Dynamic pricing strategies in the presence of demand shocks

Omar Besbes∗
Columbia University

Denis Saure†
University of Chile

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Abstract

We study the pricing problem of a retailer facing the prospect of a change in the demand function during a finite selling season with no inventory replenishment opportunity. In particular, the time and the form of change in demand are unknown upfront and we focus on the fundamental trade-off between collecting revenues from current demand and doing so for post-change demand, those being linked through the capacity constraint. We consider three representations of uncertainty about future demand, ranging from stochastic to adversarial views. Our analysis highlights the role dynamic pricing plays in hedging against future demand environments. Using a semi-deterministic relaxation, we show that optimal pricing strategies possess significant structure. When the time of change is independent of its form and a probabilistic description is available, we establish that the retailer can “order” the current and future demand environments to decide how to spread inventory consumption throughout the horizon. We further show that an optimal policy is always monotone up to a change and the direction of price changes depends on the aforementioned ordering. We establish that this structure extends to the case in which the time of change may be selected in an adversarial manner but not to the case in which both the time and form of change may be selected in an adversarial fashion. In the latter case, time and form of change would in general be linked and optimal policies may be non-monotonic.

Keywords: revenue management, dynamic pricing, non-stationary demand, model uncertainty

1 Introduction

Pricing decisions play a fundamental role in managing demand in many industries, and often in settings in which inventory replenishment decisions are infrequent. In this context, price-based revenue management arises naturally as a means for adjusting to the demand’s intrinsic stochastic variability. Dynamic pricing has spread to a variety of industries including airlines, hotels and retailing; see Talluri and van Ryzin (2005) for a comprehensive overview of the practice of revenue management.

The revenue management literature has traditionally considered settings in which the relationship between price and demand, referred to as the demand function, is known and stationary. A

∗Columbia Graduate School of Business, e-mail: ob2105@columbia.edu
†University of Chile, e-mail: dsaure@dii.uchile.cl
prototypical model is that of Gallego and van Ryzin (1994) which considers a finite horizon with no replenishment opportunity and the seller maximizes cumulative revenues through dynamic pricing. Nevertheless, seasonality effects and uncertainty regarding future demand are often present in practice; indeed, there are several factors that might induce shocks in demand such as the introduction of a competitive product, the release of influential product reviews or quality reports, the success of a marketing campaign, or the weather just to name a few examples. While in some settings retailers can anticipate certain aspects of future demand environments, in many practical settings these aspects are rather unpredictable.

When facing the prospect of a future change in demand, firms face multiple challenges. Real-time demand learning (see, e.g., Broder and Rusmevichientong (2010)) introduces an exploration-exploitation trade-off while uncertainty in the timing of a change brings forward the need for continuous tracking and monitoring (see, e.g., Besbes and Zeevi (2011)). In the absence of capacity constraints, it has been shown that learning and detection may be performed efficiently in the sense that one may approach the performance of a retailer unaware of the possibility of a change in demand but to whom new demand conditions are revealed once the change has taken place. However, when temporal uncertainty in demand is present in conjunction with capacity constraints, there is a “first order” problem; that of judiciously spreading inventory consumption to hedge against possible future changes. In particular, a fine trade-off between pre- and post-change revenue collection exists and dynamic pricing may play a key role in that regard, in addition to the classical roles associated with adjusting to demand variability and price experimentation.

**Main objectives and approach.** Our main focus is on the structure of dynamic pricing strategies one should adopt when future demand is uncertain. In particular, we aim to understand how the prospect of a change in the demand environment affects the retailer’s pricing decisions, the main factors influencing such decisions, and how these prescriptions vary according to different representations of future demand uncertainty. To this end, we study a family of stylized prototypical problems with finite selling horizons and fixed initial inventories where the retailer seeks to maximize the cumulative revenue by adjusting prices in a dynamic fashion. Initially operating in a known demand environment, the decision maker anticipates that the demand function might change in the future; however, neither the form nor the time of the change are known.

We analyze three informational settings. In the first setting, the retailer knows the joint distribution of the time and form of the change in demand, and maximizes the expected cumulative revenues collected over the sales horizon. In the second setting, the retailer only knows the marginal distribution of the form of demand change. Now, the retailer anticipates an adversarial realization of the time of change and attempts to minimize the worst-case difference between the revenues

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1For example, demand for hotels has been reported to rise in the aftermath of hurricanes in Florida, with a persisting effect for several months (Tan 2006).
of an oracle with access to the time of change (but no a-priori access to the form of post-change demand) and his/her performance. In other words, we focus on a minimax regret objective, where “nature” might select the time of change in an adversarial manner. In the last setting, the retailer does not have any probabilistic model for the time and form of the change demand, and anticipates adversarial realizations. Now, the retailer minimizes the difference in revenue relative to an oracle that knows both the time and form of the change in advance.

One may envision the above as a hierarchical representation of uncertainty about a potential demand change: when the firm has prior information on the likelihood of the time and form of change, s/he will use such information to maximize her/his expected revenues; when only a prior on the form of demand change is available, the retailer still plans to maximize cumulative revenues, but considering a strategy that is robust against any possible time of change realization; when no information is available, the retailer aims to maximize revenue considering strategies that are robust against any possible change in demand.

We focus on a semi-deterministic relaxation that captures the key trade-offs associated with the problem of hedging against future change in demand. In addition, the formulation enables to isolate the structure induced on dynamic pricing strategies from various other sources of dynamic price adjustments (such as experimentation or reactions to stochastic fluctuations of demand). We further comment in Section 5 on how one may combine the results of this paper with those associated with real-time detection and learning of a new demand environment.

Main results and contributions. The first contribution lies in the formulation of the problem and the ability to crisply identify the underlying pricing structure that may be induced by future changes in demand, in particular when a probabilistic description of the set of possible post-change demand functions is available. At a high level, when the time and the form of a change are independent, we show that regardless of the view the decision-maker takes on the time of change (probabilistic or adversarial), the pre- and post-change demand environments may essentially be ordered; and based on this ordering, the decision-maker decides whether to price more or less aggressively than in the absence of the possibility of a change. The ordering is quite subtle as it depends on the possible set of scenarios one may face and on the initial level of inventory. We further show that this ordering is maintained throughout the selling horizon when using an optimal policy. This also always implies a monotone pricing structure up to a change. This notion of ordering disappears when both the time and form may be selected in an adversarial fashion or are realizations from a known probability distribution but are not independent.

In the first setting, in which probabilistic descriptions of the time and form of change are available, our study relies on the analysis of the stochastic optimization problem induced for the decision-maker. We first establish monotonicity of prices for the case of independent time and form of demand change and then provide sufficient conditions for monotonicity in the general case. This
pricing structure highlights the trade-off faced by the decision-maker. If the current inventory position and the set of possible future scenarios render a post-change demand more attractive, then one should price at a higher level than in the absence of a change, to save units for later. However, while no change occurs, one should slowly decrease the price as there is less time to capitalize on the potential change. The order among pre- and post-change demand environments may be established through the computation of an index that does not depend on the optimal policy.

When the retailer anticipates an adversarial realization of the time of change, we show that when the marginal distribution of the form of demand change is stationary (we define this concept formally in Section 4), then future and present demand can again be ordered as in the first setting, and similar conclusions follow. Our results show that by anticipating an adversarial selection of the time of change the retailer is equivalently adopting a “worst” prior marginal distribution on the time of change, and we derive this distribution. When no distribution is available and “nature” may select both the time and the form of demand in an adversarial manner, the same premise with regard to a worst-case distribution holds. However, in this case, the worst possible distribution adopted under the adversarial assumption will not in general preserve the order of current and future demand throughout the selling horizon (even under an optimal policy). As a result, one will not in general hedge against a change in demand using monotone pricing policies. In addition, we further derive results on the “right” notion of performance to track in an adversarial environment when setting prices in a dynamic fashion.

Broadly speaking, our work contributes to the revenue management literature by identifying fundamental structural properties of the problem of hedging against changes, enabling to crisply understand how and when one may order current and future market environments and the implications for how aggressively one should price.

The remainder of the paper. The rest of this section reviews related work. Section 2 formulates the dynamic pricing problem under future demand uncertainty. Section 3 analyzes the case of priors on future scenarios, while Section 4 studies the case of adversarial selection of the future demand environment. Finally, Section 5 presents our conclusions. Proofs are presented in Appendix A.

Literature review. Several revenue management studies have parted from the assumption of known and stationary demand model but very few consider the conjunction of both effects. For the case of time-homogeneous demand settings, there is a growing number of studies that analyze heuristics for jointly learning and pricing and that provide fundamental lower bounds on performance. Araman and Caldentey (2010) provide a review of some of the approaches taken in the literature, including parametric and non-parametric ones (see also Wang et al. (2011) and the references therein for a recent study). In the absence of capacity constraints, Broder and Rusmevichientong (2010), den Boer and Zwart (2010) and Harrison et al. (2011) study the fine balance between exploration and exploitation.
The studies on non-stationary demand environments are much less common. Gallego and van Ryzin (1997) study dynamic pricing in a changing demand environment when the temporal evolution of the demand model is known in advance. In a related study, Levin et al. (2009) consider real-time demand learning in a dynamic pricing formulation in which strategic consumer behavior drives non-stationarity of demand. There, demand is driven by a game-theoretic discrete choice model that depends on initially unknown parameters, thus temporal evolution of demand is encoded in a parametric form.

In the absence of inventory constraints, Keller and Rady (1999) study the dynamic pricing problem of a monopolist facing future demand uncertainty. There, a set of parameters driving the demand function evolve according to a continuous time hidden Markov chain. The authors identify two regimes that are characterized by extreme or moderate experimentation levels. Besbes and Zeevi (2011) study the problem of real-time detection of a change in the demand environment and highlight the extent of experimentation required to monitor the market.

Very few studies consider the combination of temporal uncertainty in demand in conjunction with capacity constraints. Besbes and Maglaras (2009) formulate the dynamic pricing problem of a revenue-maximizing make-to-order manufacturer operating in an environment in which an unobservable market size varies stochastically over time. They devise near-optimal policies using a stochastic fluid model approximation. In a related study and closer to our setting, Chen and Farias (2010) analyze a stochastic fluid approximation of a variant of the model of Gallego and van Ryzin (1994), where the authors consider a Gaussian market share process. The authors propose heuristic re-optimized fixed price policies for the case in which the market share process is observed. The latter two studies focus on prescriptions to track (explicitly or implicitly) demand changes and to hedge against uncertainty, when demand changes over time in a “continuous” manner. These studies restrict attention to the evolution of the market size parameter and do no consider, e.g., changes in price sensitivity.

The problem we study is also related to the so-called capacity booking problem. In its simplest instance, a decision-maker with finite capacity sees two classes of customers arriving sequentially over time, each with a given (and known) willingness to pay distribution. Fares are fixed and the decision-maker needs to decide how many seats to reserve for the high fare class. The solution of this problem is given by the famous Littlewood rule (Littlewood 1972). The current paper can be seen as a capacity booking problem in which the decision-maker has dynamic pricing capabilities, and neither the arrival time nor the willingness to pay distribution of the second type are known in advance. When viewed through that lens, a related study is that of Ball and Queyranne (2009) who analyze the capacity booking problem when the order of arrivals may be generated in an adversarial fashion. The present formulation is less conservative with respect to the order of arrivals and provides pricing flexibility to the decision-maker.
2 Problem Formulation

2.1 Model Primitives

We consider the pricing problem faced by a retailer offering a single product to a stream of price-sensitive customers over a finite horizon. The retailer, initially endowed with $x$ units of the product, must select the price $p(t)$ to charge to consumers arriving at time $t$, for all $t$ in $[0, T]$, where $T$ denotes the length of the horizon. Consumers arrive to the retailer’s store according to a stationary point process with rate $\Lambda$. A consumer arriving at time $t$ purchases a unit of the product if her/his willingness to pay is greater or equal than $p(t)$; where a consumer arriving at time $t$ has a willingness to pay distributed according to $F_t$, independent of everything else.

The retailer is familiar with the initial demand environment, i.e., s/he knows $F_0$, and anticipates that $F_t = F_0$ until some unknown time $\tau \in [0, T]$; at that time, the retailer expects to face a new distribution $F_\tau$ that remains constant for the rest of the horizon (i.e, $F_t = F_\tau$ for $t \in [\tau, T]$). The time of change $\tau$ and the new distribution $F_\tau$ are not known a priori. The retailer only knows that a demand change can occur at specific times, i.e.,

$$\tau \in \{t_0, t_1, \ldots, t_N\} \ a.s.,$$

where we assume that $0 = t_0 < t_1 < \ldots < t_N = T$. We use $n \in \mathcal{N} := \{0, \ldots, N\}$ to index possible times of change in demand. Similarly, the retailer knows that the post-change willingness to pay distribution belongs to a certain class, i.e.,

$$F_\tau \in \left\{ F^k : k \in \mathcal{K} \right\} \ a.s.$$

We use $k$ to index possible future willingness to pay distributions, and let $\theta$ denote its realization. While most of our results hold for the case of an arbitrary set $\mathcal{K}$, for simplicity we assume this set has finite cardinality $K$, i.e. $\mathcal{K} := \{1, \ldots, K\}$. We assume that $F_0(0) = 0$ and $F_0(M) = 1$, and that $F^k(0) = 0$ and $F^k(M) = 1$ for $k \in \mathcal{K}$, for some finite $M > 0$. $M$ is the largest price that any customer would be willing to pay for the product.

We assume that the retailer seeks to maximize the cumulative revenue collected over the finite selling horizon by adjusting prices in a dynamic fashion. Let $v(t)$ and $v_\theta(t)$ denote the willingness to pay of a consumer arriving at time $t < \tau$ and $t \geq \tau$, respectively. We let $p$ denote a non-anticipating pricing policy\footnote{We define the class of policies we consider formally in the next section.} generating the set of prices $\{p(t), t \in [0, T]\}$. The revenue collected by applying policy $p$, as a function of the time ($\tau$) and form ($\theta$) of demand change is given by

$$\pi(p, \tau, \theta) := \int_0^\tau p(t) \mathbf{1}\{v(t) \geq p(t)\} \, dN(t) + \int_\tau^T p(t) \mathbf{1}\{v_\theta(t) \geq p(t)\} \, dN(t),$$
where $N(t)$ denotes the total number of consumers that have arrived up to time $t$ and $\mathbf{1}\{\cdot\}$ denotes the indicator function. In addition, we further restrict attention to those policies that satisfy the capacity constraint

$$
\int_0^\tau \mathbf{1}\{v(t) \leq p(t)\} \, dN(t) + \int_\tau^T \mathbf{1}\{v_\theta(t) \leq p(t)\} \, dN(t) \leq x \quad a.s.
$$

### 2.2 A Semi-deterministic Relaxation

Uncertainty in the formulation above stems from various sources: the arrival of customers is stochastic, and so is their willingness to pay; in addition, both $\tau$ and $\theta$ are initially unknown. The specific features of our formulation, and the observation above, suggest that the optimal policy might not be tractable nor provide any insight on the various drivers of pricing, as a policy needs to balance multiple roles ranging from detecting a change in demand, learning a new demand environment, to properly spread inventory consumption across the horizon to hedge against a potential future change. In the present paper, we aim to focus on the latter problem.

To isolate the challenge of hedging against a future change in the demand, we consider a semi-deterministic relaxation of the retailer’s problem. In particular, we assume that both $\tau$ and $\theta$ are revealed to the retailer at time $t = \tau$. In addition, we assume that customers arrive at a deterministic rate of $\Lambda$ customers per unit of time, and that a deterministic fraction $(1 - F_t(p(t)))$ of consumers arriving at time $t \in [0, T]$ buy a unit of the product. These assumptions eliminate the role a pricing policy plays in detecting and learning changes in the environment: revealing $\tau$ eliminates the necessity of actively monitoring the demand environment, and revealing $\theta$ eliminates the need for price experimentation to learn it.

The resulting semi-deterministic relaxation also eliminates the role pricing policies play in profiting from the inherent variability of stochastic demand, which arises even in the case of stationary demand conditions. However, this relaxation preserves the order of information revelation in the problem: the decision-maker still ignores the time $\tau$ and form $\theta$ of the change in demand up until $\tau$. Thus, the key uncertainty and trade-offs of the problem associated with hedging against a future change in demand are maintained.

**Decision-maker’s objective.** The semi-deterministic relaxation of the retailer’s problem may be rewritten as follows. For $k \in \mathcal{K}$ and $\lambda \in [0, \Lambda]$ let $p_k(\lambda)$ denote the maximum price that induces...
a sales rate of $\lambda$ when the willingness to pay is distributed according to $F^k$, i.e.,

$$\Lambda \left( 1 - F^k(p(\lambda)) \right) = \lambda.$$ 

Similarly, for $k \in K$ and $\lambda \in [0, \Lambda]$, define $r_k(\lambda)$ as the revenue per unit of time under $F^k$ when the sales rate is $\lambda$, i.e., $r_k(\lambda) = \lambda p_k(\lambda)$. In addition let $r(\lambda)$ denote the revenue per unit of time under $F_0$ when the sales rate is $\lambda$, i.e. $r(\lambda) = \lambda p(\lambda)$, where $p(\lambda)$ is such that $F_0(p(\lambda)) = 1 - \lambda/\Lambda$.

We assume that $r(\cdot)$ and $r_k(\cdot)$ for $k \in K$, are twice differentiable and strictly concave on $[0, \Lambda]$. We restrict attention to the class of non-anticipative policies (those that do not know $\tau$ and $\theta$ in advance). The cumulative revenue $\pi(p, \tau, \theta)$ achieved by an admissible policy $p$ is given by

$$\pi(p, \tau, \theta) := \int_0^\tau r(\lambda(t))dt + \int_\tau^T r(\lambda(t))dt,$$

where $\lambda(t)$ stands for the sales rate induced by price $p(t)$ at time $t$. The first and second terms on the right-hand-side above correspond to the revenue generated prior to and after the demand change, respectively. Note that, once $\tau$ and $\theta$ are realized, and contingent on the current inventory position, the retailer faces an stationary demand.

For $t \in [0, T]$, $k \in K$ and $y \leq x$, let $V_k(t, y)$ denote the optimal cumulative revenue generated by a retailer with an initial inventory of $y$ units over a horizon of length $(T - t)$, when the willingness to pay distribution during the entire horizon remains constant and equal to $F^k$. That is,

$$V_k(t, y) := \max \left\{ \int_t^T r_k(\lambda(s))ds : \int_t^T \lambda(s)ds \leq y, \lambda(s) \in [0, \Lambda] \forall s \in [t, T] \right\}.$$ 

Note that the maximization problem above corresponds to a “classical” deterministic single product dynamic pricing problem, as studied in Gallego and van Ryzin (1994). For $k \in K$, let $\lambda^*_k$ be the unique maximizer of $r_k(\cdot)$, i.e.

$$\lambda^*_k := \arg\max \{ r_k(\lambda) : \lambda \in [0, \Lambda] \} \quad k \in K.$$ 

Similarly, we let $\lambda^*$ denote the unique maximizer of $r(\cdot)$.

It is possible to establish (see Gallego and van Ryzin (1994, Proposition 2)) that the optimal revenue generated after the demand-change is given by

$$V_k(t, y) = (T - t) \min \{ \lambda^*_k , y/(T - t) \}.$$ 

(1)

The above implies that the optimal policy induces the rate

$$\lambda^*(y, t, k) := \min \{ \lambda^*_k , y/(T - t) \}$$

when the willingness to pay distribution changes to $F^k$ at time $\tau = t$, and inventory at that time is $y$. This observation implies that an admissible policy $p$ can be mapped into a unique function

\footnote{Our assumptions imply that $\lambda^*_k < \Lambda$ and $\lambda^* < \Lambda$, and thus that $r_k'(\lambda^*_k) = r'(\lambda^*) = 0$, $k \in K$.}
\( \lambda : [0, T] \rightarrow [0, \Lambda] \), where \( \lambda(t) \) corresponds to the sales rates induced by policy \( p \) at time \( t \) when no change in demand has occurred up to that instant. Let \( \mathcal{L} \) denote the set of feasible sales rates functions, i.e.,

\[
\mathcal{L} := \left\{ \lambda : \lambda(t) \in [0, \Lambda] \text{ for } t \in [0, T], \int_0^T \lambda(t) dt \leq x \right\}.
\]

With a slight abuse of notation, we define \( \pi(\lambda, \tau, \theta) \) as the revenue achieved by \( \lambda \in \mathcal{L} \) over the selling horizon as a function of the demand change realization. That is,

\[
\pi(\lambda, \tau, \theta) := \int_0^\tau r(\lambda(t)) dt + V_\theta \left( \tau, x - \int_0^\tau \lambda(t) dt \right).
\]

For any given \( \lambda \in \mathcal{L} \) the expression above can be evaluated once \( \tau \) and \( \theta \) are known. Thus, the appropriate selection of \( \lambda \in \mathcal{L} \) ultimately depends on the retailer’s assumptions regarding \( \tau \) and \( \theta \)\(^6\).

Next, we focus on three such informational settings ranging from distributional assumptions over \( \tau \) and \( \theta \) to an adversarial formulation with regard to those.

### 3 The Case of Priors on Future Scenarios

This section studies the case in which the retailer envisions both \( \tau \) and \( \theta \) as random variables and knows the likelihood associated with any possible realizations of such a pair. The retailer maximizes the expected cumulative revenue over the selling horizon.

Let \( V(t, y) \) denote the expected optimal revenue collected by a retailer with initial inventory of \( y \) units over a horizon of length \( (T - t) \), when the willingness to pay distribution during the entire horizon remains constant, but it is initially chosen randomly from the post-change class conditional on the event \( \tau = t \). That is, for \( t \in \{t_n : n \in \mathcal{N}\} \),

\[
V(t, y) := \mathbb{E}\{V_k(t, y)|\tau = t\}.
\tag{2}
\]

One can show that \( V \) is concave and differentiable in its second argument (see Lemma 1 in Appendix A.1). The retailer’s optimal pricing policy, as an element of \( \mathcal{L} \), is given by the solution to the following optimization problem

\[
J(x) := \max \left\{ \mathbb{E}\left\{ \int_0^\tau r(\lambda(t)) dt + V_\theta \left( \tau, x - \int_0^\tau \lambda(t) dt \right) \right\} : \lambda \in \mathcal{L} \right\},
\tag{3}
\]

where \( J(x) \) denotes the expected cumulative revenue generated by any optimal pricing policy, as a function of the initial inventory position \( x \).

\(^6\)Note that in the absence of capacity constraints the retailer’s problem becomes trivial, as it is in her/his best interest to induce sales rates \( \lambda^* \) and \( \lambda^*_0 \) before and after the change in demand, respectively.
The concavity of the revenue function implies that any solution to (3) must apply a constant sales rate in between any two consecutive possible demand-changing times. Thus, one can restrict attention to policies of such type, which we denote by $L'$. We let $\lambda := (\lambda_0, \ldots, \lambda_{N-1})$ represent a generic element in such a class of policies, where $\lambda_n$ denotes the sales rate that $\lambda \in L'$ applies between times $t = t_n$ and $t = t_{n+1}$, for $n \in \tilde{N} := N \setminus \{N\}$. With this, we can rewrite formulation (3) as follows.

$$J(x) = \max \left\{ \sum_{n \in \tilde{N}} \left( r(\lambda_n) \Delta_n \mathbb{P}\{\tau > t_n\} + V\left(t_n, x - \sum_{j=0}^{n-1} \lambda_j \Delta_j\right) \mathbb{P}\{\tau = t_n\} \right) : \lambda \in L' \right\},$$

where $\Delta_n := t_{n+1} - t_n$, $n \in \tilde{N}$. Note that no additional constraints are imposed on the $\lambda_n$’s, i.e. one is not worried about having $\lambda_n > \Lambda$ for some $n \in \tilde{N}$ (one might justify that $\lambda_n \leq \lambda^*$ for all $n \in \tilde{N}$).

The formulation above maximizes a continuously differentiable concave function subject to affine constraints (inventory constraint and non-negativity of rates), thus there exists an optimal vector of sales rates and the Karush-Kuhn-Tucker conditions (see, e.g. Boyd and Vandenberghe (2004)) are necessary and sufficient to characterize this vector.

An important case we analyze is that in which form and time of demand change are independent, as formalized in the following assumption.

**Assumption 1.** The form $\theta$ and time $\tau$ of demand change are such that

$$\mathbb{P}\{\theta = k | \tau = t_n\} = \mathbb{P}\{\theta = k | \tau = t_0\} \quad k \in K, \ n \in N.$$

Assumption 1 implies independence through “stationarity” of the conditional distribution of $\theta$. In Section 3.2 we extend our results to the general case of correlated random variables.

The next result highlights the significant structure that the optimal policy possesses when Assumption 1 holds.

**Theorem 1** (Monotonicity of the optimal strategy). Let Assumption 1 hold. The optimal sales rates are always monotonic. In particular, if $\lambda_0 \geq x/T$, then the sequence of sales rates $(\lambda_n : n \in \tilde{N})$ is non-increasing. Conversely, if $\lambda_0 \leq x/T$, then the sequence of sales rates $(\lambda_n : n \in \tilde{N})$ is non-decreasing.

Theorem 1 establishes that the possibility of a future change in demand induces a monotone pricing structure up to a change. In general, prices will be different along time until the realization of change. The monotonicity direction depends on the initial optimal sales rate and how it
compares to the rate that would uniformly deplete inventory over the course of the horizon $x/T$. In proving Theorem 1, we further show that the optimal sales rates always stay either above or below the corresponding depletion rates, and that these depletion rates are also always monotone along an optimal path. This implies that the retailer is consistent in her/his attitude towards demand uncertainty along an optimal path. For example, if the retailer initially depletes inventory aggressively (i.e. at a pace that is not sustainable), then later in the horizon such a bias will be maintained but in a more limited fashion.

Theorem 1 states that the optimal policy is either non-increasing or non-decreasing depending on the initial sales rate, which is endogenously determined. Next, we show that comparing such a rate to the initial depletion rate is equivalent to “ordering” the current demand and future demand according to a specific criterion that does not require solving for the optimal policy.

Recall that in the absence of a potential change in demand, the optimal sales rate is constant and given by $\min \{ \lambda^*, x/T \}$. Define $\rho$ as the difference between the marginal value of a unit of inventory absent demand change and that expected under new demand conditions. That is,

$$
\rho(x, T) := r'(\min \{ \lambda^*, x/T \}) - V_x(0, x). 
$$

Broadly speaking, $\rho$ quantifies the difference in marginal revenue associated with not having a change in demand at all, or having it immediately: to see this note that (See proof of Lemma 2)

$$
V_x(0, x) = \mathbb{E} \left\{ r'_{\theta} (\min \{ \lambda^*_\theta, x/T \}) \right\}.
$$

Intuitively speaking, a non-negative $\rho$ indicates that one should impose an initial sales rate above the depletion rate if one is to balance current and future instantaneous marginal revenues. Similar conclusion holds when $\rho$ is non-positive (note this is always the case when $\lambda^* < x/T$). This, when combined with Theorem 1 suggests that the direction of monotonicity of optimal sales rate can be determined from the sign of $\rho$. The next proposition formalizes this intuition.

**Proposition 1** (monotonicity direction). Let Assumption 1 hold. Then $\lambda_0 \geq x/T$ if and only if $\rho \geq 0$. Hence, the sequence of optimal sales rates is non-increasing if $\rho \geq 0$ and non-decreasing if $\rho \leq 0$.

Proposition 1 establishes that a retailer for whom current demand conditions compare favorably to expected future conditions will sell aggressively initially, adjusting such behavior through the selling season, but without changing his preference for current demand conditions. (The same observation holds for the case of a retailer that “prefers” future conditions over current conditions.) Consider for instance, a retailer that depletes inventory when no change is expected to occur: if the (expected) marginal value of a unit under future demand is low enough, then the retailer will initially deplete inventory faster than when no change is expected, and will reduce such a rate as
time progresses if no change occurs; on the other hand, if the (expected) marginal value of a unit under future demand is above some threshold, then the retailer will deplete inventory slower than when no change is expected, and will increase this rate as time progresses, if no change occurs. It is important to note that such an ordering of the different demand environments not only depends on the possible realizations of demand functions but also on the inventory on hand. We illustrate this in Example 2 below.

3.1 Examples

We illustrate how Proposition 1 can be used to determine the shape of the sequence of optimal sales rates when \( \tau \) and \( \theta \) are independent.

Example 1 (The case of unknown price sensitivity). Consider the case in which

\[
F^k(p) = 1 - \beta ((a - e_k p)^+) \quad p \in \mathbb{R}_+, 
\]

where \( \beta(\cdot) \in [0,1] \) is an arbitrary increasing continuously differentiable function, \( e_k > 0 \) for all \( k \in K \), and \( z^+ \) denotes \( \max\{z,0\} \), and assume that \( F_0(\cdot) = F^1(\cdot) \). We assume that \( \beta(0) = 0 \), and that \( r(\lambda) \) admits a unique maximizer \( \lambda^* > x/T \). One can check that \( r_k(\lambda) = r(\lambda)(e_1/e_k) \) for all \( k \in K \) and \( \lambda \in [0,\Lambda] \), hence \( \lambda_k^* = \lambda^* \) for all \( k \in K \). Using Proposition 1 one can check that \( \lambda_0 \geq x/T \) if and only if

\[
e_1 \mathbb{E}\{e_{\theta}^{-1}\} \leq 1. \tag{5}
\]

The above implies that one can have both increasing or decreasing sales rates strategies even when one constrains price sensitivity, for example, to increase on average (i.e., \( \mathbb{E}\{e_{\theta}\} \geq e_1 \)) in the event of a change in demand conditions. Indeed, in such a case, the decision-maker might still want to sell in a slow fashion initially and save units for post-change demand. Figure 1 illustrates the case \( \beta(x) = x/10 \), and \( a = \Lambda = 10 \). The instances depicted in Figure 1 consider \( x = 10 \) and \( T = 10 \), with \( \tau \) uniformly distributed in \( \{0,1,\ldots,10\} \), and independent of \( \theta \). There, \( K = 3, e_1 = 1.0, e_2 = 2.5 \) and \( e_3 = 0.5 \). Graph (a) corresponds to the setting \( \mathbb{P}\{\theta = 2\} = 1 - \mathbb{P}\{\theta = 3\} = 3/10 \), and graph (b) to \( \mathbb{P}\{\theta = 3\} = 1 - \mathbb{P}\{\theta = 2\} = 3/10 \). Solid lines indicate the pre-change sales rates along the optimal path, for both settings. Dotted lines depict the depletion rate at each decision point along the optimal path. While both instances are such that \( \mathbb{E}\{e_{\theta}\} \geq e_1 \), the one in panel (a) is such that \( \mathbb{E}\{e_{\theta}^{-1}\} > e_1^{-1} \), hence the monotonicity of the optimal sales rates is as the one predicted by (5). The instance in panel (b) on the other hand is such that \( \mathbb{E}\{e_{\theta}^{-1}\} < e_1^{-1} \), and results in decreasing optimal sales rates. Note that, in both cases, depletion rates are either always above or below the optimal rates, as expected.

Example 2 (The case of unknown multiplicative factor/market size). Consider the case of

\[
F^k(p) = 1 - e_k \exp (-p) \quad p \in \mathbb{R}_+, 
\]
where $e_k > 0$, $k \in K$, with $e_1 = 1$, and assume that $F_0(\cdot) = F^1(\cdot)$. In this setting one has that

$$r_k (\lambda) = \lambda \left( \ln \Lambda + \ln e_k - \ln \lambda \right) \quad \lambda \in [0, \Lambda e_k], \ k \in K.$$ 

One can show that $\lambda^*_k = \lambda^* e_k$, thus, for $\Lambda/e > x/T$, one has that

$$\rho = (\ln \Lambda - 1 - \ln (x/T)) P \{ \Lambda e_k/e < x/T \} - E \{ 1 \{ \Lambda e_k/e > x/T \} \ln e_k \}.$$ 

Thus, if $e_k > 1$ for all $k \in K$, then the optimal sales rates are non-increasing or non-decreasing depending on whether $E \{ \ln e_k \}$ is positive or negative. However, in general, the sign of $\rho$ depends on $x/T$ as well. To illustrate this, consider the case in which $\Lambda = 1$, $T = 10$, $K = 3$ with $e_2 = 2.5$, $e_3 = 0.1$, and $P \{ \theta = 3 \} = P \{ \theta = 2 \} = 0.5$. Figure 2 illustrates the optimal pre-change sales rates for different values of $x$: graph (a) corresponds to $x = 1$ (for which $\rho > 0$), and graph (b) corresponds to $x = 2$ (for which $\rho < 0$). Solid lines indicate the pre-change sales rates along the optimal path, for both settings. Dotted lines depict the depletion rate at each decision point along the optimal path. We see that the view of the future market vis-a-vis the current one depends critically on the amount of inventory on hand.

### 3.2 Correlated Time and Form of Change

When Assumption fails to hold, optimal sales rates are not necessarily monotonic, as the following example illustrates.
Figure 2: Example 2. The graphs (a) and (b) illustrate pre-change sales rates along the optimal path for instances where $x = 1$ and $x = 2$, respectively.

Example 3 (non-monotonicity). Consider a setting with a time horizon $T = 10$ and initial inventory $x = 10$. As in Example 1, we assume that $F^k(p) = 1 - (1 - e_k p/10)^+$, $p \in \mathbb{R}_+$, that $F_0 = F^1$, $e_k > 0$, $k \in K$ with $e_1 = 1$, and that $\Lambda = 10$. In addition, suppose that $\tau$ is uniformly distributed in $\{0, 1, \ldots, 10\}$, and that $\theta$ is such that

$$
P \{ e_\theta = 1/2 + (T - t_n)/5 \mid \tau = t_n \} = 1, \quad \text{for all } n \in \mathcal{N}.
$$

Figure 3 depicts the pre-change sales rates and the corresponding depletion rates along the optimal path in this setup. In this example, optimal sales rates are initially decreasing, when the possibility of a change in demand implies switching to a more elastic demand. When a change in demand implies switching to a less elastic demand we observe the sales rates become increasing. Note that depletion and optimal rates cross.

While optimal sales rates need not to be monotonic in general, the above suggests that local monotonicity properties might be derived from those of the continuation function $V$. In particular, Example 3 suggests that monotonicity of optimal sales rates will occur when the expectation about future demand gets progressively better or worse with the time of change, relative to an initially worse or better initial demand environment, respectively. With this in mind, consider the following definition.
**Figure 3**: Example 3. The graph illustrates the pre-change sales rates and corresponding depletion rates along the optimal path in Example 3.

**Definition 1** (Monotone expected future marginal profit). *We say expected future marginal profit is non-decreasing (non-increasing) if, for any fixed $z \in \mathbb{R}_+$, $V_x(t, z (T-t))$ is a non-decreasing (non-increasing) function of $t$."

When $\tau$ and $\theta$ are independent one can check that (see proof of Lemma 2) 

$$V_x(t, z (T-t)) = V_x(0, z T) \quad \forall z \in \mathbb{R}_+,$$

for $t < T$, hence expected future marginal profit is constant. Consider the setting of Example 1: one can show that 

$$V_x(t, z (T-t)) = 1 \{z \leq \lambda^*\} \mathbb{E} \{e_1/e_{\theta} | \tau = t\} \quad \forall z \in \mathbb{R}_+,$$

hence expected future marginal profit is non-decreasing (non-increasing) when, for example, $e_{\theta}^{-1}$ conditional on $\tau$ is first-order stochastically increasing (decreasing) in $\tau$.

Our next result uses this notion of evolution on post-change demand priors to establish sufficient conditions for monotonicity of optimal sales rates.

**Theorem 2** (Monotonicity of the optimal strategy continued). *Suppose expected future marginal profit is non-decreasing and that $r'(\lambda) \geq V_x(t_{N-1}, \lambda \Delta_{N-1})$ for all $\lambda \in [0, \lambda^*]$, then the sequence of optimal sales rates is non-increasing. Conversely, if expected future marginal profit is non-increasing and $r'(\lambda) \leq V_x(t_{N-1}, \lambda \Delta_{N-1})$ for all $\lambda \in [0, \lambda^*]$, then the sequence of optimal sales rates is non-decreasing.*

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8That is, the distribution of $\theta_k$ conditional on $\tau = t$ is first-order stochastically larger/lower than that conditional on $\tau = s$ for $t \leq s$. 

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Broadly speaking, this result says that monotonicity holds when future changes preserve the ordering established in condition [4] for all periods.

4 The case of adversarial selection of demand change

This section considers settings in which the retailer has incomplete knowledge about the stochastic behavior of the change in demand, and plans her/his pricing strategy based on an adversarial framework. We first analyze the case in which the retailer only knows the behavior of \( \theta \) (for each possible time of change), but not that of \( \tau \). Then, we study the case in which the probabilistic descriptions of both \( \theta \) and \( \tau \) are unavailable, and these values are assumed to be selected in an adversarial fashion.

4.1 Adversarial selection of time of change

Here, we assume the retailer knows the marginal distribution of \( \theta \) conditional on \( \tau = t_n \), for all \( n \in \mathcal{N} \), but does not know the marginal distribution of \( \tau \). In this situation, the retailer opts to hedge against an adversarial realization of \( \tau \). Thus, from the retailer’s perspective, \( \tau \) is not a random variable but rather an unknown quantity.

For a given feasible sales rate function \( \lambda \) in \( \mathcal{L} \), let \( \pi(\lambda, \tau) \) denote the expected revenues generated by \( \lambda \) when demand changes at time \( t = \tau \). This is,

\[
\pi(\lambda, \tau) := \mathbb{E}\left\{ \pi(\lambda, \tau, \theta) \mid \tau \right\}.
\]

(This expectation only requires knowing the conditional distribution of \( \theta \).) Define \( D(\tau) \) as the maximal revenues attainable by a retailer that knows \( \tau \) at time \( t = 0 \). This is

\[
D(\tau) := \sup \{\pi(\lambda, \tau) : \lambda \in \mathcal{L}\}.
\]

We assume that the retailer, who does not possess such a piece of information, aims to minimize the revenue loss relative to this semi-oracle revenue while considering all possible realizations of \( \tau \). For that, define \( R(x) \) as the minimum revenue loss (across all feasible sales rate functions) relative to \( D(\tau) \), when \( \tau \) is selected in an adversarial fashion. That is

\[
R(x) := \inf \left\{ \sup \{D(t_n) - \pi(\lambda, t_n) : n \in \mathcal{N}\} : \lambda \in \mathcal{L}\right\}.
\]

We refer to \( R(x) \) as the retailer’s minimax regret when the initial inventory is \( x \). Note that this formulation can be seen as a Stackelberg game between the decision-maker and “nature”, where the former first selects a policy and then the latter counters such a policy by selecting the worst possible time of change.
As in Section 3, one can restrict attention to rate functions in $L'$. With this, one can rewrite (6) as

$$R(x) = \min \left\{ \max_{n \in \mathcal{N}} \left\{ D(t_n) - \sum_{j=0}^{n-1} r(\lambda_j) \Delta_j - V \left( t_n, x - \sum_{j=0}^{n-1} \lambda_j \Delta_j \right) \right\} : \lambda \in L' \right\}. \quad (7)$$

The formulation above minimizes a convex function subject to affine constraints. We next show that one may derive further insights on the structure of an optimal policy by establishing a strong connection between Formulations (3) and (7). For that, consider the following equivalent formulation of (7).

$$\min \ z \quad (8a)$$

$$s.t. \quad D(t_n) - \sum_{j=0}^{n-1} r(\lambda_j) \Delta_j - V \left( t_n, x - \sum_{j=0}^{n-1} \lambda_j \Delta_j \right) \leq z, \quad n \in \mathcal{N} \quad (8b)$$

$$z \in \mathbb{R}, \lambda \in L'. \quad (8c)$$

Proposition 2. The objective function in Problem (7) is convex, and any optimal solution coincides with an optimal solution of Problem (3) when

$$\mathbb{P} \{ \tau = t_n \} = \mu_n \quad n \in \mathcal{N}, \quad (9)$$

where the probability distribution $\mu := (\mu_0, \ldots, \mu_N)$ is the vector of shadow prices associated with the constraints (8b).

Note that, in the current setting, the retailer cannot solve formulation (3) as the marginal distribution of $\tau$ is unknown. Proposition 2 essentially says that an adversarial view about $\tau$ is equivalent to assuming a “worst-case” marginal distribution for $\tau$ and solving (3) using such a proxy. The result implies that both Theorems 1 and 2 hold in this setting (note that those results are independent of the marginal distribution of $\tau$). In this context, the next corollary, which we state without proof, is a natural extension of Proposition 1 to the case of adversarial selection of the time of change.

Corollary 1 (Monotonicity of optimal strategy in an adversarial environment). Suppose Assumption 1 holds. Then the optimal sales rates associated with Problem (7) are always monotonic. Moreover, if $\rho \geq 0$, then such a sequence is non-increasing (with $\rho$ defined as in (4)); conversely, if $\rho \leq 0$, the sequence is non-decreasing.

Note that one would have $\mu_n = 0$ when the corresponding constraint (8b) is not tight, therefore demand will not change at $t = t_n$ under the worst-case marginal distribution; this implies that $\lambda_n = \lambda_{n-1}$ for such $n$. Hence, the optimal sales rates will be held constant in between periods an adversary would not select for changing demand, and $\mu$ only assigns positive probability to periods in which the regret is maximal under the optimal solution. This is illustrated in the following example.
Example 4 (adversarial time of change). Consider the setting in Example 1 in which the retailer does not know the marginal distribution of $\tau$. Figure 4 depicts the optimal sales rates and the regret associated with it as a function of the time of change when $P\{\theta = 2\} = 1 - P\{\theta = 3\} = 3/10$. There, graph (a) depicts the optimal pre-change sales rates under distributional (solid line) and adversarial (dotted line) selections of the time of change. Graph (b) on the other hand depicts the regret attained along the optimal path for both the case of distributional (solid line) and adversarial (dotted line) beliefs as a function of the time of change $\tau$. In the distributional setting, $\tau$ is distributed uniformly over the possible times of change. We see that the optimal sales rates are constant between $t = 0$ and $t = t_3$, when the regret associated with the optimal solution peaks; afterwards, optimal sales rates increase, and the associated regret remains constant. Inspection of the shadow prices of constraints (8b) reveals that $\mu_0 = \mu_1 = \mu_2 = 0$, as expected.

Sub-game perfection. Note that sales rates applied after the worst-case selection of $\tau$ still play a key role in countering nature: subsequent actions by the retailer should not result in a higher future regret. This means that, when envisioned as a Stackelberg game between the retailer and nature, the resulting equilibrium must be sub-game perfect. However, sub-game perfection must be achieved while considering a consistent performance criterion throughout the horizon.

To illustrate the above, consider for instance a retailer facing a stochastic demand using the policy emanating from solving (7) in a heuristic fashion. Such an approach would call for resolving

Figure 4: Solution to Example 4. The graphs (a) and (b) illustrate the optimal sales rates and associated regret for the setup in Example 1, respectively. The solid line represents the distributional case and the dotted line represents the adversarial case.
at some intermediate time $t$, conditional on a change not occurring, when having an inventory of $y$ units. In doing so, the retailer might be tempted to minimize the regret in this new setup (the regret-to-go), i.e. solving for $R(y)$ using a modified horizon $T' = T - t$. The next example shows that this is in fact suboptimal.

**Example 5** (Minimal regret-to-go). Consider the setting in Example 4. Figure 5 compares the policy and regret (as a function of the time of change) achieved by the optimal policy (dotted line) to that of a retailer that at each time $t = t_n$ resolves after updating the inventory and horizon (solid line).

Figure 5: **Example 5**. The graphs (a) and (b) compare the sales rates and regret associated with the optimal policy (dotted lines) with those of the policy that each period minimizes the regret-to-go (solid lines), respectively.

The myopic retailer is inadvertently modifying the performance criterion in each period, comparing his/her revenues against the performance of an oracle who took the same decisions he/she took in the past. In this regard, suboptimality of the resulting policy stems precisely from inconsistent adjustments to the performance criterion. We show next that a “consistent” retailer would, in each period, instead focus on minimizing a partial regret-to-go.

Formally, for $n \in \mathcal{N}$ and $y \in [0, x]$, define $R_n(y)$ as the partial regret experienced by the decision-
maker when s/he has \( y \) units of inventory at time \( t_n \). That is,

\[
R_n(y) := \min \left\{ \max \left\{ D_j - \sum_{i=n}^{j-1} r_i \Delta_i - V(t_j, y - \sum_{i=n}^{j-1} \lambda_i \Delta_i) : n \leq j \leq N \right\} : \lambda \in \mathcal{L}_n'(y) \right\},
\]

where \( \mathcal{L}_n'(y) := \left\{ \lambda \in \mathcal{L}' : \sum_{j=n}^{N-1} \lambda_j \Delta_j \leq y \right\} \) (note that this definition discards revenues collected up to time \( t_n \)).

**Proposition 3 (Dynamic formulation).** The sequence of partial regret functions \( R_n(\cdot) \) satisfies the following recursion

\[
R_n(y) = \max \left\{ D(t_n) - V(t_n, y), \min \left\{ R_{n+1}(y - \lambda \Delta_n) - r(\lambda) \Delta_n : 0 \leq \lambda \leq \frac{y}{(T - t_n)} \right\} \right\},
\]

\[
R_N(y) = D(T), \tag{10}
\]

for all \( y \in [0, x] \). In addition, Problems (7) and (10) are equivalent in the following sense: i.) \( R_0(x) = R(x) \) for all \( x \geq 0 \); and ii.) the sales rates function \( \lambda \in \mathcal{L}' \) solving (10) corresponds to the optimal sales rates function for (7).

The dynamic formulation (10) says that the retailer selects the purchase rate for period \( n \) considering that the next period may be the one that nature selects for the demand change if this choice yields the maximum (partial) regret accumulated through the rest of the horizon. This formulation lends a “physical” significance to the rates selected in each period.

### 4.2 Adversarial Selection of the time and form of demand change

Here, we assume the retailer does not know the joint or marginal distributions of \( \tau \) and \( \theta \). In this situation, we assume the retailer opts to hedge against an adversarial realization of both.

Let \( D(\tau, \theta) \) denote the optimal revenues level attained by a retailer that knows upfront the time \( \tau \) and form \( \theta \) of demand change. That is,

\[
D(\tau, \theta) := \sup \left\{ \pi(\lambda, \tau, \theta) : \lambda \in \mathcal{L} \right\}.
\]

Such level of revenue is, of course, unlikely to be achieved by any feasible policy. We assume the retailer looks to solve the following problem

\[
R(x) := \inf \left\{ \max \{ D(n, k) - \pi(\lambda, n, k) : n \in \mathcal{N}, k \in \mathcal{K} \} : \lambda \in \mathcal{L} \right\}. \tag{11}
\]

We will refer to \( R(x) \) as the retailer’s mini-max regret when the initial inventory is \( x \). Like formulation (6), the above can be seen as a game between the retailer and nature, but now the latter selects both the time and type of demand change in an adversarial manner.
Equivalent formulation. As in Section 4.1, one can restrict attention to rate functions in $L'$. Thus, one can write (11) as

$$R(x) = \min \left\{ \max_{n \in \mathcal{N}, k \in \mathcal{K}} \left\{ D(t_n, k) - \sum_{j=0}^{n-1} r(\lambda_j) \Delta_j - V_k \left( t_n, x - \sum_{j=0}^{n-1} \lambda_j \Delta_j \right) \right\} : \lambda \in \mathcal{L}' \right\}$$

(12)

As in the previous section, one may evaluate $R(x)$ by solving the following alternative formulation.

$$\min z \quad \text{(13a)}$$
$$\text{s.t.} \quad D(t_n, k) - \sum_{j=0}^{n-1} r(\lambda_j) \Delta_j - V_k \left( t_n, x - \sum_{j=0}^{n-1} \lambda_j \Delta_j \right) \leq z, \quad n \in \mathcal{N}, \ k \in \mathcal{K} \quad \text{(13b)}$$
$$z \in \mathbb{R}, \lambda \in \mathcal{L}'. \quad \text{(13c)}$$

The next result extends that of Proposition 2 to this setting.

**Corollary 2.** The objective function in Problem (13) is convex, and any optimal solution coincides with the optimal solution of Problem (3) when

$$\mathbb{P} \{ \tau = t_n, \ \theta = k \} = \mu_{n,k} \quad n \in \mathcal{N}, \ k \in \mathcal{K}, \quad \text{(14)}$$

where the probability distribution $\mu := (\mu_{n,k} : n \in \mathcal{N}, \ k \in \mathcal{K})$ is the vector of shadow prices associated with the constraints (13b).

According to Proposition 2, solving (13) is equivalent to considering a “worst-case” distribution for $\tau$ and $\theta$ and solving (3) using such a proxy. In the present case, this worst-case distribution does not necessarily satisfy the requisites of Theorem 2, suggesting that the optimal policy might fail to be monotonic. This follows from the fact that no structure can be imposed a priori on $\mu$. This is, condition (14) does not necessarily correspond to neither the case of independent $\theta$ and $\tau$, nor to the case of monotone expected future marginal revenue. This point is illustrated next.

**Example 6** (non-monotonicity). Consider the setting of Example 4. Figure 6 shows the optimal sales rate and associated regret when no distributional assumption is made over $\theta$ nor $\tau$. Graph (a) depicts the pre-change sales rates along the optimal path, and graph (b) depicts the revenue loss (as a function of the time of change) relative to an oracle retailer that knows the time of change and whether $\theta = 2$ (dotted line) or $\theta = 3$ (solid line). As in Section 4.1, one can show that $\mu_{n,k} = 0$ for all pairs $(n, k) \in \mathcal{N} \times \mathcal{K}$ such that the regret associated with the event $\{\tau = t_n, \ \theta = k\}$ is not maximal. In this instance in particular, one has that the worst-case distribution assigns positive probability only to events $\{\theta = 2, \tau \in \{5,10\}\}$ and $\{\theta = 3, \tau \in \{5,6,7,10\}\}$, hence the adversary takes full advantage in selecting correlated time and form of the change.

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Figure 6: **Solution to Example 6** Graph (a) illustrates the optimal pre-change sales rates for the setup in Example 6. Graph (b) depicts the regret associated with all possible realizations of $\tau$ and $\theta$

**Remark 1.** As in section 4.1, one may construct a one dimensional dynamic programming formulation equivalent to (12). In particular, let

$$R_n(y) = \max \left\{ D(t_n, k) - V_k(t_n, y) : k \in K, \min \left\{ R_{n+1}(y - \lambda \Delta_n) - r(\lambda) \Delta_n : 0 \leq \lambda \leq \frac{y}{(T - t_n)} \right\} \right\},$$

$$R_N(y) = \max \{ D(T, k) : k \in K \},$$

for all $y \in [0, x]$. One can show that formulations (12) and (15) are equivalent in the sense of Proposition 3. In this regard, a retailer solving (12) dynamically over time still should focus on minimizing the partial regret-to-go.

## 5 Conclusion

This paper has analyzed the role price-based revenue management plays in distributing inventory over the horizon in settings when there is uncertainty regarding future demand. To do so, we considered a semi-deterministic relaxation of a prototypical revenue management setting that enabled to preserve the uncertainty surrounding future demand while isolating the role that dynamic pricing plays for trading off pre- and post-change revenues. Our results shed light on dynamic pricing settings with non-stationary and uncertain demand.

The semi-deterministic relaxation abstracted away from the issues associated with adjusting to stochastic fluctuations of demand, as well as learning and monitoring market changes. In general,
with the foundation this paper provides, we anticipate that one may be able to design prescriptions for general stochastic settings in which all challenges are present at once. This would involve combining the present trade-off with the exploration-exploitation tension introduced by the need to learn the market (see, e.g., Broder and Rusmevichientong (2010)), as well as that associated with detecting a change in demand (see, e.g., Besbes and Zeevi (2011)).

We have focused on settings in which changes in demand are infrequent, motivated by the fact that some of these are often driven by particular events. This is one end of a spectrum of possibilities for temporal uncertainty, with the other one being that of gradual and continuous changes, such as, e.g., those considered by Chen and Farias (2010) for the market parameter. Non-stationary environments are more the norm than the exception in practical revenue management settings, and yet have been quite unexplored in the literature. There are many avenues for future research, ranging from the appropriate modeling of temporal uncertainty to the delineation of the boundaries of what is possible in terms of performance when facing such environments.

References


A Proofs of Main Results

A.1 Preliminaries

In proving our results we use a pair of side lemmas. The first of these establishes the concavity and differentiability of the continuation function $V$.

**Lemma 1.** The function $V(t, \cdot)$ is concave and continuously differentiable, for all $t \in [0, T]$.

**Proof.** The result holds trivially for $t = T$, so consider $t < T$. For $y \in [0, x]$, (1) and (2) imply that

$$V_x(t, y) = \sum_{k \in K} r_k \left( \frac{y}{T-t} \right) 1 \{ \lambda^*_k (T-t) \geq y \} dP \{ \theta = k | \tau = t \}.$$  

(A-1)

Note that the term inside the integral above is continuous and non-increasing in $y$, hence so it is $V_x$. Thus, one concludes that $V(t, \cdot)$ is continuously differentiable and concave. \hfill $\square$

The second auxiliary result establishes a property of the continuation function key to establishing monotonicity of the optimal sales rates.

**Lemma 2.** If the expected future marginal profit is non-decreasing, then $V_x(t, y(T-t)) \leq V_x(s, z(T-s))$ when $y \geq z$ and $t < s$. Similarly, if the expected future marginal profit is non-increasing, then $V_x(t, y(T-t)) \geq V_x(s, z(T-s))$ when $y \leq z$ and $t < s$. 

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Proof. Consider the case of non-decreasing expected marginal profit. From the concavity of \( r(\cdot) \), one has that
\[
r_k'(y) \leq r_k'(z), \quad k \in \mathcal{K}.
\]
Similarly, one has that \( 1 \{ \lambda^*_k \geq y \} \leq 1 \{ \lambda^*_k \geq z \}, \; k \in \mathcal{K} \). Thus, one has that
\[
V_x(t, y(T - t)) \overset{(a)}{\leq} V_x(s, y(T - s)) \leq V_x(z, y(T - s)),
\]
where \((a)\) follows since expected marginal profit is non-decreasing, and \((b)\) follows from the observations above and \([A-1]\).

The case of non-increasing expected marginal profit uses the same arguments; hence we omit its proof.

The reader can check that, when Assumption 1 holds, expected future marginal profit is both non-increasing and non-decreasing.

A.2 Proofs of Results in Section 3

One can solve for the optimal sales rates by solving the following optimization problem.

\[
J(x) := \max \left\{ \sum_{n \in \tilde{N}} \left[ r(\lambda_n) \Delta_n \mathbb{P} \{ \tau > t_n \} + V(t_n, y_n) \mathbb{P} \{ \tau = t_n \} \right] : \sum_{n \in \tilde{N}} \lambda_n \Delta_n \leq x, \; \lambda_n \geq 0 \; n \in \tilde{N} \right\},
\]

where for \( \lambda \in \mathcal{L}' \) we define
\[
y_n := x - \sum_{j < n} \lambda_j \Delta_j, \quad n \in \mathcal{N}.
\]

As mentioned in Section 3, the formulation above maximizes a continuously differentiable concave function subject to affine constraints (inventory constraint and non-negativity of rates), thus Karush-Kuhn-Tucker conditions (henceforth, KKT) are necessary and sufficient to characterize the optimal sales rates. Moreover, by Weistrauss Theorem, there exits at least one optimal set of sales rates.

The KKT conditions are
\[
r'(\lambda_n) \mathbb{P} \{ \tau > t_n \} - \sum_{j > n} V_x(t_j, y_j) \mathbb{P} \{ \tau = t_j \} + \gamma_n = \gamma \quad n \in \tilde{N} \quad \text{(A-2a)}
\]
\[
\sum_{n \in \tilde{N}} \lambda_n \Delta_n \leq x \quad \text{(A-2b)}
\]
\[
\lambda_n \geq 0 \quad n \in \tilde{N} \quad \text{(A-2c)}
\]
\[
\gamma_n \lambda_n = 0 \quad n \in \tilde{N} \quad \text{(A-2d)}
\]
\[
\gamma \left( \sum_{n \in \tilde{N}} \lambda_n \Delta_n - x \right) = 0 \quad \text{(A-2e)}
\]
where $\gamma_n \geq 0$ is the KKT multiplier associated with non-negativity of $\lambda_n$, $n \in \tilde{N}$, and $\gamma \geq 0$ is KKT the multiplier associated with the capacity constraint.

**Proof of Theorem 1.** Let $\lambda := (\lambda_n : n \in \tilde{N})$ denote a set of optimal sales rates, and $y_n$ the inventory position at time $t_n$ when the retailer uses such a policy, for $n \in \tilde{N}$. We prove the result in four steps. In the first three steps, we show that optimal sales rates are always monotonic. Then, we show that the direction of the monotonicity depends on the relation between $\lambda_0$ and $x/T$.

**Step 1: Monotonicity of the optimal sales rate when $\gamma = 0$.** We will prove by induction on $n$ that $\lambda_n \leq \lambda_{n+1}$ for all $n < N - 1$. Consider first the case of $n = N - 2$. Condition (A-2a) for $N - 1$ becomes

$$r'(\lambda_{N-1})P\{\tau = t_N\} + \gamma_{N-1} = 0.$$ 

Since $r'(0) > 0$ and $\gamma_{N-1} \geq 0$, one must have that $\gamma_{N-1} = 0$ and $\lambda_{N-1} = \lambda^*$. If $\gamma_{N-2} > 0$, then $\lambda_{N-2} = 0$ and the base case holds. Therefore, assume that $\gamma_{N-2} = 0$: condition (A-2a) for $n = N - 2$ becomes

$$r'(\lambda_{N-2})P\{\tau > t_{N-2}\} - V_x(t_{N-1}, y_{N-1})P\{\tau = t_{N-1}\} = 0.$$ 

Note that $V_x(t_{N-1}, y_{N-1}) \geq 0$, hence

$$r'(\lambda_{N-1})P\{\tau > t_{N-2}\} - V_x(t_{N-1}, y_{N-1})P\{\tau = t_{N-1}\} \leq 0.$$ 

The above and the concavity of $r(\cdot)$ imply that $\lambda_{N-2} \leq \lambda_{N-1}$.

Suppose now that the induction hypothesis holds for $n + 1 \leq N - 1$. Again, we only consider the case when $\gamma_n = 0$ (otherwise the induction hypothesis holds trivially). By condition (A-2a) for $n + 1$ one has that

$$r'(\lambda_{n+1})P\{\tau > t_{n+1}\} - \sum_{j > n+1} V_x(t_j, y_j)P\{\tau = t_j\} \leq 0. \quad \text{(A-3)}$$

In addition, the induction hypothesis implies that $y_j/(T - t_j) \leq y_{j+1}/(T - t_{j+1})$, for all $j > n$. Thus, using Lemma 2 one has that

$$V_x(t_j, y_j) \geq V_x(t_{j+1}, y_{j+1}), \quad j > n.$$ 

This observation and (A-3) imply that

$$r'(\lambda_{n+1}) \leq V_x(t_{n+1}, y_{n+1}),$$

and therefore

$$(r'(\lambda_{n+1}) - V_x(t_{n+1}, y_{n+1}))P\{\tau = t_{n+1}\} + r'(\lambda_{n+1})P\{\tau > t_{n+1}\} - \sum_{j > n+1} V_x(t_j, y_j)P\{\tau = t_j\} \leq 0.$$ 

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The left-hand-side above is that of condition (A-3) for \( n \), but evaluated at \( \lambda_n = \lambda_{n+1} \). Using the concavity of \( r(\cdot) \), we conclude that \( \lambda_n \leq \lambda_{n+1} \). This proves the induction hypothesis. We conclude that, in this case, optimal sales rates are non-decreasing.

**Step 2: Monotonicity of the optimal sales rate when \( \gamma > 0 \) and \( r'(\lambda_{N-1}) \leq V_x(t_{N-1}, y_{N-1}) \).**

We will prove by induction on \( n \) that \( \lambda_n \leq \lambda_{n+1} \) for all \( n < N - 1 \). Consider first the case of \( n = N - 2 \). Condition (A-2a) for \( N - 1 \) becomes

\[
r'(\lambda_{N-1}) \mathbb{P}\{\tau = t_N\} + \gamma_{N-1} = \gamma.
\]

If \( \gamma_{N-2} > 0 \), then \( \lambda_{N-2} = 0 \) and the base case holds. Therefore, assume that \( \gamma_{N-2} = 0 \): condition (A-2a) for \( n = N - 2 \) becomes

\[
r'(\lambda_{N-2}) \mathbb{P}\{\tau > t_{N-2}\} - V_x(t_{N-1}, y_{N-1}) \mathbb{P}\{\tau = t_{N-1}\} = \gamma.
\]

However, (A-4) implies that

\[
r'(\lambda_{N-1}) \mathbb{P}\{\tau = t_N\} + (r'(\lambda_{N-1}) - V_x(t_{N-1}, y_{N-1})) \mathbb{P}\{\tau = t_{N-1}\} \leq \gamma,
\]

since the first term above is lower than \( \gamma \) and the second term is non-positive. Once again, by the concavity of \( r(\cdot) \), this implies that \( \lambda_{N-2} \leq \lambda_{N-1} \).

Suppose now that the induction hypothesis holds for \( n + 1 \leq N - 1 \). Again, we only consider the case when \( \gamma_n = 0 \) (otherwise the induction hypothesis holds trivially). By condition (A-2a) for \( n + 1 \) one has that

\[
r'(\lambda_{n+1}) \mathbb{P}\{\tau > t_{n+1}\} - \sum_{j>n+1} V_x(t_j, y_j) \mathbb{P}\{\tau = t_j\} \leq \gamma.
\]

In addition, the induction hypothesis implies that \( y_j/(T-t_j) \leq y_{j+1}/(T-t_{j+1}) \), for all \( j > n \), thus using Lemma 2, one has that

\[
V_x(t_j, y_j) \geq V_x(t_{j+1}, y_{j+1}), \quad j > n.
\]

We consider two cases.

\( \square \) Case (a): \( \gamma_{N-1} = 0 \). In this case (A-4) implies that

\[
\gamma = r'(\lambda_{N-1}) \mathbb{P}\{\tau = t_N\} \leq V_x(t_{N-1}, y_{N-1}) \mathbb{P}\{\tau = t_N\}.
\]

This observation, together with (A-6) and (A-5) imply that \( r'(\lambda_{n+1}) \leq V_x(t_{n+1}, y_{n+1}) \), therefore

\[
(r'(\lambda_{n+1}) - V_x(t_{n+1}, y_{n+1})) \mathbb{P}\{\tau = t_{n+1}\} + r'(\lambda_{n+1}) \mathbb{P}\{\tau > t_{n+1}\} - \sum_{j>n+1} V_x(t_j, y_j) \mathbb{P}\{\tau = t_j\} \leq \gamma,
\]

since the first term above is non-positive and the second term is lower than \( \gamma \). Considering condition (A-2a) for \( n \), the concavity of \( r(\cdot) \) implies that \( \lambda_n \leq \lambda_{n+1} \).
Case (b): $\gamma_{N-1} > 0$. Using the induction hypothesis one has that $\lambda_{n+1} = 0$ and $y_{N-1} = y_{n+1}$, thus

$$r'(\lambda_{n+1}) = r'(\lambda_{N-1}) \leq V_x(t_{N-1}, y_{N-1}) \overset{(a)}{=} V_x(t_{n+1}, y_{n+1}),$$

where (a) follows from Lemma 2. Following the reasoning in case (a), we have that

$$(r'(\lambda_{n+1}) - V_x(t_{n+1}, y_{n+1})) \mathbb{P}\{\tau = t_{n+1}\} + r'(\lambda_{n+1}) \mathbb{P}\{\tau > t_{n+1}\} - \sum_{j>n+1} V_x(t_j, y_j) \mathbb{P}\{\tau = t_j\} \leq \gamma,$$

since the first term above is non-positive and the second term is lower than $\gamma$. Considering condition (A-2a) for $n$, the concavity of $r(\cdot)$ implies that $\lambda_n \leq \lambda_{n+1}$.

This proves the induction hypothesis. We conclude that, in this case, optimal sales rates are non-decreasing.

**Step 3: Monotonicity of the optimal sales rate when $\gamma > 0$ and $r'(\lambda_{N-1}) \geq V_x(t_{N-1}, y_{N-1})$**

We will prove by induction on $n$ that $\lambda_n \geq \lambda_{n+1}$ for all $n < N-1$. Consider first the case of $n = N-2$ and assume $\lambda_{N-1} > 0$, and hence that $\gamma_{N-1} = 0$ (otherwise the base case holds trivially). Condition (A-2a) for $N-1$ becomes

$$r'(\lambda_{N-1}) \mathbb{P}\{\tau = t_N\} = \gamma.$$

If $\gamma_{N-2} > 0$, then $\lambda_{N-2} = 0$ and condition (A-2a) $n = N-2$ implies that

$$r'(0) \mathbb{P}\{\tau = t_N\} + (r'(0) - V_x(t_{N-1}, y_{N-1})) \mathbb{P}\{\tau = t_{N-1}\} < r'(\lambda_{N-1}) \mathbb{P}\{\tau = t_N\}.$$

However, $r'(0) \geq r'(\lambda_{N-1}) \geq V_x(t_{N-1}, y_{N-1})$, contradicting the above and therefore the fact that $\lambda_{N-2} = 0$. We conclude that $\gamma_{N-2} = 0$. Note that

$$r'(\lambda_{N-1}) \mathbb{P}\{\tau = t_N\} + (r'(\lambda_{N-1}) - V_x(t_{N-1}, y_{N-1})) \mathbb{P}\{\tau = t_{N-1}\} \geq r'(\lambda_{N-1}) \mathbb{P}\{\tau = t_N\}.$$

Considering condition (A-2a) for $n = N-2$, the concavity of $r(\cdot)$ implies that $\lambda_{N-2} \geq \lambda_{N-1}$.

Suppose now that the induction hypothesis holds for $n+1 \leq N-1$. Again, we only consider the case when $\gamma_{n+1} = 0$ (otherwise the induction hypothesis holds trivially). By condition (A-2a) for $n+1$ one has that

$$r'(\lambda_{n+1}) \mathbb{P}\{\tau > t_{n+1}\} - \sum_{j>n+1} V_x(t_j, y_j) \mathbb{P}\{\tau = t_j\} = \gamma.$$

In addition, the induction hypothesis implies that $y_j/(T-t_j) \geq y_{j+1}/(T-t_{j+1})$, for all $j > n$, thus using Lemma 2 one has that

$$V_x(t_j, y_j) \leq V_x(t_{j+1}, y_{j+1}), \quad j > n.$$
In addition, one has that condition (A-2a) for \( n = N - 1 \) implies that \( \gamma \geq r'(\lambda_{N-1}) \mathbb{P} \{ \tau = t_N \} \). Combining the observations above and the fact that \( r'(\lambda_{N-1}) \geq V_x(t_{N-1}, y_{N-1}) \), one has that

\[
r'(\lambda_{n+1}) \mathbb{P} \{ \tau > t_{n+1} \} = \gamma + \sum_{j>n+1} V_x(t_j, y_j) \mathbb{P} \{ \tau = t_j \} \\
\geq r'(\lambda_{n-1}) \mathbb{P} \{ \tau = t_N \} + \sum_{j>n+1} V_x(t_j, y_j) \mathbb{P} \{ \tau = t_j \} \\
\geq V_x(t_{n+1}, y_{n+1}) \mathbb{P} \{ \tau > t_{n+1} \}.
\]

We conclude that \( r'(\lambda_{n+1}) \geq V_x(t_{n+1}, y_{n+1}) \). With this, one has that

\[
(r'(\lambda_{n+1}) - V_x(t_{n+1}, y_{n+1})) \mathbb{P} \{ \tau = t_{n+1} \} + r'(\lambda_{n+1}) \mathbb{P} \{ \tau > t_{n+1} \} - \sum_{j>n+1} V_x(t_j, y_j) \mathbb{P} \{ \tau = t_j \} \geq \gamma,
\]

since the first term above is non-negative and the second term is equal to \( \gamma \). Considering condition (A-2a) for \( n \), the concavity of \( r(\cdot) \) implies that \( \lambda_n \geq \lambda_{n+1} \).

This proves the induction hypothesis. We conclude that, in this case, optimal sales rates are non-increasing.

**Step 4: Putting things together.** In steps 1 and 2 we have that optimal sales rates are non-decreasing, therefore one must have that \( \lambda_0 \leq x/T \), as otherwise one would violate condition (A-2b). In step 3, the optimal sales rates are non-increasing, therefore one must have that \( \lambda_0 \geq x/T \), as otherwise one would have that \( y_N > 0 \), contradicting the fact that \( \gamma > 0 \). This proves the result. \( \square \)

**Proof of Proposition 1.** The proof is organized in two steps.

**Step 1.** We first prove that \( \lambda_0 \geq x/T \) implies that \( \rho \geq 0 \). In this case Theorem 1 asserts that the optimal sales rates are non-increasing. Remember that \( \gamma \) denotes the KKT multiplier associated with the inventory constraint.

- Case (a): \( \gamma = 0 \). From step 1 in proof of Theorem 1, one has that optimal sales rates are non-decreasing, which is not possible in this case, due to the inventory constraint, unless \( \lambda_n \) is constant for all \( n \in \mathcal{N} \). In this case, condition (A-2a) for \( n = N - 1 \) implies that \( \lambda_n = \lambda^* \), and that \( \lambda^* \leq x/T \). The same condition for \( n = N - 2 \) implies that \( V_x(0, \lambda^* T) = 0 \). However, since \( \lambda^* \leq x/T \), one has that \( V_x(0, x) = 0 \) as well.

- Case (b): \( \gamma > 0 \) and \( r'(\lambda_{N-1}) \leq V_x(t_{N-1}, y_{N-1}) \). From step 2 in proof of Theorem 1 and case (a) above, one has that \( \lambda_n = x/T \) for all \( n \in \mathcal{N} \), and that \( \lambda^* \geq x/T \). In particular, condition (A-2a) for \( n = N - 2 \) implies that \( r'(x/T) = V_x(0, x) \).

- Case (c): \( \gamma > 0 \) and \( r'(\lambda_{N-1}) \geq V_x(t_{N-1}, y_{N-1}) \). From step 3 in proof of Theorem 1, one has that \( r'(\lambda_0) \geq V_x(0, x) \), and that \( \lambda^* \geq x/T \). This implies that \( r'(x/T) \geq V_x(0, x) \).

Putting the above together, we conclude that \( r'(\min \{ \lambda^*, x/T \}) \geq V_x(0, x) \).
Step 2. We now prove that $\lambda_0 \leq x/T$ implies that $\rho \leq 0$. In this case Theorem 1 asserts that the optimal sales rates are non-decreasing.

□ Case (a): $\gamma = 0$. From step 1 in proof of Theorem 1 one has that $V_x(t_n, y_n) \geq V_x(t_{n+1}, y_{n+1})$ for all $n \in \tilde{N}$, thus condition (A-2a) for $n = 0$ implies that $r'(\lambda_0) \leq V_x(0, x)$. Also, by condition (A-2a) for $n = N - 1$ and the induction hypothesis, one has that $\lambda_0 \leq \lambda^*$.

□ Case (b): $\gamma > 0$ and $r'(\lambda_{N-1}) \leq V_x(t_{N-1}, y_{N-1})$. From step 2 in proof of Theorem 1 and case (a) above, one has that $r'(\lambda_0) \leq V_x(0, x)$. In addition, condition (A-2a) for $n = N - 1$ implies that $\lambda_0 \leq \lambda^*$.

□ Case (c): $\gamma > 0$ and $r'(\lambda_{N-1}) \geq V_x(t_{N-1}, y_{N-1})$. Theorem 1 asserts that optimal sales rate are non-increasing, which is not possible due to the inventory constraint, unless $\lambda_n = x/T$ for all $n \in \tilde{N}$. Condition (A-2a) for $n = N - 1$ implies that $\lambda^* \geq x/T$, and that for $n = N - 2$ implies that $r'(x/T) = V_x(0, x)$.

Putting the above together, we conclude that $r'(\min \{\lambda^*, x/T\}) \leq V_x(0, x)$.

Proof of Theorem 2 As in proof of Theorem 1 let $\lambda := (\lambda_n : n \in \tilde{N})$ denote a set of optimal sales rates, and $y_n$ the inventory position at time $t_n$ when the retailer uses such a policy, for $n \in \tilde{N}$.

We divide the proof in two steps.

Step 1: Non-increasing expected future marginal profit. By the monotonicity of the expected future marginal profit, in this case one has that

$$V_x(t_n, y_n) \geq V_x(t_{n+1}, y_{n+1}) \text{ whenever } y_n/(T-t_n) \leq y_{n+1}/(T-t_{n+1}). \quad (A-7)$$

Therefore, the analysis in steps 1 and 2 in proof of Theorem 1 holds in this setting, provided that one invokes (A-7) instead of Lemma 2. Also, note that the setting in step 3 is contained in step 2, since one cannot have that $r'(\lambda_{N-1}) > V_x(t_{N-1}, y_{N-1})$ (note that $y_{N-1} = \lambda_{N-1} \Delta_{N-1}$, thus such a condition would violate the statement of the theorem). We conclude that the optimal sales rates are non-decreasing. Moreover, it must be the case that $\lambda_0 \leq x/T$.

Step 2: Non-decreasing expected future marginal profit. By the monotonicity of the expected future marginal profit, in this case one has that

$$V_x(t_n, y_n) \leq V_x(t_{n+1}, y_{n+1}) \text{ whenever } y_n/(T-t_n) \geq y_{n+1}/(T-t_{n+1}). \quad (A-8)$$

Therefore, the analysis in step 3 in proof of Theorem 1 holds in this setting, provided that one invokes (A-8) instead of Lemma 2. Also, note that the setting in step 2 is contained in step 3, since one cannot have that $r'(\lambda_{N-1}) < V_x(t_{N-1}, y_{N-1})$ (note that $y_{N-1} = \lambda_{N-1} \Delta_{N-1}$, thus such a condition would violate the statement of the theorem).
The above covers the case of \( \gamma > 0 \) (remember that \( \gamma \) denotes the KKT multiplier associated with the inventory constraint). Suppose now that \( \gamma = 0 \). We will prove by induction on \( n \) that \( \lambda_n = \lambda^* \) and that \( V_x(t_n, y_n) = 0 \), for \( n \in \tilde{N} \) (thus, it must be that \( x \geq \lambda^* T \)). Consider the case of \( n = N - 1 \). By condition (A-2a) for \( n = N - 1 \) one has that \( \lambda_{N-1} = \lambda^* \). Moreover, since \( y_{N-1} \geq \lambda^* \Delta_{N-1} \), one also has that \( V_x(t_{N-1}, y_{N-1}) = 0 \).

Suppose now that the induction hypothesis holds for \( n + 1 < N - 1 \). By the induction hypothesis one has that

\[
V_x(t_{n+1}, y_{n+1}) \leq V_x(t_{N-1}, \lambda^*(T - t_{n+1})) \leq V_x(t_{N-1}, y_{N-1}) = 0.
\]

Thus, condition (A-2a) for \( n = \) holds if \( \lambda_n = \lambda^* \). Note that, in this setting, the optimal sales rates are constant.

We conclude that the optimal sales rates are non-increasing. Moreover, it must be the case that either \( \lambda_0 \geq x/T \), or \( \lambda_n = \lambda^* \) for all \( n \in \tilde{N} \), in which case \( \lambda^* \leq x/T \).

\( \Box \)

### A.3 Proofs of Results in Section 4

**Proof of Proposition 2.** Note that \( V(t_n, x - \sum_{j=0}^{n-1} \lambda_j \Delta_j) \) is a concave function of \( \lambda \in L' \), for all \( n \in \mathcal{N} \): see Lemma 1. This, in conjunction with the concavity of \( r(\cdot) \) implies that the right-hand side of (8) is a convex function of \( \lambda \), for all \( n \in \mathcal{N} \). Thus, the objective function in (7) is the convex composition of convex functions, hence it is a convex function of \( \lambda \).

Let \((\tilde{z}, \tilde{\lambda})\) denote an optimal solution to Problem (8). Note that \( \tilde{z} \leq \max \{ D(t_n) : n \in \mathcal{N} \} < \infty \), and that \((\tilde{z}, x/(2T), \ldots, x(2T))\) is strictly feasible, hence Slater’s theorem states strong duality holds (see, for example, Boyd and Vandenberghe (2004)).

For \( \mu \in \mathbb{R}^{N+1}_+ \), \( z \in \mathbb{R} \), and \( \lambda \in L' \) define the Lagrangean function

\[
L(z, \lambda, \mu) := z(1 - \sum_{n=0}^{N} \mu_n) + \sum_{n=0}^{N} \mu_n \left( D(t_n) - \sum_{j=0}^{n-1} r(\lambda_j) \Delta_j - V(t_n, x - \sum_{j=0}^{n-1} \lambda_j \Delta_j) \right).
\]

Similarly, define the Lagrangean dual as

\[
\tilde{w} := \max \left\{ \Psi(\mu) := \inf \left\{ L(z, \lambda, \mu) : z \in \mathbb{R}, \lambda \in L' \right\} : \mu \in \mathbb{R}^{N+1}_+ \right\}.
\]

Let \( \tilde{\mu} \) denote the vector of Lagrangean multipliers associated with solution \((\tilde{z}, \tilde{\lambda})\), and note that \( \Phi(\mu) = -\infty \) for all \( \mu \) such that \( \sum_{n=0}^{N} \mu_n \neq 1 \). Hence, it must be the case that \( \sum_{n=0}^{N} \tilde{\mu}_n = 1 \), and \( \tilde{\mu} \) can be seen as a probability distribution with support in \( \{t_n : n \in \mathcal{N}\} \). Thus, suppose \( \mathbb{P} \) is such
that (9) holds. By strong duality, one has that

\[
\bar{w} = \min \left\{ L(z, \bar{\lambda}, \bar{\mu}) : z \in \mathbb{R}, \lambda \in \mathcal{L}' \right\}
\]

\[
= \min \left\{ \sum_{n=0}^{N} \bar{\mu}_n \left( D(t_n) - \sum_{j=0}^{n-1} r(\lambda_j) \Delta_j - V \left( t_n, x - \sum_{j=0}^{n-1} \lambda_j \Delta_j \right) \right) : \lambda \in \mathcal{L}' \right\}
\]

\[
= \mathbb{E} \{ D(\tau) \} - \max \left\{ \sum_{n \in \mathcal{N}} (r(\lambda_n) \Delta_n) \mathbb{P} \{ \tau > t_n \} + V \left( t_n, x - \sum_{j=0}^{n-1} \lambda_j \Delta_j \right) \mathbb{P} \{ \tau = t_n \} : \lambda \in \mathcal{L}' \right\}
\]

We conclude that \( \tilde{\lambda} \) (more precisely, its specification as a member of \( \mathcal{L} \)) solves (3) when (9) holds. \( \square \)

**Proof of Proposition 3.** By definition, we have \( R_0(x) = R(x) \) and \( R_N(\cdot) = D(T) \). We next establish that \( R_n(\cdot) \) necessarily satisfies the recursion (10). Indeed,

\[
R_n(y)
\]

\[
= \min \left\{ \max_{j=n+1, \ldots, N} \left\{ D(t_j) - \sum_{i=n}^{j-1} r(\lambda_i) \Delta_i - V \left( t_j, y^*_j(y, \lambda) \right) \right\} : \lambda \in \mathcal{L}'_n(y) \right\}
\]

\[
= \min \left\{ \max_{j=n+1, \ldots, N} \left\{ D(t_j) - V(t_n, y), \max_{j=n+1, \ldots, N} \left\{ D(t_j) - \sum_{i=n}^{j-1} r(\lambda_i) \Delta_i - V \left( t_j, y^*_j(y, \lambda) \right) \right\} \right\} : \lambda \in \mathcal{L}'_n(y) \right\}
\]

\[
= \max \left\{ D(t_n) - V(t_n, y) \right\}
\]

\[
\min \left\{ \min_{j=n+1, \ldots, N} \left\{ D(t_j) - \sum_{i=n}^{j-1} r(\lambda_i) \Delta_i - V \left( t_j, y^*_j(y, \lambda) \right) \right\} : \lambda \in \mathcal{L}'_{n+1}(y^*_j(y, \lambda)) \right\} : \lambda \leq \frac{y}{(T-t_n)} \right\}
\]

\[
= \max \left\{ D(t_n) - V(t_n, y) \right\}, \min \left\{ R_{n+1}(y - \lambda \Delta_n) - r(\lambda) \Delta_n \right\} : \lambda \leq \frac{y}{(T-t_n)} \right\},
\]

where \( y^*_j(y, \lambda) := y - \sum_{i=n}^{j-1} \lambda_i \Delta_i \) for \( \lambda \in \mathcal{L}' \), \( n \in \mathcal{N} \) and \( j \geq n \). This concludes the proof. \( \square \)