude that  $A_{n+1}(G) = B_{n+1}$ . Finally, we have

$$M_n(G) = \int_0^\infty (1 - G(y)^n) dy = \sum_{j=0}^\infty \int_{A_j(G)}^{A_{j+1}(G)} (1 - G(y)^n) dy = \sum_{j=0}^\infty \left( 1 - \left(\frac{\partial B_{j+1}}{\partial B_j}\right)^n \right) \partial B_j,$$

where the second equality holds since  $\lim_{i \to \infty} A_j(G) = \omega_1(G)$ , by Proposition 3.2(iii).

We are now ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. We start by showing that if for some  $\varepsilon \ge 0$ ,  $n \ge 1$ , and  $m \ge 1$  there exists a distribution F such that  $(1 + \varepsilon)A_m(F) < M_n(F)$  then the objective value of the optimization problem (1)-(4) must be negative. Note that for this distribution F it must hold that  $\omega_0(F) < \omega_1(F)$  because otherwise  $A_m(F) = M_n(F)$ , and so we must have  $A_{j+1}(F) > A_j(F)$  for all  $j \in \mathbb{N}$  by Proposition 3.2(iv).

We construct a solution  $(\delta_j)_{j \in \mathbb{N}}$  for the optimization problem as follows. For every non-negative integer j, let  $\delta_j(F) = \partial A_j(F)/\partial A_0(F)$ . We begin by showing that the sequence  $(\delta_j)_{j \in \mathbb{N}}$  satisfies (2)-(4). By construction we have  $\delta_0(F) = \partial A_0(F)/\partial A_0(F) = 1$ , that is, (4) holds. By Lemma 3.3, the sequence  $(A_j(F))_{j \in \mathbb{N}}$  satisfies properties (a)-(c). In particular, the sequence  $(\partial A_j(F))_{j \in \mathbb{N}}$  is non-increasing and therefore  $\delta_{j+1}(F) \leq \delta_j(F)$  for every integer  $j \geq 0$ , that is, (2) is satisfied. The

sequence  $(\partial A_{j+1}(F)/\partial A_j(F))_{j \in \mathbb{N}}$  is non-decreasing, and therefore  $\delta_{j+1}(F)/\delta_j(F) \ge \delta_j(F)/\delta_{j-1}(F)$ for every integer  $j \ge 1$ , that is,  $\delta_j(F)^2 \le \delta_{j-1}(F)\delta_{j+1}(F)$ , and therefore (3) is satisfied. Finally, observe that

$$0 > \frac{1}{\partial A_0(F)} \Big( (1+\varepsilon)A_m(F) - M_n(F) \Big) = (1+\varepsilon) \sum_{i=0}^{m-1} \delta_i(F) - \frac{M_n(F)}{\partial A_0(F)}$$
  
$$\geq (1+\varepsilon) \sum_{i=0}^{m-1} \delta_i(F) - \sum_{j=0}^{\infty} \left( 1 - \left(\frac{\partial A_{j+1}(F)}{\partial A_j(F)}\right)^n \right) \frac{\partial A_j(F)}{\partial A_0(F)}$$
  
$$= (1+\varepsilon) \sum_{i=0}^{m-1} \delta_i(F) - \sum_{j=0}^{\infty} \left( 1 - \left(\frac{\delta_{j+1}(F)}{\delta_j(F)}\right)^n \right) \delta_j(F),$$

where the first inequality holds by assumption and the second inequality comes from Lemma 3.4. So in particular, the last expression of the above chain which coincides with the objective in (1) must be negative.

Conversely, suppose that the value of the optimization problem (1)-(4) is negative. That is, there exists a sequence  $(\delta_i^{\star})_{j \in \mathbb{N}}$  satisfying (2)-(4) such that

$$(1+\varepsilon)\sum_{i=0}^{m-1}\delta_i^{\star} - \sum_{i=0}^{\infty} \left(1 - \left(\frac{\delta_{i+1}^{\star}}{\delta_i^{\star}}\right)^n\right)\delta_i^{\star} < 0.$$

$$\tag{7}$$

Consider the sequence  $(B_j)_{j \in \mathbb{N}}$  defined as follows:  $B_0 = 0$  and  $B_j = \sum_{i=0}^{j-1} \delta_i^*$  for every  $j \ge 1$ . In particular, we have

$$B_{j+1} = \sum_{i=0}^{j} \delta_i^{\star} > \sum_{i=0}^{j-1} \delta_i^{\star} = B_j$$

for every integer  $j \ge 1$ , and therefore the sequence  $(B_j)_{j \in \mathbb{N}}$  satisfies (a). Since the sequence  $(\delta_j^*)_{j \in \mathbb{N}}$  satisfies (2)-(3), by construction it holds directly that  $(B_j)_{j \in \mathbb{N}}$  satisfies (b)-(c), and therefore by Lemma 3.5 there exists a distribution G such that  $A_j(G) = B_j$  for every non-negative integer j, and we have

$$(1+\varepsilon)A_m(G) = (1+\varepsilon)B_m$$
  
=  $(1+\varepsilon)\sum_{i=0}^{m-1}\delta_i^*$   
 $<\sum_{i=0}^{\infty}\left(1-\left(\frac{\delta_{i+1}^*}{\delta_i^*}\right)^n\right)\delta_i^* = \sum_{i=0}^{\infty}\left(1-\left(\frac{\partial B_{i+1}}{\partial B_i}\right)^n\right)\partial B_i = M_n(G)$ 

where the last equality also holds by Lemma 3.3. This finishes the proof of the theorem.

#### 4 EXACT COMPETITION COMPLEXITY: PROOF OF THEOREM 2.1

We show next how to use Theorem 3.1 to prove the impossibility result in Theorem 2.1 about the exact competition complexity.

PROOF OF THEOREM 2.1. Letting  $\varepsilon = 0$  in Theorem 3.1, it suffices to show that the value of the optimization problem (1)-(4) is strictly negative. Consider the sequence  $(b_i)_{i \in \mathbb{N}}$  defined as follows:  $b_0 = 1$ , and  $b_1 \in (0, 1)$  to be specified later. For every  $i \in \{1, ..., m-1\}$  let

$$b_{i+1} = b_i \left(\frac{n}{n-1}\right)^{\frac{1}{n}} \left(\frac{b_i}{b_{i-1}}\right)^{\frac{n-1}{n}},$$
(8)

and for every  $i \ge m$  let  $b_{i+1} = b_i^2/b_{i-1}$ . We first show that  $(b_i)_{i\in\mathbb{N}}$  is feasible for the optimization problem (1)-(4). By construction the sequence satisfies (4). We start with the monotonicity property (2). Consider the function  $h(x) = (n/(n-1))^{1/n} x^{(n-1)/n}$  and let  $h^{(i)}$  be the function obtained from the composition of h with itself i times. From (8), we get  $b_{i+1}/b_i = h^{(i)}(b_1/b_0) = h^{(i)}(b_1)$  for every  $i \in \{0, 1, ..., m-1\}$ . Observe that h(x) is monotone increasing and continuous on  $x \in [0, 1]$ , with h(0) = 0, and therefore  $h^{(i)}$  is also monotone increasing, continuous and  $h^{(i)}(0) = 0$ , for every  $i \in \{0, 1, ..., m-1\}$ . Since we also know  $b_{j+1}/b_j = b_m/b_{m-1}$  for every  $j \ge m$ , it suffices to prove  $b_i/b_{i-1} \le 1$  for every  $i \in \{1, ..., m\}$  in order to show that the sequence  $(b_i)_{i\in\mathbb{N}}$  satisfies (2). To this end, we make any choice of  $b_1$  in a way that  $\max_{i \in \{0, 1, ..., m-1\}} h^{(i)}(b_1) \le 1$ . This implies that property (2) is satisfied.

CLAIM 2. For every 
$$x \in [0, 1]$$
 we have  $\left(\frac{n}{n-1}\right)^{\frac{1}{n}} x^{\frac{n-1}{n}} \ge x$ .

To prove the claim, consider the function  $g : \mathbb{R} \to \mathbb{R}$  given by  $g(x) = \left(\frac{n}{n-1}\right)^{\frac{1}{n}} x^{\frac{n-1}{n}} - x$ . This function is concave in the interval [0, 1] and therefore the minimum is attained in either zero or one. Since g(0) = 0 and  $g(1) = (n/(n-1))^{1/n} - 1 > 0$ , we conclude that  $g(x) \ge 0$  for every  $x \in [0, 1]$ . In particular, for every  $i \in \{1, \ldots, m-1\}$  we have

$$\frac{b_{i+1}}{b_i} = \left(\frac{n}{n-1}\right)^{\frac{1}{n}} \left(\frac{b_i}{b_{i-1}}\right)^{\frac{n-1}{n}} = g\left(\frac{b_i}{b_{i-1}}\right) + \frac{b_i}{b_{i-1}} \ge \frac{b_i}{b_{i-1}},$$

where we used the fact that  $0 \le b_i/b_{i-1} \le 1$  by the monotonicity property (2). Since  $b_{i+1}/b_i = b_m/b_{m-1} \le 1$  for every  $i \ge m$ , we conclude that (3) is also satisfied, and therefore the sequence  $(b_i)_{i\in\mathbb{N}}$  is feasible for the optimization problem (1)-(4). We now show that the objective value of the sequence  $(b_i)_{i\in\mathbb{N}}$  is strictly negative. We first observe that the objective value is equal to

$$\sum_{i=0}^{m-1} b_i - \sum_{i=0}^{m-1} \left( 1 - \left(\frac{b_{i+1}}{b_i}\right)^n \right) b_i - \sum_{i=m}^{\infty} \left( 1 - \left(\frac{b_{i+1}}{b_i}\right)^n \right) b_i$$
$$= \sum_{i=0}^{m-1} \left(\frac{b_{i+1}}{b_i}\right)^n b_i - \sum_{i=m}^{\infty} \left( 1 - \left(\frac{b_{i+1}}{b_i}\right)^n \right) b_i$$

By construction of the sequence we have

$$\sum_{i=0}^{m-1} b_i \left(\frac{b_{i+1}}{b_i}\right)^n = b_1^n + \frac{n}{n-1} \sum_{i=1}^{m-1} b_i \left(\frac{b_i}{b_{i-1}}\right)^{n-1}$$
$$= b_1^n + \frac{n}{n-1} \sum_{i=1}^{m-1} b_{i-1} \left(\frac{b_i}{b_{i-1}}\right)^n = b_1^n + \frac{n}{n-1} \sum_{i=0}^{m-2} b_i \left(\frac{b_{i+1}}{b_i}\right)^n,$$

and therefore

$$b_1^n = \sum_{i=0}^{m-1} b_i \left(\frac{b_{i+1}}{b_i}\right)^n - \frac{n}{n-1} \sum_{i=0}^{m-2} b_i \left(\frac{b_{i+1}}{b_i}\right)^n$$
$$= b_{m-1} \left(\frac{b_m}{b_{m-1}}\right)^n + \sum_{i=0}^{m-2} b_i \left(\frac{b_{i+1}}{b_i}\right)^n - \frac{n}{n-1} \sum_{i=0}^{m-2} b_i \left(\frac{b_{i+1}}{b_i}\right)^n$$
$$= b_{m-1} \left(\frac{b_m}{b_{m-1}}\right)^n - \frac{1}{n-1} \sum_{i=0}^{m-2} b_i \left(\frac{b_{i+1}}{b_i}\right)^n.$$

By rearranging terms we conclude that

$$\sum_{i=0}^{m-1} b_i \left(\frac{b_{i+1}}{b_i}\right)^n = b_{m-1} \left(\frac{b_m}{b_{m-1}}\right)^n + \sum_{i=0}^{m-2} b_i \left(\frac{b_{i+1}}{b_i}\right)^n$$
$$= b_{m-1} \left(\frac{b_m}{b_{m-1}}\right)^n + (n-1) \left(b_{m-1} \left(\frac{b_m}{b_{m-1}}\right)^n - b_1^n\right)$$
$$= nb_{m-1} \left(\frac{b_m}{b_{m-1}}\right)^n - (n-1)b_1^n.$$

Let  $\gamma = b_m/b_{m-1}$ . We have  $\gamma < 1$ ,  $b_m = \gamma b_{m-1}$  and inductively  $b_{m+i} = \gamma^{i+1}b_{m-1}$  for every non-negative *i*. Therefore, overall, the objective value of the sequence is equal to

$$\begin{split} &\sum_{i=0}^{m-1} \left(\frac{b_{i+1}}{b_i}\right)^n b_i - \sum_{i=m}^{\infty} \left(1 - \left(\frac{b_{i+1}}{b_i}\right)^n\right) b_i \\ &= nb_{m-1}\gamma^n - (n-1)b_1^n - (1-\gamma^n)\sum_{i=0}^{\infty}\gamma^{i+1}b_{m-1} \\ &= nb_{m-1}\gamma^n - (n-1)b_1^n - \frac{(1-\gamma^n)\gamma}{1-\gamma}b_{m-1} \\ &= nb_{m-1}\gamma^n - (n-1)b_1^n - b_{m-1}\sum_{i=1}^n\gamma^i \\ &= -(n-1)b_1^n - b_{m-1}\left(\sum_{i=1}^n\gamma^i - n\gamma^n\right) < 0, \end{split}$$

which concludes the proof.

#### 5 APPROXIMATE COMPETITION COMPLEXITY: PROOF OF THEOREM 2.2

In this section we show how to use Theorem 3.1 to derive Theorem 2.2 about the approximate competition complexity. More specifically, in what follows we show how to optimally solve the optimization problem (1)-(4). Given  $\varepsilon > 0$  and positive integer  $n \ge 2$ , let  $(\rho_{\varepsilon,j})_{j \in \mathbb{N}}$  be the sequence defined by the following recurrence:

$$\rho_{\varepsilon,1} = 1$$
, and  
 $(n-1)\rho_{\varepsilon,j-1}^n - \varepsilon = n\rho_{\varepsilon,j}^{n-1}$  for every  $j \ge 2$  such that  $(n-1)\rho_{\varepsilon,j-1}^n - \varepsilon > 0$ .

For fixed  $\varepsilon$  and n we say that  $\rho_{\varepsilon,j}$  is well defined if  $(n-1)\rho_{\varepsilon,j-1}^n - \varepsilon > 0$ . Since  $\rho_{\varepsilon,j}$  is decreasing in j, we know that if  $\rho_{\varepsilon,m}$  is well defined, then so is  $\rho_{\varepsilon,j}$  for  $j \le m$ .

As a first step, we will show that the optimal value of (1)-(4) can be obtained in terms of the sequence  $(\rho_{\varepsilon,j})_{j\in\mathbb{N}}$ . To prove this result we require a few propositions. Consider the function  $\Gamma_{n,m}^{\varepsilon}: \mathbb{R}^{m-1}_+ \to \mathbb{R}$  defined by

$$\Gamma_{n,m}^{\varepsilon}(x) = \varepsilon + x_1^n + \sum_{i=1}^{m-2} x_i \left(\varepsilon + \left(\frac{x_{i+1}}{x_i}\right)^n\right) - x_{m-1}(n-1-\varepsilon).$$

PROPOSITION 5.1. Let  $\varepsilon > 0$ , and let  $n \ge 2$  and  $m \ge 3$  be two positive integers such that  $\rho_{\varepsilon,m}$  is well defined. Then,  $\Gamma_{n,m}^{\varepsilon}$  is convex over  $\mathbb{R}^{m-1}_+$  and it has a unique minimizer Y in this region, given by

$$Y_1 = \rho_{\varepsilon,m} \quad and \quad Y_j = \prod_{k=0}^{j-1} \rho_{\varepsilon,m-k} \quad for \ every \ j \in \{2,\ldots,m-1\}.$$
(9)

Furthermore,  $\Gamma_{n,m}^{\varepsilon}(Y) = \varepsilon - (n-1)\rho_{\varepsilon,m}^{n}$ .

PROOF. We begin by proving (strict) convexity of  $\Gamma_{n,m}^{\varepsilon}$ . We proceed by induction on *m*. Observe first that when m = 3, we have that  $\Gamma_{n,3}^{\varepsilon}(x_1, x_2) = \varepsilon + x_1^n + p(x_1, x_2) - x_2(n - 1 - \varepsilon)$ , where  $p(y, z) = y(\varepsilon + (z/y)^n)$ . The Hessian of *p* is

$$\nabla^2 p(y,z) = n(n-1)z^{n-2}y^{1-n} \begin{pmatrix} z^2/y^2 & -z/y \\ -z/y & 1 \end{pmatrix}$$

and this is a positive semidefinite matrix for every  $(y, z) \in \mathbb{R}^2_+$ , since one eigenvalue is equal to zero, and the other is  $n(n-1)z^{n-2}y^{1-n}((z/y)^2+1) > 0$ . In particular, p is convex over  $\mathbb{R}^2_+$ . Since the function  $\varepsilon + x_1^n - x_2(n-1-\varepsilon)$  is also convex over  $\mathbb{R}^2_+$ , we conclude that  $\Gamma_{n,3}^{\varepsilon}$  is convex over  $\mathbb{R}^2_+$ . Now consider an integer value m > 3, and observe that

$$\Gamma_{n,m+1}^{\varepsilon}(x_1,\ldots,x_m) = p(x_{m-1},x_m) - (x_m - x_{m-1})(n-1-\varepsilon) + \Gamma_{n,m}^{\varepsilon}(x_1,\ldots,x_{m-1}),$$

and therefore the convexity follows by the inductive step, that is,  $\Gamma_{n,m}^{\varepsilon}$  convex over  $\mathbb{R}^{m-1}_+$ , together with p convex over  $\mathbb{R}^2_+$ . Every minimizer y of  $\Gamma_{n,m}^{\varepsilon}$  over  $\mathbb{R}^{m-1}_+$  is a solution to the system given by  $\nabla \Gamma_{n,m}^{\varepsilon} = 0$ , that is,

$$(n-1)\left(\frac{y_2}{y_1}\right)^n - \varepsilon = ny_1^{n-1},\tag{10}$$

$$(n-1)\left(\frac{y_{i+1}}{y_i}\right)^n - \varepsilon = n\left(\frac{y_i}{y_{i-1}}\right)^{n-1} \quad \text{for every } i \in \{2, \dots, m-2\},\tag{11}$$

$$n-1-\varepsilon = n \left(\frac{y_{m-1}}{y_{m-2}}\right)^{n-1}, \text{ and } y \in \mathbb{R}^{m-1}_+.$$
 (12)

The above system has a unique solution and therefore this proves the first part.

To finish the proof, we show that Y defined in (9) is strictly positive, satisfies the system (10)-(12), and  $\Gamma_{n,m}^{\varepsilon}(Y) = \varepsilon - (n-1)\rho_{\varepsilon,m}^{n}$ . Since  $\rho_{\varepsilon,j}$  is well-defined for all  $j \leq m$ , we have  $\rho_{\varepsilon,j} = ((n-1)\rho_{\varepsilon,j-1}^{n} - \varepsilon)^{1/(n-1)} > 0$ . This implies that  $Y \in \mathbb{R}^{m-1}_+$ . Next observe that  $Y_2 = \rho_{\varepsilon,m}\rho_{\varepsilon,m-1}$  and therefore  $Y_2/Y_1 = \rho_{\varepsilon,m-1}$ . Then, we have

$$(n-1)(Y_2/Y_1)^n - \varepsilon = (n-1)\rho_{\varepsilon,m-1}^n - \varepsilon = n\rho_{\varepsilon,m}^{n-1} = nY_1^{n-1},$$

and therefore (10) is satisfied. Similarly, for every  $j \in \{2, ..., m-2\}$ , we have  $Y_j/Y_{j-1} = \rho_{m-j+1}$  and  $Y_{j+1}/Y_j = \rho_{m-j}$ . Then, we have

$$(n-1)(Y_{j+1}/Y_j)^n - \varepsilon = (n-1)\rho_{\varepsilon,m-j}^n - \varepsilon = n\rho_{\varepsilon,m-j+1}^{n-1} = n(Y_j/Y_{j-1})^{n-1},$$

and therefore (11) is satisfied. Finally, since  $Y_{m-1}/Y_{m-2} = \rho_{\varepsilon,2}$ , we have

$$n-1-\varepsilon = (n-1)\rho_{\varepsilon,1}^{n} - \varepsilon = n\rho_{\varepsilon,2}^{n-1} = n(Y_{m-1}/Y_{m-2})^{n-1},$$

and therefore (12) is satisfied. We now evaluate  $\Gamma_{n,m}^{\varepsilon}(Y)$ . The vector Y satisfies (10)-(12) and therefore

$$(n-1)\sum_{i=1}^{m-2} Y_i \left(\frac{Y_{i+1}}{Y_i}\right)^n + (n-1)Y_{m-1} - \varepsilon \sum_{i=1}^{m-1} Y_i = nY_1^n + n\sum_{i=2}^{m-1} Y_i \left(\frac{Y_i}{Y_{i-1}}\right)^{n-1}$$
$$= nY_1^n + n\sum_{i=2}^{m-1} Y_{i-1} \left(\frac{Y_i}{Y_{i-1}}\right)^n$$
$$= nY_1^n + n\sum_{i=1}^{m-2} Y_i \left(\frac{Y_{i+1}}{Y_i}\right)^n.$$

By subtracting the first term of the left hand side we get

$$\sum_{i=1}^{m-2} Y_i \left(\frac{Y_{i+1}}{Y_i}\right)^n = (n-1)Y_{m-1} - \varepsilon \sum_{i=1}^{m-1} Y_i - nY_1^n,$$

and by rearranging terms we obtain that

$$\sum_{i=1}^{m-2} Y_i \left( \varepsilon + \left( \frac{Y_{i+1}}{Y_i} \right)^n \right) = (n-1-\varepsilon) Y_{m-1} - n Y_1^n.$$

Therefore, the minimum of  $\Gamma_{n,m}^{\varepsilon}$  over  $\mathbb{R}^{m-1}_+$  is equal to

$$\varepsilon + Y_1^n + (n-1-\varepsilon)Y_{m-1} - nY_1^n - (n-1-\varepsilon)Y_{m-1} = \varepsilon - (n-1)Y_1^n.$$

The proof follows since we have  $Y_1 = \rho_{\varepsilon,m}$ .

**PROPOSITION 5.2.** Let  $\varepsilon > 0$ , let  $n \ge 2$  and  $m \ge 3$  be two positive integers such that  $\rho_{\varepsilon,m}$  is well defined, and let Y be as defined in (9). Then, the following holds:

- (a) For every  $j \in \{1, ..., m-1\}$  we have that  $Y_{j+1} \leq Y_j$ . (b) For every  $j \in \{2, ..., m-1\}$  we have that  $Y_j^2 \leq Y_{j-1}Y_{j+1}$ .

**PROOF.** Observe that for every  $k \in \{1, \ldots, m-1\}$ , we have  $Y_{m-k+1}/Y_{m-k} = \rho_{\varepsilon,k}$ . For k = 1 we have  $Y_m/Y_{m-1} = \rho_{\varepsilon} = 1$ . From the definition of the recurrence, we have  $(n-1)\rho_{\varepsilon,k-1}^n \ge (n-1)\rho_{\varepsilon,k-1}^n - \varepsilon = 0$  $n\rho_{\varepsilon,k}^{n-1}$  for every  $k \in \{2, \ldots, m-1\}$ . By induction, if  $\rho_{\varepsilon,k-1} \leq 1$ , we have  $\rho_{\varepsilon,k}^{n-1} \leq (n-1)/n$  and therefore  $\rho_{\varepsilon,k} \leq 1$ . This concludes part (a). Since for every  $j \in \{1, \dots, m-1\}$  we have  $Y_{j+1}/Y_j = \rho_{\varepsilon,m-j}$ , to prove part (b) it suffices to show  $\rho_{\varepsilon,k+1} \leq \rho_{\varepsilon,k}$  for every  $k \in \{1, \ldots, m-2\}$ . From the construction of the recurrence, for every  $k \in \{1, ..., m - 2\}$  it holds that

$$\rho_{\varepsilon,k} \ge \left(\frac{n}{n-1}\right)^{\frac{1}{n}} \rho_{\varepsilon,k+1}^{(n-1)/n}$$

By (a) we have  $\rho_{\varepsilon,k+1} \in [0, 1]$ , which together with Claim 2 implies that

$$\left(\frac{n}{n-1}\right)^{\frac{1}{n}}\rho_{\varepsilon,k+1}^{(n-1)/n} \ge \rho_{\varepsilon,k+1}$$

Therefore we conclude that  $\rho_{\varepsilon,k+1} \leq \rho_{\varepsilon,k}$ . This proves part (b).

PROPOSITION 5.3. For every sequence  $(\delta_j)_{j \in \mathbb{N}}$  satisfying (2)-(4) for which  $\delta_m / \delta_{m-1} < 1$ , there exists a sequence  $(\beta_i)_{i \in \mathbb{N}}$  satisfying (2)-(4), and such that the following holds:

(a) For every  $j \in \{0, 1, ..., m-1\}$  we have  $\delta_j = \beta_j$ , and  $\beta_m / \beta_{m-1} < 1$ .

(b) 
$$\sum_{i=m-1}^{\infty} \left(1 - \left(\frac{\delta_{i+1}}{\delta_i}\right)^n\right) \delta_i \leq \beta_{m-1} \sum_{i=0}^{n-1} \left(\frac{\beta_m}{\beta_{m-1}}\right)^i.$$

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PROOF. Suppose we are given  $(\delta_j)_{j \in \mathbb{N}}$  satisfying (2)-(4) for which  $\delta_m/\delta_{m-1} < 1$ . We claim that then there exists a sequence  $(\beta_j)_{j \in \mathbb{N}}$  satisfying (2)-(4) such that (a) holds and furthermore (i)  $\beta_j \ge \delta_j$  for all  $j \ge m$  and (ii)  $\beta_j/\beta_{j-1} = \beta_m/\beta_{m-1}$  for all  $j \ge m$ .

If  $(\delta_j)_{j \in \mathbb{N}}$  does not already satisfy these properties, then it must be because of (ii). In particular, there must be a smallest index  $j \ge m$  such that  $\delta_{j+1}/\delta_j > \delta_m/\delta_{m-1}$ . We next describe a procedure that maintains all properties, but extends (ii) so that it holds for one more index. Applying this procedure iteratively, we obtain  $(\beta_j)_{j \in \mathbb{N}}$ .

Given  $(\delta_j)_{j \in \mathbb{N}}$  satisfying (2)-(4), let  $k(\delta) \ge m$  be the first value j such that  $\delta_{j+1}/\delta_j > \delta_m/\delta_{m-1}$ . In particular, we have  $\delta_j/\delta_{j-1} = \delta_m/\delta_{m-1}$  for every  $j \in \{m, \ldots, k(\delta)\}$ . Consider the sequence  $(D_j)_{j \in \mathbb{N}}$  defined as follows:  $D_j = \delta_j$  for every  $j \in \{0, 1, \ldots, m-1\}$ ,

$$D_m = \delta_{m-1} \left(\frac{\delta_{k(\delta)+1}}{\delta_{m-1}}\right)^{\frac{1}{k(\delta)-m+2}}$$

 $D_j = D_m (D_m / \delta_{m-1})^{j-m}$  for every  $j \in \{m+1, \dots, k(\delta)\}$ , and  $D_j = \delta_j$  for every  $j \ge k(\delta) + 1$ . Observe that from the construction it holds directly that  $D_{j+1}/D_j = D_m / \delta_{m-1}$  for every  $j \in \{m, \dots, k(\delta) - 1\}$ . Furthermore, we have

$$\frac{\delta_{k(\delta)+1}}{D_{k(\delta)}} = D_m \left(\frac{D_m}{\delta_{m-1}}\right)^{k(\delta)-m+1} \cdot \frac{1}{D_m} \left(\frac{\delta_{m-1}}{D_m}\right)^{k(\delta)-m} = \frac{D_m}{\delta_{m-1}},$$

and therefore, we have  $D_{j+1}/D_j = D_m/D_{m-1}$  for every  $j \in \{m - 1, ..., k(\delta)\}$ . By construction, the sequence  $(D_j)_{j \in \mathbb{N}}$  satisfies (2)-(4) and  $D_m/D_{m-1} < 1$ . We show next that  $D_j \ge \delta_j$  for every  $j \in \{m, ..., k(\delta)\}$ . Since  $\delta_{j+1}/\delta_j \ge \delta_m/\delta_{m-1}$  for every  $j \in \{m, ..., k(\delta)\}$ , we have

$$\left(\frac{\delta_m}{\delta_{m-1}}\right)^{k(\delta)-m+1} \leq \prod_{j=m}^{k(\delta)} \frac{\delta_{j+1}}{\delta_j} = \frac{\delta_{k(\delta)+1}}{\delta_m},$$

which implies that  $\delta_m \leq \delta_{m-1}(\delta_{k(\delta)+1}/\delta_{m-1})^{\frac{1}{k(\delta)-m+2}} = D_m$ . For  $j \in \{m+1,\ldots,k(\delta)\}$  we proceed by induction:

$$D_j = D_{j-1} \frac{D_m}{\delta_{m-1}} \ge \delta_{j-1} \frac{D_m}{\delta_{m-1}} \ge \delta_{j-1} \frac{\delta_m}{\delta_{m-1}} = \delta_j \cdot \frac{\delta_{j-1}}{\delta_j} \cdot \frac{\delta_m}{\delta_{m-1}} = \delta_j,$$

where the first equality holds by construction of the sequence, the first inequality holds by the inductive hypothesis, the second inequality holds since  $D_m \ge \delta_m$ , and the last equality follows since  $\delta_j/\delta_{j-1} = \delta_m/\delta_{m-1}$  for every  $j \in \{m, \ldots, k(\delta)\}$ .

In the remainder we will prove part (b) using the existence of a sequence  $(\beta_j)_{j \in \mathbb{N}}$  for which (a) holds as well as (i) and (ii). To this end we need the following definition and claim. For every sequence  $(\delta_j)_{j \in \mathbb{N}}$  let

$$\mathcal{R}(\delta) = \sum_{i=m-1}^{\infty} \left( 1 - \left( \frac{\delta_{i+1}}{\delta_i} \right)^n \right) \delta_i.$$

CLAIM 3.  $\mathcal{R}$  is non-decreasing in  $\delta_i$  for every  $i \geq m$ .

Before proving Claim 3, we show how together with the properties of the sequence  $(\beta_j)_{j \in \mathbb{N}}$  it implies property (b). Namely,

$$\begin{split} \sum_{i=m-1}^{\infty} \left(1 - \left(\frac{\delta_{i+1}}{\delta_i}\right)^n\right) \delta_i &\leq \sum_{i=m-1}^{\infty} \left(1 - \left(\frac{\beta_{i+1}}{\beta_i}\right)^n\right) \beta_i \\ &= \left(1 - \left(\frac{\beta_m}{\beta_{m-1}}\right)^n\right) \sum_{i=m-1}^{\infty} \beta_i \\ &= \beta_{m-1} \left(1 - \left(\frac{\beta_m}{\beta_{m-1}}\right)^n\right) \sum_{i=0}^{\infty} \left(\frac{\beta_m}{\beta_{m-1}}\right)^i = \beta_{m-1} \sum_{i=0}^{n-1} \left(\frac{\beta_m}{\beta_{m-1}}\right)^i, \end{split}$$

where the inequality holds by Claim 3 and (i), the first equality holds by (ii), and the second equality holds because (ii) implies  $\beta_i = \beta_{m-1} (\beta_m / \beta_{m-1})^{i-m+1}$  for  $i \ge m-1$ . It remains to prove Claim 3. Consider the function  $\varphi : \mathbb{R}^2_+ \to \mathbb{R}_+$  such that  $\varphi(x, y) = (1 - (y/x)^n)x$ . In particular, the derivative of  $\mathcal{R}$  with respect to  $\delta_i$ , with  $i \ge m$ , is equal to

$$\begin{split} \frac{\partial \varphi}{\partial y}(\delta_{i-1},\delta_i) &+ \frac{\partial \varphi}{\partial x}(\delta_i,\delta_{i+1}) = -n\left(\frac{\delta_i}{\delta_{i-1}}\right)^{n-1} + 1 + (n-1)\left(\frac{\delta_{i+1}}{\delta_i}\right)^n \\ &= n\left(\frac{\delta_{i+1}}{\delta_i}\right)^{n-1}\left(\frac{1}{n}\left(\frac{\delta_i}{\delta_{i+1}}\right)^{n-1} + \left(1 - \frac{1}{n}\right)\frac{\delta_{i+1}}{\delta_i} - \left(\frac{\delta_i^2}{\delta_{i+1}\delta_{i-1}}\right)^{n-1}\right) \\ &\geq n\left(\frac{\delta_{i+1}}{\delta_i}\right)^{n-1}\left(1 - \left(\frac{\delta_i^2}{\delta_{i+1}\delta_{i-1}}\right)^{n-1}\right) \geq 0, \end{split}$$

The first inequality holds because for any  $p \in [0, 1]$  we have that  $(1 - 1/n)p + 1/(np^{n-1}) \ge 1$ , and  $\delta_{i-1} \le \delta_i \le \delta_{i+1}$  for every  $i \ge m$ , and the second holds since  $(\delta_j)_{j \in \mathbb{N}}$  satisfies constraint (3). This concludes the proof of the claim.

The following lemma relates the optimal value of the optimization problem (1)-(4) with the sequence  $(\rho_{\varepsilon,j})_{j \in \mathbb{N}}$ .

LEMMA 5.4. Let  $\varepsilon > 0$ , and let  $n \ge 2$  and  $m \ge 3$  be two positive integers such that  $\rho_{\varepsilon,m}$  is well defined. Then, the value of the optimization problem (1)-(4) is equal to  $\varepsilon - (n-1)\rho_{\varepsilon,m}^n$ .

PROOF. Consider  $Y \in \mathbb{R}^{m-1}_+$  as defined in (9). For every  $\alpha \in (0, 1)$ , consider the sequence  $(\mathcal{Y}_j(\alpha))_{j\in\mathbb{N}}$  defined as follows:  $\mathcal{Y}_0(\alpha) = 1$ ,  $\mathcal{Y}_j(\alpha) = Y_j$  for every  $j \in \{1, \ldots, m-1\}$  and  $\mathcal{Y}_j(\alpha) = \alpha^{m-j+1}Y_{m-1}$  for every  $j \ge m$ . Thanks to Proposition 5.1 and Proposition 5.2, for every  $\alpha \in (0, 1)$  the sequence  $(\mathcal{Y}_j(\alpha))_{j\in\mathbb{N}}$  satisfies (2)-(4). Furthermore, the objective value (1) of the sequence is equal

to

$$(1+\varepsilon) \sum_{i=0}^{m-1} \mathcal{Y}_i(\alpha) - \sum_{i=0}^{\infty} \left( 1 - \left(\frac{\mathcal{Y}_{i+1}(\alpha)}{\mathcal{Y}_i(\alpha)}\right)^n \right) \mathcal{Y}_i(\alpha)$$

$$= \varepsilon + Y_1^n + (1+\varepsilon)Y_{m-1} + \sum_{i=1}^{m-2} \left( \varepsilon + \left(\frac{Y_{i+1}}{Y_i}\right)^n \right) Y_i - \sum_{i=m-1}^{\infty} \left( 1 - \left(\frac{\mathcal{Y}_{i+1}(\alpha)}{\mathcal{Y}_i(\alpha)}\right)^n \right) \mathcal{Y}_i(\alpha)$$

$$= \varepsilon + Y_1^n + (1+\varepsilon)Y_{m-1} + \sum_{i=1}^{m-2} \left( \varepsilon + \left(\frac{Y_{i+1}}{Y_i}\right)^n \right) Y_i - (1-\alpha^n) \sum_{i=0}^{\infty} \mathcal{Y}_{m+i-1}(\alpha)$$

$$= \varepsilon + Y_1^n + \sum_{i=1}^{m-2} \left( \varepsilon + \left(\frac{Y_{i+1}}{Y_i}\right)^n \right) Y_i - Y_{m-1} \left( \left( (1-\alpha^n) \sum_{i=0}^{\infty} \alpha^i \right) - 1 - \varepsilon \right)$$

$$= \Gamma_{n,m}^{\varepsilon}(Y) + Y_{m-1} \left( n - (1-\alpha^n) \sum_{i=0}^{\infty} \alpha^i \right)$$

where the last equality holds by Proposition 5.1. In particular, the feasibility of  $(\mathcal{Y}_j(\alpha))_{j \in \mathbb{N}}$  for every  $\alpha \in (0, 1)$  implies that the value of the optimization problem (1)-(4) is upper bounded by

$$\varepsilon - (n-1)\rho_{\varepsilon,m}^{n} + Y_{m-1} \inf_{\alpha \in (0,1)} \left\{ n - (1-\alpha^{n}) \sum_{i=0}^{\infty} \alpha^{i} \right\} = \varepsilon - (n-1)\rho_{\varepsilon,m}^{n}.$$
(13)

Let  $(\delta_j)_{j \in \mathbb{N}}$  be any sequence satisfying (2)-(4). We denote by  $\mathcal{V}(\delta)$  the objective value (1), which by rearranging terms, is equal to

$$\mathcal{V}(\delta) = \Gamma_{n,m}^{\varepsilon}(\delta_1, \dots, \delta_{m-1}) + n\delta_{m-1} - \sum_{i=m-1}^{\infty} \left(1 - \left(\frac{\delta_{i+1}}{\delta_i}\right)^n\right)\delta_i.$$

Now either  $\delta_{i+1}/\delta_i = 1$  for all  $i \ge m-1$  in which case  $\mathcal{V}(\delta) = \Gamma_{n,m}^{\varepsilon}(\delta_1, \dots, \delta_{m-1}) + n\delta_{m-1} \ge \min_{x \in \mathbb{R}^{n-1}_+} \Gamma_{n,m}^{\varepsilon}(x) = \varepsilon - (n-1)\rho_{\varepsilon,m}^n$ , where the last inequality holds by Proposition 5.1. Otherwise, by Proposition 5.3, there exists a sequence  $(\beta_j)_{j \in \mathbb{N}}$  satisfying (2)-(4) for which the following holds:

$$\begin{aligned} \mathcal{V}(\delta) &= \Gamma_{n,m}^{\varepsilon}(\delta_{1},\ldots,\delta_{m-1}) + n\delta_{m-1} - \sum_{i=m-1}^{\infty} \left(1 - \left(\frac{\delta_{i+1}}{\delta_{i}}\right)^{n}\right) \delta_{i} \\ &= \Gamma_{n,m}^{\varepsilon}(\beta_{1},\ldots,\beta_{m-1}) + n\beta_{m-1} - \sum_{i=m-1}^{\infty} \left(1 - \left(\frac{\delta_{i+1}}{\delta_{i}}\right)^{n}\right) \delta_{i} \\ &\geq \Gamma_{n,m}^{\varepsilon}(\beta_{1},\ldots,\beta_{m-1}) + n\beta_{m-1} - \beta_{m-1} \sum_{i=0}^{n-1} \left(\frac{\beta_{m}}{\beta_{m-1}}\right)^{i} \\ &\geq \min_{x \in \mathbb{R}^{m-1}_{+}} \Gamma_{n,m}^{\varepsilon}(x) + \beta_{m-1} \left(n - \sum_{i=0}^{n-1} \left(\frac{\beta_{m}}{\beta_{m-1}}\right)^{i}\right) \\ &\geq \varepsilon - (n-1)\rho_{\varepsilon,m}^{n} + \beta_{m-1} \left(n - \sum_{i=0}^{n-1} \left(\frac{\beta_{m}}{\beta_{m-1}}\right)^{i}\right), \end{aligned}$$

where the second equality holds by property (a) in Proposition 5.3, the first inequality holds by property (b) in Proposition 5.3, and the last inequality again holds by Proposition 5.1. Observe

that for every  $(\beta_j)_{j \in \mathbb{N}}$ , the last term of the above inequality can be lower bounded by zero, and therefore, we get that  $\mathcal{V}(\delta) \geq \varepsilon - (n-1)\rho_{\varepsilon,m}^n$  also in this case. This, together with the upper bound in (13), concludes the proof of the lemma.

As a second step, we study the recurrence  $(\rho_{\varepsilon,j})_{j \in \mathbb{N}}$  to find the point in which it becomes nonpositive. More specifically, by Theorem 3.1 and Lemma 5.4, our aim is to find the greatest index m for which  $\rho_{\varepsilon,m}$  is well defined, or equivalently the unique m for which  $(n-1)\rho_{\varepsilon,m}^n - \varepsilon \leq 0$ . To understand this problem we consider a differential equation that will serve as an upper bound to our recurrence relation. Recall the definition of  $\phi(\varepsilon) = \int_0^1 1/(y(1 - \log(y)) + \varepsilon) dy$ . Given a value  $\varepsilon > 0$ , consider the following ordinary differential equation:

$$y'(t) = y(t)(\log(y(t)) - 1) - \varepsilon \text{ for every } t \in (0, \phi(\varepsilon)),$$
(14)

$$y(0) = 1.$$
 (15)

We define  $y(\phi(\varepsilon)) = \lim_{t \uparrow \phi(\varepsilon)} y(t)$  as the continuous extension of y in  $\phi(\varepsilon)$ .

The following lemma, whose proof appears in the full version, summarizes our results for the differential equation and  $\phi(\varepsilon)$ .

LEMMA 5.5. For every  $\varepsilon > 0$ , the differential equation (14)-(15) has a unique solution  $y_{\varepsilon}$ . Furthermore, the following holds:

- (a) For every  $t \in [0, \phi(\varepsilon))$  we have  $y'_{\varepsilon}(t) < 0$ . In particular,  $y_{\varepsilon}$  is decreasing and invertible on  $[0, \phi(\varepsilon))$  and  $y_{\varepsilon}(\phi(\varepsilon)) = 0$ .
- (b) For every integer  $n \ge 2$ , and every  $j \in \mathbb{N}$  for which  $\rho_{\varepsilon,j}$  is well-defined, we have

$$\frac{n-1}{n}\rho_{\varepsilon,j}^n - \frac{\varepsilon}{n} \le y_{\varepsilon}\left(\frac{j}{n}\right).$$

- (c) For every  $\delta \in (0, \phi(\varepsilon))$ , there exists  $n_0$  such that for every  $n \ge n_0$  we have  $(n-1)\rho_{\varepsilon,k}^n \varepsilon > 0$ , where  $k = \lfloor (\phi(\varepsilon) - \delta)n \rfloor$ .
- (d) We have  $\phi(\varepsilon) = \Theta(\log \log 1/\varepsilon)$ .

We are now ready to prove Theorem 2.2.

PROOF OF THEOREM 2.2. Fix  $\varepsilon > 0$  and consider the non-trivial case where  $n \ge 2$ . We begin with the first part of the theorem. By Lemma 5.4 it suffices to find the largest index j for which  $\rho_{\varepsilon,j}$  is well defined. Suppose for a contradiction that for some  $m \ge \phi(\varepsilon)n$ ,  $\rho_{\varepsilon,m}$  is well defined but  $(n-1)\rho_{\varepsilon,m}^n - \varepsilon > 0$ . Define  $\varepsilon' > 0$  such that  $m/n = \phi(\varepsilon')$ . Note that such an  $\varepsilon'$  exists and  $\varepsilon' \le \varepsilon$  because  $\phi$  is monotone and continuous. Moreover, a simple inductive argument shows that  $\rho_{\varepsilon',j} \ge \rho_{\varepsilon,j}$  for all j, and that the former is well defined whenever the latter is well defined. Using this,

$$\frac{n-1}{n}\rho_{\varepsilon,m}^n - \frac{\varepsilon}{n} \leq \frac{n-1}{n}\rho_{\varepsilon',m}^n - \frac{\varepsilon'}{n} \leq y_{\varepsilon'}\left(\phi(\varepsilon')\right) = 0,$$

where the second inequality holds by Lemma 5.5(b) and the final equality holds by Lemma 5.5(a). This yields a contradiction.

By Lemma 5.5 (d) we have  $\phi(\varepsilon) = \Theta(\log \log 1/\varepsilon)$ . The second part of the theorem holds by Lemma 5.4 and Lemma 5.5(c). This finishes the proof of the theorem.

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