A sufficient condition for self-clique graphs

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Abstract

The clique graph K(G) of a graph G is the intersection graph of the cliques of G. If $G \cong K(G)$ then G is a self-clique graph. We describe a sufficient condition for a graph to be self-clique.

Key words: clique graph, clique-Helly graph, self-clique graph, 2-self-clique graph.

1 Introduction

Denote by G a connected, finite graph, with vertex set $V(G)$ and edge set $E(G)$. Represent by $\delta(G)$ the minimum degree among the vertices of G. Also, $g(G)$ denotes the *girth* of G, that is, the length of the shortest cycle of G, if any. The *k-th power* of G is the graph G^k, whose vertices are those of G, two

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of them being adjacent whenever their distance in G is at most k. A *complete* of G is a subset of vertices inducing a complete subgraph of G. A *clique* is a maximal complete. Represent by $\mathcal{C}(G)$ the collection of cliques of G. The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. Say that G is *self-clique* when G is isomorphic to $K(G)$ and denote $G \cong K(G)$. More generally, G is $2-self-clique$ when $G \cong K(K(G))$. Clearly, if G is self-clique then it is 2-self-clique.

In this paper we prove the following sufficient condition for a graph G to be self-clique. If $\delta(G) \geq 2$ and $g(G) \geq 6k+1$ then G^{2k} is a self-clique graph, $k \geq 1$.

Clique graphs have been studied for some time. However, not much is known about self-clique graphs. Balakrishnan and Paulraja [1] described examples of graphs of this class. Escalante [4] proved that a clique-Helly graph G is 2-self-clique if and only if there are no distinct vertices $v, w \in V(G)$, such that the closed neighbourhood of v contains that of w. As a special case, [4] concludes that if G is triangle free and $\delta(G) \geq 2$ then it is always 2-self-clique and, in addition, it is self-clique if and only if G consists of a simple cycle. The latter has been extended by Chia [2], by characterizing the self-clique graphs whose cliques have all size 2, except precisely for one clique. The problem of characterizing self-clique graphs remains open, even restricted to clique-Helly graphs. Finally, self-clique graphs have also been studied by Escalante and Toft [5].

2 The Theorem

In this section, we prove the mentioned sufficient condition for self-clique graphs. The following notation is used. Let G be a graph and $v, w \in V(G)$. Denote by $P_{v,w}$ a path between v and w in G. Also, represent by $P^e_{v,w}$ the edges of $P_{v,w}$. Let $T(v)$ represent a *shortest path tree* of G. That is, $T(v)$ is a tree with root v, containing exactly the vertices of G and such that the path in $T(v)$ between v and any vertex w is a shortest $P_{v,w}$ path in G. The *level* of a vertex w in $T(v)$ is the length of the path $P_{v,w}$ in $T(v)$. Denote by $T_k(v)$ the rooted tree obtained from $T(v)$, by removing the vertices of it, which lie at levels $> k$. Denote by $D_k(v)$ the subset of vertices of G, at distance $\leq k$ from v. Clearly, $D_k(v) = V(T_k(v))$. In general, say that a rooted tree is *uniform (relative to its leaves)* when all its leaves belong to a same level.

Theorem 1 *Let G be a graph, where $\delta(G) \geq 2$ and $g(G) \geq 6k+1$, $k \geq 1$. Then G^{2k} is a self-clique graph.*

PROOF. Let G be a graph satisfying the hypothesis, for some $k \geq 1$. The first part of the proof consists of showing that $\mathcal{C}(G^{2k}) = \dot{\bigcup}_{v \in V(G)} D_k(v)$. With this purpose, examine $D_j(v) \subseteq V(G)$, for an integer j. Denote by $G_j(v)$ and $G_j^{2k}(v)$ the subgraphs induced by $D_j(v)$ in G and G^{2k}, respectively. We describe below $G_j(v)$ and $G_j^{2k}(v)$.
Clearly, $T_j(v)$ is a spanning tree of $G_j(v)$, because all vertices of $T_j(v)$ lie at distance $\leq j$ from v. We study the possible alternatives for the existence of edges $e \in E(G_j(v)) \setminus E(T_j(v))$. Let $e = (u, w)$. We know that u and w must belong either to the same level or to adjacent levels in $T_j(v)$, because $T_j(v)$ is a shortest path tree. Let ℓ be the largest between the levels of u and v. Adding e to $T_j(v))$ would produce a cycle of length $\leq 2\ell + 1$. Since $g(G) \geq 6k + 1$, we conclude that $\ell \geq 3k$. Consequently, no such edges e may exist at levels $< 3k$. Therefore, $G_j(v)$ is a tree isomorphic to $T_j(v)$, for $j < 3k$. Moreover, since $\delta(G) \geq 2$ it follows that v has at least two children in $T(v)$. In addition, $\delta(G) \geq 2$ and $g(G) \geq 6k + 1$ imply that $T_j(v)$, and consequently $G_j(v)$, are uniform trees, for $j \leq 3k$.
Next, examine G_j^{2k}. Clearly, G_j^{2k} is a complete graph for $j \leq k$, since the distance, in G, between two vertices of $D_j(v)$ is $\leq 2j$. Consequently, $D_k(v)$ is a complete of G^{2k}. Moreover, we prove that $D_k(v)$ must be a clique of G^{2k}. Suppose it is not. Then there exists some vertex $w \in V(G) \setminus D_k(v)$, such that w is adjacent in G^{2k} to all vertices of $D_k(v)$. Let ℓ be the level of w in $T(v)$. Clearly, $\ell \leq 2k$, otherwise v and w are not adjacent in G^{2k}. Also, $\ell > k$, otherwise $w \in D_k(v)$. Choose a vertex $u \in V(G)$ at the level k of $T(v)$ and not belonging to the path between v and w in $T(v)$. We know that this choice is possible. Clearly, $u \in D_k(v)$, implying that $(u, w) \in E(G^{2k})$. Let us study the alternatives for a shortest path $P_{u,w}$ in G. If $P_{u,w}$ contains v then $P_{u,w} = P_{u,v} \cup P_{v,w}$. In this case, the length of $P_{u,w}$ is $\ell + k > 2k$, which contradicts $u, w \in E(G^{2k})$. If P does not contain v then P contains solely vertices located at levels $\geq k$ in $T(v)$. Since $G_j(v) \cong T_j(v)$ for $j < 3k$, it follows that the length of $P_{u,w}$ is $\geq 5k - \ell + 1 > 2k$. The latter contradicts, again, $u, w \in E(G^{2k})$. Consequently, no such w may exist, which assures $\mathcal{C}(G^{2k}) \supseteq \bigcup_v D_k(v)$.
In the sequel, we prove that $\mathcal{C}(G^{2k}) \subseteq \cup D_k(v)$. Let M be a clique of G^{2k}. We know that for each pair $v, w \in M$, there exists a path $P_{v,w}$ of length $\leq 2k$ in G. Denote by M^* the set of vertices contained in all such paths $P_{v,w}$, for all $v, w \in M$. Clearly, $M^* \supseteq M$ and we prove that $M^* \subseteq M$. Let $v, w \in M$ and $u \in P_{v,w}$. We have to show that $u \in M$. Choose $z \in M$ and consider $P_{v,z}$. Since $P_{v,w}$ and $P_{v,z}$ are both of length $\leq 2k$ and $g(G) \geq 6k+1$, we know that the subgraph induced in G by $P_{v,w} \cup P_{v,z}$ is a tree. Furthermore, the symmetric difference $P_{v,w}^e \oplus P_{v,z}^e$ defines a path between z and w in G. If $u \in P_{v,z}$ then G contains a path between u and z, of length $\leq 2k$, meaning that $u \in M$, as required. Otherwise, suppose $u \notin P_{v,z}$. Then $u \in P_{v,w} \oplus P_{v,z}$. We show that $P_{v,w} \oplus P_{v,z}$ has length $\leq 2k$. To start, we know that $P_{v,w} \oplus P_{v,z}$ has length $\leq 4k$. Since $z, w \in M$, there exists a path $P_{w,z}$ of length $\leq 2k$. If $P_{v,w} \oplus P_{v,z} \neq P_{w,z}$ then $(P_{v,w} \oplus P_{v,z}) \cup P_{w,z}$ defines a cycle of length $\leq 6k$, contradicting the girth of G. Consequently, $P_{v,w} \oplus P_{v,z} = P_{v,z}$. That is, $P_{u,z} \subseteq P_{w,z}$, implying

that $u \in M$. Consequently, $M^* = M$. We are now ready to examine the subgraph G', induced in G by M. Since $M^* = M$, it follows that G' contains a $P_{v,w}$ path of length $\leq 2k$, for each pair $v, w \in M$. That is, the diameter of G' is $\leq 2k$. We know that any graph H containing some cycle satisfies $g(H) \leq 2diam(H) + 1$ [3]. That is, if G' contains a cycle then $g(G') \leq 4k + 1$, a contradiction. Consequently, G' is a forest. Moreover, a path connects every pair of vertices of G'. The latter implies that G' is a tree. Let v denote a center of G'. Then G' contains a path $P_{v,w}$ of length $\leq k$, for each $w \in M$. Because $P_{v,w}$ is the unique path of length $\leq k$ in G, it follows that $M \subseteq D_k(v))$. Furthermore, if $w \in D_k(v)$ and $z \in M$, the union of the paths $P_{v,w}$ and $P_{v,z}$ in $T_k(v)$ corresponds to a $P_{w,z}$ path of length $\leq 2k$. Consequently, $w \in M$, implying that $M = D_k(v)$. Therefore $\mathcal{C}(G^{2k}) \subseteq \bigcup D_k(v)$.
The conclusion is that $\mathcal{C}(G^{2k}) = \bigcup D_k(v)$. Even more, the union is disjoint. To show the latter assertion, let $v, w \in V(G)$ and $v \neq w$. We need to show that $D_k(v) \neq D_k(w)$. If $w \notin D_k(v)$ then $D_k(v) \neq D_k(w)$, trivially. Suppose $w \in D_k(v)$ and choose a vertex u, located at the level $k+1$ in $T(v)$ and such that the path $P_{v,u}$ in $T(v)$ contains w. Such a choice is possible, otherwise G contains a cycle of length $\leq k$. Then $u \notin D_k(v)$. However, $u \in D_k(w)$, because the distance between w and u in $P_{v,u}$ is $\leq k$. Consequently, $D_k(v) \neq D_k(w)$, meaning that $\bigcup_{v \in V(G)} D_k(v)$ is a disjoint union.
The last step of the proof is to verify that $G^{2k} \cong K(G^{2k})$. Define the bijection assigning each vertex $v \in V(G^{2k})$ to the clique $D_k(v)$ of G^{2k}. Suppose $(v, w) \in E(G^{2k})$. Let P be a shortest $P_{v,w}$ path in G and denote by u a center of $P_{v,w}$. Then v and w are at distance $\leq k$ from u. That is, $u \in D_k(v) \cap D_k(w)$, meaning that $D_k(v)$ and $D_k(w)$ correspond to adjacent vertices in $K(G^{2k})$. Conversely, when $(v, w) \notin E(G^{2k})$ then v and w are at distance $> 2k$ in G. Consequently, G can not contain a vertex at distance $\leq k$ from both v and w. Therefore $D_k(v) \cap D_k(w) = \emptyset$, implying that $D_k(v)$ and $D_k(w)$ are not adjacent in $K(G^{2k})$.

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