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A two-stage stochastic programming model for transportation network protection

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ABSTRACT

Network protection against natural and human-caused hazards has become a topical research theme in engineering and social sciences. This paper focuses on the problem of allocating limited retrofit resources over multiple highway bridges to improve the resilience and robustness of the entire transportation system in question. The main modeling challenges in network retrofit problems are to capture the interdependencies among individual transportation facilities and to cope with the extremely high uncertainty in the decision environment. In this paper, we model the network retrofit problem as a two-stage stochastic programming problem that optimizes a mean-risk objective of the system loss. This formulation hedges well against uncertainty, but also imposes computational challenges due to involvement of integer decision variables and increased dimension of the problem. An efficient algorithm is developed, via extending the well-known L-shaped method using generalized benders decomposition, to efficiently handle the binary integer variables in the first stage and the nonlinear recourse in the second stage of the model formulation. The proposed modeling and solution methods are general and can be applied to other network design problems as well.

1. Introduction

Transportation systems are critical infrastructure systems, whose smooth operation is important for maintaining normal functions of our society. However, these spatially distributed systems are also vulnerable to large scale urban disasters, such as earthquakes, hurricanes, flood, and bio/chemical/nuclear hazards. For example, the 1994 Northridge earthquake caused damages to 286 state highway bridges, of which seven major ones collapsed [1]. A damaged transportation system directly affects the effectiveness of post-disaster rescue and repair activities, and also causes huge socio-economic losses [2]. Despite the unpredictable nature of disasters in terms of location, time, and magnitude, retrofit appears as one of the effective mitigation methods from an engineering perspective. Again using the 1994 Northridge earthquake as an example, the highway bridges that had been retrofitted survived the earthquake even though some were within 100 m of collapsed structures. Some bridges, which were under seismic risk but not retrofitted, were damaged in the earthquake [3]. This empirical experience naturally raises a question: How should limited resources be allocated to competing facilities for retrofit so that the total loss of the entire transportation system caused by future earthquakes is minimized?

Federal highway administration (FHWA) seismic retrofit manual [4] states that retrofit decisions are made according to seismic hazard and the importance of individual components. The importance is mainly judged by the daily traffic volume that a highway segment carries, and some other subjective judgments such as its connectivity to critical facilities. However, individual components in a transportation system are actually not independent of each other. Any change in one component of the system may cause redistribution of the traffic and thus affect the traffic on other remote components as well. Therefore, a rigorous retrofit decision should be made at a system level, where a spatially distributed transportation system may be modeled as a network and the interrelations between different components can be captured by network flow theories. Such system issues are not currently considered in seismic retrofit practice due primarily to the lack of adequate system-based evaluation and decision tools [4].

Another challenge in retrofit decision making is the extremely high uncertainty induced by the nature of most disasters, which makes deterministic modeling techniques less relevant. However, most existing research in disaster mitigation is still scenario specific [5,6]. For example, an easy way is to consider a single representative disaster scenario. The computed solution is then evaluated in a set of possible disaster scenarios. Such efforts can only tell us how sensitive a chosen strategy is to environment uncertainty, but not the explicit optimal strategy that hedges well against a whole range of possible hazards. Moving beyond current scenario analysis based

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approaches to arrive at a more rigorous stochastic approach is another focus of this paper.

This paper will introduce a mathematical model for supporting retrofit decision making, considering both interdependency among individual network components and the uncertain nature of disaster occurrence. In this model, the benefit of retrofit is only quantified as savings in reconstruction and travel delay costs. Other broader socio-economics impacts are not included at this stage of our research. In some sense, the problem of making retrofit decisions at network level falls into the general category of network design problems (NDP), which have been intensively studied in the field of transportation engineering and planning [7,8]. However, the key difference between the two problems is that the retrofit problem faces an uncertain network configuration. More specifically, a given design decision in standard NDP results in a deterministic new network configuration, while in retrofit problems there can be many possible post-disaster network configurations resultant from a given retrofit decision, depending on the disaster scenario that is actually realized.

In order to cope with the high uncertainty involved in this problem, we built the model in a two-stage mean-risk stochastic programming framework, which aims to take into account both expected costs and the risk. Stochastic programming was first introduced by Dantzig [9] to handle mathematical programming with uncertainty, and was further developed both in theory and computational aspects by subsequent work (e.g. [10–12]). Examples of applications of stochastic programming in transportation and operations research include stochastic network routing [13,14] and capacity expansion [15]. Traditional stochastic programming is risk neutral in the sense that it focuses only on optimizing the expected value. In the context of disaster mitigation where extremely severe consequence should be avoided, we can include some risk measures in the model to improve the robustness of the solution.

The deterministic equivalent form of our formulation is a mixed-integer nonlinear program. In some cases, standard commercial solvers can be used directly to solve the equivalent form. However, due to the large number of variables and constraints involved in this problem, direct usage of standard solvers is inefficient, and sometimes infeasible. On the other hand, the special structure of the problem naturally leads to consideration of decomposition methods for efficient solution schemes. In regards of these considerations, we designed a solution algorithm that exploits the ideas from L-shaped method [11] and generalized Bender's decomposition [16]. We will also show that the proposed algorithm achieves finite convergence.

The organization of this paper is as follows. Section 2 focuses on the modeling and solution methods, including modeling assumptions and the proposed stochastic programming formulation for the network retrofit problem, the solution algorithm, and discussions on the convergence of this solution algorithm. Numerical examples and some computational results will be given in Section 3, followed by discussions on advantages, limitations, and possible future extensions of this research in Section 4.

2. Methodologies

2.1. Mathematical model

The proposed model is general and in principle can be used to address the question of how to protect or strengthen any type of network under limited resources. However, for the convenience of discussion, we only focus on seismic risks. The network retrofit problem is stated as: Which bridges should be retrofitted for given budget constraints and hazard estimates in order to reduce the potential system damage quantified by the total structural and travel delay loss? In the framework of two-stage stochastic programming, the first stage of the retrofit problem is to make retrofit decisions before the earthquake happens, while the second stage is to evaluate the total loss due to a realized earthquake including repair cost and increased travel delay in the network. The second stage cost (recourse cost) is a random variable dependent on the first-stage retrofit decision and the particular realization of bridge damages.

A common approach that considers a weighted mean-risk criterion [17] will be adopted to build a risk-averse model, in which the objective function consists of the expected cost and some dispersion statistic that can be used as an estimate of risk. In this paper, risk will be measured by central semideviation defined as $\delta_p[Y] = (\mathrm{E}[(Y - \mathrm{E}Y)_+^p])^{1/p}$. It has been mathematically proven [17] that the mean-risk objective $E[Y] + \eta\delta_p[Y]$ is convexity-preserving for all $p \geqslant 1$ and $\eta \in [0, 1]$.

2.1.1. Modeling assumptions

2.1.1.1. Network notations and flow assumptions. Let us denote a transportation network as $G(N, A)$, where N is the set of nodes of size n and A is the set of network links of size m. Denote $\overline{A}$ ($\overline{A} \subset A$) as the set of links that are subject to earthquake hazards and thus the candidates for retrofit. The size of $\overline{A}$ is $\overline{m}$. The binary decision variable u_a is 1 if link a is to be retrofitted and 0 otherwise. For each commodity[1] $k \in \{1 \ldots K\}$, $x^k \in R_+^m$ is the link flow vector, and $q^k \in R^n$ is the vector of demands and supplies of commodity k at each node. Denote f_a as the total flow on link a, i.e.,$f_a = \sum_{k=1}^K x_a^k$, $\forall a \in A$. In transportation network literature, traffic is often assumed to be in equilibrium condition, where no traveler can gain more by simply changing her own routing decisions. This assumption works well in a normal situation in which travelers can learn about and adapt to day-to-day traffic condition. However, how to model travelers routing behavior in a sudden changing environment (such as following a catastrophic disaster) is still arguable. In this paper, we assume that traffic flow can be controlled to achieve system optimal condition. The total costs estimated under this assumption can be considered as a lower bound of the costs in reality.

2.1.1.2. Damage scenarios. Seismic damage to a structure (highway bridges in this study) is usually classified into five categories, ranging from no damage to complete collapse. Advanced structural analysis can lead to probabilistic assessment of structural damage for a given earthquake, in terms of a set of discrete probabilities associated with each of the five damage categories. Seismologists, on the other hand, have predictions to the probabilities of various earthquake occurrences. The two sets of probabilistic estimations from earthquake-structural engineers and seismologists can be combined to prepare the damage prediction. For simplicity, we only consider a binary damage state, with 1 indicating being damaged and 0 otherwise. This assumption is merely for the convenience of discussion. It can be easily relaxed without changing the structure of the proposed model, as long as the data for supporting the more detailed analysis is available. Let the random vector $\boldsymbol{\xi}$ describe the uncertain events of link damages without considering any retrofit decision. Each realization of $\boldsymbol{\xi}$, ξ, and the corresponding probability $p(\xi)$ define a damage *scenario*. In this paper we represent the uncertainty in link damages with a given finite set of damage scenarios ξ^l, $l = 1, \ldots, L$, each with probability $p_l = prob[\boldsymbol{\xi} = \xi^l]$. In this work, we consider the discrete scenarios and the associated probabilities are given. In some applications, the random variables may follow continuous probability distributions, in which case Monte Carlo sampling techniques may be used to generate a finite set of discrete random scenarios.

[1] In transportation network literature, the flow between each origin-destination pair is often considered as one commodity. Different commodities represent travel between different origin-destination pairs.

2.1.1.3. Post-retrofit damage states. A random vector Ξ is introduced to represent the random event of link damages following an earthquake given any retrofit implementation. We assume that if a link is retrofitted, its probability of being damaged is zero. A more realistic way is to assume reduced but nonzero damage probabilities for retrofitted links. However, this choice would make the problem fall in the class of stochastic programming problems with decision dependent random events, which rely heavily on heuristic methods to solve problems with realistic sizes due to computational difficulties [18]. The relationship between the pre-retrofit link damage state ξ, the retrofit decision u, and the post-retrofit damage state $\Xi(\xi, u)$ is described as

$$\Xi(\xi, u)_a = \begin{cases} \xi_a(\xi_a - u_a), & \forall a \in \bar{A}, \\ 0, & \forall a \in A \backslash \bar{A}. \end{cases} \tag{1}$$

For a scenario, if the pre-retrofit damage state of link a is 1 ($\xi_a = 1$), but the link is retrofitted ($u_a = 1$), the value of $\Xi(\xi, u)_a$ is 0, indicating that the link will be intact under this scenario. On the other hand, if the link is not retrofitted but its pre-retrofit damage state is 1, the value of $\Xi(\xi, u)_a$ would be 1, indicating that the link will be damaged. If the pre-retrofit damage state of link a is 0 ($\xi_a = 0$), then the link is always in good condition no matter whether it is retrofitted or not.

2.1.2. Model formulation

With the above assumptions, the network retrofit problem is formulated in a two-stage mean-risk stochastic programming framework as follows.

Network retrofit problem (*NRP*):

$$\min_u \quad 0 + \mathop{E}_{\xi \in \Xi_d}\{Q(u, \xi)\} + \eta \mathop{E}_{\xi \in \Xi_d}\left\{\left[Q(u, \xi) - \mathop{E}_{\xi \in \Xi_d}\{Q(u, \xi)\}\right]_+\right\} \tag{2}$$

$$\text{s.t.} \quad \langle c_1, u\rangle \leqslant B, \tag{3}$$

$$u \in \{0, 1\}^{\overline{m}}, \tag{4}$$

with

$$Q(u, \xi) := \min_{x^k} \quad \langle c_2, \Xi(\xi, u)\rangle + \gamma\langle f, t(f)\rangle \tag{5}$$

$$\text{s.t.} \quad Wx^k = q^k, \quad \forall k = 1 \ldots K, \tag{6}$$

$$x^k \leqslant (e - \Xi(\xi, u))M, \quad \forall k = 1 \ldots K, \tag{7}$$

$$f = \sum_{k=1}^{K} x^k, x^k \in R_+^m, \tag{8}$$

where c_1 is the retrofit cost vector, B is the total budget for retrofitting, and c_2 is the repair cost vector. Vector e has all entries 1, i.e., $e_a = 1, \forall a \in A$. The link travel time t depends on the link flow f. Their relationship is usually described by a non-decreasing function such as the bureau of public roads (BPR) function. The notation W represents the node-link adjacency matrix, and M is an arbitrarily large positive number.

Condition (3) represents the budget constraint. Condition (4) simply restricts u to be binary. Expression (5) states the second-stage cost, which includes the repair cost term $\langle c_2, \Xi(\xi, u)\rangle$ and the weighted flow cost $\gamma\langle f, t(f)\rangle$, where γ is a weight coefficient converting time to monetary value. This cost becomes known once the earthquake hazard has been realized, thus is the recourse cost quantifying the effectiveness of the first-stage decision. Condition (6) is flow conservation constraint for second stage problem. Condition (7) restricts the link flow to zero if the link is damaged by the earthquake. Finally expression (2) describes our objective as to minimize the weighted sum of mean and risk of second stage cost. The weighting factor η trades off expected cost with risk. Note that the first stage retrofit cost is incorporated in the budget constraint, instead of contributing to the total system cost. System modelers may consider adding retrofit cost directly to the objective function. This modeling choice would not change the structure of the model.

For simplicity of the model presentation, we omit the superscripts in x^k and q^k in the rest of this paper. Hence constraints (6) and (7) are denoted as follows:

$$Wx = q, \tag{9}$$

$$x \leqslant (e - \Xi(\xi, u))M. \tag{10}$$

Under the assumption of finite discrete distributions of the uncertain parameters, used in this work, the deterministic equivalent program (DEP) of this formulation is a mixed-integer nonlinear program. As the size of the network and the number of damage scenarios increase, the DEP can become prohibitively large. Difficulties in solving large scale testing problems through direct usage of commercial software (e.g. Cplex 10.0 and SBB with nonlinear sub-solvers) motivate us to use alternative solution methods based on decomposition and exploiting the problem structure that can handle large size problems with reasonable computing and memory requirements.

2.2. Solution method

Van Slyke and Wets [11] introduced the L-shaped decomposition algorithm for stochastic linear programs, which greatly reduced the computational efforts required to generate a solution. The procedure takes advantage of the fact that the second-stage value function is convex and piecewise linear on a polyhedral domain, thus may be represented by a finite number of so-called feasibility and optimality cuts. It then proceeds to generating these cuts by solving successive linear programming problems. We follow this general approach to obtain a solution method based on the basic ideas of decomposition, linearization, and successive approximation. Our method is a special case of generalized Bender's decomposition, which we review next.

2.2.1. Decomposition

Benders decomposition (BD) method has been a classical method for solving large scale stochastic programming problems through decomposition and cutting planes method [20]. It was originally designed by Benders [19] to solve mix-integer linear problems, and later extended to nonlinear programs by Geoffrion [16]. Before presenting our solution algorithm, we review the basic ideas of generalized BD [16] for nonlinear problems.

BD is appropriate for problems with complicating variables, which, when temporarily held constant, render the remaining problem more tractable. It decomposes the problem into two parts through the projection of original problem onto the space of complicating variables. For example, consider the following problem:

$$\min f_1(u) + f_2(u, x) \quad \text{s.t. } G(x, u) \leqslant 0, \quad x \in X, \ u \in U. \tag{11}$$

Assume that $f_1(u), f_2(u, x)$, and $G(x, u)$ are convex functions, and that X is a convex set. Let the vector u represent the complicating variables. The projection of problem (11) onto the u-space is

$$\min f_1(u) + v(u) \quad \text{s.t. } u \in U \cap V, \tag{12}$$

where

$$v(u) := \inf_x \{f_2(u, x)\} \quad \text{s.t. } G(x, u) \leqslant 0, \quad x \in X, \tag{13}$$

and

$$V := \{u | G(x, u) \leqslant 0, \text{ for some } x \in X\}. \tag{14}$$

Note that V is the set of induced constrains, which restricts u to guarantee that $v(u)$ is feasible. Function $v(u)$ is the objective value of the optimization problem parameterized by u, which is called the value function. Both $v(u)$ and V are convex since they are projections of a convex function and a convex set, respectively. By the designation of u as complicating variables, evaluating $v(u)$ is much easier than solving problem (11). Problem (12)–(14) can simply be reformulated as

$$\min_{\theta,u} f_1(u)+\theta \quad \text{s.t. } \theta \geqslant v(u), \quad u \in U \cap V. \tag{15}$$

The original problem (11) is equivalent to problem (15) (see [16, Theorem 2.1]). Problem (15) can be solved by a cutting-plane method which explores the approximate representation of the convex set V and convex function $v(u)$.

2.2.2. Problem reformulation and relaxation

The structure of the two-stage formulation of problem (NRP) suggests a natural decomposition scheme: the network retrofit decisions are complicating variables, and once these are fixed, the sub-problem is a convex min-cost multicommodity network flow problem, for which efficient algorithms are available in the literature [21]. To see this, we simply rewrite the formulation (2)–(8) in a compact way:

Reformulated network retrofit problem $(R-NRP)$:

$$\min_{\theta,u} \theta \quad \text{s.t. } \theta \geqslant EQ(u)+\eta DQ(u)_+, \quad u \in U \cap V \tag{16}$$

with

$$p_l = prob[\boldsymbol{\xi} = \xi^l], \quad l=1,\dots,L, \tag{17}$$

$$EQ(u) = E\{Q(u,\boldsymbol{\xi})\} = \sum_{l=1}^{L} p_l Q(u,\xi^l), \tag{18}$$

$$DQ(u)_+ = DQ(u)_+ = \sum_{l=1:Q(u,\xi^l)>EQ}^{L} p_l[Q(u,\xi^l)-EQ], \tag{19}$$

$$Q(u,\xi^l) := \min_x \{\langle c_2, \Xi(\xi^l,u)\rangle + \gamma\langle f, t(f)\rangle | x \leqslant (e-\Xi(\xi^l,u))M, x\in X\}, \tag{20}$$

$$X := \{x|Wx=q, x\in R_+^m\}, \tag{21}$$

$$U := \{u|\langle c_1,u\rangle \leqslant B, u\in\{0,1\}^{\overline{m}}\}, \tag{22}$$

$$V := \bigcap_{l=1}^{L} V(\xi^l) = \bigcap_{l=1}^{L}\{u|x\leqslant(e-\Xi(\xi^l,u))M, \text{ for some } x\in X\}. \tag{23}$$

Note that U is a binary set, X is a polyhedral set, and V is a convex set. We refer to expression (20) and (21) as *sub-problem* $(SP(u,\xi^l))$, where $Q(u,\xi^l)$ is the value function of this sub-problem. Set V defines the induced constraints to retrofit decisions such that the second stage min-cost network flow sub-problem $SP(u,\xi^l)$ is feasible. One way of representing the induced constraints V and $V(\xi^l)$ is to solve the following optimization problem given $u=\hat{u}$:

Feasibility sub-problem $(FSP(\hat{u},\xi^l))$:

$$Q_0(\hat{u},\xi^l) := \min_{x,s} \quad \|s\|_1 \tag{24}$$

$$\text{s.t.} \quad x \leqslant (e-\Xi(\xi^l,\hat{u}))M+s, \tag{25}$$

$$x\in X, \quad s\geqslant 0, \tag{26}$$

where s is the slack variable, and $Q_0(\hat{u},\xi^l)$ is the value function of this minimization problem. Problem $FSP(\hat{u},\xi^l)$ is always feasible through constraints relaxation. If $Q_0(\hat{u},\xi^l)>0$, it means that problem $(SP(\hat{u},\xi^l))$ is infeasible for this particular choice of $\hat{u}$ and ξ^l. Therefore an alternative way of expressing constraint (23) is

$$0 \geqslant Q_0(u,\xi^l), \quad \forall l=1,\dots,L. \tag{27}$$

Then problem $(R-NRP)$ is equivalent to the following master problem with associated sub-problems $SP(u,\xi^l)$ and $FSP(u,\xi^l)$:

Master problem (M):

$$\min_{\theta,u} \theta \quad \text{s.t. } \theta\geqslant EQ(u)+\eta DQ(u)_+, 0\geqslant Q_0(u,\xi^l), \forall l=1,\dots,L, u\in U. \tag{28}$$

The optimal solution of problem (M) includes u^* and θ^*, which tells the first-stage optimal solution and the objective value to the original problem (NRP), respectively.

For each given scenario l, functions $Q_0(u,\xi^l)$ and $Q(u,\xi^l)$ are convex as the inf-projections on convex sets of convex functions defined by $FSP(u,\xi^l)$ and $SP(u,\xi^l)$, respectively. Function $EQ(u)+\eta DQ(u)_+$ is convex because function $Q(u,\xi^l)$ is convex and the mean-risk objective is convexity-preserving. The master problem (M) is solved through relaxation and outer linearization to $EQ(u)+\eta DQ(u)_+$ and $Q_0(u,\xi^l)$. At iteration step k we solve the following relaxed master problem M^k:

$$\min_{\theta,u} \quad \theta \tag{29}$$

$$\text{s.t.} \quad 0\geqslant Q_0(u^v,\xi^l)+\langle w_0^{v,l}, u-u^v\rangle, \quad \forall v\leqslant k: SP(u^v,\xi^l) \text{ infeasible}, \tag{30}$$

$$\theta \geqslant [EQ(u^v)+\eta DQ(u^v)_+] + \langle w^v, u-u^v >, \quad \forall v\leqslant k: SP(u^v,\xi^l) \text{ feasible for } l=1\dots L, \tag{31}$$

$$u\in U, \tag{32}$$

where $w_0^{v,l}\in\partial Q_0(u^v,\xi^l)$ and $w^v\in\partial\{EQ(u^v)+\eta DQ(u^v)_+\}$, and $\partial\{\cdot\}$ represents the subgradient.

Let (u^v,θ^v) be the solution to the master problem M^k. We then check for every ξ^l if the sub-problem $SP(u^v,\xi^l)$ is infeasible, namely $Q_0(u^v,\xi^l)>0$. If the l-th sub-problem is infeasible, we add a constraint $0\geqslant Q_0(u^v,\xi^l)+\langle w_0^{v,l}, u-u^v\rangle$ to the relaxed master problem M^k. This constraint is also called a *feasibility cut*. If the sub-problem $SP(u^v,\xi^l)$ is feasible for every ξ^l, we then proceed to check the optimality of current solution to M^k. If $[EQ(u^v)+\eta DQ(u^v)_+]-\theta^v$ is larger than certain tolerance $\varepsilon>0$, we add the constraint $\theta\geqslant[EQ(u^v)+\eta DQ(u^v)_+]+\langle w^v, u-u^v\rangle$ to M^k, which is also called an *optimality cut*. The algorithm proceeds by solving the relaxed master problem M^k and sub-problems (including $FSP(u^v,\xi^l)$ and $SP(u^v,\xi^l)$) iteratively. The optimal objective value of problem M^k, i.e. θ^v, defines a nondecreasing sequence of lowerbounds of the optimal objective value of the original problem (NRP), and the values of $EQ(u^v)+\eta DQ(u^v)_+$ defines a sequence of upper bounds. Thus the algorithm terminates when the gap between the upper and lower bounds is within the predefined tolerance ε. The detailed solution algorithm is provided in Section 2.2.3.

The computation of the solution involves evaluation of the subgradients $\partial\{EQ(u^v)+\eta DQ(u^v)_+\}$ and $\partial Q^0(u^v,\xi^l)$. A derivation for obtaining these subgradients is given in the appendix.

2.2.3. Solution procedure

The detailed procedure for obtaining a numerical solution to the network retrofit problem is as follows.

Benders decomposition (BD)-based Algorithm:

Step 0: Initialization. Set $v=0$, $k=0$.

Step 1: If $v=0$, let u^v be any feasible point in the domain U, and θ^v be $-\infty$. Otherwise, solve the relaxed master problem M^k. Denote the current optimal solution as (u^v, θ^v).

Step 2: For $l=1\ldots L$, solve feasibility sub-problem ($FSP(u^v, \xi^l)$).

(1) If $Q_0(u^v, \xi^l)>0$, it means that sub-problem $SP(u^v, \xi^l)$ is infeasible. The feasibility cut $0 \geqslant Q_0(u^v, \xi^l) + \langle w_0^{v,l}, u-u^v\rangle$ is generated and added to the problem M^k. Let $k=k+1$, $v=v+1$. Return to Step 1.
(2) Otherwise, if $Q_0(u^v, \xi^l)=0$ for all $l=1\ldots L$, go to Step 3.

Step 3: For $l=1\ldots L$, solve the sub-problem ($SP(u^v, \xi^l)$).

This problem must be feasible, since we have already passed the feasibility test.

(1) If sub-problem is unbounded, then the original problem (*NRP*) is unbounded. Thus stop the process.
(2) If sub-problem is bounded, we have the following cases:
   (a) if $[EQ(u^v)+\eta DQ(u^v)_+]-\theta^v \leqslant \varepsilon$, then stop. The solution (θ^v, u^v) is the optimal solution of problem (*NRP*);
   (b) otherwise, the optimality cut $\theta \geqslant [EQ(u^v)+\eta DQ(u^v)_+] + \langle w^v, u-u^v\rangle$ is generated and added to problem M^k. Let $k=k+1$, $v=v+1$. Return to Step 1.

The *finite convergence* of our algorithm is a direct consequence of the finiteness of the discrete feasible set U and the fact that no u^v can ever repeat itself in a solution to the master problem (M) (see [16, Theorem 2.4]).

3. Numerical examples

Numerical experiments were implemented using two case studies. The first case study has a relatively small size, and is used to validate the proposed solution procedure and to test the numerical efficiency of the algorithm. The second case study uses realistic seismic risk and cost data, and is included to demonstrate potential real world applications of the proposed methodologies.

3.1. Case study I: Sioux Fall City network

The first case study uses the well known Sioux Fall City road network (24 nodes and 76 links) [22] as shown in Fig. 1. It is assumed that six bi-directional highway bridges (labeled as A–F) are under potential threaten from future earthquakes and thus need to be retrofitted. The possible damage scenarios of these six bridges are considered as input data to the model. Here we use the independent probabilities given in Table 1 to generate a total of $2^6=64$ damage scenarios for the random vector $\boldsymbol{\xi}$ in problem (*NRP*). Note that assumption of independent probabilities is only for the convenience of generating test data. Probabilities of damage scenarios generated with consideration of correlation between individual bridge damage states can be used in the same manner as an input to the model. The *BPR* function is in the form of $t_a^0[1+\alpha(x_a/c'_a)^\beta]$, where t_a^0 and x_a are free flow travel time and flow for link a, respectively, c'_a is the "practical capacity'' of link a and is set to be 90% of the design capacity. The values of other model parameters are: $c_{1a}=1$, $c_{2a}=1.5$, $\gamma=1$, $\alpha=0.15$, and $\beta=4$. At this point we temporarily set η to zero and focus on the numerical performance of the decomposition algorithm. Later, η will be

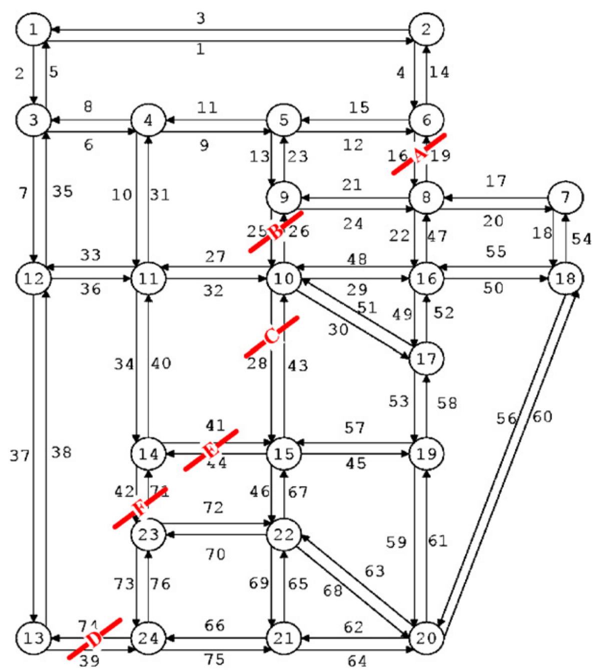

Fig. 1. Sioux Fall city network.

Table 1
Independent probability of bridge damage for generating the set of damage scenarios.

Bridge	A	B	C	D	E	F
Probability of damage	0.1	0.1	0.4	0.5	0.8	0.7

increased to show how consideration of risk may affect the retrofit strategies.

We consider two approaches to solve problem (*NRP*): using commercial solvers to solve the DEP directly or running the BD-based algorithm presented in Section 2.3.4. We investigate the efficiency of using these two approaches for different forms of link performance function ($\beta=4$ and 1). The DEP of the problem (*NRP*) considered is a mixed-integer nonlinear program with more than 110,000 variables (76 links $\times$ 24 origins 64 scenarios $=116,736$). The commercial package GAMS SBB[2] (Simple Branch and Bound) solver is used to solve the DEP directly.

3.1.1. Results on the efficiency of the solution method

When the budget for retrofit is set to be sufficient for only two bridges, there are 15 (C_6^2) possible retrofit solutions, in which case all possible retrofit solutions can be easily enumerated. We use the results from this enumeration as a benchmark for validating the accuracy of the proposed solution algorithm. Directly solving the DEP using commercial optimization solvers and solving the problem using the BD-based algorithm both return the correct solution (to retrofit bridges D and E). However, the computational efficiency resulting from the decomposition method is much better than solving (DEP) directly. For linear link performance functions, $t_a^0[1+\alpha(x_a/c'_a)]$, solving (*DEP*) directly using GAMS SBB solver took 3012 s; the BD-based algorithm, with CONOPT solving sub-problems, solved the problem in 290 s. When BPR link performance functions are considered, solving (*DEP*) directly using SBB solver took 22,817 s; the BD-based

[2] SBB is a GAMS solver for mixed integer nonlinear programming problems. It needs a nonlinear programming solver such as CONOPT to run.

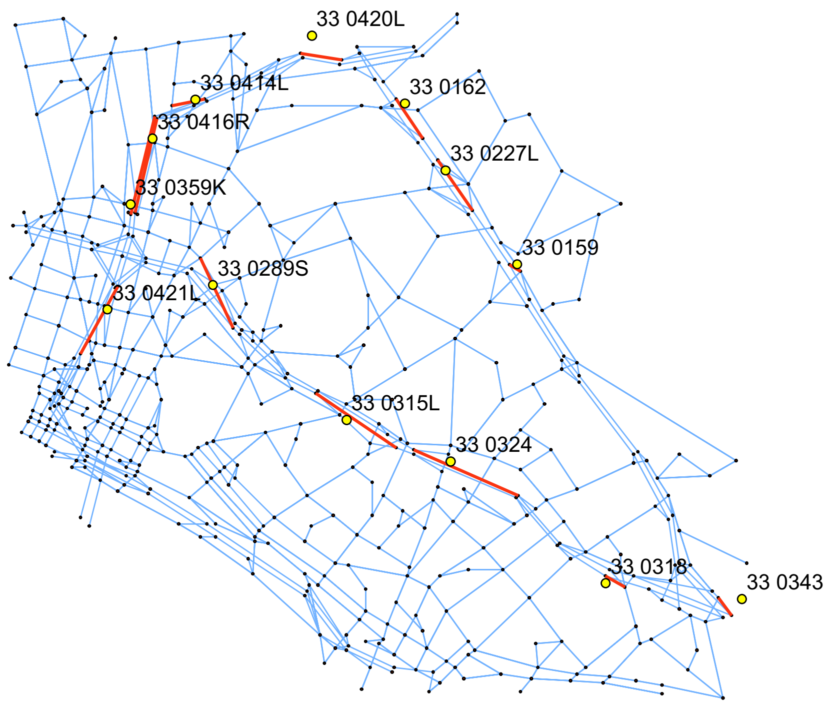

Fig. 2. Alameda County road network.

Table 2
Model input data: damage scenarios and cost data.

ID	National bridge index (NBI)	Replacement cost	Retrofit cost	Scenarios					
				1	2	3	4	5	6
A	33 0343	$833,833	$208,458	0	0	0	1	1	0
B	33 0318	$1,144,154	$286,038	0	1	0	1	1	0
C	33 0159	$1,024,100	$256,025	0	1	0	1	1	0
D	33 0324	$1,806,588	$451,647	0	1	1	1	1	0
E	33 0227L	$706,420	$176,605	0	1	0	1	1	0
F	33 0162	$1,980,990	$495,247	0	1	1	1	1	0
G	33 0315L	$3,878,490	$969,622	0	1	0	1	1	0
H	33 0420L	$1,737,450	$434,362	0	0	0	0	1	0
I	33 0289S	$489,940	$122,485	0	1	0	1	1	0
J	33 0414L	$1,361,008	$340,252	1	1	1	1	1	0
K	33 0416R	$9,007,614	$2,251,903	1	1	1	1	1	0
L	33 0359K	$1,746,030	$436,507	0	1	0	1	1	0
M	33 0421L	$5,871,690	$1,467,922	1	1	1	1	1	0
	Probability of each damage scenario (%)			7.6	11.3	6.2	7.7	1.6	66

algorithm, with CONOPT solving sub-problems, solved the problem in 859 s.[3]

Additional numerical experiments are conducted to test the performance of the BD-based algorithm in problems of different size. The performance of the algorithm is measured by the number of optimality cuts since it determines the number of NLP sub-problems to be solved. The problem difficulty is reflected by the size of the solution space of the first-stage integer variables, since these integer variables are the major complicating factors in the NRP problem. We now let retrofit decisions to be associated with each directional link, thus increasing the number of first-stage decision variables from six to 12. To speed up the experiment, only the 10 most likely scenarios are included in this test. It is observed that the rate of increase of the number of optimality cuts is smaller than the increase rate of the number of possible first-stage solutions. For example, as the number of possible retrofit solution increased about 40 times from C_{12}^1 to C_{12}^4, the number of optimality cuts only increased about six times from 9 to 61. This observation suggests that BD-based algorithms may be a favorable choice for problems where the first-stage integer variables are the major complicating factors. Numerical results from this case study also demonstrate that solving NRP problems through decomposition is much more efficient than direct use of commercial solvers as long as the problem size is nontrivial.

3.1.2. Results on the sensitivity of the solution

When η increases within the interval [0,1], the retrofit solution remains the same (to retrofit bridges D and E). When the mean term is removed from the objective, or equivalently when the risk term is weighed very highly, the retrofit solution changes to bridges C and D. This risk-averse solution trades off 4% increase in expected cost (from 46.4 to 48.3) with 18% reduction in the semi-deviation (from 1.5 to 1.2) by comparison with the risk-neutral solution ($\eta = 0$). In

[3] All the numerical results reported in this paper were computed using a Windows XP Dell Workstation with dual Intel(R) Xeon(R) CPU (2.40 GHz) and 3.5 GB RAM.

Table 3
Numerical results from the Alameda case study.

Retrofit budget (M$)	Number of feasible retrofit policies	Optimal retrofit policy	Total retrofit ost (M$)	Expected second-stage costs EQ (M$)	Number of optimality cuts	Computing time
0.5	15	33 0414L 33 0289S	0.463	12.442	10	1 h 14 min 18 s
4	1205	33 0416R 33 0421L	3.97 33 0159	7.688	98	9 h 42 min 34 s

this particular case study, the effect of risk consideration on retrofit strategy is observable, but not significant. However, the example demonstrates that (1) an optimal solution based on mean criterion may not be the most reliable; (2) different risk preference may affect retrofitting strategies.

3.2. Case study II: Alameda County network

The second case study uses a sub-network of California Alameda County road network (including highways and major local streets) as shown in Fig. 2, which includes 510 nodes, 1424 links, and 2401 origin–destination pairs.

Thirteen highway bridges in the study area are found vulnerable while being evaluated under 31 potential earthquake events that are likely to affect Alameda County. Most of these earthquake events are not severe enough to cause functional damage to the bridges. After aggregating all no-damage scenarios, we have a total of six damage scenarios to consider. The probabilities of these damage scenarios are computed based on the Poisson arrival rates of the 31 earthquake events and a 10-year planning horizon. Table 2 provides information of the damage scenarios and the retrofit and replacement costs of each candidate bridge. The structure damage estimation was provided by Prof. Anne Kiremidjian's research group at Stanford University. The replacement costs were provided to us by California Department of Transportation. The retrofit cost of a bridge is estimated to be one-forth of the corresponding replacement cost.

Parameter γ converts 2-h peak time delay to yearly (assuming reconstruction of a bridge takes one year) dollar value. It is set as $(\frac{1}{60})*8*365*20=973.3$, where $(\frac{1}{60})$ is to convert minutes to hours, 365 is to convert daily to yearly value, 20 is the average value of time for travelers in the study area, and 8 is the two-peak-hour conversion factor to daily impact estimated for the San Francisco Bay Area.[4] Link performance function is the same as the BPR function used in the previous case study.

In this particular case study, the risk-neutral and risk-averse models return the same solution. The computational experience reported below is based on setting $\eta=0$. Each sub-problem in the BD-based algorithm is a min-cost flow problem with more than 400K (1424 links $\times$ 49 origins $\times$ 6 scenarios $=418,656$) continuous variables. Two retrofit budgets are considered: 0.5M$ and 4M$, resulting in 15 and 1205 possible retrofit solutions, respectively. In both cases, the solution algorithm converged within a finite number of optimality cuts. Detailed numerical results are given in Table 3. From an engineering perspective, the following observations are made:

1. Retrofit strategy may change completely as budget varies. This suggests that a commonly used engineering approach that picks retrofit bridges based on their ranks may be questionable.

[4] This conversion factor is estimated based on peak duration and daily vehicle hours in year 2006 provided by Metropolitan Transportation Commission (unpublished).

Table 4
Performance of wait-and-see and the stochastic programming solutions.

	Wait-and-see policy	Scenario cost with perfect information (M$)	Expected cost over all scenarios (M$)
Scenario 1	33 0416R 33 0421L	5.81	7.90
Scenario 2	33 0359K 33 0421L 33 0289S 33 0315L 33 0159 33 0318 33 0324	17.51	8.79
Scenario 3	33 0416R 33 0421L	9.61	7.90
Scenario 4	33 0414L 33 0359K 33 0421L 33 0289S 33 0315L 33 0343 33 0324	18.32	8.70
Scenario 5	33 0414L 33 0359K 33 0421L 33 0289S 33 0315L 33 0343 33 0324	20.06	18.25
Scenario 6	None	4.45	13.02
Stochastic program	33 0416R 33 0421L 33 0159		7.69

2. Retrofit program has a positive impact on the society. For example, as the retrofit budget increases by 3.5M$ from 0.5M$ to 4M$, the total system cost (retrofit cost plus the expected repairing and time delay costs) decreases from 12.9M$ to 11.7M$. The gained benefit is about 10%.

3.2.1. Stochastic programming approach vs. wait-and-see approach

Wait-and-see approach [23] is a commonly used deterministic approach which seeks an optimal solution for each scenario, as if we could wait and see the realization of random events and then make decisions accordingly. Since wait-and-see approach provides a set of scenario-dependent solutions, simple heuristic rules are often used to aggregate these solutions to a single one that can be implemented.

Wait-and-see policies $u(\xi)$ for all the scenarios are reported in Table 4, given 4M dollars of retrofit budget. The "scenario cost with perfect information'' $Q(u(\xi)$ reports the recourse cost of each scenario when the corresponding wait-and-see policy is followed. This is the least possible cost for each scenario. The "expected cost over all scenarios'' evaluates the performance of the wait-and-see policies in an expected sense. As expected, the stochastic programming solution provides the least expected cost compared with wait-and-see policies. The difference ranges from 210 K to 10.56M dollars.

The wait-and-see solution [23], defined as $WS = E_{\xi}[Q(u(\xi), \xi)]$, is 5.03M$. Expected recourse cost from stochastic programming solution is 7.69M$. Therefore, the expected value of perfect information (EVPI) is $7.69 - 5.03 = 2.66$M$. The EVPI of stochastic programming solution suggests that effort in improving estimates of uncertain parameters is worthwhile, even though stochastic programming model may be less sensitive to imperfect information than its deterministic correspondents.

3.2.2. Value of stochastic programming solutions

Stochastic programming approach explicitly considers the entire range of uncertain scenarios thus hedging better against uncertainty than its deterministic correspondents. However, it also increases computational complexity dramatically. The concept of value of stochastic programming solution (VSS) [23] can be used to justify whether the extra effort on modeling and solving stochastic programming is worthwhile.

The VSS compares the value obtained by a commonly used engineering approach with the stochastic programming solution, defined by $VSS = EEV - SP$. Here u^* is the solution of the engineering approach, for instance the solution for the most likely scenario, with $EEV := E_{\xi}(Q(u^*, \xi))$ and $SP := \min_u E\xi Q(u, \xi)$. In general, a bigger VSS indicates higher benefit from using stochastic programming approach. In this case study, the VSS is 1.1M$, by comparison with the wait-and-see solution to the most likely scenario (Scenario 2), the relative gain is 12.6%. This relatively large value justifies use of more sophisticated modeling techniques and the extra computational efforts.

4. Discussion

The main contribution of this paper is the new formulation of the network retrofit problem in risk-averse stochastic programming framework, and the development of a solution algorithm that can solve the problem without excessive memory and computing time requirements. Nevertheless, we are still at an early stage of this research where the focus is mainly on theory and model development. Several issues arising from retrofit practice have yet to be considered. For example, from a construction view point, bridges are often grouped during a retrofit project and thus the retrofit decisions would be made over clusters instead of individual bridges. If the clusters are predefined, then the proposed formulation is still suitable. However, if clustering decision needs to be made simultaneously with the retrofit decision, this requirement would impose one more layer of complexity to the model. Other practical issues include considerations of the convenience and safety of detour during construction.

From computational view point, immediate extensions of this work include implementing the model to networks of larger sizes and find stable scenario reduction methods. In addition, there are several modeling related questions to be further investigated. In this study, we made a simplified assumption on network flow, i.e., the flow conforms to system optimum. Consideration of other behavior assumptions, such as travelers learning or user equilibrium, may change the structure of the problem. For example, assumption of equilibrium traffic condition leads to a stochastic mathematical programming model with equilibrium constraints (SMPEC). SMPEC is an important but computationally difficult problem, whose properties and solution schemes are still being explored (see e.g. [24–26]). The risk measure defined by positive mean deviation did not seem to play an important role in the case studies included here. In another word, the optimal stochastic programming solution happens to perform well in terms of both expectation and risk. However, such an observation is data specific and should not be generalized. In the future work, we may explore other risk-averse criteria, such as maximizing the probability of achieving a certain goal, guaranteeing certain level of reliability (also called the chance constrained model), or minimizing the regret in the worst case scenario (also called robust model) [27]. Understanding of how decision makers' risk preferences might affect their choices and eventually impact the effectiveness of the entire society will have significant policy implications.

The discussion of this paper is focused on highway networks. However, the modeling and solution methods are general and can be tailored to other transportation modes and a broad range of lifeline systems that can be analyzed as networks. It is our hope that this study will attract more research effort into this important subject of strategic resource allocation for critical infrastructure protection and hazard prevention.

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Appendix A. Subgradient of optimal value function

We first derive the subgradient in the general case and then apply it to the present problem. Define the value function of a convex program as

$$v(u) = \min_x \quad \{f_0(x, u)\} \tag{33}$$

$$\text{s.t.} \quad f_i(x, u) \leqslant 0, \quad i = 1, \dots, s, \tag{34}$$

$$f_i(x, u) = 0, \quad i = s+1, \dots, m, \tag{35}$$

$$(x, u) \in X \times U \subset R^{n_1} \times R^{n_2}, \tag{36}$$

where X is a closed convex set, for $i = 1, \dots, s$, the functions $f_i : R^{n_1} \times R^{n_2} \to R$ are convex and for $i = s+1, \dots, m$, the functions $f_i : R^{n_1} \times R^{n_2} \to R$ are affine. We note that the associated Lagrangian function to this program is

$$L(x, u, \lambda, \pi) = f_0(x, u) + \sum_{i=1}^{s} \lambda_i f_i(x, u) + \sum_{i=s+1}^{m} \pi_i f_i(x, u).$$

To evaluate $v(\hat{u})$, we may just fix u to be $\hat{u}$ and solve the resultant optimization problem. We have the following lemma on the subgradient of $v(u)$ at $\hat{u}$.

Lemma 1 (*Subgradient of optimal value function*). *Assume strong duality holds for $v(\hat{u})$ (for example in the case of satisfying Slater constraint qualification, i.e. $\exists$ feasible $(\hat{x})$, which in addition satisfies $f_i(\hat{x}, \hat{u}) < 0$, $i = 1, \dots, s$). Let x^*, λ^*, and π^* denote the optimal primal and dual solutions for $v(\hat{u})$, then*

$$\nabla_u L(x^*, \hat{u}, \lambda^*, \pi^*) = \nabla_u f_0(x^*, \hat{u}) + \sum_{i=1}^{s} \lambda_i^* \nabla_u f_i(x^*, \hat{u}) + \sum_{i=s+1}^{m} \pi_i^* \nabla_u f_i(x^*, \hat{u})$$

is a subgradient of value function $v(u)$ *at* $\hat{u}$, *i.e.*,

$$\nabla_u L(x^*, \hat{u}, \lambda^*, \pi^*) \in \partial v(\hat{u}).$$

Proof. First note that $\nabla_x L(x^*, \hat{u}, \lambda^*, \pi^*) = 0$ from the KKT conditions for problem (1). The Lagrangian function $L(x, u, \lambda, \pi)$ is convex with respect to x and u

$$\begin{aligned}
&\Rightarrow L(x, u, \lambda, \pi) \geqslant L(x^*, \hat{u}, \lambda, \pi) \\
&\quad + \nabla_u L(x^*, \hat{u}, \lambda, \pi)^{\mathrm{T}}(u - \hat{u}) \\
&\quad + \nabla_x L(x^*, \hat{u}, \lambda, \pi)^{\mathrm{T}}(x - x^*) \\
&\quad = L(x^*, \hat{u}, \lambda, \pi) \\
&\quad + \nabla_u L(x^*, \hat{u}, \lambda, \pi)^{\mathrm{T}}(u - \hat{u}) \\
&\Rightarrow \inf_x L(x, u, \lambda, \pi) \geqslant L(x^*, \hat{u}, \lambda, \pi) \\
&\quad + \nabla_u L(x^*, \hat{u}, \lambda, \pi)^{\mathrm{T}}(u - \hat{u}).
\end{aligned}$$

From strong duality we have $v(u) = \sup_{\lambda \geqslant 0, \pi} \inf_x L(x, u, \lambda, \pi) \geqslant \inf_x L(x, u, \lambda, \pi)$. Thus we have $v(u) \geqslant \inf_x L(x, u, \lambda, \pi) \geqslant L(x^*, \hat{u}, \lambda, \pi) + \nabla_u L(x^*, \hat{u}, \lambda, \pi)^{\mathrm{T}}(u - \hat{u})$. Evaluating the both sides of last equation at λ^*, π^* yields $v(u) \geqslant L(x^*, \hat{u}, \lambda^*, \pi^*) + \nabla_u L(x^*, \hat{u}, \lambda^*, \pi^*)^{\mathrm{T}}(u - \hat{u})$.

Invoking strong duality again we have that $L(x^*, \hat{u}, \lambda^*, \pi^*) = v(\hat{u})$. Substituting this in the previous inequality gives $v(u) \geqslant v(\hat{u}) + \nabla_u L(x^*, \hat{u}, \lambda^*, \pi^*)^{\mathrm{T}}(u - \hat{u})$, which is the definition of subgradient, thus $\nabla_u L(x^*, \hat{u}, \lambda^*, \pi^*) \in \partial v(\hat{u})$. □

Applying the above lemma directly to the evaluation of $\partial Q_0(u^v, \xi^l)$ and $\partial Q(u^v, \xi^l)$ yields

$$\nabla_u \left[\|s\|_1 + \sum_{i \in \bar{A}} \lambda_i^* (x_i^* - \xi_i^l M u_i^v - M + M(\xi_i^l)^2) \right] = \begin{bmatrix} -\xi_{\bar{A}_1}^l \lambda_{\bar{A}_1}^* M \\ \vdots \\ -\xi_{\bar{A}_{\bar{m}}}^l \lambda_{\bar{A}_{\bar{m}}}^* M \end{bmatrix} \in \partial Q_0(u^v, \xi^l),$$

$$\nabla_u \left[\langle c_2, \Xi \rangle + \gamma \langle x^*, t(x^*) \rangle + \sum_{i \in \bar{A}} \lambda_i^* (x_i^* - \xi_i^l M u_i^v - M + M(\xi_i^l)^2) \right] = \begin{bmatrix} -\xi_1^l (c_{2_{\bar{A}_1}} + \lambda_{\bar{A}_1}^* M) \\ \vdots \\ -\xi_{\bar{A}}^l (c_{2_{\bar{A}_{\bar{m}}}} + \lambda_{\bar{A}_{\bar{m}}}^* M) \end{bmatrix} \in \partial Q(u^v, \xi^l).$$

According to [17], the subgradient of mean-semideviation function is calculated as

$$s \in \partial \{EQ(u) + \eta DQ(u)_+\}.$$

Note

$$\pi(\xi^l) = \begin{bmatrix} -\xi_1^l (c_{2_{\bar{A}_1}} + \lambda_{\bar{A}_1}^* M) \\ \vdots \\ -\xi_{\bar{A}}^l (c_{2_{\bar{A}_{\bar{m}}}} + \lambda_{\bar{A}_{\bar{m}}}^* M) \end{bmatrix} \in \partial Q(u^v, \xi^l).$$

Then $s = E[\pi] + \eta E_{Q(u,\xi) \geqslant EQ(u)}[(\pi(\xi) - E(\pi))]$.

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