Product Assortment and Price Competition under Multinomial Logit Demand

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Abstract

The role of assortment planning and pricing in shaping sales and profits of retailers is well documented and studied in monopolistic settings. However, such a role remains relatively unexplored in competitive environments. In this paper, we study equilibrium behavior of competing retailers in two settings: *i.*) when prices are exogenously fixed, and retailers compete in assortments only; and *ii.*) when retailers compete jointly in assortment and prices. For this, we model consumer choice using a multinomial Logit, and assume each retailer selects products from a predefined set, and face a display constraint. We show that when the sets of products available to retailers do not overlap, there always exists one equilibrium that pareto dominates all others, and that such an outcome can be reached through an iterative process of best responses. A direct corollary of our results is that competition leads a firm to offer a broader set of products compared to when it is operating as a monopolist, and to broader offerings in the market compared to a centralized planner. When some products are available to all retailers, i.e., assortments might overlap, we show that display constraints drive equilibrium existence properties.

Keywords: assortment planning, competition, choice models, multinomial Logit, pricing

1 Introduction

Assortment planning decisions are fundamental drivers of consumers' purchase decisions and ultimately of a retailer's profitability. Retailers face significant challenges to understand the mapping from assortment decisions to consumer behavior as this mapping should synthesize complex aspects of purchase decisions such as, for example, substitution behavior, consumers' collection and aggregation of information, consumer heterogeneity, and the effect of competition. A key input in most assortment models is a consumer choice model. In this regard, and despite its documented deficiencies, the multinomial Logit model (MNL) of consumer choice has been widely used in the economics, operations and marketing literatures (see Ben-Akiva and Lerman (1985), Train (2002),

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and Guadagni and Little (1983)), and also in practice. Thus, it is important to study its properties and its implications on decision-making of firms in competitive settings, which predominate in practice. However, the theory on competitive outcomes in assortment and/or prices appears underdeveloped. For example, recently assortment and pricing decisions under the MNL have been analyzed in Misra (2008), but only best-responses, specific to the *joint* selection of assortment and prices, were studied. In particular, no results on general equilibrium properties, such as existence or uniqueness, or the structure of the equilibrium set are known.

The present paper aims to develop a general framework that enables to analyze equilibrium outcomes from competition in assortment-only or in joint assortment and pricing decisions. In particular, the goals of the present paper are two-fold: our main objective is, given the widespread use of the MNL model, to advance the theory pertaining to equilibrium outcomes under this model. In addition, we also aim to derive insights on assortment and pricing actions in competitive settings using the MNL model, complementing those in the existing literature.

To this end, we analyze a model of assortment and price competition in a duopolistic setting. On the consumers' side, we assume that customers select from the set of products offered by both retailers according to an MNL model. On the firms's side, each retailer has access to a set of products from which to select an assortment and is constrained by limited display capacity. The set of products from which retailers may select products are general and may overlap. In particular, we differentiate between *common* products, i.e., those that are available to select in an assortment for both competitors, and *exclusive* products, those that are unavailable to a firm's competitors. We first analyze competition when product prices are exogenous, a setting we refer to as assortment-only competition. We then analyze competition when prices are endogenous, i.e., are decided by the retailers. In this case, we consider a general formulation when firms face arbitrary minimum margins, and analyze the setting in which firms simultaneously select assortments and prices. Unless otherwise stated, we analyze the interactions between retailers as a game and focus on pure strategy Nash equilibria.

Regarding our first goal above, our results may be summarized as follows:

i.) We establish that, when retailers have access only to exclusive products, an equilibrium in pure strategies always exists for both assortment-only and joint assortment and price competition.

ii.) We prove that, in those settings, when multiple equilibria exist, an equilibrium will always pareto dominate all others. In other words, all retailers would prefer to settle at the same equilibrium. Moreover, we show that this equilibrium arises naturally starting from a monopolistic setup, when retailers periodically react through best-responses to the competitors' last observed offerings.

iii.) We establish that when retailers have access to exclusive and common products, an equilibrium in pure strategies is guaranteed to exist as long as there are no shelf space constraints. When shelf space constraints bind, it is possible that the number of non-pareto dominated equilibria grows exponentially with the retailers' display capacity. In addition, while an equilibrium in mixed strategies always exists, it is now possible that an equilibrium in pure strategies fails to exist.

Many of these results extend to the case of an oligopoly with an arbitrary number of firms and we comment on how to do so along the way.

Regarding our second goal, our results allow to measure the impact of competition on offerings in the market. For example, when retailers compete only through exclusive products, the introduction of a product by a retailer leads the competitor to also broaden her assortment. Also, we show that the competitor might be better off reducing the breadth of the assortment offered, so as to induce the retailer to reduce her assortment as well, which leads both retailers to increase their profit (this is the case when the original equilibrium is not pareto dominant). In terms of comparisons with monopolists, our analysis indicate that a retailer will offer a broader set of products than if she were operating as a monopolist (with the same capacity), and that the set of retailers will jointly offer a broader set of products than if a central planner (facing the same display constraints) coordinated the assortment decisions. In this regard, our results complement those from studies in assortment selection and pricing in different competitive settings (e.g. Cachon et al. (2008), Dukes et al. (2009) and Coughlan and Shaffer (2009)). Our analysis also indicates that when retailers have access to both common and exclusive products, the interactions between retailers may take a fundamentally different form. The presence of common products introduces an interplay between retailers' decisions at the product level: a product that otherwise should not be included in an assortment might be so if the competitor includes it in her assortment, and vice versa. We show that this interplay is softened when display constraints are absent since there is no longer an opportunity cost associated with offering a given product. In particular, we show that display constraints are the driver of the possibility of equilibrium non-existence in pure strategies.

Assortment optimization is in general a complex combinatorial problem. Thus, characterizing and analyzing properties of the outcome of competition among retailers may appear to be a daunting task a priori. This is probably why most formulations in the existing literature abstract away from this combinatorial structure. The main contribution of the present paper is to establish that under an MNL model, the problems of assortment-only and assortment and price competition are actually amenable to analysis, despite the combinatorial nature of the problems solved by the retailers. In this regard, our work contributes to similar work on equilibria computation in large structured games (see e.g. Immorlica et al. (2011)). From a methodological viewpoint, the analysis builds on the idea of computing best responses via a problem transformation. Such an approach has been previously used in various settings when faced with a combinatorial optimization problem with a rational objective function. It was, for example, used by Dantzig et al. (1966) for finding the minimal cost to time ratio cycle in a network, by Megiddo (1979) for computational complexity results on the optimization of rational objective functions, and more recently by Rusmevichientong et al. (2010) in the closely related context of monopolistic assortment optimization with Logit demand. The current paper leverages this transformation in a novel fashion, and shows that it can serve as one of the building blocks for a unified framework to analyze a *competitive* setting. In addition, the resulting framework is shown to be fairly flexible, enabling one to, e.g., incorporate endogenous prices.

Literature review. Misra (2008) studies joint assortment and price competition of retailers offering exclusive products with MNL demand and in the presence of display constraints, and conducts an empirical study to analyze the impact of competition on assortment size and prices. The analytical results obtained focus only on best response analysis and do not provide equilibrium existence or uniqueness results, which may be obtained through our framework. Furthermore, the present paper also develops theory for the case of assortment-only competition. We also refer the reader to Draganska and Jain (2006) and Draganska et al. (2009) for empirical investigations of assortment and pricing strategies in oligopolistic markets.

Additional dimensions of competition as well as alternative consumer choice models have also been analyzed. See Anderson and de Palma (2006), Symeonidis (2009), Hopp and Xu (2008), Cachon and Kök (2007), Kök and Xu (2011). The present paper is the first, to the best of our knowledge, to study assortment-only competition and to provide a framework that applies to both the latter and joint assortment and pricing competition.

The possibility of offering overlapping assortments has been considered before in the literature. The challenges introduced by common products are highlighted in Cachon et al. (2008) when modeling competition with consumers that sequentially search for products (see also Iyer and Kuksov (2012) for an analysis of the role of search cost in competitive environments), and by Dukes et al. (2009) in a competitive setting dominated by a retailer. Similarly, Coughlan and Shaffer (2009) highlights the interaction between common and exclusive products in the context of price match guarantees when retailers compete in price and assortment.

The interplay between product introduction and price competition has been studied in Thomadsen (2012) to highlight that a rival may benefit from a firm introducing an additional product. Price-only competition under choice models has been studied and is still an active area of research. Anderson et al. (1992) study oligopoly pricing for single-product firms under Logit demand and study pricing and assortment depth for multi-product firms in a duopoly with a nested Logit demand, restricting attention to symmetric equilibria. When firms offer a single product and customers' choice is described by an attraction model, Bernstein and Federgruen (2004) establish existence and uniqueness of an equilibrium for profit maximizing firms and Gallego et al. (2006) generalize this result for different cost structures. Gallego and Wang (2014) study price competition under nested the logit model. For the Logit model, Konovalov and Sándor (2009) provide guarantees for the existence and uniqueness of an equilibrium for affine cost functions when firms may have multiple products. Allon et al. (2013) provide conditions that ensure existence and uniqueness of an equilibrium under MNL demand with latent classes.

While the studies above focus on assortment competition, there is a large body of work that focuses on monopolistic assortment optimization, and approaches to compute optimal strategies given the combinatorial nature of the problem. The problem of assortment planning has often been studied in conjunction with inventory decisions, starting with the work of van Ryzin and Mahajan (1999), who consider Logit demand and assume customers do not look for a substitute if their choice is stocked out. They identify a tractable set of candidates that contains the optimal assortment. Maddah and Bish (2007) study a similar model, where in addition, the retailer could select prices; see also Aydin and Ryan (2000) for a study in the absence of inventory considerations. More recently, dynamic multi-period assortment optimization has been analyzed; see, e.g., Caro et al. (2012). The case of customers looking for substitutes if their choice is stocked out, known as stock-out based substitution, was studied in Smith and Agrawal (2000), Mahajan and van Ryzin (2001) and more recently Goyal et al. (2009). We also refer the reader to Rooderkerk et al. (2013) and references therein for a recent study of attribute-based assortment optimization.

In the present work, we do not consider inventory decisions and assume that products that are included in a retailer's assortment are always available when requested; hence stock-out based substitution does not arise. In particular, we focus on the case in which the retailers face display constraints. Such a setting with Logit demand and fixed prices in a monopolistic context has been studied in Chen and Hausman (2000), where the authors analyze mathematical properties of the problem, and in Rusmevichientong et al. (2010), where the authors provide an efficient algorithm for finding an optimal assortment. Fisher and Vaidyanathan (2009) also study assortment optimization under display constraints and highlight how such constraints arise in practice. When demand is generated by a mixture of Logit, Miranda Bront et al. (2009) show that when the number of classes is sufficiently high, the assortment optimization problem is NP-Hard (see also Rusmevichientong et al. (2014)). A review of the literature on monopolistic assortment optimization and of industry practices can be found in Kök et al. (2008).

The remainder of the paper. Section 2 formulates the model of competition. Sections 3 and 4 present our analysis of the assortment only and joint assortment and price competition settings, respectively. Section 5 presents our concluding remarks. Proofs are relegated to Appendix A.

2 Model of Assortment and Price Competition

We next describe the setting in which retailers compete and the demand model considered, and then present two competitive settings: one where retailers compete on assortments in which prices are predetermined and one in which retailers compete on both assortments and prices.

Setting. We consider duopolistic retailers that compete in product assortment and pricing decisions. We index retailers by 1 and 2, and whenever we use n to denote a retailer's index, we use m to denote her/his competitor's index (e.g., if $n = 1$, then $m = 2$).

We assume retailer n has access to a subset $\mathcal{S}_n$ of products, from which she or he must select her or his product assortment. In addition, we assume that, due to display space constraints, retailer n can offer at most $C_n \geq 1$ products. Such display constraints have been used and motivated for various settings in previous studies (see, e.g., Rusmevichientong et al. (2010), Misra (2008), and Fisher and Vaidyanathan (2009)). Without loss of generality, we assume that $C_n \leq |\mathcal{S}_n|$, where $|A|$ denotes the cardinality of a set A. We let $\mathcal{S}$ denote the set of all products, i.e., $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$, and denote its elements by $\{1, \ldots, S\}$. For each product $i \in \mathcal{S}$ and $n = 1, 2$, we let $c_{n,i} \geq 0$ denote the marginal cost to retailer n resulting from acquiring a unit of the product, which is assumed constant.

We say that product i is *exclusive* to retailer n if it belongs to $\mathcal{S}_n$ but not to $\mathcal{S}_m$; we denote the set of exclusive products for retailer n by $\mathcal{S}_n \setminus \mathcal{S}_m$, where $A \setminus B := A \cap B^c$ stands for the set difference between sets A and B, and the complement of a set is taken relative to $\mathcal{S}$. Similarly, we say that product i is *common* if it is available to both retailers, i.e., if i belongs to $\mathcal{S}_1 \cap \mathcal{S}_2$. An example of exclusive products would be private labels and of common products would be national brands.

For $n \in \{1, 2\}$, we define $\mathcal{A}_n$ to be the set of feasible assortment selections for retailer n, i.e.,

$$\mathcal{A}_n := \{A \subseteq \mathcal{S}_n \,:\, |A| \leq C_n\} .$$

We let A_n denote the assortment selection and $p_n := (p_{n,1}, \ldots, p_{n,S})$ the vector of prices offered by retailer n. Note that p_n specifies a price for all products in $\mathcal{S}$, for notational convenience: it should be clear that only prices that correspond to the assortment selection of the retailer matter. In addition, we will omit the dependence of various quantities on the price decisions when possible, making them explicit only when deemed necessary.

Demand model and retailers' objective. We assume that customers have perfect information about product assortments and prices offered by both retailers. (Here, we ignore search costs as, e.g., in Thomadsen (2012); see, e.g., Kuksov and Villas-Boas (2010) for a study that accounts for such costs.) We assume that customer t assigns a utility $U_{n,i}(t)$ to buying product i from retailer n, and utility $U_{n,0}(t)$ to not purchasing any product, where

$$
\begin{aligned}
U_{n,i}(t) &:= \mu_{n,i} - \alpha_{n,i}\, p_{n,i} + \xi_i^t, \qquad n = 1, 2 \\
U_{n,0}(t) &:= \mu_0 + \xi_0^t.
\end{aligned}
$$

In the above, $\mu_{n,i}$ represents the adjusted mean utility associated with buying product i from retailer n. Similarly, $\alpha_{n,i} > 0$ is a parameter of price sensitivity. To obtain MNL demand, we assume that $\left\{\xi_i^t : i \in \mathcal{S} \cup \{0\}\right\}$ are i.i.d. random variables following a standard Gumbel distribution. Note that these random variables, which represent idiosyncratic shocks to utility, are independent of the retailer n and hence consumers identify common products as such. (Considering idiosyncratic shocks of the form $\xi_{i,n}^t$ would lead to MNL demand with exclusive only products, a special case of analysis. We discuss a setting with Nested Logit demand in Section 5).

Without loss of generality, we set $\mu_0 := 0$. Customers are utility maximizers; customer t computes the best option from each retailer, $i_n \in \operatorname{argmax}\{U_{n,i}(t) : i \in A_n \cup \{0\}\}$, for $n \in \{1, 2\}$, and then selects option i that belongs to $\operatorname{argmax}\{U_{1,i_1}(t), U_{2,i_2}(t)\}$. Note that the assumption above implies that utility maximization may be attained simultaneously at a common product offered by both retailers *with positive probability* (e.g., when $\mu_{1,i} - \alpha_{1,i}\, p_{1,i} = \mu_{2,i} - \alpha_{2,i}\, p_{2,i}$). In such a case, we assume customers select any of the retailers, with equal probability.

Remark 1. Note that in our model, if customers do not prefer any retailer, a retailer offering a lower price for a common product will capture the whole market for that product. Thus, we have assumed customers have perfect information and are rational. An alternative interpretation of the model is that products are sold at an intermediary, which remains unknown to the final customer, and the demand is fulfilled by the cheaper supplier. Section 5 explores a demand model in which market share is not fully captured by a single retailer).

For $n = 1, 2$ define the *attraction factor* of product $i \in \mathcal{S}_n$ when offered by retailer n as follows

$$\nu_{n,i} := e^{\mu_{n,i} - \alpha_{n,i}\, p_{n,i}}.$$

The above setup leads to MNL demand where the customers' consideration set is obtained after eliminating options that are strictly dominated. These are product-retailer pairs such that the same product is offered by the competitor and provides a higher utility when bought from the competitor. In particular, one can show that for given assortment and price decisions $\{(A_n,\, p_n) \,:\, n = 1, 2\}$, the probability that a customer elects to purchase product $i \in A_n$ from retailer n, $q_{n,i}$, is given by

$$q_{n,i}(A_n, p_n, A_m, p_m) \quad := \quad \frac{\nu_{n,i} \,\Big(\mathbf{1}\,\{i \notin A_m\} + \,\delta_{n,i}\; \mathbf{1}\{i \in A_m\}\Big)}{1 + \sum_{i \in A_n \setminus A_m} \nu_{n,i} + \sum_{i \in A_n \cap A_m} (\delta_{n,i}\; \nu_{n,i} + \delta_{m,i}\; \nu_{m,i}) + \sum_{i \in A_m \setminus A_n} \nu_{m,i}},$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function and

$$\delta_{n,i} := \mathbf{1}\,\{\nu_{n,i} > \nu_{m,i}\} + \frac{1}{2}\mathbf{1}\,\{\nu_{n,i} = \nu_{m,i}\}$$

defines the split of product i's market share between the retailers (when offered by both). The expected profit per customer for retailer n, is then written as

$$\pi_n(A_n, p_n, A_m, p_m) = \sum_{i \in A_n} (p_{n,i} - c_{n,i})\; q_{n,i}(A_n, p_n, A_m, p_m).$$

Each retailer's objective is to maximize her expected profit per customer, given the competitor's decision.

3 Assortment-only Competition: Main Results

In this section, prices are assumed to be predetermined and not under the control of retailers. We further assume that all products have a positive profit margin (i.e. $p_{n,i} > c_{n,i}$). This accommodates settings where, e.g., prices are set by the manufacturers/service providers and not the retailers. Here and throughout the paper, we will abstract away from strategic interactions between retailers and manufacturers.

Given retailer m's assortment decision, retailer n selects an assortment so as to maximize her/his expected profit per customer subject to the display constraint on the number of products that can be offered. Mathematically, the problem that retailer n solves can be written as follows

$$\max_{A_n \in \mathcal{A}_n} \left\{ \pi_n(A_n, A_m) \right\}, \tag{1}$$

where we have omitted the dependence of the expected profit on prices. In problem (1) the retailer attempts to find the best set of products to offer among a combinatorial number of possibilities in $\mathcal{A}_n$. We say that a feasible assortment A_n is a best response to A_m if A_n maximizes the profit per customer for retailer n, i.e., if A_n solves problem (1). Given that there is a finite number of feasible assortments, there always exists at least one best response to each assortment $A_m \in \mathcal{A}_m$. We say that an assortment pair (A_1, A_2) is an equilibrium if A_n is a best response to A_m for $n = 1, 2$. Formally, this corresponds to the concept of a *pure strategy Nash equilibrium.*

3.1 Best response correspondence

Our analysis of the best response correspondence relies on a simple equivalent formulation of the profit maximization problem. This idea of focusing on such an equivalent formulation when faced with a combinatorial problem with an objective function in the form of a ratio has previously been used in various settings as mentioned in the introduction. The reader is in particular referred to Gallego et al. (2004) and Rusmevichientong et al. (2010) for applications to monopolistic assortment optimization with Logit demand. The treatment that follows develops the appropriate modifications for the *competitive* setting we study. Consider the following problem

$$\max \quad \lambda \tag{2a}$$

$$\text{s.t.} \quad \max_{A \in \mathcal{A}_n} \Bigg\{ \sum_{i \in A \setminus A_m} (p_{n,i} - c_{n,i} - \lambda)\, \nu_{n,i} + \sum_{i \in A \cap A_m} \Big(\delta_{n,i}(p_{n,i} - c_{n,i}) - \lambda\Big)\, (\delta_{n,i}\, \nu_{n,i} + \delta_{m,i}\, \nu_{m,i}) - \lambda \sum_{i \in A_m \setminus A} \nu_{m,i} \Bigg\} \geq \lambda. \tag{2b}$$

Lemma 1. *Problems* (1) *and* (2) *are equivalent in the following sense: the optimal values for both problems are equal and an assortment is optimal for problem* (1) *if and only if it maximizes the left-hand-side of* (2b) *when* $\lambda = \lambda^*$, *where* λ^* *corresponds to the optimal objective function of* (2).

Lemma 1 exploits the rational form of the profit function, by first finding any assortment that surpasses a given profit level, and then looking for the highest profit level attainable. Hence, in theory, one could solve for the best response to A_m by solving the maximization in (2b) for all possible values of λ, and then selecting any assortment maximizing the left-hand-side of (2b) for λ^*. This way of envisioning solving (2) will prove useful in the equilibrium analysis we conduct for this setting, as well as throughout the rest of the paper. We now outline how to solve the maximization in (2b). To that end, for $i \in \mathcal{S}_n$, define

$$\theta_{n,i}(\lambda) := \begin{cases} (p_{n,i} - c_{n,i} - \lambda)\, \nu_{n,i} & \text{if } i \notin A_m, \\ \delta_{n,i}\left((p_{n,i} - c_{n,i} - \lambda)\, \nu_{n,i} + \lambda\, \nu_{m,i}\right) & \text{if } i \in A_m. \end{cases} \tag{3}$$

Given this, formulation (2) can be rewritten as

$$\max\left\{\lambda\in\mathbb{R}:\max_{A\in\mathcal{A}_n}\left\{\sum_{i\in A}\theta_{n,i}(\lambda)\right\}\geq\lambda\Big(1+E_m(A_m)\Big)\right\},\tag{4}$$

where $E_m(A)$, referred to as the *attractiveness* of an assortment A, is defined as follows

$$E_m(A):=\sum_{i\in A}\nu_{m,i},\quad m=1,2.$$

For given non-overlapping assortment offerings, A_1 and A_2, this quantity is related to an aggregate measure of the market share of retailer n, $E_n(A_n)/(1+E_1(A_1)+E_2(A_2))$. One can solve the inner maximization in (4) by ordering the products in $\mathcal{S}_n$ according to the corresponding values of $\theta_{n,i}(\lambda)$, from highest to lowest, and selecting the maximum number of products in the assortment (up to C_n) with positive values of $\theta_{n,i}(\lambda)$.

Product ranking. As already highlighted in the monopolistic setting by Rusmevichientong et al. (2010), the $\theta_{n,i}(\lambda)$-ranking for an optimal value of λ need not to coincide with the ranking of the profit margins (the latter does appear in the absence of capacity constraints: see van Ryzin and Mahajan (1999)). In addition, note that in the competitive setting under analysis, the product ranking according to the $\theta_{n,i}$'s (and hence the selected assortment) will vary depending on the value of λ and on which products are included in the competitor's assortment A_m. This last observation implies that, for a fixed value of λ, a product that is not "appealing" (i.e., a product that is not included in a best response) if not offered by the competitor might become appealing when the latter offers it. This can be seen from the second case in (3) where $\theta_{n,i}(\lambda)$ might increase by $\lambda\,\nu_{m,i}$ when product i is offered by retailer m. This gain can be interpreted as the value of profiting from product i without having to expand the consideration set of products.

3.2 The Case of Exclusive Products

This section studies the case of retailers having only *exclusive products*, i.e., $\mathcal{S}_1\cap\mathcal{S}_2=\emptyset$. We begin by specializing the best response computation to this setting, and then study equilibrium behavior. In this setting, one has that for each product $i\in\mathcal{S}_n$

$$\theta_{n,i}(\lambda)=(p_{n,i}-c_{n,i}-\lambda)\nu_{n,i},$$

independent of A_m, thus the solution to the inner maximization in (4) depends on A_m only through $E_m(A_m)$. Define $\lambda_n(e)$ as retailer n's expected profit per customer when retailer m offers assortment A_m with attractiveness e. That is

$$\lambda_n(e):=\max\left\{\lambda\in\mathbb{R}\,:\,\max_{A\in\mathcal{A}_n}\left\{\sum_{i\in A}\theta_{n,i}(\lambda)\right\}\geq\lambda(1+e)\right\}.\tag{5}$$

Similarly, let $a_n(e)$ denote retailer n's best response correspondence to assortments with attractiveness e, i.e.,

$$a_n(e) := \operatorname*{argmax}_{A \in \mathcal{A}_n} \left\{ \sum_{i \in A} \theta_{n,i}(\lambda_n(e)) \right\}. \tag{6}$$

The next result establishes monotonicity properties of the best response correspondence in terms of attractiveness and profit level.

Proposition 1 (best response properties)**.** *Suppose that* $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.

i.) *Retailer n's best response profit is decreasing in the attractiveness of its competitor's assortment, i.e.,* $\lambda_n(e)$ *is decreasing in e.*

ii.) *The attractiveness of retailer n's best response assortments is non-decreasing in the attractiveness of the competitor's assortment, e, in the following sense: for any* $e > e' \geq 0$, $E_n(a') \leq E_n(a)$ *for all* $a \in a_n(e)$ *and* $a' \in a_n(e')$.

Proposition 1.*i.*) states that a retailer's (optimized) profits will decrease if the competitor increases the attractiveness/breadth of its offerings, which is in line with intuition. Proposition 1.*ii.*) provides an important qualitative insight: if one retailer increases the attractiveness of the products it is offering, then so will the other one.

The conclusions of Proposition 1 are usually obtained in the context of supermodular games. However, it is worth noting that, in general, it is not clear whether one could obtain a supermodular representation of the assortment game with the exception of the case where margins are equal across products. (In the latter case, the game can be seen to be log-supermodular on the discrete lattice of possible attractiveness levels induced by all feasible assortments.) However, once properties outlined in Proposition 1 are at hand, one can establish existence and ordering of equilibria in a similar fashion as is usually performed for supermodular games (see e.g. Vives (2000)), which we do next. The following result guarantees that an equilibrium exists.

Theorem 1 (equilibrium existence)**.** *Suppose that* $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. *Then there always exists an equilibrium in assortment decisions.*

The proof of this result rests on the fact that assortments with higher attractiveness levels lead the competitor to also offer assortments with higher attractiveness. Since there is a finite number of possible attractiveness levels to offer, attractiveness of best responses will necessarily settle at a certain level, with the corresponding assortments forming an equilibrium.

Theorem 1 establishes the existence of an equilibrium, but leaves open the possibility of having multiple equilibria. While there might indeed exist multiple equilibria, we show next that if such a case occurs, both retailers will prefer the same equilibrium.

Proposition 2 (best equilibrium)**.** *Suppose that* $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ *and that multiple equilibria exist. Then, one equilibrium pareto dominates all equilibria, and such an equilibrium minimizes the attractiveness of the offerings of each retailer among all equilibria.*

In other words, when multiple equilibria exist, retailers would prefer to select one with the least breadth of offerings. The result is a direct consequence of the relationship between profit level and attractiveness of the offering, established in Proposition 1.*i*.).

Remark 2 (arbitrary number of firms)**.** Proposition 1, Theorem 1 and Proposition 2 can be generalized to an arbitrary number of retailers and we briefly indicate how one might do so in the proofs of those results.

The next example illustrates the possibility of multiple equilibria.

Example 1 (multiple equilibria)**.** Let $\mathcal{S}_n := \{i_1^n, i_2^n, \ldots\}$ be such that $\nu_{n,i_j^n} > \nu_{n,i_{j+1}^n}$ for all $j < C_n$, $n = 1, 2$, and suppose that for all i in $\mathcal{S}_n \setminus \{i_1^n\}$, $p_{n,i} - c_{n,i} = r_n$, where

$$r_1 := \frac{\left(p_{1,i_1^1} - c_{1,i_1^1}\right)\ \nu_{1,i_1^1}}{1 + \nu_{1,i_1^1} + \sum_{j \le C_2} \nu_{2,i_j^2}}, \quad r_2 := \frac{\left(p_{2,i_1^2} - c_{2,i_1^2}\right)\ \nu_{2,i_1^2}}{1 + \nu_{2,i_1^2} + \nu_{1,i_1^1}}.$$

In other words, for each retailer, all products except the one with the highest attraction factor have the same profit margin, and the latter is strictly lower than that of the former. The construction above is such that when retailer 1 selects the assortment $\left\{i_1^1\right\}$, retailer 2's best response is any assortment of the type $\left\{i_1^2, \ldots i_j^2\right\}$ for $j \le C_2$. Similarly, when retailer 2 selects the assortment $\left\{i_j^1 : j \le C_2\right\}$, retailer 1's best response is any assortment of the type $\left\{i_1^1, \ldots i_j^1\right\}$ for $j \le C_1$. This, in conjunction with Proposition 1 *ii.*) implies that

$$(\{i_1^1\}, \{i_1^2, \ldots i_j^2\}),\ j \le C_2, \quad (\{i_1^1, \ldots i_j^1\}, \{i_1^2, \ldots i_{C_2}^2\}),\ j \le C_1,$$

are all equilibria.

Note that the number of equilibria in Example 1 is $C_1 + C_2 - 1$, and that on each of these equilibria, retailers offer different total attractiveness, thus they are not trivially equivalent. The following result shows that such a number is the highest possible in the setting.

Theorem 2 (bound on the number of equilibria). *Suppose that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ and let $\mathcal{E}_n$ denote all possible attraction levels offered in a best response by retailer n, i.e.,*

$$\mathcal{E}_n := \{E_n(a) \in \mathbb{R}_+ \,:\, a \in a_n(e)\,,\, e \geq 0\}\,, \qquad \textit{for } n = 1, 2.$$

There are at most $|\mathcal{E}_1| + |\mathcal{E}_2| - 1$ equilibria in which retailers offer different attraction levels.

A priori, a trivial bound on the number of fundamentally different equilibria is the number of combinations of best response attractiveness levels, $|\mathcal{E}_1|\,|\mathcal{E}_2|$. Theorem 2 provides a significantly sharper bound. The proof of Theorem 2 relies on the strong monotonicity property established in Proposition 1 *ii.*), which enables one to eliminate a large set of equilibrium candidates. Back to Example 1, we see that $\mathcal{E}_n = C_n$, $n = 1, 2$, thus we conclude that the bound in Theorem 2 is tight.

The next result, which we state without proof, illustrates the use of Theorem 2.

Corollary 1 (Sufficient condition for a unique equilibrium). *Suppose that all products offered by a given retailer have the same margin, i.e., $p_{n,i} - c_{n,i} = r_n$ for all $i \in \mathcal{S}_n$, $n = 1, 2$, where r_1 and r_2 are given positive constants. Then, retailers offered the same attraction level in all equilibria.*

3.2.1 Tatônement stability

Proposition 1 has strong implications for stability. Consider the following discrete-time interaction dynamics: suppose that initially one of the retailers operates as a monopolist until a competitor enters the market, and that retailers adjust their assortment decisions on a periodic basis by taking turns, always reacting optimally to their competitors' last observed offering (that is, retailers behave myopically, and adjust their decisions periodically without anticipating the competitors' reaction). Formally, starting with offerings $A_1^0 \in a_1(0)$ and $A_2^0 = \emptyset$, retailer n offers in period t the assortment

$$A_n^t \in a_n\left(E_m\left(A_m^{t-1}\right)\right). \tag{7}$$

One can show that such a best-response process will converge to an equilibrium that pareto dominates all others, as described in Proposition 2. We formalize this next.

Corollary 2. *Let (A_1, A_2) denote a pareto-dominant equilibrium. The tatônement process $\left\{(A_1^t, A_2^t) : t \geq 1\right\}$ is such that $E_n(A_n^t) = E_n(A_n)$ for all $t \geq \tilde{t}$ for some $\tilde{t} < \infty$, $n = 1, 2$.*

The result above, which we state without proof, follows directly from Proposition 1.*i.*), the fact that the attraction offered by a monopolist is below that offered by a pareto-dominant equilibrium, and that a pareto-dominant equilibrium minimizes (among equilibria) the attraction offered by both retailers.

From above, one can envision a pareto-dominant equilibrium as arising naturally as the outcome of an iterative best-response process that starts in the absence of competition. If one starts the best-response iteration process from arbitrary assortments, then convergence to an equilibrium is guaranteed but not necessarily to one that pareto dominates all others. To see this, suppose we start with A_2^0 arbitrary: if $E_2(A_2^0) > E_2(A_2^1)$, then the recursive application of Proposition 1 *ii*) implies that the sequences of total attractiveness of the assortments offered by both retailers (excluding that of A_0^1) will be non-increasing, i.e. $E_n(A_n^k) \geq E_n(A_n^{k+1})$, $k \geq 1$, $n = 1, 2$. Similarly, if $E_2(A_2^0) < E_2(A_2^1)$ then the sequences of attractiveness will non-decreasing. This observation, together with the finiteness of the product sets, imply convergence to an equilibrium. (The fact that convergence to the pareto-dominant equilibrium is not guaranteed follows directly from the (possible) existence of multiple equilibria.)

3.2.2 Competitive outcome versus monopolist and centralized solutions

Let (A_n^{comp}, A_m^{comp}) denote an equilibrium that pareto dominates all others. Let (A_n^{cent}, A_m^{cent}) denote the optimal pair of assortments to offer by a central planner, i.e.,

$$(A_n^{cent}, A_m^{cent}) \in \underset{(A_n, A_m) \in \mathcal{A}_n \times \mathcal{A}_m}{\operatorname{argmax}} \left\{ \pi_n(A_n, A_m) + \pi_m(A_m, A_n) \right\}.$$

Finally, let A_n^* an optimal assortment for a monopolist that does not face any competition, i.e.,

$$A_n^* \in \underset{A_n \in \mathcal{A}_n}{\operatorname{argmax}} \left\{ \pi_n(A_n, \emptyset) \right\}.$$

We have the following result, which we state without proof.

Corollary 3. *Suppose that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.*

i.) $E_n\left(A_n^*\right) \leq E_n\left(A_n^{comp}\right)$.

ii.) $E_n\left(A_n^{cent}\right) + E_m\left(A_m^{cent}\right) \leq E_n\left(A_n^{comp}\right) + E_m\left(A_m^{comp}\right)$.

In particular, *i.*), which follows from Proposition 1, implies that a retailer operating as a monopolist with some fixed capacity (i.e. with its competitor offering no products) will increase her/his offerings in terms of attractiveness when a competitor enters the market. *ii.*) establishes that competing retailers will jointly offer a broader offering, resulting in a higher probability of purchase, relative to a setting in which decisions are coordinated by a central planner aiming to maximize total profits, and facing similar capacity constraints. To see this, note that the central planner achieves higher profits than those achieved by any of the retailers in a potential equilibrium; a close inspection of the proof of Proposition 1.*ii*.) reveals that the attractiveness of the solution to the assortment maximization in (2b) is non-increasing in the level λ; this implies that the attractiveness of the products offered by the central planner would never be higher than the joint attractiveness of the products offered by the competing retailers in equilibrium.

3.3 The Case of both Exclusive and Common Products

We now turn to the case when retailers may offer the same products in their respective assortments, i.e., when $\mathcal{S}_1 \cap \mathcal{S}_2$ is not empty. Our next result shows that an equilibrium is guaranteed to exist when retailers do not face display constraints.

Theorem 3 (equilibrium existence with ample capacity)**.** *Suppose that $C_n = |\mathcal{S}_n|$ for $n = 1, 2$. Then an equilibrium always exists.*

The proof of this result is constructive: we establish that the tatônement process described in Section 3.2 converges to an equilibrium provided that initially both retailers offer all common products. In addition, it is possible to establish that the tatonement process (7) with $A_1^0 \in a_1(0)$ and $A_2^0 = \emptyset$ is guaranteed to converge to an equilibrium. Thus, as in section 3.2.1, one can envision such a limit equilibrium as arising naturally as the outcome of an iterative best-response process that starts in the absence of competition. However, in this setting, such an equilibrium does not necessarily pareto dominate all others.

It is possible to find alternative conditions that will ensure the existence of an equilibrium. For example, it is possible show that conditions 1 and 2 below each ensure existence of an equilibrium.

Condition 1. Monotonic margins: $\nu_{n,i} \geq \nu_{n,i+1},$ and $p_{n,i} - c_{n,i} \geq p_{n,i+1} - c_{n,i+1}$ for all $i \in \mathcal{S}_n$, $n = 1, 2$, and $\nu_{n,i} > \nu_{m,i}$ for all $i \in B_n \cap \mathcal{S}_m$, for some $B_n \in \mathcal{P}_n(C_n)$, $n = 1, 2$. Here, $\mathcal{P}_n(C_n)$ is the set of "popular assortments" defined as (see, e.g., Kök et al. (2008)):

$$\mathcal{P}_n(C) := \left\{ \{\sigma_n(1), \cdots, \sigma_n(C)\} : \sigma_n \text{ is a permutation of } \mathcal{S}_n \text{ s.t. } \nu_{n,\sigma_n(1)} \geq \nu_{n,\sigma_n(2)} \geq \cdots \geq \nu_{n,\sigma_n(|\mathcal{S}_n|)} \right\}.$$

Condition 2. Equal margins: $p_{n,i} - c_{n,i} = r_n,$ for all $i \in \mathcal{S}_n, n = 1, 2$, where r_1 and r_2 are given positive constants, and $\mathcal{S}_1 = \mathcal{S}_2$, $C_1 = C_2$, and $\nu_{1,i} = \nu_{2,i}$ for all $i \in \mathcal{S}_1$.

The impact of display constraints. In general, when common products are available and display constraints are present, the structural results of the previous section fail to hold, as we illustrate through the following example.

Example 2 (non-existence of equilibrium in pure strategies.)**.** Consider a setting with two retailers, each having access to the same three products $\mathcal{S}_1 = \mathcal{S}_2 = \{1, 2, 3\}$, and with display capacities $C_1 = 2$, $C_2 = 1$. Suppose that prices and costs are uniform across products and retailers and given by $p_{n,1} = p_{n,2} = p_{n,3} = p > 1$ and $c_{n,1} = c_{n,2} = c_{n,3} = p - 1$ for $n = 1, 2$, and that the remaining parameters are such that $\nu_{n,1} = 1.1, \nu_{n,2} = 1.01, \nu_{n,3} = 1$. Table 1 depicts the rewards for each retailer for feasible pairs of assortment decisions (A_1, A_2).

		A_1					
		$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
A_2	$\{1\}$	$(0.262, 0.262)$	$(0.354, 0.325)$	$(0.355, 0.323)$	$(0.177, 0.502)$	$(0.177, 0.500)$	$(0.268, 0.489)$
	$\{2\}$	$(0.325, 0.354)$	$(0.251, 0.251)$	$(0.336, 0.332)$	$(0.162, 0.516)$	$(0.246, 0.511)$	$(0.168, 0.500)$
	$\{3\}$	$(0.323, 0.355)$	$(0.332, 0.336)$	$(0.250, 0.250)$	$(0.243, 0.513)$	$(0.161, 0.516)$	$(0.166, 0.502)$

Table 1: Illustration of non-existence of equilibrium. Each entry in the table corresponds the profit of retailer 1 and retailer 2 as a function of the assortments selected for the setup in Example 2.

One can verify that no equilibrium exists. Intuitively, the latter stems from the fact that retailer 1, with a capacity of 2, will always prefer to incorporate in its assortment the product that retailer 2 is offering, while retailer 2 prefers to offer a product not offered by the competitor. Recalling the discussion following the definition of $\theta_{n,i}(\lambda)$ in (3), the current example illustrates how a product gains in appeal (measured by $\theta_{n,i}(\lambda)$) when offered by the competitor. In this setting, this prevents the possibility of an equilibrium.

The focus above was on equilibria in pure strategies. Since each retailer has a finite number of alternatives *the assortment game will always admit a mixed-strategy equilibrium* (see, e.g., Fudenberg and Tirole (1991, §1.3.1)): in the setting of Example 2, one can check that retailer 1 offering $A_1 = \{1, 2\}$ with probability 0.51 and $A_1 = \{1, 3\}$ with probability 0.49, and retailer 2 offering $A_2 = \{2\}$ with probability 0.35 and $A_2 = \{3\}$ with probability 0.65 constitutes a mixed-strategy Nash equilibrium.

In addition to the above, even when an equilibrium in pure strategies exists, the presence of common products may also preclude the existence of a pareto dominant equilibrium, as highlighted by the following example.

Example 3 (exponential number of non-pareto dominated equilibria)**.** Consider the following setup. Suppose that the set of available products is common to both retailers ($\mathcal{S}_1 = \mathcal{S}_2$) and has $S = 2C$ elements, where $C_1 = C_2 = C$, that all products are priced at the same price p, that their marginal cost is zero, and that other parameters are such that $\nu_{1,i} = \nu_{2,i}$ for all $i \in \mathcal{S}$, and

$$\nu_{n,1} > \nu_{n,2} > \ldots > \nu_{n,S}, \quad \text{and} \quad \nu_{n,1} < \frac{3}{2}\nu_{n,S}.$$

Define $A^* := \mathcal{P}_1(C_1)$ and suppose that $3\ E_1(A^*) \leq 1$. This condition corresponds to assuming that the maximum share any retailer can achieve (under any scenario) is below 25%. Under the setup above, we show that if retailer 2 offers an arbitrary selection of products A_2, then the best response of retailer 1 is to offer the set of C products with the highest $\nu_{n,i}$'s in $\mathcal{S}_1 \setminus A_2$. Recalling (2), retailer 1 solves for the maximal λ such that

$$\max_{A_1 \subseteq \mathcal{S}} \left\{ E_1(A_1 \setminus A_2)\ (p - \lambda) + E_1(A_1 \cap A_2)\ (\frac{p}{2} - \lambda) - \lambda E_2(A_2 \setminus A_1) \right\} \geq \lambda.$$

Given any assortment offered by retailer 2, A_2, the revenue of retailer 1, λ, is bounded by the revenue of a monopolist (with display capacity C), i.e., $\lambda \leq p(E_1(A^*))(1 + E_1(A^*))^{-1}$. This, in conjunction with the market share condition above, implies that $\lambda \leq p/4$.

For any products $j \in \mathcal{S}_1 \setminus A_2$ and $j' \in A_2$, given that $\nu_{2,j'} < (3/2)\nu_{2,j}$, it will always be the case that $\nu_{2,j}(p - \lambda) \geq (3/4)\nu_{2,j}p > \nu_{2,j'}\ p/2$. Hence $\theta_{2,j}(\lambda) > \theta_{2,j'}(\lambda)$ and the best response of retailer 1 to A_2 will never include any product in A_2. In addition, since $\lambda \leq p/4$, retailer 1 will always include C products in $\mathcal{S}_1$ not offered by the competitor.

Given the above, one can verify that any pair of assortments (A_1, A_2) that belongs to the set

$$\{A_1 \subseteq \mathcal{S}, A_2 = \mathcal{S} \setminus A_1\ :\ |A_1| = C\}$$

is an equilibrium. It is also possible to see that one can choose the $\nu_{n,i}$'s so that all equilibria yield different profits to the retailers and are non-pareto dominated. The cardinality of the set above is $\binom{2C}{C}$. This illustrates that in general, even when prices are uniform across products, the number of non-pareto dominated equilibria may be exponential in the capacities of the retailers in contrast with what was observed in the case of exclusive products.

4 Joint Assortment and Price Competition: Main Results

We now turn attention to the case where in addition to assortment decisions, retailers also set prices for the products they offer. We follow a parallel exposition to that of Section 3 by separating the analysis for the case of exclusive products and that of both exclusive and common products.

We assume throughout this section that prices are restricted to be greater or equal than $c_{n,i} + r_{n,i}$ for any product i and retailer n, where $r_{n,i}$ is a minimal margin imposed by the product manufacturer. This assumption reflects the fact that minimal prices are commonly imposed directly or indirectly by manufacturers through, e.g., a Manufacturer's Suggested Retail Price (MSRP).

When price is an additional lever, various forms of competition may arise. We focus on the case in which assortments and prices are selected simultaneously by the firms. Thus, best responses are computed as unilateral deviations in both assortment and prices.

4.1 The Case of Exclusive Products

We start with the case of retailers having only exclusive products, i.e., $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. It turns out that equilibrium prices can be related to assortment selections through the profit attained by each retailer. This is, given fixed assortment selections, each firm solves a classical multi-product pricing problem with constraints on the prices. In particular, one can show that retailer n will set the price of product i (if offered) to

$$p^*_{n,i} = c_{n,i} + \max\left\{\frac{1}{\alpha_{n,i}} + \lambda_n, r_{n,i}\right\}, \tag{8}$$

where λ_n is the equilibrium profit retailer n achieves (which will be shown to be well defined in the proof of Theorem 4). Relationship (8) can be seen to be an expression of *equal margins* across offered products, with the modifications to account for the differentiated minimal margins imposed. Margins, adjusted to differences in price sensitivities, will be equal provided that the retailer's profit is relatively large compared to the minimum margins. Variants of such a property have previously appeared in various related settings; see, e.g., Anderson et al. (1992).

Following the analysis in Section 3.2, and using the observation above, one can show that the best response of a retailer still depends only on the competitor's offered attractiveness (which now depends on the competing assortment and prices). In particular, retailer n's expected profit per customer as a function of the competing attractiveness level e is given by (5), but considering that

$$\theta_{n,i}(\lambda) = \left(p^*_{n,i}(\lambda) - c_{n,i} - \lambda\right) \exp\left\{\mu_{n,i} - \alpha_{n,i}\, p^*_{n,i}(\lambda)\right\}.$$

Thus, one has that the best response correspondence $a_n(e)$, whose elements are now assortment and price vector pairs, is given by

$$a_n(e) := \operatorname*{argmax}_{A \in \mathcal{A}_n} \left\{\sum_{i \in A} \theta_{n,i}(\lambda_n(e))\right\} \times \left\{p^*_n(\lambda_n(e))\right\}.$$

In such a setting, in which prices are set by the firms, one may establish parallel results to Proposition 1 and Theorem 1, which were established when prices were exogenously fixed.

Theorem 4 (equilibrium existence and pareto dominance)**.** *Suppose that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. Then, there always exists a pure-strategy Nash equilibrium in assortments and prices. Moreover, there is always one equilibrium that pareto dominates all others.*

To establish this result, we first establish that prices are uniquely defined by assortment selections. We then show that similar monotonicity properties as those established in Proposition 1 hold in the current setting in which prices may be adjusted. These properties in turn imply the existence of a pure strategy Nash equilibrium and that one equilibrium pareto dominates all others. Moreover, such properties can be used to extend the comparison of outcomes under competitive versus a monopolist or a centralized setting to obtain a result similar to that in Corollary 3. We also note that the analysis of tatônement stability presented in Corollary 2 generalizes to this setting.

From (3) and (8) one observes that $\theta_i > 0$ for every product $i \in \mathcal{S}_n$. This, in turn, implies that retailers will always use their full capacity in any equilibrium. This stands in contrast with the case in which prices were exogenously set. In particular, in the absence of display constraints, all products are offered and the equilibrium is unique. The next result, which we state without proof, formalizes this.

Corollary 4. *Suppose that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. In equilibrium, retailers always use their full capacity. In particular, in the absence of display constraints, there exists a unique equilibrium in which firm n offers all products in $\mathcal{S}_n$, $n = 1, 2$.*

When no minimum margins are imposed by the manufacturers and price sensitivity is the same across products and retailers, it is possible to establish the existence of a unique equilibrium. The analysis of best responses in such a setting has already appeared in the literature (see, e.g., Misra (2008)); however, no equilibrium results were provided. The present approach establishes existence and uniqueness of an equilibrium, but also illustrates along the way the general applicability of the framework we use to, e.g., assortment-only competition.

4.2 The Case of both Exclusive and Common Products

When retailers may offer the same products, equilibrium prices depend not only on the profit obtained by each retailer, but also on the competitor's offering.

As in the case of competition with exclusive only products, one can show that for given fixed assortment selections and competitor's prices, retailer n will set the price of an exclusive product following (8). Now, for a common product i offered by both retailers, it is possible to show that one can, without loss of generality for the equilibrium outcome in terms of profits, restrict attention to the following candidate equilibrium prices

$$p^*_{n,i} = \max\left\{\min\left\{\frac{\mu_{n,i}-\mu_{m,i}}{\alpha_{n,i}} + \frac{\alpha_{m,i}}{\alpha_{n,i}}(c_{m,i}+r_{m,i}), c_{n,i} + \frac{1}{\alpha_{n,i}} + \lambda_n\right\}, c_{n,i}+r_{n,i}\right\},$$

where λ_n is the equilibrium profit retailer n achieves. The difference between the expression above and that in (8) follows from the fact whenever possible, retailers will undercut the competitor's prices so as to capture the full market for a product. In this regard, it is implicit in the formula above that retailers price marginally below the price that makes a customer indifferent between buying from either retailer as long as it yields a positive profit.

We note that if the minimal margins are sufficiently high, then the retailers will face a problem where prices are effectively fixed. Thus, the results in Section 3.3 imply that it is not possible to ensure existence of an equilibrium under when common products are available in conjunction with display constraints. We show, however, that as in the analysis of assortment competition with fixed prices, a equilibrium is guaranteed to exist when retailers do not face display constraints.

The next result ensures equilibria existence is guaranteed in the absence of display constraints

Proposition 3. *In the assortment and price competition setting with common products, suppose retailers do not face display constraints. Then, there always exists a pure-strategy Nash equilibrium.*

As in the case of simultaneous competition with exclusive only products, the result above rests on the fact that a best response to any offering involves offering all products. Thus, any equilibrium in the pricing game where all products are offered is also an equilibrium in the simultaneous assortment and pricing competition setting.

5 Extensions and Additional Challenges

The present paper has analyzed an assortment game in which firms face display constraints and consumer demand is driven by a multinomial Logit model. We have characterized equilibrium behavior, showing that significant structure is present in such a problem. The approach taken to analyze the game is fairly general and has been extended to cases in which prices are endogenous.

The need for a better understanding/modeling of the case of common products. The current study has highlighted that the presence of common products may lead to significant differences in equilibrium behavior. The possibility of common products has been assumed away in most of the literature. For example, when the choice model is a nested Logit in which customers first select a retailer, the product utility shocks are assumed to be independent once a retailer has been chosen. While such assumptions are appropriate for settings in which consumers do not search across retailers due to loyalty or costs of search, it becomes inappropriate if consumers perform some data collection before selecting a product to purchase. In particular, it appears that common products ought to be treated differently than exclusive products.

The present paper has analyzed one case in this spectrum in which the product utility shock is identical across retailers. Nonetheless, the proposed framework provides enough flexibility to incorporate many variations of the base setting. For example, one could accommodate the hypothesis that products that are offered by more than one retailer become more attractive to consumers. For that, we would consider a formulation in which

$$\delta_{n,i} := \mathbf{1}\left\{\nu_{n,i} > \nu_{m,i}\right\} + \beta \mathbf{1}\left\{\nu_{n,i} = \nu_{m,i}\right\},$$

where $0.5 \leq \beta \leq 1$ reflects the potential increase in product attractiveness when it is offered by multiple firms. In such a setting, $\beta = 0.5$ corresponds to the setting analyzed in this paper, and $\beta = 1$ reduces to the case of exclusive-only products. One can show that the results in Sections 3.3 and 4.2 continue to hold in such a setting, after minor modifications to their proofs. (Note that cases with exclusive-only products are not affected by this modification.)

An important avenue for future research is to further one's understanding of customer choice behavior in the face of both common and exclusive products and to understand the implications of such behavior on equilibrium outcomes.

Other forms of competition or operational constraints. In the present paper, we have studied simultaneous assortment and price competition. Another interesting direction to analyze the impact of the type of competition (simultaneous versus sequential) on the type of outcomes one observes. Similarly, one might test the flexibility of the approach to incorporate new operational constrains. For example, one can show that most results in our analysis hold when the display constraint is replaced by a similar constraint of the type $A_n \geq C_n$, that might arise in setting in which managers must fulfill minimum assortment *diversity* requirements.

General choice models. In the case of assortment-only competition, our analysis relies on the attraction form of the demand model and the particular Logit assumption plays a key role in the joint assortment and pricing analysis. One should highlight the challenges one faces under more general models. Next, we examine how the analysis in this paper changes under popular variations of MNL demand.

5.1 The Case of Nested Logit Demand

As pointed out in Remark 1, our demand model is such that it is possible for a retailer to capture the whole market for a product offered by both retailers. This feature follows from the fact that idiosyncratic shocks to utility are independent of the retailer. Let us analyze the case where such shocks depend on the retailer as well, i.e.

$$U_{n,i}(t) \quad := \quad \mu_{n,i} - \alpha_{n,i}\, p_{n,i} + \xi^t_{n,i}, \qquad n = 1,2.$$

(We assume that ξ^t_0 does not depend of the retailer, as it represents an outside alternative). As mentioned in Section 2, assuming that $\left\{\xi^t_{n,i} : n = 1,2,\, i \in \mathcal{S}_n,\, t \geq 1\right\}$ are i.i.d. random variables a standard Gumbel distribution would recover MNL demand at the expense of eliminating common products. Let us consider then the case of nested demand, where consumers first select the product to buy, and then the retailer to buy from. In terms of the utility specification above, assume that $\left(\xi^t_{n,i} : n = 1,2,\, i \in \mathcal{S}\right)$ has a GEV distribution of the form

$$\mathbb{P}\left\{\xi^t_{n,i} \leq x_{n,i}, n = 1,2,\, i \in \mathcal{S}\right\} = \exp\left\{-\sum_{i\in\mathcal{S}}\left(e^{-x_{1,i}/\gamma_i} + e^{-x_{2,i}/\gamma_i}\right)^{\gamma_i}\right\},$$

where $\gamma_i \in [0,1]$ relate to the importance of a product choice over retailer choice. This gives rise to a Nested Logit demand (see e.g. Train (2002)), where in a first stage customers choose a product according to an aggregate measure of the utility they perceive from buying from the retailers offering such a product, and in a second stage they choose the retailer according to a traditional MNL model. In particular, the probability that a customer elects to purchase product $i \in A_n$ from retailer n, $q_{n,i}$ is

$$q_{n,i}(A_n, p_n, A_m, p_m) = q^1_i(A_n, p_n, A_m, p_m) \cdot q^2_{n,i},$$

where

$$q^1_i(A_n, p_n, A_m, p_m) := \frac{\tilde{\nu}_i}{1 + \sum_{j\in A_n\cup A_m}\tilde{\nu}_j}, \qquad q^2_{n,i} := \frac{\nu_{n,i}^{1/\gamma_i}}{\nu_{n,i}^{1/\gamma_i} + \nu_{m,i}^{1/\gamma_i}},$$

where $\tilde{\nu}_i := \left(\nu_{1,i}^{1/\gamma_i} + \nu_{1,i}^{1/\gamma_i}\right)^{\gamma_i}$. In this model, products capture a larger market share when offered by both retailers, however the split between the retailers depends on the value of γ_i (the case of $\gamma_i = 0$ corresponds to our base definition of common products, and the case of $\gamma_i = 1$ corresponds to the case of exclusive products). Note that the modification above only affects settings with common products: for a exclusive product $i \in \mathcal{S}_n$, one has that $\tilde{\nu}_i = \nu_{n,i}$ and $q^2_{n,i} = 1$, thus recovering the models of Sections 3.2 and 4.1.

One can show that the setting above corresponds to an extension of our base model in which

$$\delta_{n,i} = \nu_{n,i}^{1/\gamma_i - 1} \left(\nu_{1,i}^{1/\gamma_i} + \nu_{2,i}^{1/\gamma_i} \right)^{\gamma_i - 1}.$$

With this equivalence at hand, one can show that the results in Section 3.3 for assortment-only competition continue to hold under Nested Logit demand, and only minor modifications to their proofs are required. For the case of assortment and price competition in Section 4.2, while the results continue to hold, a different set of proving techniques is required. In particular, while a closed-form expression for the optimal price $p^*_{n,i}$ is not available, the discontinuities in the payoff functions introduced by the original definition of $\delta_{n,i}$ now disappear, allowing to prove Proposition 3 using standard fixed-point results.

5.2 The Case of Mixed Logit Demand

While it has been shown that under mixed Logit demand, under some conditions and for given assortments, existence and uniqueness of an equilibrium in prices can be guaranteed (see Allon et al. (2013)), the assortment problem becomes (theoretically) intractable, as highlighted in Rusmevichientong et al. (2014) where it is shown that the monopolist's problem is in general NP-Hard, even in the absence of display capacities. One may also show that structural properties presented here will not continue to hold for such models.

Consider, for example, a setting with assortment-only competition and exclusive products, where a fraction ρ_n of the consumers are loyal to retailer n, meaning that they only consider products offered by retailer n when making the purchase decision, $n = 1, 2$. In this setting the expected profit per customer for retailer n, $\widetilde{\pi}_n(A_n, A_m)$, is given by

$$\widetilde{\pi}_n(A_n, A_m) = \rho_n \ \pi_n(A_n, \emptyset) + (1 - \rho_n - \rho_m) \ \pi_n(A_n, A_m).$$

This is a special instance of mixed logit demand. We provide below an example with exclusive products in which an equilibrium fails to exist for the assortment competition game. Consider a setting with two retailers, each with a capacity of $C_1 = C_2 = 1$; retailer 1 has access to $\mathcal{S}_1 = \{1, 2\}$ while retailer 2 has access to $\mathcal{S}_2 = \{3, 4\}$. Parameters are set such that $\nu_{1,1} = 9.8$, $\nu_{1,2} = 4.6, \nu_{2,3} = 9.1$, $\nu_{2,4} = 6.5$, $p_{1,1} - c_{1,1} = 0.4$, $p_{1,2} - c_{1,2} = 0.2$, $p_{2,3} - c_{2,3} = 0.9$ and $p_{2,4} - c_{2,4} = 0.8$. In addition, consider $\rho_1 = \rho_2 = 0.3$ and $\beta_1 = \beta_2 = 0$. Table 2 depicts the rewards for each retailer for feasible pairs of assortment decisions (A_1, A_2). One can verify that no equilibrium exists.

We have shown that the analysis in our paper extend to the case of Nested Logit demand, and that it does not for the case of mixed Logit demand. Generalizing the class of models for which

		A_2	
		{3}	{4}
A_1	{1}	$(0.897, 0.575)$	$(0.985, 0.571)$
	{2}	$(0.900, 0.657)$	$(0.977, 0.680)$

Table 2: Illustration of non-existence of equilibrium for the case of multiple segments. Each entry in the table corresponds the profit of retailer 1 (row) and retailer 2 (column) as a function of the assortments selected.

assortment games are amenable to analysis is an important theoretical direction of research.

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A Proofs

Proof of Lemma 1. Problem (1) is equivalent to finding the maximal value of λ such that

$$\max_{A\in\mathcal{A}_n}\left\{\frac{\sum_{i\in A}\nu_{n,i}\left(\mathbf{1}\left\{i\notin A_m\right\}+\delta_{n,i}\mathbf{1}\left\{i\in A_m\right\}\right)\left(p_{n,i}-c_{n,i}\right)}{1+\sum_{i\in A\setminus A_m}\nu_{n,i}+\sum_{i\in A\cap A_m}\left(\delta_{n,i}\ \nu_{n,i}+\delta_{m,i}\ \nu_{m,i}\right)+\sum_{i\in A_m\setminus A}\nu_{m,i}}\right\}\quad\geq\quad\lambda,$$

and the quantity on the left-hand-side above is exactly the expected profit per customer associated with a best response from retailer n to retailer m's assortment selection of A_m. One obtains (2) from rearranging the terms in the above. Hence the value of problems (2) and (1) are equal. Moreover, any assortment that maximizes the left-hand-side in (2b) for a given λ leads to an expected profit of at least λ, hence such an assortment is a best response to A_m when computed for λ equal to the solution to (2). □

Proof of Proposition 1. For any $e\geq 0$, by the definition of $a_n(e)$ and $\lambda_n(e)$, one has that for any assortment $a\in a_n(e)$,

$$\sum_{i\in a}\nu_{n,i}(p_{n,i}-c_{n,i}-\lambda_n(e))=\lambda_n(e)(1+e).$$

Proof of part 1. Consider now any e,e' such that $0\leq e'<e$. It is necessarily the case that

$$\max_{A_n\in\mathcal{A}_n}\left\{\sum_{i\in A_n}\nu_{n,i}(p_{n,i}-c_{n,i}-\lambda_n(e))\right\}>\lambda_n(e)(1+e').$$

Since both $\lambda \mapsto \max_{A_n \in \mathcal{A}_n} \left\{ \sum_{i \in A_n} \nu_{n,i}(p_{n,i} - c_{n,i} - \lambda) \right\}$ and $\lambda \mapsto \lambda(1+e')$ are continuous in λ, one has that

$$\max_{A_n \in \mathcal{A}_n} \left\{ \sum_{i \in A_n} \nu_{n,i}(p_{n,i} - c_{n,i} - \lambda) \right\} > \lambda(1+e')$$

for all λ in a neighborhood of $\lambda_n(e)$. Noting that the left-hand-side above is decreasing in λ and the right-hand-side is increasing in λ, it is necessarily the case that $\lambda_n(e') > \lambda_n(e)$. This completes the proof of part 1.

Proof of part 2. Fix e, e' such that $0 \le e' < e$ and define $\Delta\lambda := \lambda_n(e') - \lambda_n(e)$, which is positive by part $i.$). Let a' be any best response to an assortment with attractiveness e', i.e., $a' \in a_n(e')$. Then, recalling the definition of the $\theta_{n,i}$'s in (3) and the discussion that followed, it is necessarily the case that

$$\theta_{n,i}(\lambda_n(e')) \ge \theta_{n,j}(\lambda_n(e')) \text{ for any } i \in a',\ j \in \mathcal{S}_n \setminus a'. \tag{A-1}$$

Also, since $\theta_{n,i}(\lambda) = \nu_{n,i}(p_{n,i} - c_{n,i} - \lambda)$ in the case of exclusive products, one has that for all $a \in a_n(e)$, for any $i \in a,\ j \in \mathcal{S}_n \setminus a$,

$$\theta_{n,i}(\lambda_n(e')) + \nu_{n,i}\ \Delta\lambda \ = \ \theta_{n,i}(\lambda_n(e)) \ \ge \ \theta_{n,j}(\lambda_n(e)) \ = \ \theta_{n,j}(\lambda_n(e')) + \nu_{n,j}\ \Delta\lambda. \tag{A-2}$$

Combining (A-1) and (A-2), we conclude that for any pair $(a, a') \in a_n(e) \times a_n(e')$,

$$\nu_{n,i} \ge \nu_{n,j}, \quad \forall\, i \in a \setminus a',\ j \in a' \setminus a. \tag{A-3}$$

For any given $\lambda \in \mathbb{R}_+$, all assortments a solving the maximization in (4) consist only of products with non-negative values of $\theta_{n,i}(\lambda)$. Since $\theta_{n,i}(\lambda)$ is strictly decreasing in λ for all products, higher values of λ translate into fewer products with non-negative $\theta_{n,i}(\lambda)$. Thus, the cardinality of the assortments solving the maximization in (4) is non-increasing in λ. Since $\lambda_n(e') > \lambda_n(e)$, one has that $|a'| \le |a|$, for any $a' \in a_n(e')$ and $a \in a_n(e)$. This in turns implies that

$$|a' \setminus a| = |a'| - |a \cap a'| \le |a| - |a \cap a'| = |a \setminus a'|.$$

Observing that $\nu_{n,i} > 0$ for all $i \in \mathcal{S}_n$, one has that

$$\sum_{i \in a \setminus a'} \nu_{n,i} \ge |a \setminus a'| \min\left\{\nu_{n,i} \,:\, i \in a \setminus a'\right\} \overset{(a)}{\ge} |a' \setminus a| \max\left\{\nu_{n,i} \,:\, i \in a' \setminus a\right\} \ge \sum_{i \in a' \setminus a} \nu_{n,i},$$

where (a) follows from (A-3). This, in turn, implies that

$$E_n(a) = \sum_{i \in a \cap a'} \nu_{n,i} + \sum_{i \in a \setminus a'} \nu_{n,i} \ge \sum_{i \in a \cap a'} \nu_{n,i} + \sum_{i \in a' \setminus a} \nu_{n,i} = E_n(a'),$$

for all $a \in a_n(e)$ and $a' \in a_n(e')$. This concludes the proof. $\square$

Proof of Theorem 1. For any retailer n, consider the set $\{E_n(A) : A \in \mathcal{A}_n\}$ of all possible attraction levels corresponding to assortments, and denote those levels by $e_1 < e_2 < ... < e_{k_n}$. Let $Z_n := \{e_1, ..., e_{k_n}\}$ denote the ordered set of those levels. In addition, for any attraction level e^m offered by firm m, let $Y_n(e^m) = \{E_n(a) : a \in a_n(e^m)\}$ denote the set attractiveness levels corresponding to best responses to e^m. Finally, let $Y(e^1, e^2) = (Y_1(e^2), Y_2(e^1))$ denote the correspondence from $Z_1 \times Z_2$ into $Z_1 \times Z_2$. Now, note that $Z_1 \times Z_2$ is a non-empty complete lattice and that Proposition 1 *ii.*) implies that $Y(e^1, e^2)$ is a non-decreasing correspondence. These two facts, in conjunction with the fixed point result of Topkis (1998, Theorem 2.5.1), imply that $Y(e^1, e^2)$ admits a fixed point in $Z_1 \times Z_2$. Selecting the assortments that correspond to the attractiveness levels associated with this fixed points yields an equilibrium in assortment decisions and the proof is complete.

We now comment on the fact that the reasoning above extends to the case of an arbitrary number of retailers. Indeed, when there are $N \geq 2$ retailers, one can define $Y(e^1, \ldots, e^N) = (Y_1(e - e^1), \ldots, Y_N(e - e^N))$, with $e := \sum_{i=1}^{N} e^i$. By using Proposition 1 one can prove that $Y(e^1, \ldots, e^N)$ is a non-decreasing correspondence from $Z_1 \times \ldots \times Z_N$ into itself, where Z_n is defined as in the two-retailer case. Since $Z_1 \times \ldots \times Z_N$ is a non-empty complete lattice one can again use the fixed point result of Topkis (1998, Theorem 2.5.1) to establish existence of an equilibrium in assortment decisions. $\square$

Proof of Proposition 2. Following the argument in the proof of Theorem 1, Topkis (1998, Theorem 2.5.1) also yields that the set of fixed points of $Y(\cdot)$ is a nonempty complete lattice relative to $\leq$ (component-wise). Hence there exist a fixed point (e^1, e^2) such that $e^n \leq g^n$, $n = 1, 2$, for all fixed points (g^1, g^2) of $Y(\cdot)$. From proposition 1 *i.*) and noticing that fixed points of $Y(\cdot)$ map to assortment equilibria, retailer n prefers an equilibrium involving the attractiveness pair (e^1, e^2) as it minimizes e_m. This applies for both n, thus both retailers prefer the same equilibrium.

We now comment on the fact that the reasoning above extends to the case of an arbitrary number of retailers. Indeed, following the reasoning on the proof of Theorem 1, Topkis (1998, Theorem 2.5.1) establishes the existence of a fixed point $(e^1, \ldots, e^N)$ such that $e^n \leq g^n$, $n = \{1 \ldots, N\}$, for all fixed points $(g^1, \ldots, g^N)$ of $Y(\cdot)$, where N is the number of retailers. The result would then follow from the fact that $\sum_{k \neq n} e^k \leq \sum_{k \neq n} g^k$ for all fixed points $(g^1, \ldots, g^N)$ of $Y(\cdot)$, hence retailer n prefers $(e^1, \ldots, e^N)$ and this applies to all n. $\square$

Proof of Theorem 2. For $n = 1, 2$ and $e \geq 0$, let $R_n(e)$ denote the set of attractiveness levels corresponding to best responses of retailer n when retailer m offers an assortment with an attractiveness of e. That is,

$$R_n(e) := \{E_n(a) \,:\, a \in a_n(e)\}.$$

We observe that the number of different equilibria (in which retailers offer different total attractiveness) is bounded above by $\sum_{e\in\mathcal{E}_2} |R_1(e)|$. We next provide a bound on this sum. Define $\underline{\mathcal{E}}_2 := \{e \in \mathcal{E}_2 \,:\, |R_1(e)| > 1\}$. Let k denote the cardinality of $\underline{\mathcal{E}}_2$ and let us denote the elements of $\underline{\mathcal{E}}_2$ by $e_1 < ... < e_k$. One has that

$$\begin{aligned}
\sum_{e\in\mathcal{E}_2} |R_1(e)| &= \sum_{e\in\underline{\mathcal{E}}_2} |R_1(e)| + \sum_{e\in\mathcal{E}_2\setminus\underline{\mathcal{E}}_2} |R_1(e)| \\
&= \sum_{j=1}^{k} |R_1(e_j)| + |\mathcal{E}_2 \setminus \underline{\mathcal{E}}_2|.
\end{aligned}$$

Note that for any pair (e_i, e_j) with $i \neq j$, part *ii.*) of Lemma 1 implies that

$$|R_n(e_i) \cap R_n(e_j)| \leq 1.$$

In addition, the latter result, in conjunction with the fact that $|R_1(e_j)| > 1$ for all $j \in \{1, ..., k\}$ implies that $|R_1(e_j) \cap R_1(e_i)| = 0$ for any $j < i + 1$ and i, j in $\{1, ..., k\}$. Hence $\sum_{j=1}^{k} |R_1(e_j)| \leq |\mathcal{E}_1| + |\underline{\mathcal{E}}_2| - 1$. We deduce that

$$\sum_{e\in\mathcal{E}_2} |R_1(e)| \leq |\mathcal{E}_1| + |\underline{\mathcal{E}}_2| - 1 + |\mathcal{E}_2 \setminus \underline{\mathcal{E}}_2| = |\mathcal{E}_1| + |\mathcal{E}_2| - 1.$$

This completes the proof. □

Proof of Theorem 3. We prove the result by constructing a sequence of assortment pairs that yields an equilibrium. Let $\mathcal{S}_n^e$ denote the set of products that are exclusive to retailer n, i.e. $\mathcal{S}_n^e := \mathcal{S}_n \setminus \mathcal{S}_m$, $n = 1, 2$. Also, let $\mathcal{S}_{12}^c$ denote the set of common products for which no retailer has an advantage in terms of attractiveness, i.e. $\mathcal{S}_{12}^c := \{i \in \mathcal{S}_1 \cap \mathcal{S}_2 : \nu_{1,i} = \nu_{2,i}\}$. Similarly, define $\mathcal{S}_n^c$ as the set of common products for which retailer n has an advantage in terms of attractiveness. That is $\mathcal{S}_n^c := \{i \in \mathcal{S}_1 \cap \mathcal{S}_2 : \nu_{n,i} > \nu_{m,i}\}$.

Define the sequence of assortments pairs $\left\{(A_1^k, A_2^k) : k = 1, 2, \ldots\right\}$ as follows. Set $A_1^1 = A_2^1 = \mathcal{S}_{12}^c$, and for $k > 1$, let

$$\begin{aligned}
\lambda_n^k &:= \max\left\{\lambda \in \mathbb{R} : \max_{A\in\mathcal{S}_n} \sum_{i\in A} \theta_{n,i}(\lambda, A_m^{k-1}) \geq \lambda\left(1 + \sum_{i\in A_m^{k-1}} \nu_{m,i}\right)\right\} \\
A_n^k &\in \operatorname{argmax}\left\{|B| : B \in \operatorname*{argmax}_{A\in\mathcal{S}_n}\left\{\sum_{i\in A} \theta_{n,i}(\lambda_n^k, A_m^{k-1})\right\}\right\},
\end{aligned}$$

for $n = 1, 2$, where with a slight abuse of notation we make the dependence of $\theta_{n,i}(\cdot)$ on the assortment offered by retailer m explicit. This process is one in which A_n^k is a best response to A_m^{k-1}, where the best response assortment is selected to have the maximal cardinality.

Next, we show by mathematical induction that for $n = 1, 2$, $A_n^k \subseteq A_n^{k+1}$ for all $k \geq 1$.

For $k = 1$ and $n \in \{1, 2\}$, equation (3) implies that $\theta_{n,i}(\lambda, A_m^1) > 0$ for all $i \in \mathcal{S}_{12}^c$ and for all $\lambda > 0$. This, in conjunction with the fact that display constraints are absent, implies that $A_n^2 \supseteq \mathcal{S}_{12}^c = A_n^1$.

Suppose now that $A_n^j \subseteq A_n^{j+1}$ for $j =, 1, ..., k-1$. We first prove the following claim.

Claim: For $n \in \{1, 2\}$ and $k > 1$, when retailer m offers A_m^k,

$$\sum_{i \in A} \theta_{n,i}(\lambda_n^k, A_m^k) \leq \lambda_n^k \left(1 + \sum_{i \in A_m^k} \nu_{m,i}\right) \tag{A-4}$$

for any assortment $A \subseteq \mathcal{S}_n$.

To prove the claim first observe that, by the arguments above one has that $A_n^j \supseteq \mathcal{S}_{1,2}^c$ for all $j \leq k$. For any $\lambda \geq 0$, when retailer m offers A_m^k, one has that

$$\begin{aligned}
& \sum_{i \in A_n^k} \theta_{n,i}(\lambda, A_m^k) \\
= \; & \sum_{i \in A_n^k \cap \mathcal{S}_n^e} \nu_{n,i}\left(p_{n,i} - c_{n,i} - \lambda\right) + \sum_{i \in A_n^k \cap \mathcal{S}_m^c \setminus A_m^k} \nu_{n,i}(p_{n,i} - c_{n,i} - \lambda) \\
& + \sum_{i \in A_n^k \cap \mathcal{S}_n^c \cap A_m^k} \nu_{n,i}\left(p_{n,i} - c_{n,i} - \lambda\left(1 - \frac{\nu_{m,i}}{\nu_{n,i}}\right)\right) + \sum_{i \in A_n^k \cap \mathcal{S}_n^c \setminus A_m^k} \nu_{n,i}(p_{n,i} - c_{n,i} - \lambda) + \sum_{i \in \mathcal{S}_{12}^c} \frac{1}{2}\nu_{n,i}(p_{n,i} - c_{n,i}) \\
= \; & \sum_{i \in A_n^k \cap \mathcal{S}_n^e} \nu_{n,i}\left(p_{n,i} - c_{n,i} - \lambda\right) + \sum_{i \in A_n^k \cap \mathcal{S}_m^c \setminus A_m^{k-1}} \nu_{n,i}(p_{n,i} - c_{n,i} - \lambda) - \sum_{i \in A_n^k \cap \mathcal{S}_m^c \cap A_m^k \setminus A_m^{k-1}} \nu_{n,i}(p_{n,i} - c_{n,i} - \lambda) \\
& + \sum_{i \in A_n^k \cap \mathcal{S}_n^c \cap A_m^{k-1}} \nu_{n,i}\left(p_{n,i} - c_{n,i} - \lambda\left(1 - \frac{\nu_{m,i}}{\nu_{n,i}}\right)\right) + \sum_{i \in A_n^k \cap \mathcal{S}_n^c \setminus A_m^{k-1}} \nu_{n,i}(p_{n,i} - c_{n,i} - \lambda) \\
& + \sum_{i \in A_n^k \cap \mathcal{S}_n^c \cap A_m^k \setminus A_m^{k-1}} \nu_{m,i}\lambda + \sum_{i \in \mathcal{S}_{12}^c} \frac{1}{2}\nu_{n,i}(p_{n,i} - c_{n,i}),
\end{aligned}$$

where the first equality follows by definition and the second equality follows from basic simplifications. Note that for $\lambda = \lambda_n^k$, using the optimality conditions that define λ_n^k, the above becomes

$$\begin{aligned}
\sum_{i \in A_n^k} \theta_{n,i}(\lambda_n^k, A_m^k) &= \lambda_n^k \left(1 + \sum_{i \in A_m^{k-1}} \nu_{m,i} + \sum_{A_n^k \cap \mathcal{S}_n^c \cap A_m^k \setminus A_m^{k-1}} \nu_{m,i}\right) - \sum_{i \in A_n^k \cap \mathcal{S}_m^c \cap A_m^k \setminus A_m^{k-1}} \nu_{n,i}(p_{n,i} - c_{n,i} - \lambda_n^k) \\
&\leq \lambda_n^k \left(1 + \sum_{i \in A_m^{k-1}} \nu_{m,i} + \sum_{A_n^k \cap \mathcal{S}_n^c \cap A_m^k \setminus A_m^{k-1}} \nu_{m,i}\right),
\end{aligned} \tag{A-5}$$

where for the inequality, we have used the fact that $\nu_{n,i}(p_{n,i}-c_{n,i}-\lambda_n^k)=\theta_{n,i}(\lambda_n^k,A_m^{k-1})\geq 0$ for all $i\in A_n^k\setminus A_m^{k-1}$.

Now, for an arbitrary assortment $A\subseteq\mathcal{S}_n$, when retailer m offers A_m^k, one has that

$$\begin{aligned}
\sum_{i\in A}\theta_{n,i}(\lambda_n^k,A_m^k) &= \sum_{i\in A_n^k\cap A}\theta_{n,i}(\lambda_n^k,A_m^k)+\sum_{i\in A\setminus A_n^k}\theta_{n,i}(\lambda_n^k,A_m^k)\\
&\overset{(a)}{\leq} \sum_{i\in A_n^k}\theta_{n,i}(\lambda_n^k,A_m^k)+\sum_{i\in(A\setminus A_n^k)\cap(A_m^k\setminus A_m^{k-1})\cap\mathcal{S}_n^c}\Big(\theta_{n,i}(\lambda_n^k,A_m^{k-1})+\lambda_n^k\nu_{m,i}\Big)\\
&\quad+\sum_{i\in(A\setminus A_n^k)\cap A_m^{k-1}}\theta_{n,i}(\lambda_n^k,A_m^{k-1})+\sum_{i\in(A\setminus A_n^k)\cap(\mathcal{S}_n^c\setminus A_m^k)}\theta_{n,i}(\lambda_n^k,A_m^{k-1})\\
&\overset{(b)}{\leq} \lambda_n^k\left(1+\sum_{i\in A_m^{k-1}}\nu_{m,i}+\sum_{A_n^k\cap\mathcal{S}_n^c\cap A_m^k\setminus A_m^{k-1}}\nu_{m,i}+\sum_{i\in(A\setminus A_n^k)\cap(A_m^k\setminus A_m^{k-1})\cap\mathcal{S}_n^c}\nu_{m,i}\right)\\
&\leq \lambda_n^k\left(1+\sum_{i\in A_m^k}\nu_{m,i}\right),
\end{aligned}$$

where (a) follows from noting that $\theta_{n,i}(\lambda_n^k,A_m^k)\geq 0$ for all $i\in A_n^k$,[1], the fact that A_n^k contains $\mathcal{S}_{12}^c$ and the fact that $\theta_{n,i}(\lambda_n^k,A_m^k)=0$ for $i\in(A\setminus A_n^k)\cap(A_m^k\setminus A_m^{k-1})\cap\mathcal{S}_m^c$; and (b) follows from (A-5) and from noting that $\theta_{n,i}(\lambda_n^k,A_m^{k-1})<0$ for all $i\in\mathcal{S}_n\setminus A_n^k$ (since A_n^k was optimal when A_m^{k-1} was offered by retailer m). This completes the proof of the claim.

Since the claim holds for any assortment, it also holds for the assortment that maximizes the left-hand side of (A-4), which implies that $\lambda_n^{k+1}\leq\lambda_n^k$. In addition, the fact that $\theta_{n,i}(\lambda_n^k,A_m^k)\geq 0$ for all $i\in A_n^k$ together with the fact that $\theta_{n,i}(\cdot,A)$ is non-increasing implies that $\theta_{n,i}(\lambda_n^{k+1},A_m^k)\geq 0$ for all $i\in A_n^k$. We deduce that $A_n^{k+1}\supseteq A_n^k$. This concludes the induction.

Now, fix n and note that $\{A_n^k : k\geq 1\}$ is a sequence of sets such that $A_n^k\subseteq A_n^{k+1}$ and $A_n^k\subseteq\mathcal{S}_n$. Hence, it must be that after a finite number of iterations, j_n, $A_n^k=A_n^{j_n}$ for all $k\geq j_n$. The result now follows from noting that for all $k\geq\max\{j_1,j_2\}$, each assortment is a best response to the other assortment and such a pair constitutes an equilibrium. $\square$

Proof of Theorem 4. Fix an assortment A_m and a price vector p_m. One can show that retailer n's best response to such actions is given by the solution to

$$\max\left\{\lambda\in\mathbb{R}\,:\,\max_{A\in\mathcal{A}_n,\,p_n\in\mathcal{P}}\left\{\sum_{i\in A}\nu_{n,i}(p_{n,i})(p_{n,i}-c_{n,i}-\lambda)\right\}\geq\lambda\left(1+\sum_{i\in A_m}\nu_{m,i}(p_{m,i})\right)\right\},$$

[1]Indeed, $\theta_{n,i}(\lambda_n^k,A_m^k)=\theta_{n,i}(\lambda_n^k,A_m^{k-1})\geq 0$ for $i\in A_n^k\cap A_m^{k-1}$; $\theta_{n,i}(\lambda_n^k,A_m^k)=\theta_{n,i}(\lambda_n^k,A_m^{k-1})+\lambda_n^k\nu_{m,i}\geq 0$ for $i\in A_n^k\cap(A_m^k\setminus A_m^{k-1})\cap\mathcal{S}_n^c$; and $\theta_{n,i}(\lambda_n^k,A_m^k)=0$ for $i\in A_n^k\cap(A_m^k\setminus A_m^{k-1})\cap\mathcal{S}_m^c$.

where $\mathcal{P}$ denotes the set of feasible price vectors, i.e., those that satisfy the minimum margin constraint. Note that with some abuse of notation, we now make the price dependence of $\nu_{n,i}$ explicit. For a given $\lambda \in \mathbb{R}$ one can show that the inner maximization above is solved by $p^*_{n,i}(\lambda)$, defined in (8), for all i in the optimal assortment, independent of A_m. Note that the best response depends on (A_m, p_m) only through the attraction of the offered assortment/prices. With this in mind, let $\lambda_n(e)$ denote the profit attained by retailer n on a best response when retailer m offers assortment/prices with attractiveness $e > 0$. Also, let $a_n(e)$ denote retailer n's set of best responses (in assortments) when retailer m offers an attractiveness $e > 0$.

Step 1. We first establish that a retailer's profit is decreasing in the competitor's offered attractiveness. By construction one has that

$$\sum_{i \in a} \widetilde{\theta}_{n,i}(\lambda_n(e)) = \lambda_n(e)\,(1+e)\,,$$

for any $a \in a_n(e)$, where, for any $\lambda \in \mathbb{R}$,

$$\widetilde{\theta}_{n,i}(\lambda) := \nu_{n,i}(p^*_{n,i}(\lambda))(p^*_{n,i}(\lambda) - c_{n,i} - \lambda). \tag{A-6}$$

Fix e, e' such that $0 \leq e' < e$. One has that

$$\max_{a \in \mathcal{A}_n} \left\{ \sum_{i \in a} \widetilde{\theta}_{n,i}(\lambda_n(e)) \right\} > \lambda_n(e)(1+e').$$

Since both $\lambda \to \max_{a \in \mathcal{A}_n} \left\{ \sum_{i \in a} \widetilde{\theta}_{n,i}(\lambda) \right\}$ and $\lambda \to \lambda(1+e')$ are continuous in λ, one obtains that

$$\max_{a \in \mathcal{A}_n} \left\{ \sum_{i \in a} \widetilde{\theta}_{n,i}(\lambda) \right\} > \lambda(1+e')$$

for all λ in a neighborhood of $\lambda_n(e)$. The left-hand-side (right-hand-side) above is decreasing (increasing) in λ, thus it must be the case that $\lambda_n(e') > \lambda_n(e)$. We conclude that a retailer's profit is decreasing in the competitor's offered attractiveness.

Step 2. We now establish that the assortment attractiveness of a retailer's optimal decisions is increasing in the attractiveness of the competitor's assortment. Fix e, e' such that $0 \leq e' < e$ and define $\Delta = \lambda_n(e') - \lambda_n(e) > 0$. Consider $a \in a_n(e)$ and $a' \in a_n(e')$; it must be the case that

$$\begin{aligned}
&\widetilde{\theta}_{n,i}(\lambda_n(e')) \geq \widetilde{\theta}_{n,j}(\lambda_n(e')) \text{ for any } i \in a' \setminus a,\ j \in a \setminus a',\\
&\widetilde{\theta}_{n,i}(\lambda_n(e)) \geq \widetilde{\theta}_{n,j}(\lambda_n(e)) \text{ for any } i \in a \setminus a',\ j \in a' \setminus a.
\end{aligned}$$

The above implies that

$$\widetilde{\theta}_{n,i}(\lambda_n(e)) - \widetilde{\theta}_{n,i}(\lambda_n(e')) \geq \widetilde{\theta}_{n,j}(\lambda_n(e)) - \widetilde{\theta}_{n,j}(\lambda_n(e'))$$

for all $(i,j) \in (a \setminus a') \times (a' \setminus a)$. Note that $\widetilde{\theta}_{n,i}(\lambda)$ is differentiable. In particular, one has that

$$\frac{\partial \widetilde{\theta}_{n,i}(\lambda)}{\partial \lambda} = -\nu_{n,i}(p^*_{n,i}(\lambda)) < 0, \quad \forall\, \lambda \in \mathbb{R}_+,$$

and that $\nu_{n,i}(p^*_{n,i}(\cdot))$ is a continuous and non-increasing function of λ, for all $i \in \mathcal{S}_n$. The above implies that

$$\nu_{n,i}(p^*_{n,i}(\lambda_n(e)))\Delta \geq \widetilde{\theta}_{n,i}(\lambda_n(e)) - \widetilde{\theta}_{n,i}(\lambda_n(e')) \geq \widetilde{\theta}_{n,j}(\lambda_n(e)) - \widetilde{\theta}_{n,j}(\lambda_n(e')) \geq \nu_{n,j}(p^*_{n,j}(\lambda_n(e'))\Delta,$$

for any $(i,j) \in (a \setminus a') \times (a' \setminus a)$. One concludes that, for any pair $(a, a') \in a_n(e) \times a_n(e')$, $\nu_{n,i}(p^*_{n,i}(\lambda_n(e))) \geq \nu_{n,i}(p^*_{n,i}(\lambda_n(e')))$ for all $(i,j) \in (a \setminus a') \times (a' \setminus a)$. Note now that $\widetilde{\theta}_{n,i}(\lambda) > 0$ for all $\lambda \in \mathbb{R}_+$, hence it must be the case that $|a| = C_n$ for any $a \in a_n(\varepsilon)$, for all $\varepsilon \in \mathbb{R}_+$. One deduces that

$$\begin{aligned}
\sum_{i \in a} \nu_{n,i}(p^*(\lambda_n(e))) &= \sum_{i \in a \setminus a'} \nu_{n,i}(p^*(\lambda_n(e))) + \sum_{i \in a \cap a'} \nu_{n,i}(p^*(\lambda_n(e))) \\
&\geq \sum_{i \in a' \setminus a} \nu_{n,i}(p^*(\lambda_n(e'))) + \sum_{i \in a \cap a'} \nu_{n,i}(p^*(\lambda_n(e))) \\
&\geq \sum_{i \in a'} \nu_{n,i}(p^*(\lambda_n(e'))),
\end{aligned}$$

for all $(a,a') \in a_n(e) \times a_n(e')$, where the first inequality follows from the fact that $|a' \setminus a| = |a \setminus a'|$, and the second from noting that $\nu_{n,i}(p^*_{n,i}(\cdot))$ is non-increasing, for all $i \in \mathcal{S}_n$.

From steps 1 and 2, we conclude that a similar result to that of Proposition 1 holds when prices are endogenously determined. Now the result follows arguments similar to those in the proofs of Theorems 1 and Proposition 2. □

Proof of Proposition 3. We prove that retailer n's best response to any pair (A_m, p_m) involves offering all products, i.e. setting $A_n = \mathcal{S}_n$. Computing the best response to an assortment-price pair, retailer n solves the following problem.

$$\max \left\{ \lambda \in \mathbb{R} \,:\, \max_{A \in \mathcal{A}_n,\, p_n \in \mathcal{P}} \left\{ \sum_{i \in A} \theta_{n,i}(\lambda, p_n, A_m, p_m) \right\} \geq \lambda \left(1 + \sum_{i \in A_m} \nu_{m,i}(p_{m,i}) \right) \right\},$$

where $\mathcal{P}$ denotes the set of prices that are consistent with the minimum margin requirement, and

$$\theta_{n,i}(\lambda, p_n, A_m, p_m) := \begin{cases} (p_{n,i} - c_{n,i} - \lambda)\ \nu_{n,i}(p_{n,i}) & \text{if } i \notin A_m, \\ \delta_{n,i}\left((p_{n,i} - c_{n,i} - \lambda)\ \nu_{n,i}(p_{n,i}) + \lambda\ \nu_{m,i}(p_{m,i}) \right) & \text{if } i \in A_m. \end{cases}$$

Define $\theta^*_{n,i}(\lambda, A_m, p_m) := \max\{\theta_{n,i}(\lambda, p_n, A_m, p_m) : p_n \in \mathcal{P}\}$: one can see that $\theta^*_{n,i}(\cdot) \geq 0$, thus the best response to any assortment-price vector pair (A_m, p_m) can be chosen so that all products in $\mathcal{S}_n$ are offered. Thus, assuming both retailers offer all products, one can rewrite the formulation above as

$$\max\left\{\lambda \in \mathbb{R}\,:\, \sum_{i\in\mathcal{S}_n} \theta^*_{n,i}(\lambda,\mathcal{S}_m,p_m) \geq \lambda\left(1+\sum_{i\in\mathcal{S}_m}\nu_{m,i}(p_{m,i})\right)\right\}.$$

If follows that if profit levels (λ_1,λ_2) are attained in equilibrium, then exclusive products are priced as in (8), and common product i's price is given by

$$p^*_{n,i}(\lambda) = \begin{cases} \min\left\{\frac{\mu_{n,i}-\mu_{m,i}}{\alpha_{n,i}} + \frac{\alpha_{m,i}}{\alpha_{n,i}}\left(c_{m,i}+r_{m,i}\right), c_{n,i} + \max\left\{\frac{1}{\alpha_{n,i}}+\lambda, r_{n,i}\right\}\right\} & i\in\tilde{\mathcal{S}}_n \\ c_{n,i}+r_{n,i} & \text{otherwise,} \end{cases} \tag{A-7}$$

where $\tilde{\mathcal{S}}_n := \{i \in \mathcal{S}_1\cap\mathcal{S}_2 : \mu_{n,i}-\alpha_{n,i}(r_{n,i}+c_{n,i}) > \mu_{m,i}-\alpha_{m,i}(r_{m,i}+c_{m,i})\}$, $n=1,2$. Thus, any solution to the system of equations

$$\sum_{i\in\mathcal{S}_n}\theta^*_{n,i}(\lambda_n,\mathcal{S}_m,p^*_m(\lambda_m)) = \lambda_n\left(1+\sum_{i\in\mathcal{S}_m}\nu_{m,i}(p^*_{m,i}(\lambda_m))\right),$$

can be mapped to an equilibrium. The result follows from the existence of a solution to this system of equations, which we prove next.

Let $Y_n(\lambda_m)$ denote the unique solution to

$$\sum_{i\in\mathcal{S}_n}\widehat{\theta}_{n,i}(\lambda,\lambda_m) = \lambda\left(1+\sum_{i\in\mathcal{S}_m}\nu_{m,i}(p^*_{m,i}(\lambda_m))\right). \tag{A-8}$$

where we define $\widehat{\theta}_{n,i}(\lambda,\lambda_m) := \theta^*_{n,i}(\lambda_n,\mathcal{S}_m,p^*_m(\lambda_m))$. Note that, for a fixed λ_m, $\widehat{\theta}_{n,i}(\lambda,\lambda_m)$ is decreasing and positive for $\lambda\in[0,\Lambda_n]$, where Λ_n denotes retailer n's monopolist profit. On the other hand, the right hand side above increases linearly with λ and equals 0 for $\lambda=0$. We conclude $Y_n(\lambda_m)$ is well defined. Similarly, note that $\sum_{i\in\mathcal{S}_n}\widehat{\theta}_{n,i}(\lambda,\lambda_m)$ does not vary with λ_m, but the right hand side is non-increasing in λ_m. We conclude Y_n is non-decreasing. This implies that $Y_m(0)\leq Y_m(\Lambda_n)$, thus we have that

$$0 \leq Y_n(Y_m(0)) \leq Y_n(Y_m(\Lambda_n)) \leq \Lambda_n. \tag{A-9}$$

Note that equilibrium prices in (8) and (A-7) are continuous with respect to λ_n and λ_m, thus so are $Y_n(\cdot)$ and $Y_m(\cdot)$. Existence of a $\lambda^*\in[0,\Lambda_n]$ such that $Y_n(Y_m(\lambda^*))=\lambda^*$ follows from the continuity of $Y_n(Y_m(\cdot))$ and (A-9). The result follows from noting that $(\lambda^*, Y_m(\lambda^*))$ solves (A-8). $\square$

Letter to the review team and response to the reports on: Product Assortment and Price Competition with Informed Consumers (POM-May-14-OA-0266).

We are in receipt of the reports by the review team regarding submission POM-May-14-OA-0266 entitled "Product Assortment and Price Competition with Informed Consumers." We are very grateful to the review team for the effort invested in the reviews and the very detailed and constructive feedback that resulted. We have followed the suggestions of the review team and revised the manuscript accordingly. To facilitate the revision, changes to the manuscript are highlighted in blue. Below, we first summarize the changes to the paper in this revision, and then produce detailed responses to all major comments raised by the AE and the referees. We reproduce the AE and referees' reports in full in brown text.

Summary of main changes. Following the review team's feedback, we have significantly strengthened our results along various dimensions:

- We provide an analysis of the setting suggested by R1, in which products offered by both retailers are more attractive to consumers than when offered by only one retailer. Our analysis indicates that most results in the original setting continue to hold in the suggested setting.
- As part of the extensions, we now consider settings with nested Logit demand and mixed Logit demand. We provide an analysis of equilibrium behavior based on the framework studied. The results are mixed: for the case of the nested logit demand, we show most results continue to hold; for the case of mixed logit demand, we show that equilibrium may fail to exists, even in the case of exclusive-only products under assortment-only competition.
- We expanded our discussion on the possibility of multiple equilibria. In particular, we now provide an example with multiple equilibria, and develop an upper bound on the number of equilibria that might arise in each setting, which we show is tight. We illustrate the use of such a result to provide sufficient conditions for the uniqueness of equilibrium.
- We have also followed the recommendations of the review team to improve and clarify the results throughout the paper.

Response to Comments of Associate Editor

"Product Assortment and Price Competition under Multinomial Logit Demand"

The paper consider the problem of assortment and price competition among retailers, when the underlying demand model is the multinomial logit. Let me briefly review the setup for the case of 2 retailers. The authors use n to denote the retailer, and m to denote its competitor. For $n \in \{1, 2\}$, the set of available assortments for firm n is denoted by $\mathcal{A}_n = \{A \subseteq \mathcal{S}_n : |A| \leq C_n\}$ where $\mathcal{S}_n$ denote the set of products that firm n can offer. Let $\mathbf{p}_n$ be the prices offered by firm n. If $(A_n, \mathbf{p}_n)$ denote the assortment and price offered by firm n and $(A_m, \mathbf{p}_m)$ denote the decision by its competitor, then the profit for the firm n is given by:

$$\pi_n(A_n, \mathbf{p}_n, A_m, \mathbf{p}_m) = \sum_{i \in A_n} (p_{n,i} - c_{n,i})\ q_{n,i}(A_n, \mathbf{p}_n, A_m, \mathbf{p}_m).$$

where $c_{n,i}$ is the unit cost of product i for firm n, and $q_{n,i}(A_n, \mathbf{p}_n, A_m, \mathbf{p}_m)$ is the probability that a customer will purchase product i from firm n where the authors assume that the customer follows a multinomial logit model; that is,

$$q_{n,i}(A_n, p_n, A_m, p_m) \quad := \quad \frac{\nu_{n,i}\ \Big(\mathbf{1}\{i \notin A_m\} +\ \delta_{n,i}\ \mathbf{1}\{i \in A_m\}\Big)}{1 + \sum_{i \in A_n \setminus A_m} \nu_{n,i} + \sum_{i \in A_n \cap A_m} \max\{\nu_{n,i}, \nu_{m,i}\} + \sum_{i \in A_m \setminus A_n} \nu_{m,i}},$$

where $\nu_{n,i}$ denote the preference weight (aka attraction factor) associated with product i offered by firm n, and $\delta_{n,i} := \mathbf{1}\{\nu_{n,i} > \nu_{m,i}\} + 0.5\mathbf{1}\{\nu_{n,i} = \nu_{m,i}\}$ determines the splits of product i's market share when the product is offered by both retailers.

In this paper, the authors consider two typos of competition models: 1) assortment-only competition and 2) joint-assortment-and-price competition. In the assortment-only competition (Section 3), the product prices are exogenous and fixed, and the firms compete on the assortments. In the joint-assortment-and-price competition (Section 4), the firms can adjust both assortments and prices. The analysis for each type of competition is divided into 2 cases: a) the firms do not share any common products ($\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$) and b) there are common products that both firms can offer ($\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$).

Summary of Results: When the firms do not share common products with $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, also referred to as the exclusive products case, the result are very crisp and complete. For both types of competition, the authors show that there is a pure-strategy Nash equilibrium that pareto dominates all of other equilibria. Also, you can compute the pareto-dominated equilibrium using the tatonement process. Unfortunately, when $\mathcal{S} \cap \mathcal{S}_2 \neq \emptyset$, the results are extremely limited. The only main result for the setting with common products appears to be the existence of pure-strategy equilibrium, when there is no capacity constraint for both firms; that is, $C_n = |\mathcal{S}_n|$ for all n. The authors provide some examples that demonstrate the pathological behaviors that can occur when there are display constraints. I sent the paper to two experts in this area. R1 recommends a major revision, but is optimistic that the paper may be publishable. R2 recommends a minor revision. I read the paper myself, and personally, I like the paper very much. It's well written, and the analysis for the exclusive-product competition is beautiful, crisp, and complete. In my opinion, the paper offers the following contributions to the literature:

- A complete and compelling analysis in the case of exclusive products (Sections 3.1 and 4.1). This analysis extends the price-competition analysis of Misra (2008) and Wang (2012) (see Chapter 2 of the dissertation). The clean and rigorous analysis also yields important insights about the structure of the equilibrium. The sentiment is also shared by R2. I am very impressed with the sharp characterization of the equilibrium obtained by the authors in Section 3.1 and 4.1.

- The situation of exclusive products is not entirely realistic because firms often share some common products. So, I commend the authors for considering the case of common products and providing preliminary, albeit limited, results in this case.

Overall, I am quite positive about the paper, and I can see a potential path for the paper to be published in POM. However, there are still a lot of work that remains to be done, so I would characterize this as a major revision. Below is a list of major concerns that the authors should address when revising their paper.

Major Concerns

1. **Demand model:** R1 raised an important point about the split of the demand when a product is offered by both firms, which can make the product more popular. Thus, R1 has suggested that the authors modify $\delta_{n,i}$ so that

$$\delta_{n,i} := \mathbf{1}\left\{\nu_{n,i} > \nu_{m,i}\right\} + \beta \mathbf{1}\left\{\nu_{n,i} = \nu_{m,i}\right\},$$

   where $0.5 \leq \beta \leq 1$ reflects the potential increase in product popularity when it is offered by multiple firms. I agree with this point, and the authors should address this issue. Would the existing results in Sections 3.3 and 4.2 continue to hold? This modification of $\delta_{n,i}$ will not affect the analysis of the exclusive product case. So, any new results obtained here will strengthened the paper's contributions in the common-product case.

A: Let us begin interpreting R1's suggestion: in the paper $\delta_{n,i}$ represents the split of product i's market between the retailers, thus it is fundamental that $\delta_{1,i} + \delta_{2,i} = 1$. R1 suggests to consider $\beta \in [\frac{1}{2}, 1]$ to reflect the potential increase in product popularity. Because demand is driven by consumer choice, such an increment in popularity should affect the demand of other products as well. Thus, our interpretation of R1's suggestion is to consider a model in which the probability of purchase $q_{n,i}(A_n, p_n, A_m, p_m)$ is given by

$$q_{n,i} \quad := \quad \frac{\nu_{n,i}\,\Big(\mathbf{1}\left\{i \notin A_m\right\} + \ \delta_{n,i}\ \mathbf{1}\{i \in A_m\}\Big)}{1 + \sum_{i \in A_n \setminus A_m} \nu_{n,i} + \sum_{i \in A_n \cap A_m} \left(\nu_{n,i}\ \delta_{n,i} + \nu_{m,i}\ \delta_{m,i}\right) + \sum_{i \in A_m \setminus A_n} \nu_{m,i}},$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function and

$$\delta_{n,i} := \mathbf{1}\left\{\nu_{n,i} > \nu_{m,i}\right\} + \beta \mathbf{1}\left\{\nu_{n,i} = \nu_{m,i}\right\},$$

for $\beta \in [\frac{1}{2}, 1]$. As mentioned by the AE, this modification does not affect the analysis of the exclusive product case. Let us examine how this affects the results in Sections 3.3 and 4.2.

Theorem 3: the proof of this result is constructive. It begins with both retailers offering the products in $\mathcal{S}_{12}^c$, the set of common products for which no retailer has an advantage. Then it is shown that the sequence of best responses converges because $A_n^k \subseteq A_n^{k+1}$. A key element of the proof is that $\mathcal{S}_{12}^c \subseteq A_n^k$ always.

This might be the case in the new setting: for $i \in \mathcal{S}_n$, we will have that

$$\theta_{n,i}(\lambda) := \begin{cases} \left(p_{n,i} - c_{n,i} - \lambda\right)\ \nu_{n,i} & \text{if } i \notin A_m, \\ \delta_{n,i}\left(p_{n,i} - c_{n,i} - \lambda\right)\ \nu_{n,i} + \lambda\ (1 - \delta_{m,i})\nu_{m,i} & \text{if } i \in A_m. \end{cases}$$

For $i \in \mathcal{S}_{12}^c$ we have that $\delta_{n,i} = \beta$, so for $i \in \mathcal{S}_{12}^c \cap A_m$ one has that

$$\theta_{n,i}(\lambda) = \left(\beta(p_{n,i} - c_{n,i}) - \lambda(2\beta - 1)\right)\ \nu_{n,i}$$

thus it is no longer the case that $\theta_{n,i}(\lambda) \geq 0$ for all λ, so despite the absence of a display capacity, it might not be optimal to include some products in $\mathcal{S}_{12}^c$ on a best response. (Note that the case when $\beta = 1$ reduces to the exclusive products only setting.)

Nonetheless, the arguments in the proof hold with a minor modification: set $A_n^1 = \emptyset$ for $n = 1, 2$. Here it is an sketch of how to retrace the steps in the proof:

- We show that $A_n^k \subseteq A_n^{k+1}$ via mathematical induction. The base case follows because $A_n^1 = \emptyset$.
- Equation (A-4) holds because when a product $i \in A_n^{k-1} \cap \mathcal{S}_{12}^c$ is introduced in A_m^k (by the induction hypothesis $A_m^{k-1} \subseteq A_m^k$), one has that

  $$\theta_{n,i}(\lambda_n^k, A_m^{k-1}) \geq \beta\theta_{n,i}(\lambda_n^k, A_m^{k-1}) = \theta_{n,i}(\lambda_n^k, A_m^k) - \lambda_n^k \nu_{m,i}(1-\beta) \geq \theta_{n,i}(\lambda_n^k, A_m^k) - \lambda_n^k \nu_{m,i}.$$

  Now, we know that $i \in A_n^{k-1}$, thus $\theta_{n,i}(\lambda_n^k, A_m^{k-1}) \geq 0$, thus one has that

  $$0 \leq \beta\theta_{n,i}(\lambda_n^k, A_m^{k-1}) = \theta_{n,i}(\lambda_n^k, A_m^k) - \lambda_n^k \nu_{m,i}(1-\beta) \leq \theta_{n,i}(\lambda_n^k, A_m^k).$$

- From the above, one concludes that $\lambda_n^{k+1} \leq \lambda_n^k$, thus implying that a product $i \in A_n^{k-1} \cap \mathcal{S}_{12}^c$ in $A_m^k \setminus A_m^{k-1}$ is such that $\theta_{n,i}(\lambda_n^k, A_m^k) \geq 0$, thus it is in A_n^{k+1}. This proves the hypothesis (the argument for other products is as in the proof of Theorem 3).
- The result follows from the monotonicity of A_n^k.

Condition 1 guarantying the existence of an equilibrium is not affected when β is introduced.

Condition 2 still guarantees the existence of an equilibrium when β is introduced, however the arguments in the proof (which were not disclosed in the previous manuscript) no longer hold. The proof of the result for $\beta = 1/2$ showed that (A_1, A_2) is an equilibrium, where A_1 is the popular set of size C and A_2 is the best response to A_1. The proof goes through with the additional condition that if profits are not the same, then $(2\beta - 1)\nu_i \leq \nu_j$, where i is the least popular product included in $A_2 \cap A_1$ and j the most popular in $A_2^c \cap A_1^c$.

Omar: the actual condition is (assume $r = 1$ w.l.o.g.)

$$\beta\nu_i - \lambda_2(2\beta-1)\nu_i - (1-\lambda_2)\nu_j + (\lambda_1 - \lambda_2)\left(\nu_j - (2\beta-1)\nu_i\right) \geq 0.$$

The first portion is positive because retailer 2 included i in A_2 over j. The second portion is problematic: it works for β close to 0.5. I tested the result numerically. I always found that (A_1, A_2) was an equilibrium.

Examples 2 and 3 are not affected by the modification as they refer to the special cases when $\beta = 0.5$.

Proposition 4 holds true as it is still the case that the best response to any offering by a retailer involves offering all products. The proof of the result needs the following adjustment:

- Define

$$\theta_{n,i}(\lambda, p_n, A_m, p_m) := \begin{cases} (p_{n,i} - c_{n,i} - \lambda)\ \nu_{n,i}(p_{n,i}) & \text{if } i \notin A_m, \\ \delta_{n,i}\,(p_{n,i} - c_{n,i} - \lambda)\ \nu_{n,i}(p_{n,i}) + \lambda\ (1 - \delta_{m,i})\nu_{m,i}(p_{m,i}) & \text{if } i \in A_m. \end{cases}$$

One can see that with this definition it is still the case that $\theta^*(\lambda, A_m, p_m) \geq 0$, thus best responses always include all products. Assuming that both retailers offer all products one can check that equilibrium prices in (A-7) remain valid, and the equilibrium is the unique solution to (A-8). The proof follows by noting that the value of β does not affect whether (A-8) has a solution or not.

An alternative way of seeing the above, is that when retailers compete in prices, a setting with $\beta \neq \frac{1}{2}$ is equivalent to one in which market splits equally ($\beta = \frac{1}{2}$) but the attractiveness of products in $\tilde{\mathcal{S}}_1^c \cap \tilde{\mathcal{S}}_2^c$ are multiplied by β.

In the current manuscript, we include a summary of the analysis above in Section 5, when discussing possible extensions and directions of future research. We prefer this option, as said analysis heavily relies in the intuition developed with the original model for common products. We would of course consider accommodating this extension into earlier sections of the paper, were the review team strongly in favor of that option.

2. **Distribution of the random terms in the MNL:** R1 raises a question of why the authors assume that random term ξ_{it} in the utility of product i assigned by customer t is *independent* of the firm, while the authors allow the deterministic component $\mu_{n,i}$ and the price-sensitivity parameter $\alpha_{n,i}$ to be firm-dependent. What would happen if the random term depends on the firm; that is, we have $\xi_{n,i}^t$ with a dependence on the firm n? Would this simplify the problem? Again, this extension only affects the case where firms offer common products, which would enhance the contributions of the paper. Also, this issue seems to be related to R1's comment about customers not knowing their suppliers (if the customers knows their suppliers, the distribution of their utilities might be different, depending on the supplier attributes).

A: In our model, a common product is recognized as such by consumers. This is represented by the fact that a customer assigns equal idiosyncratic shocks to utility to a common product, independent of the retailer. A common interpretation of such a shock is that it represents the effect of heterogeneous preferences for unobserved (by the researcher) product's features, thus equal products should result in equal shocks.

Depending on how retailer-dependent shocks to utility are introduced in our model, one might eliminate common products: Suppose that the random term ξ_i^t depends on n, i.e. consumer t assigns utility $U_{n,i}(t)$ to buying product i from retailer n, with

$$U_{n,i}(t) \quad := \quad \mu_{n,i} - \alpha_{n,i}\, p_{n,i} + \xi_{n,i}^t, \qquad n = 1, 2.$$

(We assume that ξ_0^t does not depend of the retailer, as it represents an outside alternative).

- If we assume that $\left\{\xi_{n,i}^t : n = 1, 2,\ i \in \mathcal{S}_n,\ t \geq 1\right\}$ are i.i.d. random variables a standard Gumbel distribution, then we would recover the MNL preferences, i.e. the setting would correspond to that with exclusive only products, and the analysis in Sections 3.2 and 4.1 would apply. Hence, such an assumption greatly simplifies the analysis at the expense of eliminating common products.
- An alternative approach would be to assume that $\left(\xi_{n,i}^t : n = 1, 2,\ i \in \mathcal{S}\right)$ has a GEV distribution of the form

  $$\mathbb{P}\left\{\xi_{n,i}^t \leq x_{n,i}, n = 1, 2,\ i \in \mathcal{S}\right\} = \exp\left\{-\sum_{i \in \mathcal{S}} \left(e^{-x_{1,i}/\gamma_i} + e^{-x_{2,i}/\gamma_i}\right)^{\gamma_i}\right\},$$

  where $\gamma_i \in [0, 1]$ relate to the importance of a product choice over retailer choice. This gives rise to a Nested Logit choice model, where in a first stage customers choose a product to buy according to an aggregate measure of the utility they perceive from buying from the retailers offering such a product, and in a second stage they choose the retailer according to a traditional MNL model.

  Suppose that retailers compete only in assortments: this model can be seen as an extension of the base model in which

  $$\delta_{n,i} = \nu_{n,i}^{1/\gamma_i - 1} \left(\nu_{1,i}^{1/\gamma_i} + \nu_{2,i}^{1/\gamma_i}\right)^{\gamma_i - 1}.$$

  In this model, products capture a larger market share when offered by both retailers (pretty much in the spirit of the first extension suggested by R1), however the split between the retailers depends on the value of γ_i (the case of $\gamma_i = 0$ corresponds to our base definition of common products, and the case of $\gamma_i = 1$ corresponds to the case of exclusive products).

It is not clear whether the results in Sections 3.3 and 4.2 hold in such a setting:

Theorem 3: consider the corrected proof for the case of $\beta \neq 1/2$. The key step in the proof is to show that once a common product is offered by a retailer, it is never taken away from the offering. We have that

$$\theta_{n,i}(\lambda) := \begin{cases} (p_{n,i} - c_{n,i} - \lambda)\ \nu_{n,i} & \text{if } i \notin A_m, \\ \nu_{n,i}^{1/\gamma_i}\left(\nu_{1,i}^{1/\gamma_i} + \nu_{2,i}^{1/\gamma_i}\right)^{\gamma_i - 1}(p_{n,i} - c_{n,i}) + \lambda\ \nu_{m,i} - \lambda\ \left(\nu_{1,i}^{1/\gamma_i} + \nu_{2,i}^{1/\gamma_i}\right)^{\gamma_i} & \text{if } i \in A_m. \end{cases}$$

As we showed previsouly, the arguments in the proof hold with a minor modification: set $A_n^1 = \emptyset$ for $n = 1, 2$. Here it is an sketch of how to retrace the steps in the proof:

- We show that $A_n^k \subseteq A_n^{k+1}$ via mathematical induction. The base case follows because $A_n^1 = \emptyset$.
- Equation (A-4) holds because when a product $i \in A_n^{k-1} \cap \mathcal{S}_1 \cap \mathcal{S}_2$ is introduced in A_m^k (by the induction hypothesis $A_m^{k-1} \subseteq A_m^k$), one has that

$$\begin{aligned} \theta_{n,i}(\lambda_n^k, A_m^{k-1}) &\geq \delta_{n,i}\theta_{n,i}(\lambda_n^k, A_m^{k-1}) \\ &= \theta_{n,i}(\lambda_n^k, A_m^k) + \lambda_n^k\left(\left(\nu_{1,i}^{1/\gamma_i} + \nu_{2,i}^{1/\gamma_i}\right)^{\gamma_i} - \nu_{n,i}\delta_{n,i} - \nu_{m,i}\right) \\ &\geq \theta_{n,i}(\lambda_n^k, A_m^k) - \lambda_n^k \nu_{m,i}. \end{aligned}$$

  Now, we know that $i \in A_n^{k-1}$, thus $\theta_{n,i}(\lambda_n^k, A_m^{k-1}) \geq 0$, thus one has that

$$0 \leq \delta_{n,i}\ \theta_{n,i}(\lambda_n^k, A_m^{k-1}) = \theta_{n,i}(\lambda_n^k, A_m^k) - \lambda_n^k\left(\nu_{m,i}(1 - \delta_{m,i})\right) \leq \theta_{n,i}(\lambda_n^k, A_m^k),$$

  as it can be shown that $\delta_{n,i} \in [0, 1]$.
- From the above, one concludes that $\lambda_n^{k+1} \leq \lambda_n^k$, thus implying that a product $i \in A_n^{k-1} \cap \mathcal{S}_1 \cap \mathcal{S}_2$ in $A_m^k \setminus A_m^{k-1}$ is such that $\theta_{n,i}(\lambda_n^k, A_m^k) \geq 0$, thus it is in A_n^{k+1}. This proves the hypothesis.
- The result follows from the monotonicity of A_n^k.

Condition 1, Examples 2 and 3 are not affected.

Condition 2 requires an additional assumption, as for the case of $\beta \neq 0.5$.

Proposition 4: first, note that in equilibrium all products are offered. Thus, the result follows from proving the existence of an equilibrium in prices. In this setting, however, profit are continuous functions of the price vectors, thus equilibrium existence would follow from the application of a fixed-point theorem (after, e.g. bounding the set of prices worthy of consideration).

In the current manuscript, we include a summary of the analysis above in Section 5, when discussing possible extensions and directions of future research. We prefer this option, as said analysis heavily relies in the intuition developed with the original model for common products. We would of course consider accommodating this extension into earlier sections of the paper, were the review team strongly in favor of that option.

3. **Heterogenous customer types:** R2 has asked that the authors consider the case where there are 2 customer types, and each type only purchase from one of the firms, and possibly other variations. I think this is an interesting and important extension of the paper. Moreover, the authors may be able to leverage their existing results in the case of the exclusive product setting.

   A: Following R2's suggestion, in Section 5 we analyze a setting where a fraction of consumers are loyal to a retailer in the sense that they only consider buying products offered by said retailer. (Note that this partitions customers into three groups - the case of only loyal consumers reduces to that of two monopolists). There, we show that equilibrium existence is not guaranteed even for the case of assortment-only competition with exclusive products. Note that this is a special case of latent-class logit demand, thus it establishes a non-existence result for such a model, which is the direct extension of MNL demand to the case of consumer heterogeneity. As mentioned in Section 5, under mixed Logit demand, while under some conditions, for given assortments, existence and uniqueness of an equilibrium in prices can be guaranteed, the assortment problem becomes theoretically intractable (even the monopolist's problem is in general NP-Hard).

In addition to the above list of major concerns, I strongly encourage the authors to carefully address all of the other issues that are listed in the referee reports. I am cautiously optimistic, and I look forward to reviewing the paper's revision.

Minor Comments:

- Literature review should be updated. Some papers have already been published.

  A: We have update the list of references to reflect our best current information.

- Page 6, Line -5, "For n in $\{1, 2\}$" should be "For $n = 1, 2$" or "For $n \in \{1, 2\}$",

  A: We have corrected this inconsistency throughout the paper. Thank you.

- Page 7, Line -10, There should not be a space between the parenthesis and "e.g."

  A: We have corrected the typo. Thank you.

- Page 9, Lines 9 - 14: Perhaps, a bit of explanation here would be helpful to the reader.

  A: We now formalize the correspondence between formulations (1) and (2) in Lemma 1, whose proof can be found in the appendix. We thank the AE for ponting this issue out.

- Page 10, Line -1: Would it make sense to re-phrase the last line as $\mathbb{E}_n(a') \leq \mathbb{E}_n(a)$ for all $a' \in a_n(e')$ and $a \in a_n(e)$?

  A: We have re-phrases the result along the lines suggested by the AE.

- Page 11, Line -3: "(Arbitrary number..." should be "(arbitrary number ..." so that it's consistent with the rest of the paper.

  A: We have corrected such an inconsistency. Thank you.

- Page 23, Line -13: Should the authors mention that the second inequality follows from the fact that $\min\{\nu_{n,i} : i \in a \setminus a'\} \leq \max\{\nu_{n,i} : i \in a' \setminus a\}$ from lines 7-8 of the same page?

  A: We have modified the proof along the lines suggested by the AE. Thank you.

- Page 31: Would it make sense to put Table 1 in the main paper, instead of on page 31?

  A: We have followed your advice and put Table 1 in the main paper. Thank you.

References:

Misra, K. 2008. Understanding retail assortments in competitive markets. Working Paper, Ross School of Business.

Wang, R. 2012. *Multiproduct pricing management and design of new service products.* Ph. D. thesis, Columbia University, New York, NY.

Response to Comments of Referee 1

Referee Report: Product Assortment and Price Competition under Multinomial Logit Model

This paper studies assortment and price competition under the multinomial logit (MNL) model with the cardinality constraint, which is an increasingly popular topic in marketing, economics and operations management. The authors show that there always exists one equilibrium when the products do not overlap. They also characterize the equilibria when some products are available to multiple retailers. Comparing to the literature, the most important contribution of this paper is to investigate the effect of the common products in the competitive settings. Under the current model, when a product is offered by two firms, the market share of this product is still the same and is equally distributed between the two firms. This demand model lacks justification. When a product is available in multiple retailers, it is more popular from the customers' perspective and the total market share should be higher, comparing to the scenario offered by only a single retailer. The total demand of the product offered in two retailers (charging the same price) may be between the demands corresponding to the two scenarios treated as one product or two separate products. More specifically, it makes more sense if the $\delta_{n,i}$ function on the top of page 8 generalizes to

$$\delta_{n,i} := \mathbf{1}\left\{\nu_{n,i} > \nu_{m,i}\right\} + \beta\mathbf{1}\left\{\nu_{n,i} = \nu_{m,i}\right\}$$

where $1/2 \leq \beta \leq 1$. It is worthy to investigate whether all the analysis can still hold in generalized models, e.g., the above one or else.

A: Let us begin interpreting R1's suggestion: in the paper $\delta_{n,i}$ represents the split of product i's market between the retailers, thus it is fundamental that $\delta_{1,i} + \delta_{2,i} = 1$. R1 suggests to consider $\beta \in [\frac{1}{2}, 1]$ to reflect the potential increase in product popularity. Because demand is driven by consumer choice, such an increment in popularity should affect the demand of other products as well. Thus, our interpretation of R1's suggestion is to consider a model in which the probability of purchase $q_{n,i}(A_n, p_n, A_m, p_m)$ is given by

$$q_{n,i} \quad := \quad \frac{\nu_{n,i}\ \Big(\mathbf{1}\left\{i \notin A_m\right\} +\ \delta_{n,i}\ \mathbf{1}\{i \in A_m\}\Big)}{1 + \sum_{i \in A_n \setminus A_m} \nu_{n,i} + \sum_{i \in A_n \cap A_m} \left(\nu_{n,i}\ \delta_{n,i} + \nu_{m,i}\ \delta_{m,i}\right) + \sum_{i \in A_m \setminus A_n} \nu_{m,i}},$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function and

$$\delta_{n,i} := \mathbf{1}\left\{\nu_{n,i} > \nu_{m,i}\right\} + \beta\mathbf{1}\left\{\nu_{n,i} = \nu_{m,i}\right\},$$

for $\beta \in [\frac{1}{2}, 1]$. As mentioned by the AE, this modification does not affect the analysis of the exclusive product case. Let us examine how this affects the results in Sections 3.3 and 4.2.

Theorem 3: the proof of this result is constructive. It begins with both retailers offering the products in $\mathcal{S}_{12}^c$, the set of common products for which no retailer has an advantage. Then it is shown that the sequence of best responses converges because $A_n^k \subseteq A_n^{k+1}$. A key element of the proof is that $\mathcal{S}_{12}^c \subseteq A_n^k$ always.

This might be the case in the new setting: for $i \in \mathcal{S}_n$, we will have that

$$\theta_{n,i}(\lambda) := \begin{cases} (p_{n,i} - c_{n,i} - \lambda)\ \nu_{n,i} & \text{if } i \notin A_m, \\ \delta_{n,i}\,(p_{n,i} - c_{n,i} - \lambda)\ \nu_{n,i} + \lambda\ (1 - \delta_{m,i})\nu_{m,i} & \text{if } i \in A_m. \end{cases}$$

For $i \in \mathcal{S}_{12}^c$ we have that $\delta_{n,i} = \beta$, so for $i \in \mathcal{S}_{12}^c \cap A_m$ one has that

$$\theta_{n,i}(\lambda) = (\beta(p_{n,i} - c_{n,i}) - \lambda(2\beta - 1))\ \nu_{n,i}$$

thus it is no longer the case that $\theta_{n,i}(\lambda) \geq 0$ for all λ, so despite the absence of a display capacity, it might not be optimal to include some products in $\mathcal{S}_{12}^c$ on a best response. (Note that the case when $\beta = 1$ reduces to the exclusive products only setting.)

Nonetheless, the arguments in the proof hold with a minor modification: set $A_n^1 = \emptyset$ for $n = 1, 2$. Here it is an sketch of how to retrace the steps in the proof:

- We show that $A_n^k \subseteq A_n^{k+1}$ via mathematical induction. The base case follows because $A_n^1 = \emptyset$.
- Equation (A-4) holds because when a product $i \in A_n^{k-1} \cap \mathcal{S}_{12}^c$ is introduced in A_m^k (by the induction hypothesis $A_m^{k-1} \subseteq A_m^k$), one has that

  $$\theta_{n,i}(\lambda_n^k, A_m^{k-1}) \geq \beta\theta_{n,i}(\lambda_n^k, A_m^{k-1}) = \theta_{n,i}(\lambda_n^k, A_m^k) - \lambda_n^k\nu_{m,i}(1 - \beta) \geq \theta_{n,i}(\lambda_n^k, A_m^k) - \lambda_n^k\nu_{m,i}.$$

  Now, we know that $i \in A_n^{k-1}$, thus $\theta_{n,i}(\lambda_n^k, A_m^{k-1}) \geq 0$, thus one has that

  $$0 \leq \beta\theta_{n,i}(\lambda_n^k, A_m^{k-1}) = \theta_{n,i}(\lambda_n^k, A_m^k) - \lambda_n^k\nu_{m,i}(1 - \beta) \leq \theta_{n,i}(\lambda_n^k, A_m^k).$$

- From the above, one concludes that $\lambda_n^{k+1} \leq \lambda_n^k$, thus implying that a product $i \in A_n^{k-1} \cap \mathcal{S}_{12}^c$ in $A_m^k \setminus A_m^{k-1}$ is such that $\theta_{n,i}(\lambda_n^k, A_m^k) \geq 0$, thus it is in A_n^{k+1}. This proves the hypothesis (the argument for other products is as in the proof of Theorem 3).
- The result follows from the monotonicity of A_n^k.

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Condition 2 still guarantees the existence of an equilibrium when β is introduced, however the arguments in the proof (which were not disclosed in the previous manuscript) no longer hold. The proof of the result for $\beta = 1/2$ showed that (A_1, A_2) is an equilibrium, where A_1 is the popular set of size C and A_2 is the best response to A_1. The proof goes through with the additional condition that if profits are not the same, then $(2\beta - 1)\nu_i \leq \nu_j$, where i is the least popular product included in $A_2 \cap A_1$ and j the most popular in $A_2^c \cap A_1^c$.

Omar: the actual condition is (assume $r = 1$ w.l.o.g.)

$$\beta\nu_i - \lambda_2(2\beta - 1)\nu_i - (1 - \lambda_2)\nu_j + (\lambda_1 - \lambda_2)\left(\nu_j - (2\beta - 1)\nu_i\right) \geq 0.$$

The first portion is positive because retailer 2 included i in A_2 over j. The second portion is problematic: it works for β close to 0.5. I tested the result numerically. I always found that (A_1, A_2) was an equilibrium.

Examples 2 and 3 are not affected by the modification as they refer to the special cases when $\beta = 0.5$.

Proposition 4 holds true as it is still the case that the best response to any offering by a retailer involves offering all products. The proof of the result needs the following adjustment:

- Define

$$\theta_{n,i}(\lambda, p_n, A_m, p_m) := \begin{cases} (p_{n,i} - c_{n,i} - \lambda)\ \nu_{n,i}(p_{n,i}) & \text{if } i \notin A_m, \\ \delta_{n,i}\,(p_{n,i} - c_{n,i} - \lambda)\ \ \nu_{n,i}(p_{n,i}) + \lambda\ (1 - \delta_{m,i})\nu_{m,i}(p_{m,i}) & \text{if } i \in A_m. \end{cases}$$

One can see that with this definition it is still the case that $\theta^*(\lambda, A_m, p_m) \geq 0$, thus best responses always include all products. Assuming that both retailers offer all products one can check that equilibrium prices in (A-7) remain valid, and the equilibrium is the unique solution to (A-8). The proof follows by noting that the value of β does not affect whether (A-8) has a solution or not.

An alternative way of seeing the above, is that when retailers compete in prices, a setting with $\beta \neq \frac{1}{2}$ is equivalent to one in which market splits equally ($\beta = \frac{1}{2}$) but the attractiveness of products in $\tilde{\mathcal{S}}_1^c \cap \tilde{\mathcal{S}}_2^c$ are multiplied by β.

In the current manuscript, we include a summary of the analysis above in Section 5, when discussing possible extensions and directions of future research. We prefer this option, as said analysis heavily relies in the intuition developed with the original model for common products. We would of course consider accommodating this extension into earlier sections of the paper, were the review team strongly in favor of that option.

There is another concern about the demand model for common products. The product with higher price will receive zero demand if the same product is offered at a lower price by another retailer. In reality, it may not be the case and the retailer without any price advantage may still capture some customers.

A: Effectively, in our model, a common product is recognized as such by consumers, and thus it is bought from the cheapest retailer (after adjusting for deterministic retailer preferences). This is represented by the fact that a customer assigns equal idiosyncratic shocks to utility to a common product, independent of the retailer. A common interpretation of such a shock is that it represents the effect of heterogeneous preferences for unobserved (by the researcher) product's features, thus equal products should result in equal shocks.

To address R1's concern, we now analyze two modifications where idiosyncratic shocks to utility are dependent on the retailer, and thus consumers do not always buy products from the cheapest retailer.

Depending on how retailer-dependent shocks to utility are introduced in our model, one might eliminate common products: Suppose that the random term ξ_i^t depends on n, i.e. consumer t assigns utility $U_{n,i}(t)$ to buying product i from retailer n, with

$$U_{n,i}(t) \quad := \quad \mu_{n,i} - \alpha_{n,i}\, p_{n,i} + \xi_{n,i}^t, \qquad n = 1, 2.$$

(We assume that ξ_0^t does not depend of the retailer, as it represents an outside alternative).

- If we assume that $\left\{\xi_{n,i}^t : n = 1, 2,\, i \in \mathcal{S}_n,\, t \geq 1\right\}$ are i.i.d. random variables a standard Gumbel distribution, then we would recover the MNL preferences, i.e. the setting would correspond to that with exclusive only products, and the analysis in Sections 3.2 and 4.1 would apply. Hence, such an assumption greatly simplifies the analysis at the expense of eliminating common products.

- An alternative approach would be to assume that $\left(\xi_{n,i}^t : n = 1, 2,\, i \in \mathcal{S}\right)$ has a GEV distribution of the form

$$\mathbb{P}\left\{\xi_{n,i}^t \leq x_{n,i}, n = 1, 2,\, i \in \mathcal{S}\right\} = \exp\left\{-\sum_{i \in \mathcal{S}} \left(e^{-x_{1,i}/\gamma_i} + e^{-x_{2,i}/\gamma_i}\right)^{\gamma_i}\right\},$$

where $\gamma_i \in [0,1]$ relate to the importance of a product choice over retailer choice. This gives rise to a Nested Logit choice model, where in a first stage customers choose a product to buy according to an aggregate measure of the utility they perceive from buying from the retailers offering such a product, and in a second stage they choose the retailer according to a traditional MNL model.

Suppose that retailers compete only in assortments: this model can be seen as an extension of the base model in which

$$\delta_{n,i} = \nu_{n,i}^{1/\gamma_i - 1} \left(\nu_{1,i}^{1/\gamma_i} + \nu_{2,i}^{1/\gamma_i} \right)^{\gamma_i - 1}.$$

In this model, products capture a larger market share when offered by both retailers (pretty much in the spirit of the first extension suggested by R1), however the split between the retailers depends on the value of γ_i (the case of $\gamma_i = 0$ corresponds to our base definition of common products, and the case of $\gamma_i = 1$ corresponds to the case of exclusive products).

It is not clear whether the results in Sections 3.3 and 4.2 hold in such a setting:

Theorem 3: consider the corrected proof for the case of $\beta \neq 1/2$. The key step in the proof is to show that once a common product is offered by a retailer, it is never taken away from the offering. We have that

$$\theta_{n,i}(\lambda) := \begin{cases} (p_{n,i} - c_{n,i} - \lambda)\ \nu_{n,i} & \text{if } i \notin A_m, \\ \nu_{n,i}^{1/\gamma_i} \left(\nu_{1,i}^{1/\gamma_i} + \nu_{2,i}^{1/\gamma_i} \right)^{\gamma_i - 1} (p_{n,i} - c_{n,i}) + \lambda\ \nu_{m,i} - \lambda\ \left(\nu_{1,i}^{1/\gamma_i} + \nu_{2,i}^{1/\gamma_i} \right)^{\gamma_i} & \text{if } i \in A_m. \end{cases}$$

As we showed previsouly, the arguments in the proof hold with a minor modification: set $A_n^1 = \emptyset$ for $n = 1, 2$. Here it is an sketch of how to retrace the steps in the proof:

- We show that $A_n^k \subseteq A_n^{k+1}$ via mathematical induction. The base case follows because $A_n^1 = \emptyset$.
- Equation (A-4) holds because when a product $i \in A_n^{k-1} \cap \mathcal{S}_1 \cap \mathcal{S}_2$ is introduced in A_m^k (by the induction hypothesis $A_m^{k-1} \subseteq A_m^k$), one has that

$$\begin{aligned} \theta_{n,i}(\lambda_n^k, A_m^{k-1}) &\geq \delta_{n,i} \theta_{n,i}(\lambda_n^k, A_m^{k-1}) \\ &= \theta_{n,i}(\lambda_n^k, A_m^k) + \lambda_n^k \left(\left(\nu_{1,i}^{1/\gamma_i} + \nu_{2,i}^{1/\gamma_i} \right)^{\gamma_i} - \nu_{n,i}\delta_{n,i} - \nu_{m,i} \right) \\ &\geq \theta_{n,i}(\lambda_n^k, A_m^k) - \lambda_n^k \nu_{m,i}. \end{aligned}$$

  Now, we know that $i \in A_n^{k-1}$, thus $\theta_{n,i}(\lambda_n^k, A_m^{k-1}) \geq 0$, thus one has that

$$0 \leq \delta_{n,i}\ \theta_{n,i}(\lambda_n^k, A_m^{k-1}) = \theta_{n,i}(\lambda_n^k, A_m^k) - \lambda_n^k \left(\nu_{m,i}(1 - \delta_{m,i}) \right) \leq \theta_{n,i}(\lambda_n^k, A_m^k),$$

as it can be shown that $\delta_{n,i} \in [0,1]$.

– From the above, one concludes that $\lambda_n^{k+1} \leq \lambda_n^k$, thus implying that a product $i \in A_n^{k-1} \cap \mathcal{S}_1 \cap \mathcal{S}_2$ in $A_m^k \setminus A_m^{k-1}$ is such that $\theta_{n,i}(\lambda_n^k, A_m^k) \geq 0$, thus it is in A_n^{k+1}. This proves the hypothesis.

– The result follows from the monotonicity of A_n^k.

Condition 1, Examples 2 and 3 are not affected.

Condition 2 requires an additional assumption, as for the case of $\beta \neq 0.5$.

Proposition 4: first, note that in equilibrium all products are offered. Thus, the result follows from proving the existence of an equilibrium in prices. In this setting, however, profit are continuous functions of the price vectors, thus equilibrium existence would follow from the application of a fixed-point theorem (after, e.g. bounding the set of prices worthy of consideration).

In the current manuscript, we include a summary of the analysis above in Section 5, when discussing possible extensions and directions of future research. We prefer this option, as said analysis heavily relies in the intuition developed with the original model for common products. We would of course consider accommodating this extension into earlier sections of the paper, were the review team strongly in favor of that option.

The authors may want to argue that all the products are sold at an intermediary and the demand are fulfilled by cheaper sources. Customers do not know the suppliers; otherwise, suppliers can be part of the product attributes, which makes the common products impossible in the current model settings.

A: Following R1's feedback, we know mention such an interpretation of our model of preferences in Remark 1 in page 7. Thank you.

Besides the demand model, there remain a few other issues for authors to address, which are listed below.

(i) The cardinality constraint $|A| \leq C_n$, which may result from the space or budget constraints, is extensively studied in the literature. This paper develops a general approach for assortment and price competition under the MNL model with the same cardinality constraint. I am curious whether this approach still goes through in a different cardinality constraint, i.e.,

$|A| \geq C_n$. This constraint may be necessary in practice, e.g., a retailer may require the minimum product variety to keep market competitiveness.

A: Let us examine how the results in the paper would change in presence of the constraint $|A| \geq C_n$.

- Assortment competition, exclusive products: Proposition 1 is not affected (despite the fact that $\theta_{n,i}$ might be negative, the cardinality of the assortment solving (4) is non-increasing in λ). Thus, Theorem 1, Proposition 2 and Corollaries 2 and 3, which follows mainly from Theorem 1, are not affected as well.
- Assortment competition, common products: Theorem 3 pertains settings with no capacity constraints. Condition 1 uses the fact that $\theta_{n,i} \geq 0$ to construct an equilibrium, thus such a construction fails in the new setting. Nonetheless, a new construction would provide the existence result. Condition 2 holds trivially as the best response to offering all product is offering all products.
- Assortment and price competition: All results hold true, because when price is endogenous, best responses always include all products.

In Section 5 of the current manuscript, we discuss briefly how our results extend to the case of the display constraints suggested by R1. Thank you.

(ii) For a common product i, in the current model, the deterministic utility can be different, i.e., $\mu_{n,i} \neq \mu_{m,i}$; the price sensitivity can be different as well, i.e., $\alpha_{n,i} \neq \alpha_{m,i}$; but the random part is the same, i.e., ξ_i is the same for retailers n and m. Why do you make these assumptions? If they are all the same, the problem can be significantly simplified, especially the joint assortment and price competition.

A: As mentioned above, core to our modeling of common products is that consumers assign equal idiosyncratic shocks to utility to a common product, independent of the retailer. A common interpretation of such a shock is that it represents the effect of heterogeneous preferences for unobserved (by the researcher) product's features, thus equal products should result in equal shocks.

Earlier in this response, we addressed the issue of retailer-dependent idiosyncratic shocks to utility. Portion of that discussion is in the current manuscript (Section 5). As for the setting when the deterministic portion of the utility is independent of the retailer ($\mu_{n,i} = \mu_{m,i}$ and

$\alpha_{n,i} = \alpha_{m,i}$), our results certainly apply to such a special case. More importantly, Examples 2 and 3 correspond to such a setting, hence we conclude that despite its simplified features, equilibrium still might fail to exists in this setting, or there might be a combinatorial number of equilibria.

(iii) Proposition 2 says that there may indeed exist multiple equilibria in the case of exclusive products. Some examples will help readers to better understand this result.

A: Following R1's feedback, we now include Example 1 (immediately after Proposition 2), which shows the possibility of having multiple equilibria. Motivated by such an example, we have now developed a bound (Theorem 2) on the number of possible equilibria in which retailers offer different attractiveness levels. We illustrate the use of such result in Corollary 1, which provides sufficient conditions for uniqueness of an equilibrium. We thank R1's for pointing us in this direction.

(iv) Tatonement process converges to a pareto-dominant equilibrium if starting with $A_1^0 \in a_1(0)$ and $A_2^0 = \emptyset$; otherwise, it may converge to other equilibriums. More explanation, examples or references are needed to increase the readership here because the results are not trivial.

A: Following R1's feedback, we now include a detailed explanation of the statement. In particular, following the discussion of the tatônement process, we include the following justification:

> If one starts the best-response iteration process from arbitrary assortments, then convergence to an equilibrium is guaranteed but not necessarily to one that pareto dominates all others. To see this, suppose we start with A_2^0 arbitrary: if $E_2(A_2^0) > E_2(A_2^1)$, then the recursive application of Proposition 1 ii) implies that the sequences of total attractiveness of the assortments offered by both retailers (excluding that of A_0^1) will be non-increasing, i.e. $E_n(A_n^k) \geq E_n(A_n^{k+1})$, $k \geq 1$, $n = 1, 2$. Similarly, if $E_2(A_2^0) < E_2(A_2^1)$ then the sequences of attractiveness will non-decreasing. This observation, together with the finiteness of the product sets, imply convergence to an equilibrium. (The fact that convergence to the pareto-dominant equilibrium is not guaranteed follows directly from the (possible) existence of multiple equilibria.)

(v) Minor issue: use article "an" other than "a" in front of "MNL".

A: We have corrected the paper. Thank you.

Again, the novel part of this paper is to consider the common products in the assortment and price competition. Unfortunately, there are no enough exciting analytical results in the current version. Only limited results are derived in some special cases, e.g., without display constraints as shown in Theorem 2 and Proposition 3. The contribution is weak at this point although I understand that the problem with common products may be technically challenging. This paper investigates assortment and price competition with common products offered by multiple retailers. The technical merits are highly appreciated although there are some remaining issues. It may be publishable in the journal Production and Operations Management after revision, so I recommend Major Revision.

A: We sincerely appreciate your feedback, which as helped us improve the paper in various fronts. We hope the current manuscript as well as this response address your concerns satisfactorily.

Response to Comments of Referee 2

The authors study the problem of assortment competition between two retailers where customers' choice model is represented by MNL. They show that when the products offered by each retailer is exclusive to that retailer, an equilibrium in pure strategies always exists for both assortment-only and joint assortment and price competition.

They also show that best-response dynamics leads to a Pareto dominant equilibrium. The techniques here are based on super-modular games. On the other hand, when retailers have access to exclusive and common products, an equilibrium in pure strategies may not exists if the set of feasible assortments is constrained (e.g., by size).

One of the insights obtained from this paper is that the retailers would offer a broader set of products compared with the monopoly case. Another is that when the retailers optimize over both the prices and the assortments, all the products in the offered set have "equal margins". These are obtained for the "exclusive" case.

In short, the authors' study of the assortment planning — when products offered by each retailer are exclusive — is interesting and comprehensive. For the more general setting, they have obtained preliminary results, but a complete analysis of the general case appears to be quite complicated. In my opinion, this paper makes a nice contribution to the assortment planing literature and POM journal. I recommend a minor revision and ask the authors to explore the case of heterogeneous customer types.

A: We sincerely appreciate your feedback. Our objective here is to advance the assortment literature by incorporating the realistic feature of common products. We hope that this first step motivates further research on the subject.

Major comment: Please discuss what happens in a setting with heterogeneous customer types. For instance, under what conditions there exist an equilibrium where the customers are partitioned into two groups and in each group the customers purchase products only from one of the retailers?

A: Following your suggestion, in Section 5 we analyze a setting where a fraction of consumers are loyal to a retailer in the sense that they only consider buying products offered by said retailer. (Note that this partitions customers into three groups - the case of only loyal consumers reduces to that of two monopolists). There, we show that equilibrium existence is not guaranteed even for the case of assortment-only competition with exclusive products. Note that this is a special case of latent-class logit demand, thus it establishes a non-existence result for such a model, which is the direct extension of MNL demand to the case of consumer heterogeneity. As mentioned in Section 5, under mixed Logit demand, while under some conditions, for given assortments, existence and uniqueness of an equilibrium in prices can be guaranteed, the assortment problem becomes theoretically intractable (even the monopolist's problem is in general NP-Hard).

As a side note, we would like to mention that in the current manuscript we extend our results to the case when demand is driven by a Nested Logit model. The details of such a setting can be found in our response to the AE. While such a demand model does not incorporate heterogeneity, it does advance in addressing some of the deficiencies of MNL demand.

We hope that this extension, together with the analysis of the case of latent-class logit address your concerns satisfactorily.

Minor Comments:

* p6, l18: "Competitive setting" $\rightarrow$ "Setting"

* p11, l11: Add references for "supermodular games"

* The following paper seems related:

[Immorlica, Kalai, Lucier, Moitra, Postlewaite, Tennenholtz, "Dueling Algorithms", STOC, 2011]

A: We have implemented all comments in the paper. Thank you.