Robust Capacity Expansion of Network Flows*

Fernando Ordóñez† and Jiamin Zhao‡

November 2006

Abstract

We consider the problem of expanding arc capacities in a network subject to demand and travel time uncertainty. We propose a robust optimization approach to obtain capacity expansion solutions that are insensitive to this uncertainty. Our results show that, under reasonable assumptions for network flow applications, such robust solutions can be computed by solving tractable conic linear problems. For example, the robust solution for a multicommodity flow problem is obtained by solving a linear program if the problem has a single source and sink per commodity and the uncertainty in demand and travel time is given by independent bounded polyhedral sets. Preliminary computational results show that the robust solution is attractive, as it can reduce the worst case cost by more than 20%, while incurring a 5% loss in optimality when compared to the optimal solution of a representative scenario.

Key Words: Network flow, capacity expansion, robust optimization, uncertainty.

---

*supported through METRANS grant # 0306.

†Industrial and Systems Engineering, University of Southern California, GER-240, Los Angeles, CA 90089-0193, USA, email: fordon@usc.edu

‡Oracle Corporation, Redwood Shores, CA 94065, USA, email: jiamin.zhao.zhao@oracle.com

1 Introduction

A natural problem in many network applications is where to increase arc capacity so that the overall network routing/transmission cost is reduced. There exists substantial research on capacity expansion (or capacity planning) problems in different domains, such as manufacturing [26], electric utilities [22], telecommunications [20], inventory management [18], and transportation [21]. This diverse body of work includes some common elements: these problems expand the capacity of a network flow problem and consider uncertainty in the data. In this paper we consider a classic network flow problem with additional decision variables for arc capacity expansion. More precisely, we represent a network with n nodes and m arcs using its arc-incidence matrix $N \in \Re^{n \times m}$, a vector $u \in \Re^m$ of initial arc capacities, and a demand-supply vector $b \in \Re^n$. The capacity expansion problem is given by

$$\begin{aligned} z_D(b,c) = \quad \min_{x,y} \quad & c^T x \\ \text{s.t.} \quad & Nx = b \\ & x \leq u + y \\ & d^T y \leq q \\ & x,\ y \geq 0\ , \end{aligned} \tag{1}$$

where we minimize a linear transportation cost with coefficients $c \in \Re^m$ by selecting feasible arc flow variables $x \in \Re^m$ that satisfy new arc capacities defined by the arc expansion variables $y \in \Re^m$. Expanding the capacity of arc e incurs a linear cost d_e with a total budget for investment of q.

This basic network flow model can be enhanced, using multiple time periods, non-linear latency cost functions, etc.; see [2] for a description of different network flow problems. In particular, this problem is closely related to the mixed-integer network design problem, which is obtained by dropping the budget q, considering y as integer variables, and including the capacity expansion cost $d^T y$ in the objective.

In our work we consider that travel times c and demands b are uncertain, which is observed in many applications and is particularly relevant for planning problems. For example, Problem (1) can be used to plan freight routes on a transportation network. The uncertain traffic conditions lead to an uncertain transportation cost, variability in demand for freight leads to uncertainty in the demand vector b, and increasing the number of vehicles on a certain route amounts to increasing the capacity on arcs.

Uncertainty in capacity expansion problems can be traced back to [13]. A standard method to represent the uncertainty in optimization problems is via discrete uncertainty scenarios, an approach used in stochastic optimization [10]; see [1] for its use in capacity planning. Previous robust optimization methodology in the literature either considers discrete uncertainty scenarios as in [23] or are defined for discrete optimization problems with continuous uncertainty in the objective function coefficients; see [19]. Additional, problem specific, methods for robust optimization with scenario uncertainty are described in [17] for uncapacitated network design, [14] for routing in a network with failures, and in [24] for capacity planning in a manufacturing application. Main drawbacks of scenario based uncertainty models are that the distribution of the uncertainty must be known and that a prohibitive number of scenarios might be required to represent the uncertainty accurately. For robust discrete optimization, these methods are limited to uncertainty in the objective function, in addition they either lead to NP-hard formulations for problems whose deterministic version is polynomially solvable, or use solution techniques that are specific to each problem.

In this paper we use a robust optimization model (in the sense of Ben-Tal and Nemirovski [6]) to find a solution to the capacity expansion problem with uncertainty in demand and travel time. We use *conic uncertainty sets* to represent this uncertainty and show that the robust solution can be obtained by solving tractable *conic linear problems*. These conic problems include linear programming (LP), second order cone, and semidefinite programs that can be solved efficiently with current interior point methods.

For example, we find that a robust solution is obtained as the optimal solution to a conic linear problem for a multicommodity flow problem with single source and sink per commodity, interval uncertainty on demand, and independent conic uncertainty set on the cost vector. Linear, convex quadratic, and semi-definite constraints form different types of conic uncertainty sets, which lead to tractable conic linear problems. We discuss these conic sets in more detail in Section 3. Two recent papers [3, 8] also consider robust optimization for network flow and design problems using a different type of uncertainty set referred to as *budget of uncertainty*. In [8] the authors consider the network flow problem over arbitrary networks with deterministic demand and a budget of uncertainty set on travel time. In [3] the authors investigate robust solutions for the adaptive network flow and network design problems. That work considers budget of uncertainty sets on demand and shows that the robust problem is tractable on totally ordered networks or arborescences.

Our work, in contrast, considers arbitrary networks and identifies convex uncertainty sets on demand and travel time that lead to tractable robust problems. We find that conic uncertainty sets in travel time (which includes bounded polyhedra and ellipsoids) and independent interval uncertainty in the demand of a single source-sink pair make the robust problem tractable regardless of the underlying network. We also provide examples that show that relaxing these assumptions on the demand uncertainty leads to non-convex problems. In addition, we present computational results comparing the robust solution to the deterministic optimal solution for fixed nominal data. These results show that the robust solution can improve significantly in the worst case, while incurring a modest loss in optimality on the nominal instance.

The structure of the paper is as follows: in the next section we describe the robust optimization approach as it pertains to our problem. In Section 3 we present the robust capacity expansion problem, the types of uncertainty sets considered, and investigate the reason for the difficulty of this problem. In Section 4 we identify the conditions that make it possible to solve the robust capacity expansion problem efficiently. We present our computational results in Section 5 and provide concluding remarks in Section 6.

2 Robust Optimization Methodology

The robust optimization approach was introduced for convex optimization in [6]. This methodology has generated research on a number of applications such as structural design [5], least-square optimization [11], portfolio optimization [12, 15], and supply chain management problems [9] to name a few.

The robust solution is defined as the solution that achieves the best worst case objective function value. Consider the following optimization problem under uncertainty:

$$\begin{aligned} \min_{u,v} \quad & f(u,v,w) \\ \text{s.t.} \quad & g(u,v,w) \leq 0 \ , \end{aligned}$$

where the uncertainty parameter w belongs to a closed bounded and convex uncertainty set $w \in \mathcal{U}$. The robust solution is obtained by solving the following robust counterpart problem (RC):

$$\begin{aligned} z_{RC} = \quad \min_{u,v,\gamma} \quad & \gamma \\ \text{s.t.} \quad & f(u,v,w) \leq \gamma \ \text{ for all } w \in \mathcal{U} \\ & g(u,v,w) \leq 0 \ \text{ for all } w \in \mathcal{U} \ . \end{aligned} \tag{2}$$

An attractive feature of this approach is that the complexity of solving problem (RC) is, for very general cases, the same as the complexity of the original problem. For example, when the original problem is an LP, from [7] we know that Problem (2) above is equivalent to an LP when $\mathcal{U}$ is a polyhedron and to a quadratically constrained convex program when $\mathcal{U}$ is a bounded ellipsoidal set. In addition, the size of the resulting problem (RC) is bounded by a polynomial in the original problem's dimensions, which implies a polynomial method for the robust solution.

The robust counterpart for a stochastic problem with recourse, dubbed the adjusted robust counterpart problem (ARC), is introduced in [4]. In a problem with recourse, some of the decision variables u are decided a priori, while the rest v can adjust to the outcome of the uncertainty, which yields the following (ARC) problem:

$$
\begin{aligned}
z_{ARC} = \quad & \min_{u,\gamma} \quad \gamma \\
& \text{s.t.} \qquad \text{for all } w \in \mathcal{U} \text{ exists } v \ : \ \begin{cases} f(u,v,w) \leq \gamma \\ g(u,v,w) \leq 0 \ . \end{cases}
\end{aligned} \tag{3}
$$

Clearly $z_{ARC} \leq z_{RC}$, since selecting one v that is feasible for all $w \in \mathcal{U}$ is a possibility for (ARC). However, we do not retain the nice complexity results, as Theorem 3.5 of [16] shows that the (ARC) problem of an LP with polyhedral uncertainty is NP-hard. In our work we focus on identifying the conditions on the uncertainty set that yield a tractable (ARC) for the robust capacity expansion problem.

3 The Robust Capacity Expansion Problem

We formulate the robust counterpart for Problem (1) assuming the demand b and travel times c belong to given uncertainty sets, $b \in \mathcal{U}_b$ and $c \in \mathcal{U}_c$, where the sets $\mathcal{U}_b$ and $\mathcal{U}_c$ are closed, convex, and bounded. Given this uncertainty, it is natural to separate the decision variables, deciding investment variables y prior to observing the traffic conditions (realizations of b and c), and letting the traffic flow x adapt to these conditions. Thus, our problem is a stochastic problem with recourse and the robust capacity expansion problem (RCEP) is obtained by substituting the capacity expansion Problem (1) in the (ARC) Problem (3):

$$
\begin{aligned}
z_{ARC} = \quad & \min_{y,\gamma} \quad \gamma \\
& \text{s.t.} \quad d^T y \le q \\
& \qquad\; y \ge 0 \\
& \qquad\; \text{for all } c \in \mathcal{U}_c,\ b \in \mathcal{U}_b \text{ exists } x \ : \ \begin{cases} Nx = b \\ 0 \le x \le u + y \\ c^T x \le \gamma \ . \end{cases}
\end{aligned} \tag{4}
$$

Proposition 1 *The (RCEP) Problem (4) is equivalent to Problem (5) below, in that both problems have the same optimal solution y^* and $z_{ARC} = z_R$.*

$$
\begin{array}{llll}
z_R = \min_y & \max_{c,b} & \min_x & c^T x \\
\quad d^T y \le q & c \in \mathcal{U}_c & \text{s.t.} & Nx = b \\
\quad y \ge 0 & b \in \mathcal{U}_b & & 0 \le x \le u + y \ .
\end{array} \tag{5}
$$

Proof: For any $b \in \mathcal{U}_b$ and vector $y \ge 0$ let $X(b, y) = \{x \mid Nx = b,\ 0 \le x \le u+y\}$ be the set of feasible flows. Then the last constraint in problem (4) becomes: for all $c \in \mathcal{U}_c$ and $b \in \mathcal{U}_b$, there exists $x \in X(b, y)$ and $c^T x \le \gamma$, which is equivalent to: for all $c \in \mathcal{U}_c$ and $b \in \mathcal{U}_b$ we have $\gamma \ge \min_{x \in X(b,y)} c^T x$. This is equivalent to $\gamma \ge \max_{c,b} \min_{x \in X(b,y)} c^T x$, which concludes the proof. ■

3.1 Uncertainty Sets

The uncertainty model considers a demand and travel time that belong to closed, convex, and bounded uncertainty sets. There are no distribution assumptions over these sets and they model an independent uncertainty between demand and travel time. Such uncertainty sets can represent, for example, the confidence intervals of the uncertain quantities.

Here we present conditions on the uncertainty sets that help obtain a tractable (RCEP). We begin with the assumption that all travel times in the uncertainty set have non-negative arc costs, that is if $c \in \mathcal{U}_c$ then $c_e \geq 0$ for all arcs e. In addition, we make the following assumption on the uncertainty sets, which is equivalent to the *relatively complete recourse* assumption of the stochastic programming literature [10]:

Assumption 1 *For every $b \in \mathcal{U}_b$ and $c \in \mathcal{U}_c$, the network flow problem* $\min\{\ c^T x \mid Nx = b,\ 0 \leq x \leq u\ \}$ *is feasible.*

This assumption implies that the network flow LP has an optimal solution for all $c \in \mathcal{U}_c$ and $b \in \mathcal{U}_b$, since $c \geq 0$ guarantees that the objective is bounded.

In this paper, we consider conic uncertainty sets, in particular those from linear or ellipsoidal constraints. A polyhedral set in $\Re^h$ formed by the intersection of p hyperplanes (linear constraints) is given by $\mathcal{U} = \{x \mid Mx \leq g\}$, where M is a $p \times h$ matrix and $g \in \Re^p$. Ellipsoidal uncertainty sets in $\Re^h$ are given by $\mathcal{U} = \{x \mid x = x^0 + \sum_{l=1}^{L} \xi_l x^l,\ \xi \in \mathcal{X}\}$, where $\mathcal{X} = \{\xi \mid \exists w, A\xi + Bw \geq_{\mathcal{K}} d\}$ and the constraint $a \geq_{\mathcal{K}} b$ means the vector $a - b \in \mathcal{K}$, for some regular cone $\mathcal{K}$. Note that the ellipsoidal set reduces to the polyhedral set if $h = L$, $x^0 = 0$, x^l is the l-th canonical vector, $A = -M$, $B = 0$, $d = -g$, and $\mathcal{K}$ is the p dimensional positive orthant. A similar transformation shows that ellipsoidal sets include sets defined by any conic linear system $\mathcal{U} = \{x \mid -Mx \leq_{\mathcal{K}} -g\}$, which includes linear, second order, and semidefinite constraints. See [6] for a detailed discussion and motivation of ellipsoidal sets.

Below we provide other examples of ellipsoidal sets. Let $\mathcal{K}^*$ denote the positive polar of cone $\mathcal{K}$ and $e \in \Re^h$ the vector of all ones, $\Re^h_+$ the h dimensional positive orthant, and $\mathcal{L}^{h+1} = \{(x_1, \bar{x}) \in \Re^{h+1} \mid \bar{x} \in \Re^h,\ \|\bar{x}\|_2 \leq x_1\}$ the $h+1$ dimensional second order cone.

- $\mathcal{U}$ is an ellipse centered at x^0 with axes $x^1, \ldots, x^L$ if the set $\mathcal{X} = \{\xi \mid \|\xi\|_2 \leq 1\}$, which is given by the conic constraints $\mathcal{X} = \{\xi \mid \exists w,\ (w, \xi) \in \mathcal{L}^{L+1},\ -w+1 \in \Re_+\}$.

- $\mathcal{U}$ is a box centered at x^0, with edges in directions $x^1, \ldots, x^L$ if the set $\mathcal{X} = \{\xi \mid \|\xi\|_\infty \leq 1\}$, given by the conic constraints $\mathcal{X} = \{\xi \mid -\xi + e \in \Re^L_+,\ \xi + e \in \Re^L_+\}$.

- $\mathcal{U}$ is the convex combination of discrete uncertainties $x^0, x^0 + x^1, \ldots, x^0 + x^L$ if the set $\mathcal{X} = \{\xi \mid \|\xi\|_1 \leq 1,\ \xi \geq 0\}$, given by the conic constraints $\mathcal{X} = \{\xi \mid \xi \in \Re^L_+,\ -e^T\xi + 1 \in \Re_+\}$.

3.2 Difficulty of Solving (RCEP)

Here we investigate whether there exists any special structure in (RCEP) that can guarantee a polynomial solution to this instance of the NP-hard Problem (ARC). We begin defining the worst case cost of investment decision y by $\phi(y)$. Thus, problem (RCEP) is simply $\min\{\phi(y) \mid d^T y \leq q,\ y \geq 0\}$, and the worst case cost is given by

$$\begin{array}{rlll} \phi(y) = & \max_{c,b} & \min_x & c^T x \\ & c \in \mathcal{U}_c & \text{s.t.} & Nx = b \\ & b \in \mathcal{U}_b & & x \leq u + y \\ & & & x \geq 0\ . \end{array} \tag{6}$$

Theorem 1 *Under Assumption 1, $\phi(y)$ is a convex function in y.*

Proof: Assumption 1 implies that the network flow in the innermost minimization problem is feasible and has an optimal solution. Therefore we replace the innermost minimization problem with its dual, which attains the same objective value, obtaining:

$$\begin{array}{rll} \phi(y) = & \max_{c,b,\lambda,\pi} & b^T\lambda - (u+y)^T\pi \\ & \text{s.t.} & N^T\lambda - \pi \leq c \\ & & b \in \mathcal{U}_b,\ c \in \mathcal{U}_c,\ \pi \geq 0\ . \end{array} \tag{7}$$

It is straightforward to show from this last expression that $\phi(y)$ is a convex function in y since the maximum of a sum is less than the sum of maximums. ∎

Theorem 1 shows that the (RCEP) is the minimization of a convex function over a simplex, thus it can be NP-hard only when evaluating $\phi(y)$ cannot be done in polynomial time. The non-linear term $b^T\lambda$ in the objective of Problem (7) is the challenging aspect of this problem. For example, for deterministic demand, i.e. $\mathcal{U}_b = \{b\}$, the objective becomes linear and computing the value of $\phi(y)$, and thus solving (RCEP), can be done in polynomial time. We study this case in detail in the beginning of the next section.

The following examples, which only consider uncertainty in the demand, illustrate that evaluating $\phi(y)$ can indeed be a difficult problem.

Example 1: Consider the network given in Figure 1, with fixed cost vector c and an investment y that yields the capacities on the figure. In this example the only uncertain parameter is the total amount of supply and demand at nodes 1 and 3. This demand and supply pair is parameterized by $\delta \in [-1, 1]$. The minimum cost flow for this example for each value of δ is exactly $4 + 3|\delta|$ and thus it is maximized for $\delta \in \{-1, 1\}$.

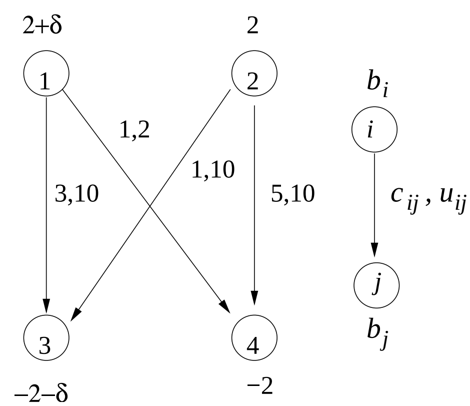


Figure 1: Difficult to evaluate $\phi(y)$. Multiple sources and sinks, $\delta \in [-1, 1]$.

Example 2: Consider the network given in Figure 2, where again we have a fixed cost c and an investment y that yields the capacities on the figure. Now, the demands at nodes 2 and 3 are parameterized by $\delta \in [-1, 1]$. The minimum cost flow of this problem as a function of δ has an objective function value of $12 + 4|\delta|$ and thus it is also maximized for $\delta \in \{-1, 1\}$.

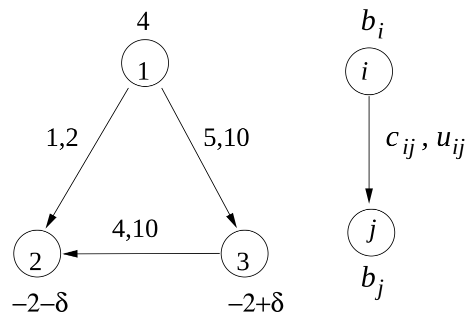


Figure 2: Difficult to evaluate $\phi(y)$. Multiple sink uncertainty, $\delta \in [-1, 1]$.

Both examples maximize a convex function to evaluate $\phi(y)$. Although these are simple one dimensional examples, they illustrate the potential difficulty in finding the demand b that defines the worst case. Example 1 is based on what is known as the "more for less paradox": in increasing the supply in node 1 from 1 to 2 (and increasing the demand at node 3 accordingly), we actually reduce the total cost as we replace the expensive flow on $(2, 4)$ with cheaper flow on $(1, 4)$. This stops when the supply at 1 increases above 2 units, since then the arc $(1, 4)$ is saturated and the extra flow is sent on an expensive arc increasing the total cost. A similar phenomenon occurs in Example 2, where we use the capacity of the low cost arc to switch from decreasing the total cost to increasing it.

4 Solving (RCEP)

We now present three cases in which the (RCEP) can be formulated as a conic linear problem and thus solved in polynomial time by interior point algorithms. The cases are: fixed demand, single commodity with uncertain demand, and a multicommodity problem with uncertain demand.

4.1 Case of Deterministic Demand

In the case of deterministic demand, in other words $\mathcal{U}_b = \{b\}$, the set of feasible flows is fixed since the uncertainty only affects the cost vector. In this case, besides the (RCEP) obtained from (ARC), we can define the following standard robust problem (RC), as in Problem (2), by also deciding the routing x prior to the realization of the uncertainty:

$$\begin{aligned} z_{RC} = \quad \min_{x,y,\gamma} \quad & \gamma \\ \text{s.t.} \quad & Nx = b \\ & x \leq u + y \\ & d^T y \leq q \\ & x, y \geq 0 \\ & \text{for all } c \in \mathcal{U}_c \quad c^T x \leq \gamma \ . \end{aligned} \tag{8}$$

It is well known that Problem (RC) is a tractable problem when the uncertainty set $\mathcal{U}_c$ is a polyhedral or ellipsoidal set [6, 7]. In the case of polyhedral uncertainty for example, if $\mathcal{U}_c = \{c \mid Mc \leq g,\ c \geq 0\}$ then from weak LP duality we have that the last constraint in Problem (8) is equivalent to the inequalities: $\gamma \geq g^T w,\ M^T w \geq x,\ w \geq 0$, which makes Problem (RC) the following linear program:

$$\begin{aligned} \min_{y,x,w} \quad & g^T w \\ \text{s.t.} \quad & Nx = b \\ & x \leq u + y \\ & x \leq M^T w \\ & d^T y \leq q \\ & x, w, y \geq 0 \end{aligned} \tag{9}$$

In fact (RCEP) is also equivalent to Problem (9) if Assumption 1 holds, $\mathcal{U}_b = \{b\}$, and $\mathcal{U}_c = \{c \mid Mc \leq g, c \geq 0\}$. In this case $\phi(y)$ in Problem (7) is an LP whose dual combined with the outer minimization on y yields Problem (9).

We now show that this equivalence between the robust counterpart (RC) and the adjusted robust counterpart (RCEP) of the capacity expansion problem holds regardless of the uncertainty set $\mathcal{U}_c$ considered. This states that for deterministic demand there is no change in the robust solution by conducting the routing decisions after the uncertainty is revealed. To solve (RCEP) we need only to solve (RC), which is a tractable problem for polyhedral or ellipsoidal uncertainty sets.

Theorem 2 *If $\mathcal{U}_b = \{b\}$ then (RCEP) is equivalent to (RC).*

Proof: Problem (RC), defined in (8), is equivalent to

$$\begin{array}{rll} z_{RC} = & \min_{y,x} & \max_c \ c^T x \\ & Nx = b & c \in \mathcal{U}_c \\ & x \leq u + y & \\ & d^T y \leq q & \\ & x, y \geq 0 \ . & \end{array}$$

Let $X(y) = \{x \mid Nx = b,\ 0 \leq x \leq u + y\}$. Since the function $f(x, c) = c^T x$ is bilinear and the convex uncertainty set $\mathcal{U}_c$ is bounded, then we have that for any feasible y

$$\min_{x \in X(y)} \max_{c \in \mathcal{U}_c} c^T x = \max_{c \in \mathcal{U}_c} \min_{x \in X(y)} c^T x \ ,$$

see for example Corollary 37.3.2 in [25]. Substituting this saddle point equivalence in the expression for (RC) above, we obtain Problem (5) with deterministic demand. ∎

4.2 Uncertain Demand

As pointed out in Examples 1 and 2, evaluating the worst case cost function $\phi(y)$ can be a non-convex problem under demand uncertainty. We now identify the conditions on the uncertainty set $\mathcal{U}_b$ under which we can evaluate $\phi(y)$ efficiently. The key observation for this result is that for single source and single sink, an optimal routing is greedy sending every additional unit of flow along a path of minimum cost out of paths with remaining available capacity. In this case, if the total flow increases, then so does the value of the minimum cost solution and there is no “more for less paradox.”

A slightly broader case is to consider multiple sinks and a single source, or equivalently multiple sources and a single sink, with demand uncertainty only in a single source and sink pair. We describe the methodology in the case with multiple sinks and single source, and omit the analogous multiple source/single sink case. Let s be the single source, and assume a demand $\bar{b}$ with uncertainty in s and one fixed sink node $t \neq s$. This implies the following demand uncertainty set

$$\mathcal{U}_b = \{b \mid b = \bar{b} + \delta(e_s - e_t),\ \delta \in [0, \bar{\delta}]\}\ , \tag{10}$$

where $\bar{\delta} > 0$ and $e_i \in \Re^n$ is the i-th canonical vector. Note that we assume the demand uncertainty does not change whether a node is source or sink, thus $\bar{b}_s \geq 0$ and $\bar{b}_i \leq 0$ for any $i \neq s$. Under the conditions of a single source and sink pair being affected by the uncertainty and not allowing changes of source to sink and vice-versa, the only demand uncertainty set possible is of the form (10).

Theorem 3 *Consider a network flow problem with a single source s and an uncertainty set $\mathcal{U}_b$ given by Equation (10). Then $\phi(y)$ is the following convex optimization problem*

$$\begin{array}{rll} \phi(y) = & \max_{\lambda,\pi,c} & \bar{b}^T\lambda + \bar{\delta}(\lambda_s - \lambda_t) - (u+y)^T\pi \\ & \text{s.t.} & N^T\lambda - \pi - c \leq 0 \\ & & c \in \mathcal{U}_c \\ & & \pi \geq 0\ . \end{array} \tag{11}$$

Proof: Under the uncertainty set $\mathcal{U}_b$ the definition of $\phi(y)$ becomes

$$\begin{array}{rlll} \phi(y) = & \max_c & \max_\delta & \min_x c^T x \\ & c \in \mathcal{U}_c & \delta \in [0,\bar{\delta}] & Nx = \bar{b} + \delta(e_s - e_t) \\ & & & 0 \leq x \leq u + y\ . \end{array}$$

The proof is based in showing that the function

$$\begin{array}{rlcl} \Gamma(\delta) = & \min_x\ c^T x & = & \max_{\lambda,\pi}\ \bar{b}^T\lambda + \delta(\lambda_s - \lambda_t) - (u+y)^T\pi \\ & Nx = \bar{b} + \delta(e_s - e_t) & & N^T\lambda - \pi \leq c \\ & 0 \leq x \leq u + y & & \pi \geq 0 \end{array}$$

is an non-decreasing function of δ. If $\Gamma(\delta)$ is non-decreasing it implies that $\phi(y) = \max_{c \in \mathcal{U}_c} \Gamma(\bar{\delta})$ completing the proof. From Assumption 1 we have that function $\Gamma(\delta)$ is well defined for all $\delta \in [0, \bar{\delta}]$. The complementary slackness optimality conditions state that $x_{ij}(\lambda_i - \lambda_j - \pi_{ij} - c_{ij}) = 0$ for all arcs (i, j), which combined with a positive flow from s to t implies that $\lambda_s - \lambda_t \geq 0$ at optimality for all $\delta > 0$. Therefore we can restrict the dual problem only to variables that satisfy $\lambda_s - \lambda_t \geq 0$. Then, for any dual feasible point (λ, π) and $\delta \leq \delta'$, the dual objective function $D(\lambda, \pi, \delta)$ satisfies $D(\lambda, \pi, \delta) \leq D(\lambda, \pi, \delta')$, which implies $\Gamma(\delta) \leq \Gamma(\delta')$. ■

Remark 1 *The conditions (a) single source s and (b) uncertainty set $\mathcal{U}_b$ given by (10) are necessary and sufficient for $\phi(y)$ to be a convex optimization problem.*

Proof: The theorem proves the sufficiency of these conditions. For the necessity we show that if either of the conditions does not hold, then evaluating $\phi(y)$ is not a convex problem. Example 1 considers a network with multiple sources and sinks and uncertainty only in a single source and sink pair, i.e. violates only condition (a). Example 2 considers a network with a single source and uncertainty among the sink nodes, i.e. only condition (b) does not hold. In both examples we evaluate $\phi(y)$ by maximizing a convex function, which is not a convex optimization problem. ■

The following two corollaries present the (RCEP) for problems with demand uncertainty given by (10) and polyhedral or ellipsoidal uncertainty sets on travel time. We omit the proofs as both simply take the dual of Problem (11) with the appropriate $\mathcal{U}_c$.

Corollary 1 *Consider a network flow problem with a single source s and an uncertainty set $\mathcal{U}_b$ given by Equation (10). If $\mathcal{U}_c = \{c \mid Mc \leq g,\ c \geq 0\}$, then (RCEP) is equivalent* *to the following LP:*

$$\begin{aligned}\min_{y,x,w} \quad & g^T w \\ \text{s.t.} \quad & Nx = \bar{b} + \bar{\delta}(e_s - e_t) \\ & x \leq u + y \\ & x \leq M^T w \\ & d^T y \leq q \\ & x, w, y \geq 0 \ . \qquad \blacksquare\end{aligned}$$

Corollary 2 *Consider a network flow problem with a single source s and an uncertainty set $\mathcal{U}_b$ given by Equation (10). Let $\mathcal{U}_c = \{c \mid c = c^0 + \sum_{l=1}^{L} \xi_l c^l,\ \xi \in \mathcal{X}\}$, with $\mathcal{X} = \{\xi \mid \exists w, A\xi + Bw \geq_{\mathcal{K}} d\}$, and let $\mathcal{C} = [c^1, \ldots, c^L]$. Then (RCEP) is equivalent to the conic linear program*

$$\begin{aligned}\min_{y,x,z} \quad & (c^0)^T x - d^T z \\ \text{s.t.} \quad & Nx = \bar{b} + \bar{\delta}(e_s - e_t) \\ & x \leq u + y \\ & \mathcal{C}^T x + A^T z = 0 \\ & B^T z = 0 \\ & d^T y \leq q \\ & x, y \geq 0,\ z \geq_{\mathcal{K}^*} 0 \ . \qquad \blacksquare\end{aligned}$$

4.3 Multicommodity Flow

A relevant model for network flow applications is the multicommodity flow problem with a single source and single sink for each commodity. In this section we show that the (RCEP) is also tractable for a multicommodity flow problem, in which each commodity has a single source and sink with demand uncertainty independent per commodity. The (RCEP) is also tractable for the slightly more general conditions where each commodity has either multiple sources or sinks with a single source - sink pair affected by demand uncertainty, in other words where each commodity has demand uncertainty given by (10). However, we omit this result here for clarity of exposition.

Assuming that commodity k has a source s^k and a sink t^k, and the amount to be sent is uncertain, but bounded, we can define the demand uncertainty set by

$$\mathcal{U}_b = \left\{ (b^1, \ldots, b^K) \mid b^k = \delta^k \left(e_{s^k} - e_{t^k} \right), \ \delta^k \in [\delta_l^k, \delta_u^k], \text{ for all } k \in 1, \ldots, K \right\} , \tag{12}$$

where we assume that $\delta_l^k \geq 0$ for all $k = 1, \ldots, K$. In other words, the demand uncertainty does not allow a supply node to become a demand node.

Theorem 4 *Consider the multicommodity flow problem, where each commodity has a single source s^k and single sink t^k and that $\mathcal{U}_b$ is given by Equation (12). Then $\phi(y)$ is the following convex optimization problem*

$$\begin{aligned} \phi(y) = \quad \max_{\lambda,\pi,c} \quad & \sum_{k=1}^{K} \delta_u^k (\lambda_{s^k}^k - \lambda_{t^k}^k) - (u+y)^T \pi \\ \text{s.t.} \quad & N^T \lambda^k - \pi - c^k \leq 0 \qquad\qquad k = 1, \ldots, K \\ & (c^1 \ldots c^K) \in \mathcal{U}_c \\ & \pi \geq 0 \ . \end{aligned}$$

Proof: This proof is analogous to the proof of Theorem 3, and requires showing that the inner minimization problem

$$\begin{aligned} \Gamma(\delta^1 \ldots \delta^K) = \quad \min_x \quad & \sum_{k=1}^{K} (c^k)^T x^k \\ \text{s.t.} \quad & N x^k = \delta^k (e_{s^k} - e_{t^k}) \quad k = 1 \ldots K \\ & \sum_{k=1}^{K} x^k \leq u + y \\ & x^k \geq 0 \qquad\qquad\qquad k = 1 \ldots K \end{aligned}$$

is a non-decreasing function of $(\delta^1 \ldots \delta^K)$. ■

The following two corollaries show that problem (RCEP) is tractable for the multicommodity flow problem with single source and sink and for polyhedral and ellipsoidal cost uncertainty sets. We omit the proofs as they are easily derived from Theorem 4.

Corollary 3 *Consider the multicommodity flow problem, where each commodity has a single source and sink, and $\mathcal{U}_b$ is given by Equation (12). If $\mathcal{U}_c = \{(c^1 \dots c^K) \mid M^k c^k \leq g^k, c^k \geq 0\}$, then the (RCEP) problem is equivalent to the following LP:*

$$\begin{array}{lll}
\min_{y,x,w} & \sum_{k=1}^{K} (g^k)^T w^k & \\
\text{s.t.} & N x^k = \delta_u^k \left(e_{s^k} - e_{t^k}\right) & k = 1, \dots, K \\
 & x^k \leq (M^k)^T w^k & k = 1, \dots, K \\
 & \sum_{k=1}^{K} x^k \leq u + y & \\
 & d^T y \leq q & \\
 & x^k, w^k, y \geq 0 \ . & \blacksquare
\end{array}$$

Corollary 4 *Consider the multicommodity flow problem, where each commodity has a single source and sink, and $\mathcal{U}_b$ is given by Equation (12). If $\mathcal{U}_c = \{(c^1 \dots c^K) \mid c^k = c^{k0} + \sum_{l=1}^{L} \xi_l c^{kl},\ \xi \in \mathcal{X}\}$, with $\mathcal{X} = \{\xi \mid \exists w, A\xi + Bw \geq_{\mathcal{K}} d\}$, and let $\mathcal{C}^k = [c^{1k} \dots c^{Lk}]$. Then (RCEP) is equivalent to the following conic linear program*

$$\begin{array}{lll}
\min_{y,x,z} & \sum_{k=1}^{K} (c^{k0})^T x^k - z^T d & \\
\text{s.t.} & N x^k = \delta_u^k \left(e_{s^k} - e_{t^k}\right) & k = 1, \dots, K \\
 & \sum_{k=1}^{K} \left(\mathcal{C}^k\right)^T x^k + A^T z = 0 & \\
 & B^T z = 0 & \\
 & \sum_{k=1}^{K} x^k \leq u + y & \\
 & d^T y \leq q & \\
 & x^k, y \geq 0,\ \ z \geq_{\mathcal{K}_*} 0 \ . & \blacksquare
\end{array}$$

5 Computational Experiments

We now present computational experiments that compare the robust solution to the deterministic solution obtained for *nominal data*, i.e. data that is representative of the uncertainty set. These experiments serve to illustrate the conditions under which a robust solution is preferable to a deterministic solution.

For each experiment described in this section, we compute four values: z_D the optimal value of the deterministic solution, z_R the optimal value of the robust solution, z_{wc} the worst case value of the deterministic solution, and z_{ac} the objective value of the robust solution for the nominal data. We obtain $z_D = z_D(\bar{b}, \bar{c})$ as the optimal objective function value of Problem (1) for the nominal data $\bar{b} \in \mathcal{U}_b$ and $\bar{c} \in \mathcal{U}_c$. Let y_D be the optimal investment strategy for the deterministic problem. The value z_R is obtained by solving the appropriate tractable characterization of (RCEP), either Corollary 1 or 3 depending on the form of the uncertainty sets. Let y_R be the optimal robust investment strategy. The worst case value $z_{wc} = \phi(y_D)$ is obtained from the appropriate tractable characterization of Problem (7) (either Theorem 3 or 4). Finally the cost of the robust solution for the nominal data, z_{ac}, is obtained by solving

$$\begin{array}{ll} \min & \bar{c}^T x \\ \text{s.t.} & Nx = \bar{b} \\ & 0 \leq x \leq u + y_R \ . \end{array}$$

We compare the performance of the robust and deterministic solution through the following two ratios:

$$r_{wc} = \frac{z_{wc} - z_R}{z_R} \quad \text{and} \quad r_{ac} = \frac{z_{ac} - z_D}{z_D} \ .$$

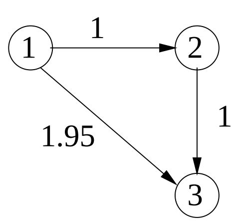
The quantity r_{wc} is the relative improvement of the robust solution in the worst case and r_{ac} is the relative loss of optimality of the robust solution on the nominal data.

5.1 A Triangular Network

Our first example consists of a scalable triangular network shown in Figure 3, formed by repeating r times a simple network on three nodes shown in the left of Figure 3. The nominal travel times appear on the arcs of the 3-node network and are set to 1.95 on diagonal and 1 on non-diagonal arcs; arcs have capacity $u_e = 1$ and a rate of investments $d_e = 2$ for diagonal and $d_e = 1$ for non-diagonal arcs. We consider an uncertainty set on travel times given by

$$\mathcal{U}_c = \left\{ c \mid 0.5\bar{c} \leq c \leq 1.5\bar{c}, \quad \sum_{(i,j)\ \text{non-diag}} 4c_{ij} + \sum_{(i,j)\ \text{diag}} c_{ij} = 9.95r \right\} .$$

There is a deterministic demand of δ units of flow from the top left to the lower right and a total investment of q. This example is constructed so straight paths using diagonal arcs are shorter but with higher variability than the longer alternate paths using horizontal and vertical arcs.



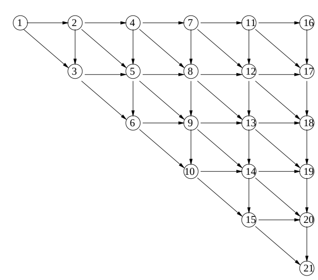


Figure 3: Triangular network. Three node building block and a 21 node example, $r = 15$.

In Figure 4 we plot r_{wc} and r_{ac} for the 3-node network. We present the ratios for $q \in [2, 3]$ in 0.25 increments, for each q considering $\delta \in [1, 3]$ in increments of 0.1. Note that the robust solution is able to achieve more than a 20% reduction in the value of the worst case with a smaller than 2% increase in the value for the nominal data. This occurs for cases with a flow δ bigger than 2.25; in addition this worst case benefit improves for larger values of investment budget q. Note that, for some fixed investments q, the benefit

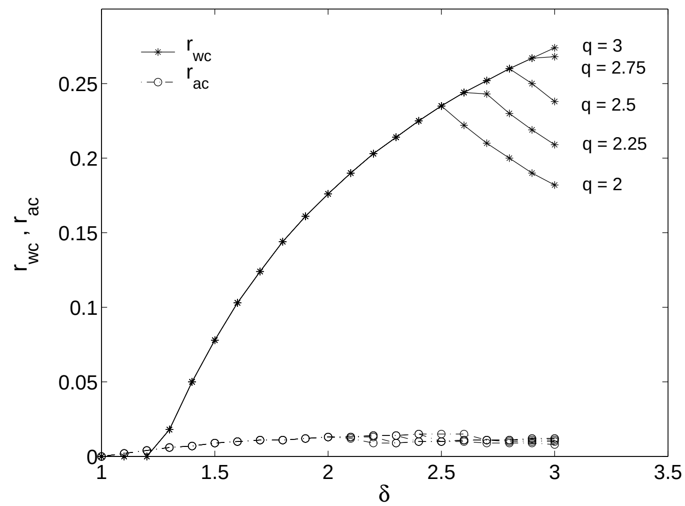


Figure 4: Comparison of robust and deterministic solutions for 3-node network. Investments $q \in \{2,\ 2.25,\ 2.5,\ 2.75,\ 3\}$ with $\delta \in [1, 3]$ in 0.1 increments for each q

of the robust solution in the worst case, r_{wc}, increases and then decreases with δ. Clearly for δ close to 1, the flow can be sent on either of the existing paths, thus both investment solutions are comparable. As the flow increases however, the investment becomes crucial in routing the flow. For large enough flows, for example $\delta = 2.5$ for the case $q = 2$, all the new capacity is installed in the best path, and as the flow keeps increasing the benefit of the robust solution decreases, as it must route flow through the other path.

In Figure 5 we plot the ratios r_{wc} and r_{ac} obtained for the 21-node network for different values of the investment $q \in \{10, 30, 50, 70, 80\}$, for each q considering different total flows $\delta \in [1, 10]$ in 0.2 increments. Similar to the 3-node network example, the robust solution is better than the deterministic solution in the worst case, in some cases by about 20%, while it is never worse than 2.5% of the deterministic solution on the nominal data. Here we also observe that r_{wc} decreases for flows larger than a certain level and that this drop can be substantial. Note that some investment-flow combinations are infeasible, for example $q = 10$ and $\delta = 10$. For large flows δ, the only feasible investment strategy for the robust and deterministic solutions becomes to expand the bottleneck arcs near the source and sink nodes, which are used by all paths from 1 to 21.

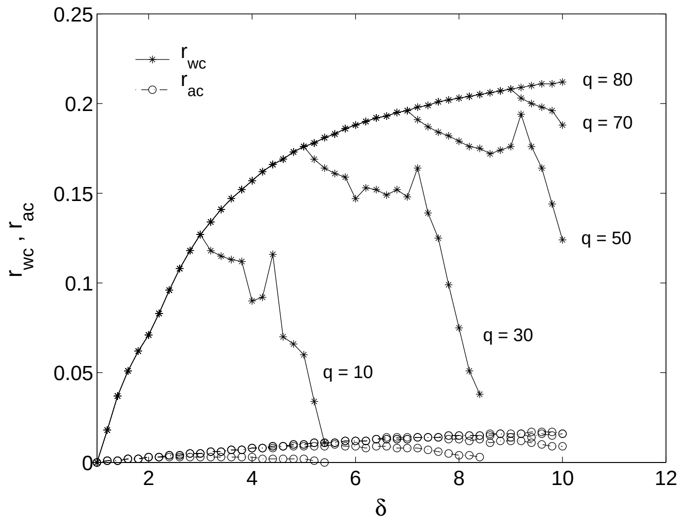


Figure 5: Comparison of robust and deterministic solutions for 21-node network. Investments $q \in \{10, 30, 50, 70, 80\}$ with $\delta \in [1, 10]$ in 0.2 increments for each q

5.2 A Transportation Network

We now consider a multicommodity flow problem with cost and demand uncertainty on a planar network. The network is given in Figure 6 with the nominal travel times depicted. In this example we plan routes from sources at either node 2 or node 5 to be delivered in either nodes 1, 4, or 8 in a rush hour situation, where traffic along the corridor 2-5-7-8 and 4-5 is subject to travel time uncertainty. The nominal demand values are 1000 units of flow from node 2 to node 4, 500 units from node 2 to node 1, and 1000 from node 2 to node 8. We also have 500 units of flow from node 5 to node 8. The network has uniform arc capacity of 900, and there is a total budget of 2000 units of arc capacity to distribute. We consider that the travel times on arcs $(2, 5), (5, 4), (5, 7)$, and $(7, 8)$ are subject to uniform box uncertainty and can vary either up or down by $\mu\%$. The demand is also under uniform box uncertainty, and all commodities can have

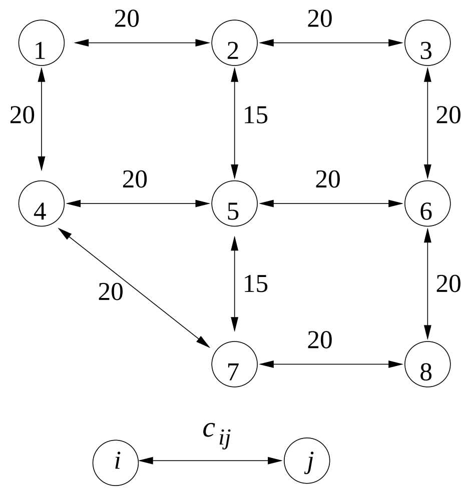


Figure 6: Transportation network problem.

their demand/supply vary by $\mu_b\%$ up or down.

In Figure 7 we present the ratios r_{wc} and r_{ac} for different values of total investment budget $q \in [0, 3000]$ in 500 increments as we vary the amount of uncertainty in travel time $\mu \in [0.1, 1]$. We first consider deterministic demand, i.e. $\mu_b = 0$. We observe that for small investments and uncertainty levels in travel times there is no clear benefit of the robust solution, as the relative loss in the nominal instance is comparable to, sometimes greater than, the relative improvement in the worst case. However as investment budgets and uncertainty levels increase the robust solution becomes more attractive. For instance, when $\mu \geq 0.6$ and $q \geq 1500$, the robust solution obtains more than 15% improvement in the worst case incurring less than a 7% overhead for the nominal data. In Figure 8 we plot the same results for the different values of μ as we vary the investment budget q. Here we observe that we reach the best improvement in r_{wc} starting with $q = 1500$, regardless of the uncertainty in travel time. The same does not happen for r_{ac} which increases as more budget is available until a sharp drop.

In Figure 9 we plot how demand uncertainty affects the performance ratios. For a fixed investment budget $q = 2000$ and different $\mu \in [0.2, 1]$ in 0.2 increments, we plot

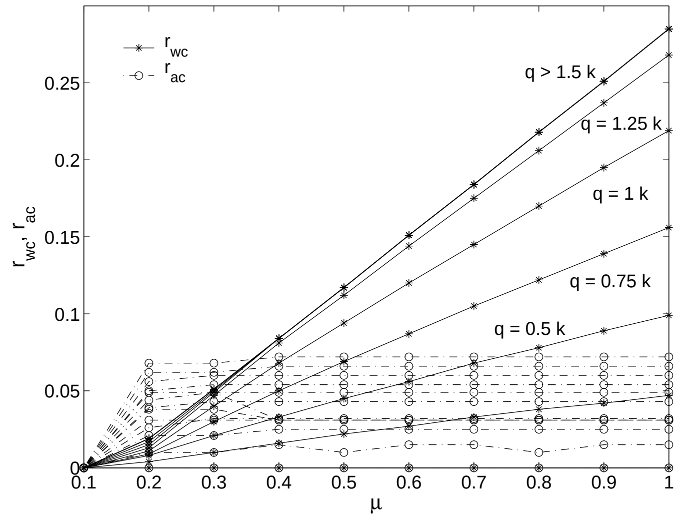


Figure 7: Comparison of robust and deterministic solutions for transportation network as a function of $\mu \in [0.1, 1]$, for different $q \in [0, 3000]$ in 500 increments, with $\mu_b = 0$.

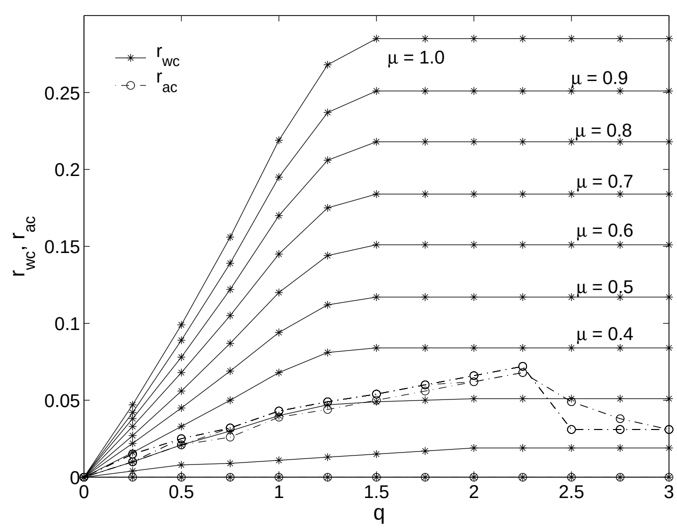


Figure 8: Comparison of robust and deterministic solutions for transportation network as a function of $q \in [0, 3000]$, for different $\mu \in [0.1, 1]$ in 0.1 increments, with $\mu_b = 0$.

the value of r_{wc} and r_{ac} as a function of $\mu_b \in [0, 0.5]$. There is no significant change in r_{ac}. The effect on r_{wc} of changing μ_b varies for different values of the uncertainty μ. For small μ (0.2 or 0.4) there is a small improvement in r_{wc} as μ_b increases. However for $\mu \geq 0.6$, the ratio r_{wc} increases and then decreases for increasing μ_b. This effect is similar to our prior observation that r_{wc} for a robust solution under travel time uncertainty can decrease with the total flow, see Figure 5, and the fact that demand uncertainty leads to routing the maximum possible demand.

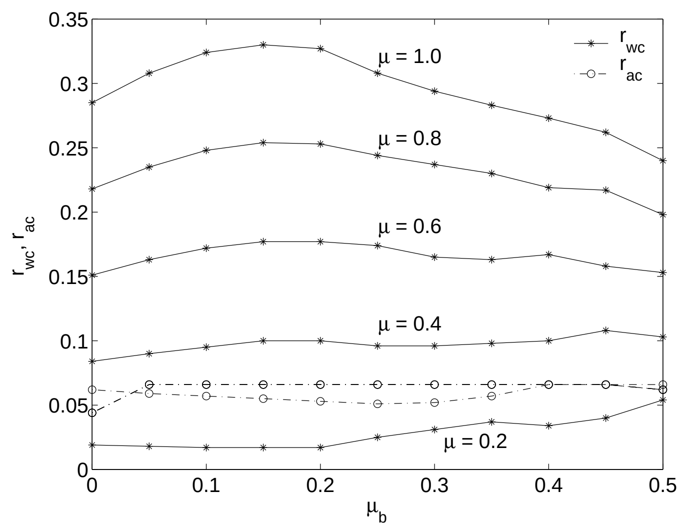


Figure 9: Sensitivity to demand uncertainty $\mu_b \in [0, 0.5]$ for transportation network, different values of $\mu \in \{0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0\}$ and $q = 2000$.

6 Conclusions

The robust capacity expansion problem (RCEP) we consider in this paper, Problem (4), is the basis of an approach to decide capacity expansions for network flow problems that finds a robust solution with respect to the uncertainty in demands and travel times. Here we exploit the structure of the capacity expansion problem to show that

(RCEP) is a tractable problem under conditions that are reasonable for network flow applications: for a multicommodity flow problem with a single source and sink per commodity, independent uncertainty on the demand and travel times given by conic uncertainty sets (e.g. polyhedral or ellipsoidal), non-negative travel times, and feasible flow routes under any uncertain outcome of the demand. We identify a tractable (RCEP) by showing the problem can be formulated as a conic linear problem, which can be solved with interior point methods in polynomial time. The computational results obtained indicate that the robust solution can reduce substantially the worst case cost while incurring a small loss of optimality with respect to the optimal deterministic solution for a nominal uncertainty data. In particular, our examples showed that the robust solution becomes more attractive as the uncertainty in travel times and available budget increases.

The methodology presented here can be applied to the classic network design problem, as the integer variables are part of the outermost problem, yielding a mixed-integer robust counterpart problem. The present work on the linear capacity expansion problem and our preliminary computational results indicate that the robust solution can be attractive for certain network instances. This suggests both that the mixed-integer network design problem might also have an attractive robust solution and that these robust solutions could be efficient in practice. Extensions to considering more complicated demand uncertainty models, or correlations among the demand and travel time uncertainties, do not appear to be straightforward and are a topic for future research.

References

[1] S. Ahmed, A. J. King, and G. Parija. A multi-stage stochastic integer programming approach for capacity expansion under uncertainty. *Journal of Global Optimization*, 26(1):3–24, 2003.

[2] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin. *Network Flows: Theory, Algorithms, and Applications*. Prentice Hall, New Jersey, 1993.

[3] A. Atamtürk and M. Zhang. Two-stage robust network flow and design under demand uncertainty. Technical Report BCOL.04.03, IEOR, University of California–Berkeley, December 2004.

[4] A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376, 2004.

[5] A. Ben-Tal and A. Nemirovski. Robust truss topology design via semidefinite programming. *SIAM Journal on Optimization*, 7(4):991–1016, 1997.

[6] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.

[7] A. Ben-Tal and A. Nemirovski. Robust solutions of uncertain linear programs. *Operations Research Letters*, 25(1):1–13, 1999.

[8] D. Bertsimas and M. Sim. Robust discrete optimization and network flows. *Mathematical Programming*, 98(1-3):49–71, 2003.

[9] D. Bertsimas and A. Thiele. A robust optimization approach to supply chain management. Technical report, MIT, LIDS, November 2003.

[10] J. R. Birge and F. Louveaux. *Introduction to Stochastic Programming.* Springer Verlag, New York, 1997.

[11] L. El-Ghaoui and H. Lebret. Robust solutions to least-square problems to uncertain data matrices. *SIAM Journal on Matrix Analysis and Applications*, 18(4):1035–1064, 1997.

[12] L. El-Ghaoui, M. Oks, and F. Oustry. Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. *Operations Research*, 51(4):543–556, 2003.

[13] A. Ferguson and G. Dantzig. The allocation of aircraft to routes - an example of linear programming under uncertain demand. *Management Science*, 3(1):45–73, 1956.

[14] M. Ferris and A. Ruszczynski. Robust path choice in networks with failures. *Networks*, 35(3):181–194, 2000.

[15] D. Goldfarb and G. Iyengar. Robust portfolio selection problems. *Mathematics of Operations Research*, 28(1):1–38, 2003.

[16] E. Guslitser. Uncertainty-immunized solutions in linear programming. Master's thesis, Minerva Optimization Center, Technion, 2002. http://iew3.technion.ac.il/Labs/Opt/index.php?4.

[17] G. Gutiérrez, P. Kouvelis, and A. Kurawarwala. A robustness approach to uncapacitated network design problems. *European Journal of Operational Research*, 94(2):362–376, 1996.

[18] V. N. Hsu. Dynamic capacity expansion problem with deferred expansion and age-dependent shortage cost. *Manufacturing & Service Operations Management*, 4(1):44–54, 2002.

[19] P. Kouvelis and G. Yu. *Robust Discrete Optimization and its Applications*. Kluwer Academic Publishers, Norwell, MA, 1997.

[20] M. Laguna. Applying robust optimization to capacity expansion of one location in telecommunications with demand uncertainty. *Management Science*, 44(11):S101–S110, 1998.

[21] T. Magnanti and R. Wong. Network design and transportation planning: Models and algorithms. *Transportation Science*, 18(1):1–55, 1984.

[22] S. Malcolm and S. Zenios. Robust optimization for power systems capacity expansion under uncertainty. *Journal of the Operational Research Society*, 45(9):1040–1049, 1994.

[23] J. M. Mulvey, R. J. Vanderbei, and S. A. Zenios. Robust optimization of large-scale systems. *Operations Research*, 43(2):264–281, 1995.

[24] D. Paraskevopoulos, E. Karakitsos, and B. Rustem. Robust capacity planning under uncertainty. *Management Science*, 37(7):787–800, 1991.

[25] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, New Jersey, 1997.

[26] F. Zhang, R. Roundy, M. Çakanyildrim, and W. T. Huh. Optimal capacity expansion for multi-product, multi-machine manufacturing systems with stochastic demand. *IIE Transactions*, 36(1):23–36, 2004.