Optimal Timing of a Multi-Stage Project Under Market Uncertainty

René Caldentey
University of Chicago, Rene.Caldentey@chicagobooth.edu

Francisco Castro
Columbia University, fcastro19@gsb.columbia.edu

Rafael Epstein
University of Chile, repstein@dii.uchile.cl

Denis Sauré
University of Chile, dsaure@dii.uchile.cl

We study the problem of how to optimally time the sequential execution of N stages of a project when payoffs are triggered at the beginning of the execution of each stage and depend on a market price that evolves stochastically over time. At any point in time, except for when a stage is under execution, the decision-maker might choose to start execution of the next stage, wait for market conditions to evolve, or abandon the project altogether. This setting arises in various industries, including real estate and mining. We use a dynamic programming formulation of the problem to find upper and lower bounds on the optimal expected net return and develop a family of approximations for which the optimal policy is shown to be of the threshold-type. We also derive approximating policies whose performance is tested numerically. Finally, the methodology is applied to a real case in the context of mid-term planning of operations at an underground copper mine.

Key words: Real Options, Optimal Stopping Time, Approximate Dynamic Programming

1. Introduction

We consider the problem faced by a decision-maker who must decide the timing at which to execute a project consisting of N sequential stages. Execution of a stage takes time, cannot be interrupted until completion, and incurs on a instantaneous cost. Initiating execution of a stage triggers the collection of a payoff, which is assumed to depend on the value of a stochastically evolving market indicator or price. Specifically, we assume that said profit is a linear function of the market price at the time the stage begins execution, although this also accommodates settings where payoffs are collected continuously over time (see Section 5). At any point in time, except for when a stage is being executed, the decision-maker can opt to start the execution of the next stage, wait for the market to evolve, or shut down the project altogether. Waiting for the market to evolve while maintaining the capacity to execute the project is costly; we consider a constant marginal maintenance cost (per unit of time). Shutting the project down involves incurring on a fixed cost, which might represent either a penalty due to non-completion of commitments or a residual benefit associated with liquidating the project's assets. The problem thus consists in adaptively timing the execution of the project so as to maximize its expected economic value.

This problem arises in numerous real-world application areas. Take, for example, the case of a large-scale real-estate development project that must follow a given geographically ordered sequence of lots. The project manager develops the project one lot at the time: after completing one such a lot, depending on the state of the real-estate market, the manager either continues developing a new lot or waits until conditions improve; if prices are very low and not expected to rise in the near future, the project manager may abandon further development and sell off the undeveloped lots for a residual benefit.

Another example is the mid-term planning of mining operations. Long-term plans typically divide a mine into various phases that ought to be extracted in a given sequence. The payoff derived from extracting one such a phase depends on the metal's market price, which is typically modeled as a stochastic process. Assuming that both the extraction cost and the metal content of the phases are known, the mine planner may suspend operations if the market price of the metal is too low and wait until it improves, while in the meantime incurring a cost for maintaining the mine's production capacity. If the operator decides to shut down the mine permanently, it will incur environmental mitigation costs but will also have the option of attaining a residual benefit by selling the property.

In our analysis we assume that the underlying market price follows geometric Brownian motions (GBM), which is commonly used, both in academia and practice, to model the stochastic evolution of commodities and capital stock. There exists extensive research on the modeling of commodity prices. In this regard, we note that while more sophisticated models are widely used for specific assets (e.g. models that exhibit regression to the mean, for the case of commodities such as gold and copper), the GBM assumption usually cannot be statistically discarded, or remains true after a change of measure argument (usually involving assumptions on the market price of risk and/or convenience yields; see (Shreve 2004, Chapter 5)).

We formulate the problem of optimally timing the execution of a project as a dynamic program where periods are given by each of the project stages and the state variable is the underlying market price, thus solving each stage is equivalent to solving an optimal stopping problem. We construct upper and lower bounds to solve the problem approximately, develop a family of approximations based on these bounds that can be solved by a threshold-type policy, and specify a procedure for calculating such thresholds.

The principal contributions of this paper are the identification and formalization of a type of decision-making problem under uncertainty that is frequently encountered in many industries, and the proposal of a family of efficient easy-to-compute heuristic policies based on an approximate dynamic programming approach (Powell 2011). In this regard, we note that the problem under study is a complex one and while we have not found an analytic solution, by exploiting the structure and properties of our formulation we are able to devise an algorithm that can find high quality approximate solutions.

The remainder of this paper is organized as follows. In Section 1.1 we discuss the related literature. In Section 2 we describe our model and formulate the problem as a sequential optimal stopping problem. In Section 3 we use optimal stopping theory to propose a solution. We also derive some properties for the value function and compute upper and lower bounds for the optimal value. In Section 4 we generalize these bounds to a family of approximations that share the bounds' structure. For these approximations we provide a result ensuring that the optimal policies are of the threshold type. We also discuss how to compute the thresholds and develop a procedure to efficiently compute optimal policies. In Section 5 we use a test instance to demonstrate the quality of our proposed approximations and then apply our methodology to the mid-term planning of a real-world mining project. Our conclusions and suggestions for future research are presented in Section 6. Proofs are relegated to an appendix.

1.1. Related Literature

Our approach to solving the optimal timing of a project can be viewed as an application of real options theory (see Dixit and Pindyck (1994), Trigeoris (1996)) when the underlying market price is correlated to some tradable security (e.g., a futures market for the output of a mining project). Previous work in this area analyze settings similar to ours in various application areas: valuing investments in natural resources (see McDonald and Siegel (1985) and Brennan and Schwartz (1985) for early work on the subject), flexibility in manufacturing systems (e.g. Triantis and Hodder (1990)), inventory management (*e.g.*

Ritchken and Tapiero (1986)), technology licensing (*e.g.* Ziedonis (2007)), just to name a few examples. (See Brennan and Trigeorgis (2000), Lumley and Zervos (2001), Pyndick (1978, 1981) for further applications of the methodology.)

There exists ample work on the application of the real options approach to valuating investment projects. For example, Dixit (1989) examines a firm's entry and exit decisions when the output price follows a GBM, and shows that the optimal investment policy is characterized by a pair of trigger prices for entry and exit. McDonald and Siegel (1985) develop and study a methodology for valuing risky investment projects where there is an option of temporarily and costlessly shutting down production whenever variable costs exceed operating revenues. In McDonald and Siegel (1986) the same authors consider the optimal timing of investment in a project whose benefits and investment cost behave stochastically over time. They obtain the optimal investment policy and the value of the investment option in closed-form, and also analyze the scrapping decision. Closer to our setting, Schwartz et al. (2001) develop a real options model for valuing a multiple-stage exploration phase of a natural resource, followed by investment and extraction phases. The authors consider a timing option for the investment phase and closure, opening and abandonment options for the extraction phase.

The application of the approach to the development of real estate projects is rather scant. Titman (1985) provides a model for pricing vacant lots and provides intuition regarding the conditions under which it is rational to defer construction to a later date. Guthrie (2009), who uses a real options approach and a binomial tree to evaluate a real-estate project consisting of five separate stages with the options of continuing, suspending or abandoning development.

The model we this paper can be seen as an extension of that in Caldentey et al. (2017), which studies the optimal timing of a natural resource's extraction. Because such a model does not consider the option to shut down operations, the structure of an optimal policy (an its analysis) is greatly simplified, which allows the development of a set of tractable policies. Our work leverages the ideas in that work to incorporate the shutdown option. While the structure of the optimal policies changes significantly, we are still able to characterize its structure, and to extend the policies to the case when shut down is possible.

2. Model Formulation

Consider a sequential project consisting of N stages, where for notational convenience we let N denote the first stage in the project, and 1 the last one. The decision-maker ought to decide when to start executing each stage of the project, considering that stage i must be completed before beginning execution of stage $i-1$, and that it takes T_i units of time to execute stage i. We also assume that once started, execution of a stage can not be interrupted. Letting τ_i denote the time at which stage i starts its execution, the above implies that the execution times of the project must satisfy

$$\tau_i \geq \tau_{i+1} + T_{i+1} \quad \text{and} \quad \tau_N \geq 0. \tag{1}$$

When execution of stage i begins, the decision-maker incurs on a cost of C_i, and generates a payoff that is proportional to the current market price. In this respect, we let $\{S_t\}_{t\geq 0}$ denote the market price process, where S_t is the market price at time t. We assume that $\{S_t\}_{t\geq 0}$ evolves as a GBM and, therefore, it solves the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 = S, \tag{2}$$

where $\mu > 0$ is the drift coefficient, $\sigma > 0$ is the process volatility, S_0 is the (known) market price at time $t = 0$, and B_t is a one-dimensional Brownian motion with respect to an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the natural filtration generated by the $\{S_t\}$ process and consider timing policies satisfying (1), where each τ_i is an $\mathcal{F}_t$-stopping time. We let

$$W_i(S) \triangleq R_i\, S - C_i$$

denote the payoff triggered by starting execution of stage i when the market price is S, where the positive constant R_i denotes a marginal return associated with stage i. (Later in Section 5 we derive the functions $W_i(\cdot)$ from first principles in the context of mid-term planning of a mining project.)

At any point in time, if no stage is undergoing execution, the decision-maker might choose to start execution of the next stage of the project, to wait for market conditions to evolve, or to abandon the project altogether. If further processing is temporarily suspended, a waiting cost M is incurred for each time unit of waiting (which reflects the expense of maintaining production capacity). If the project is to be abandoned, a shutdown cost C_0 is incurred (which if negative might represent a residual benefit).

We assume that profits are discounted at a exogenous and fixed rate $r > 0$. The decision-maker's goal is to maximize the expected net present value of the cumulative payoff by timing the execution of the stages of the project, considering suspension (waiting) and shutdown (abandonment) decisions. In the sequel we assume that $\mu < r$, i.e. payoffs are discounted at a faster rate than the expected grow of the market price (otherwise a policy that waits indefinitely to execute stage N is optimal).

We formulate the problem faced by the decision-maker via dynamic programming. In our formulation, periods correspond to the various stages of the project, and the state variable is the current market price. That is, we let $V_i(S)$ denote the expected maximum net present value when stage $i+1$ has already been completed but stage i has not, and the market price is currently S. The value of the project is therefore $V_N(S_0)$. The Bellman equation for our formulation is

$$(\mathcal{P}) \quad V_i(S) = \sup_{\tau>0} \mathbb{E}\left[e^{-r\tau} \max\left\{ W_i(S_\tau) + e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\tau+T_i})|\mathcal{F}_\tau], -C_0 \right\} - \frac{1-e^{-r\tau}}{r} M \,\middle|\, S_0 = S \right]$$

$$\text{s.t.} \quad dS_t = \mu S_t dt + \sigma S_t dB_t \tag{3a}$$

$$\tau \text{ is a valid } \mathcal{F}_t - \text{stopping time} \tag{3b}$$

$$\text{and } V_0(S) = -C_0 \text{ for all } S. \tag{3c}$$

The optimal strategy takes into account the options to wait, continue executing the project (i.e., continuing to the next stage) or abandon the project. If the option to continue is chosen, the profit is given by

$$W_i(S_\tau) + e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\tau+T_i})|\mathcal{F}_\tau],$$

which includes the profit obtained from the execution of stage i plus that associated with the decisions relating to the following stages, T_i time units later. If the decision-maker opts for abandonment, shutdown cost C_0 is incurred. The last term in the decision-maker's objective represents the cost of maintaining execution capacity until τ, at which point either stage i begins execution or the project is abandoned.

3. Properties of the Value Function and Bounds

Next, we use optimal stopping theory to identify properties of the solution for problem $(\mathcal{P})$, and develop lower and upper bounds for $V_i(S)$, which we exploit to solve the problem approximately.

3.1. Value Function Analysis

Solving for $V_i(S)$ is equivalent to solving an optimal stopping problem. Furthermore, it can be shown that a solution to this class of problems is of the threshold type (see, e.g. Øksendal (2007)), with optimal stopping time given by

$$\tau^* = \inf\{t \geq 0 : S_t \notin (S_i^a, S_i^b)\}.$$

We will see below that this structure implies that it is optimal to shut down the project if the market price falls below S_i^a, and to start executing stage i if the market price rises above S_i^b; on the other hand, if the market price lies in the interval (S_i^a, S_i^b), then it is optimal to wait until either the price drops below S_i^a or rises above S_i^b. (The interval (S_i^a, S_i^b) is called the *continuation* region.)

The verification theorem (Øksendal 2007) tells us that for $(\mathcal{P})$ to have a solution, one must have that

$$-rV_i(S) + \mu S \frac{\partial}{\partial S} V_i(S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} V_i(S) = M, \quad S_i^a < S < S_i^b, \tag{4}$$

where S_i^a and S_i^b satisfy the following boundary conditions:

$$V_i(S_i^a) = -C_0, \tag{5a}$$

$$V_i(S_i^b) = W_i(S_i^b) + e^{-rT_i}\mathbb{E}[V_{i-1}(S_{T_i}) | S_0 = S_i^b], \tag{5b}$$

$$\frac{\partial}{\partial S} V_i(S_i^a) = 0, \tag{5c}$$

$$\frac{\partial}{\partial S} V_i(S_i^b) = R_i + e^{-rT_i} \frac{\partial}{\partial S}\mathbb{E}[V_{i-1}(S_{T_i}) | S_0 = S]\Big|_{S=S_i^b}. \tag{5d}$$

Conditions in (5), known as value matching and smooth pasting, are smoothness and continuity conditions. The solution of partial differential equation (4) is known and given by

$$V_i(S) = B_i S^{\lambda_1} + D_i S^{\lambda_2} - M/r, \quad S_i^a < S < S_i^b,$$

where

$$\lambda_{1,2} \triangleq \frac{1}{\sigma^2}\Big[\frac{1}{2}\sigma^2 - \mu \pm \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}\Big], \quad \text{and} \quad \lambda_2 < 0 < 1 < \lambda_1.$$

Finally, $V_i(S)$ satisfies

$$V_i(S) = \begin{cases} -C_0 & \text{if } S \le S_i^a \\ B_i S^{\lambda_1} + D_i S^{\lambda_2} - M/r & \text{if } S_i^a < S < S_i^b \\ W_i(S) + e^{-rT_i}\mathbb{E}[V_{i-1}(S_{T_i})|S_0 = S] & \text{if } S_i^b \le S. \end{cases} \tag{6}$$

The above corroborates the structure of the optimal policy. The project must be abandoned and the shutdown cost paid when the market price falls below some threshold. Stage i should start execution as soon as the market price rises above some other threshold, which triggers immediate collection of the payoff $W_i(S)$ and posterior collection of the optimal discounted payoffs from future execution of the project's remaining stages.

The terms $B_i S^{\lambda_1}$ and $D_i S^{\lambda_2}$ in (6) associated to the interval (S_i^a, S_i^b) are related to the expected discount for the time taken by the market price process to exit that interval (see Ross (1996) Proposition 8.4.1). Finally, the $-M/r$ term can be interpreted as the discounted cost of maintaining capacity for an indefinite period without either abandoning the project or executing any further stage.

Constants S_i^a, S_i^b, B_i and D_i are the solution to the non-linear system (5)-(6). Unfortunately, there is no closed formula for these constants, even for the simplest case of $i = 1$.

3.2. Properties of $V_i(S)$

In order to simplify the exposition, we introduce the following notation. For two vectors X and α define

$$X_{k,i}^{+} \triangleq \sum_{h=k+1}^{i} X_h, \quad \alpha_{k,i}^{\times} \triangleq \prod_{h=k+1}^{i} \alpha_h \quad \text{and} \quad (\alpha^{\times} X)_{k,i}^{+} \triangleq \sum_{h=k+1}^{i} \alpha_{h,i}^{\times} X_h.$$

The following proposition characterizes the asymptotic behavior of $V_i(S)$.

PROPOSITION 1. *For all $i \ge 1$ and $S > 0$,*

$$(\mathcal{R}_i S - \mathcal{C}_i) \le V_i(S) \le G_i(S) + (\mathcal{R}_i S - \mathcal{C}_i),$$

where

$$G_i(S) \triangleq \frac{i(i+1)}{2} \cdot S_i^m \cdot R_i^m \cdot \exp\left(-\frac{1}{\sigma^2} \min_{1\le k\le i} \frac{1}{T_{k,i}^+} \left[\ln\left(\frac{S}{S_i^m}\right) + \{r-\rho+\frac{\sigma^2}{2}\} T_{k,i}^+ \right]^2 \right),$$

$$\mathcal{R}_i \triangleq \sum_{h=1}^{i} e^{-(r-\mu)T_{h,i}^+} R_h, \quad \mathcal{C}_i \triangleq \sum_{h=0}^{i} e^{-rT_{h,i}^+} C_h, \quad S_i^m \triangleq \max_{1\le k\le i} S_k^b \quad \textit{and} \quad R_i^m \triangleq \max_{1\le k\le i} R_k.$$

Clearly, $G_i(S) > 0$ *and* $G_i(S) \to 0$ *when* $S \to \infty$.

This proposition implies that, roughly speaking, when S is sufficiently high the decision-maker executes all remaining stages of the project without suspensions or abandonment. In this case the value is approximately $\mathcal{R}_i S - \mathcal{C}_i$, where $\mathcal{R}_i S$ and $\mathcal{C}_i$ correspond to the net present value and cost of uninterruptedly executing all remaining stages, respectively, starting at a price S. The next proposition provides further insight on the value function.

PROPOSITION 2. *For all* $i \ge 1$, *the function* $V_i(S)$ *is convex and increasing.*

The growth of $V_i(S)$ reflects the intuitive notion that the higher the asset's market price, the higher the project's expected value. Next, we approximate $V_i(S)$ by constructing two functions $V_i^U(S)$ and $V_i^L(S)$, which we show are upper- and lower-bounds for the value function. Interestingly, in Section 4.1 we show that they satisfy a similar set of optimality conditions as those in (5) but without the conditional expectation, which is a property that significantly simplify their analysis.

3.3. Bounds on the Value Function

Given the convexity of $V_i(S)$ and recalling Jensen's inequality, we can construct a lower bound by replacing the stochastic evolution of the price process by a deterministic counterpart. In particular, we replace the evolution of the market price while a stage is being executed by its expected trajectory. This lower-bound function, which we denote by $V_i^L(S)$, is defined as follows.

$$V_i^L(S) \triangleq \sup_{\tau\ge 0} \mathbb{E}\Big[e^{-r\tau} \max\big\{ W_i(S_\tau) + e^{-rT_i} V_{i-1}^L(e^{\mu T_i} S_\tau), -C_0 \big\} - \frac{1-e^{-r\tau}}{r} M \Big| S_0 = S \Big]$$
$$\text{s.t.} \quad \text{conditions in (3).}$$

Below, in Proposition 3, we show that this approximation indeed constitutes a valid lower bound to the value function.

To define an upper bound function we rely on the following observation. Consider stage $i-1$ and suppose that at time τ the price is S_τ. Then, waiting for a period of time T as part of a strategy is no better than the optimal strategy. That is,

$$e^{-rT}V_{i-1}(S_{\tau+T}) - \int_0^T e^{-rs}M ds \le V_{i-1}(S_\tau).$$

We provide formal verification of this inequality in the proof of Proposition 3. Given this result, we define the following upper bound function

$$V_i^U(S) \triangleq \sup_{\tau\ge 0} \mathbb{E}\Big[e^{-r\tau}\max\{W_i(S_\tau)+V_{i-1}^U(S_\tau)+\int_0^{T_i} e^{-rt}M dt, -C_0\} - \frac{1-e^{-r\tau}}{r}M\Big| S_0 = S\Big]$$
$$\text{s.t.} \quad \text{conditions in (3).}$$

The following proposition establishes that the above are valid lower- and upper- bounds to the value function.

Proposition 3. *Assume that $C_0 < M/r$. Then, for all $i \ge 1$,*

$$V_i^L(S) \le V_i(S) \le V_i^U(S).$$

Note that condition $C_0 < M/r$ implies that it is economically convenient to shut down the project rather than simply suspending operations indefinitely. In the sequel we assume that this is always the case, so that the shutdown option has economic sense.

4. Approximations and Algorithm

In this section we present a new family of functions $\mathcal{V}_i(S)$ that generalizes the lower and upper bounds $V_i^L(S)$ and $V_i^U(S)$. We also set out an algorithm for computing this family of approximations and analyze their asymptotic behavior when the market price tends to infinity.

4.1. (α,η,γ)-Approximations

For a given set of non-negative constants (α,η,γ) we recursively define the function

$$\mathcal{V}_i(S) \triangleq \sup_{\tau\ge 0} \mathbb{E}\Big[e^{-r\tau}\max\{W_i(S_\tau)+\gamma_i+\alpha_i\mathcal{V}_{i-1}(\eta_i S_\tau), -C_0\} - \frac{1-e^{-r\tau}}{r}M\Big| S_0 = S\Big] \quad (7a)$$
$$\text{s.t.} \quad \text{conditions in (3).} \quad (7b)$$

Note this function generalizes the bounds V_i^L and V_i^U: if α_i and η_i are equal to 1 and γ_i is equal to $\int_0^{T_i} e^{-rt} M dt$, then we obtain $V_i^U(S)$; similarly, if α_i is equal to e^{-rT_i}, η_i is equal to $e^{\mu T_i}$ and γ_i is equal to 0, then we get $V_i^L(S)$.

As with $(\mathcal{P})$, the problem associated with $\mathcal{V}_i(S)$ is also a sequential optimal stopping problem. Making use once again of the verification theorem, we propose the following solution for (7):

$$\mathcal{V}_i(S) = \begin{cases} -C_0 & \text{if } S \le \mathcal{S}_i^a \\ \mathcal{B}_i S^{\lambda_1} + \mathcal{D}_i S^{\lambda_2} - M/r & \text{if } \mathcal{S}_i^a < S < \mathcal{S}_i^b \\ R_i S - (C_i - \gamma_i) + \alpha_i \mathcal{V}_{i-1}(\eta_i S) & \text{if } \mathcal{S}_i^b \le S, \end{cases} \tag{8}$$

where the thresholds $\mathcal{S}_i^a, \mathcal{S}_i^b$ and the constants $\mathcal{B}_i, \mathcal{D}_i$ are solutions of the following non-linear system

$$\mathcal{B}_i \mathcal{S}_i^{a\lambda_1} + \mathcal{D}_i \mathcal{S}_i^{a\lambda_2} - M/r = -C_0 \tag{9a}$$

$$\mathcal{B}_i \mathcal{S}_i^{b\lambda_1} + \mathcal{D}_i \mathcal{S}_i^{b\lambda_2} - M/r = (R_i \mathcal{S}_i^b - (C_i - \gamma_i)) + \alpha_i \mathcal{V}_{i-1}(\eta_i \mathcal{S}_i^b) \tag{9b}$$

$$\lambda_1 \mathcal{B}_i \mathcal{S}_i^{a\lambda_1 - 1} + \lambda_2 \mathcal{D}_i \mathcal{S}_i^{a\lambda_2 - 1} = 0 \tag{9c}$$

$$\lambda_1 \mathcal{B}_i \mathcal{S}_i^{b\lambda_1 - 1} + \lambda_2 \mathcal{D}_i \mathcal{S}_i^{b\lambda_2 - 1} = R_i + \alpha_i \eta_i \mathcal{V}'_{i-1}(\eta_i \mathcal{S}_i^b). \tag{9d}$$

To prove the validity of (8) we must find the conditions under which (9) has a solution. Next, we first characterize such conditions for the case when $i = 1$, and then analyze the case when $i = 2$. We then use the insight gained to propose an efficient algorithm for the general case of $i > 2$.

The case of $i = 1$. In the following theorem we fully characterize $\mathcal{V}_1$, the simplest case of a project with just one stage.

THEOREM 1. *Consider a project with a single stage. The system* (9) *has a solution such that* $0 < \mathcal{S}_1^a < \mathcal{S}_1^b$ *and* $\mathcal{B}_1, \mathcal{D}_1 > 0$*. Furthermore,* $\tau^* = \inf\{t \ge 0 : S_t \notin (\mathcal{S}_1^a, \mathcal{S}_1^b)\}$ *is an optimal stopping time for* $\mathcal{V}_1(S)$ *and its solution is given by* (8).

The case of $i = 2$. In order to compute $\mathcal{V}_i$ for $i \ge 1$ we have to know $\mathcal{V}_{i-1}$ evaluated at $\eta_i \mathcal{S}_i^b$. Since S_i^b is unknown, we must iterate (8) using $\mathcal{V}_{i-1}(\eta_i \mathcal{S}_i^b)$ to identify what interval

$\eta_i \mathcal{S}_i^b$ belongs to. Next, we show how to solve the case for two stages, which can be directly generalized to the case of i stages. Consider the solution to the single-stage project

$$\mathcal{V}_1(S) = \begin{cases} -C_0 & \text{if } S \leq \mathcal{S}_1^a \\ \mathcal{B}_1 S^{\lambda_1} + \mathcal{D}_1 S^{\lambda_2} - M/r & \text{if } \mathcal{S}_1^a < S < \mathcal{S}_1^b \\ R_1 S - (C_1 - \gamma_1) - \alpha_1 C_0 & \text{if } \mathcal{S}_1^b \leq S. \end{cases}$$

For computing $\mathcal{V}_2(S)$ we propose the following candidate function:

$$\mathcal{V}_2(S) = \begin{cases} -C_0 & \text{if } S \leq \mathcal{S}_2^a \\ \mathcal{B}_2 S^{\lambda_1} + \mathcal{D}_2 S^{\lambda_2} - M/r & \text{if } \mathcal{S}_2^a < S < \mathcal{S}_2^b \\ R_2 S - (C_2 - \gamma_2) + \alpha_2 \mathcal{V}_1(\eta_2 S) & \text{if } \mathcal{S}_2^b \leq S. \end{cases}$$

To calculate S_2^a, S_2^b, B_2, D_2 it is sufficient to know $\mathcal{V}_1(\eta_2 S_2^b)$. We therefore propose a method that iterates assuming $\eta_2 \mathcal{S}_2^b$ belongs to one of the three intervals defined for $\mathcal{V}_1(S)$, after which we solve the system defined by (9) and then check whether the assumption is satisfied; if that is not the case, then we repeat the procedure and check for a second interval and, if need be, for the third. For example, if we assume that $\eta_2 \mathcal{S}_2^b \leq \mathcal{S}_1^a$, we then solve the problem of the second stage using the fact that $\mathcal{V}_1(\eta_2 S_2^b)$ is equal to $-C_0$. Here we can apply Theorem 1 using (9) evaluated for this situation and calculate $\mathcal{B}_2, \mathcal{D}_2, \mathcal{S}_2^a$ and $\mathcal{S}_2^b$. The value function $\mathcal{V}_2(S)$ is then given by

$$\mathcal{V}_2(S) = \begin{cases} -C_0 & \text{if } S \leq \mathcal{S}_2^a \\ \mathcal{B}_2 S^{\lambda_1} + \mathcal{D}_2 S^{\lambda_2} - \frac{M}{r} & \text{if } \mathcal{S}_2^a < S < \mathcal{S}_2^b \\ (R_2 S - (C_2 - \gamma_2) - \alpha_2 C_0 & \text{if } \mathcal{S}_2^b \leq S \leq \mathcal{S}_1^a/\eta_2 \\ (R_2 S - (C_2 - \gamma_2) + \alpha_2(\mathcal{B}_1(\eta_2 S)^{\lambda_1} + \mathcal{D}_1(\eta_2 S)^{\lambda_2} - \frac{M}{r}) & \text{if } \mathcal{S}_1^a/\eta_2 < S < \mathcal{S}_1^b/\eta_2 \\ (R_2 S - (C_2 - \gamma_2)) + \alpha_2(R_1 \eta_2 S - (C_1 - \gamma_1) - \alpha_1 C_0) & \text{if } \mathcal{S}_1^b/\eta_2 \leq S. \end{cases} \tag{10}$$

If solution (10) contradicts the initial assumption that $\eta_2 \mathcal{S}_2^b \leq \mathcal{S}_1^a$, we assume that $\mathcal{S}_1^a < \eta_2 \mathcal{S}_2^b < \mathcal{S}_1^b$. In this case, $\mathcal{V}_1(\eta_2 S_2^b) = \mathcal{B}_1(\eta_2 S_2^b)^{\lambda_1} + \mathcal{D}_1(\eta_2 S_2^b)^{\lambda_2} - M/r$ and the solution obtained is the same as (10) except that $\mathcal{S}_2^b > \mathcal{S}_1^a/\eta_2$, meaning that the third interval disappears. If this solution also contradicts the initial assumption that $\mathcal{S}_1^a < \eta_2 \mathcal{S}_2^b < \mathcal{S}_1^b$, we assume that $\eta_2 \mathcal{S}_2^b \geq \mathcal{S}_1^b$. In this case, $\mathcal{V}_1(\eta_2 S_2^b) = R_1(\eta_2 S_2^b) - (C_1 - \gamma_1) - \alpha_1 C_0$ and the solution is the same as (10) with the exception that $\mathcal{S}_1^b/\eta_2 < \mathcal{S}_2^b$ and, thus, the third and fourth intervals disappear.

<u>The case of $i > 2$.</u> As mentioned above, in order to solve for $\mathcal{V}_i$ we need to know where $\eta_i \mathcal{S}_i^b$ lies. Because we do not know this upfront, we need to iterate the procedure above conjecturing where $\eta_i \mathcal{S}_i^b$ lies. The number of iterations is then at most that of the number of intervals use to define the function $\mathcal{V}_i$. The following proposition characterizes the number of intervals in the characterization of function V_i (as in (10)).

PROPOSITION 4. *Let m_i be the number of intervals defining the function V_i and l_i be the index of the corresponding interval if the price equals $\eta_{i+1}\mathcal{S}_{i+1}^b$. Then*

$$m_i = 2 + (m_{i-1} - l_{i-1} + 1).$$

Note that m_i is at least 3 and at most $2i + 1$.

To facilitate the notation we introduce a family of matrices Y^i, each one associated with the intervals used in defining function $\mathcal{V}_i(S)$. The rows of Y^i represent the pieces of $\mathcal{V}_i$ while the columns are the 6 constants that define each function: R_i, C_i, $\mathcal{B}_i$, $\mathcal{D}_i$, M/r and C_0. As an example, for the function $\mathcal{V}_2$ given by (10), the matrix is

$$Y^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & C_0 \\ 0 & 0 & \mathcal{B}_2 & \mathcal{D}_2 & M/r & 0 \\ R_2 & (C_2 - \gamma_2) & 0 & 0 & 0 & \alpha_2 C_0 \\ R_2 & (C_2 - \gamma_2) & \alpha_2 \eta_1^{\lambda_1} \mathcal{B}_1 & \alpha_2 \eta_1^{\lambda_2} \mathcal{D}_1 & \alpha_2 M/r & 0 \\ (R_2 + \alpha_2 \eta_2 R_1) & ((C_2 - \gamma_2) + \alpha_2 (C_1 - \gamma_1 + \alpha_1 C_0)) & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We also introduce the family of matrices P^i and the family of vectors $v^i(S)$ and $I^i(S)$. The P^i rows represent the $\mathcal{V}_i$ thresholds, the $v^i(S)$ rows represent the values at which $\mathcal{V}_i$ is evaluated and each $I^i(S)$ column indicates the piece of $\mathcal{V}_i$ that S belongs to. For the case of (10),

$$P^2 = \begin{pmatrix} -\infty & \mathcal{S}_2^a \\ \mathcal{S}_2^a & \mathcal{S}_2^b \\ \mathcal{S}_2^b & \mathcal{S}_1^a/\eta_2 \\ \mathcal{S}_1^a/\eta_2 & \mathcal{S}_1^b/\eta_2 \\ \mathcal{S}_1^b/\eta_2 & \infty \end{pmatrix}, \quad v^2(S) = \begin{pmatrix} S \\ -1 \\ S^{\lambda_1} \\ S^{\lambda_2} \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad I^2(S) = \begin{pmatrix} \mathbf{1}_{\{(P^2_{1,1}, P^2_{1,2})\}} \\ \mathbf{1}_{\{(P^2_{2,1}, P^2_{2,2})\}} \\ \mathbf{1}_{\{(P^2_{3,1}, P^2_{3,2})\}} \\ \mathbf{1}_{\{(P^2_{4,1}, P^2_{4,2})\}} \\ \mathbf{1}_{\{(P^2_{5,1}, P^2_{5,2})\}} \end{pmatrix},$$

so that

$$\mathcal{V}_2(S) = I^2(S)^\top Y^2 v^2(S).$$

Using this notation we can write (9) in compact form. Generalizing the results of Theorem 1 we propose the V-approx algorithm (Algorithm 1 below) for computing the value of $\mathcal{B}_i, \mathcal{D}_i, \mathcal{S}_i^a$ and $\mathcal{S}_i^b$, thus giving the value of the approximation function $\mathcal{V}_i$.

Algorithm 1 V-approx

Require: $\{(R_k),(C_k),(\alpha_k,\eta_k,\gamma_k)\}_{k=1}^{i}, M/r, C_0$

1: Create Y^1, P^1 and solve the System (9) for $k=1$. Set $k=2$.

2: Compute $m =$ number of rows of Y^{k-1}.

3: **for** $l = 1 \ldots m$ **do**

4: Create $\tilde{Y}^l, \tilde{P}^l$, both of which are zero matrices with $2+(m-l+1)$ rows, and 6 and 2 columns, respectively.

5: Set $\tilde{Y}^l_{1,\cdot} = (0\;0\;0\;0\;0\;C_0)$, $\tilde{Y}^l_{2,\cdot} = (0\;0\;B_k\;D_k\;M/r\;0)$, $\tilde{P}^l_{1,\cdot} = (-\infty\;\mathcal{S}_k^a)$ and $\tilde{P}^l_{2,\cdot} = \left(\mathcal{S}_k^a\;\mathcal{S}_k^b\right)$.

6: **for** $j = 1 \ldots (m-l+1)$ **do**

7: Set

$$\begin{array}{ll}
\tilde{Y}^l_{(2+j),1} = R_k + \alpha_k \cdot \eta_k \cdot Y^{k-1}_{(l-1+j),1} & \tilde{Y}^l_{(2+j),2} = C_k + \alpha_k \cdot Y^{k-1}_{(l-1+j),2} \\
\tilde{Y}^l_{(2+j),3} = \alpha_k \cdot \eta_k^{\lambda_1} \cdot Y^{k-1}_{(l-1+j),3} & \tilde{Y}^l_{(2+j),4} = \alpha_k \cdot \eta_k^{\lambda_2} \cdot Y^{k-1}_{(l-1+j),4} \\
\tilde{Y}^l_{(2+j),5} = \alpha_k \cdot Y^{k-1}_{(l-1+j),5} & \tilde{Y}^l_{(2+j),6} = \alpha_k \cdot Y^{k-1}_{(l-1+j),6} \\
\tilde{P}^l_{(2+j),2} = P^{k-1}_{(l-1+j),2}/\eta_k. &
\end{array}$$

8: If $j=1$ then $\tilde{P}^l_{(2+j),1} = \mathcal{S}_k^b$, Else $\tilde{P}^l_{(2+j),1} = P^{k-1}_{(l-1+j),1}/\eta_k$.

9: **end for**

10: Solve System (9) using Y^l. If $\eta_k \mathcal{S}_k^b \in (P^{k-1}_{l,1}, P^{k-1}_{l,2})$ then Break and define $l^* = l$.

11: **end for**

12: Set $Y^k = \tilde{Y}^{l^*}$ and $P^k = \tilde{P}^{l^*}$. If $k = i$ Stop, Else $k = k+1$ and go to step 2.

Ensure: $\{Y^k, P^k\}_{k=1}^{i}$.

The following corollary bounds the number of calculations required by the algorithm.

Corollary 1. *Consider a project consisting of i stages. The V-approx algorithm terminates in no more than $(3i-2)$ iterations each of which solves exactly one non-linear system of the same type as (9).*

4.2. Asymptotic Approximations

Following the ideas behind Proposition 1, one can shown that if S is sufficiently large, then $\mathcal{V}_i(S)$ is a linear function. More precisely, when the market price is above $\bar{\mathcal{S}}_i \triangleq \max_{1\leq k\leq i}\{\mathcal{S}_k^b/\eta_{k,i}^{\times}\}$, we have that

$$\mathcal{V}_i(S) = ((\alpha\eta)^{\times}R)_i^+ S - (\alpha^{\times}(C-\gamma))_i^+ - \alpha_{0,i}^{\times}C_0, \quad \text{for all } S \geq \bar{\mathcal{S}}_i.$$

Specializing this result for the case $\alpha_i = \eta_i = 1$, we have that

$$V_i^U(S) = R_i^+ S - (C-\gamma)_i^+ - C_0, \quad \text{for all } S \geq \bar{\mathcal{S}}_i.$$

In similar fashion, if we specialize for the case $\alpha_i = e^{-rT_i}$, $\eta_i = e^{\mu T_i}$ and $\gamma_i = 0$, we have that

$$V_i^L(S) = \mathcal{R}_i S - \mathcal{C}_i, \quad \text{for all } S \geq \bar{\mathcal{S}}_i.$$

This asymptotic behavior of V_i^L is the same as that presented by V_i described in Proposition 1. With this result we can now calculate a new function for the asymptotic behavior of $V_i(S)$. First, we know that for a sufficiently large S, $V_{i-1}(S) \approx V_{i-1}^L(S)$ and therefore $\mathbb{E}[V_{i-1}(S_{T_i})|S_0 = S] \approx \mathbb{E}[V_{i-1}^L(S)|S_0 = S]$.

Using the observation above and (6) we find a new asymptotic approximation for V_i, which we refer to as $\widehat{V_i^L}$, defined by

$$\widehat{V_i^L}(S) \triangleq \begin{cases} -C_0 & \text{if } S \leq \widehat{S_i^a} \\ \widehat{B}_i S^{\lambda_1} + \widehat{D}_i S^{\lambda_2} - M/r & \text{if } \widehat{S_i^a} < S < \widehat{S_i^b} \\ (R_i + e^{-(r-\mu)T_i}\mathcal{R}_{i-1})S - (C_i + e^{-rT_i}\mathcal{C}_{i-1}) & \text{if } \widehat{S_i^b} \leq S, \end{cases}$$

where $\widehat{B}_i, \widehat{D}_i$ and $\widehat{S_i^a}$ and $\widehat{S_i^b}$ are calculated by imposing the smooth pasting and value matching conditions as in (9). The convergence of V_i and V_i^L, as S becomes large, suggests that $\widehat{V_i^L}$ is a good quality approximation. Similarly, we can construct an approximation $\widehat{V_i^U}$ based on the asymptotic behavior of V_i^U. In such a case, however, there is no result suggesting that this approximation is of good quality. As we will see later, numerical examples confirm these hypotheses.

5. Numerical Experiments

We first illustrate the approximations and algorithms in the previous sections by means of numerical examples. In the next section, we apply our methods to a case study.

5.1. Synthetic examples.

Let us now consider a project with 5 stages, where $C_0 = 5$, $M = 3.8$, $\mu = 0.057$, $\sigma = 0.233$ and $r = 0.12$. The parameters of each stage are summarized in Table 1.

Stage	R_i	C_i	T_i
1	0.25	14	1.2
2	0.3	9	1.6
3	0.4	16	1
4	0.32	10	2
5	0.35	12.25	0.7

Table 1 Parameters for the 5-stage project example.

Using the V-approx algorithm, we compute $V_5^U(S)$ and $V_5^L(S)$. In Figure 1 the resulting approximations are compared with $V_5(S)$, which is computed numerically. We observe that all three functions are asymptotically linear and that $V_5^L(S)$ converges to $V_5(S)$.

Figure 2 compares $\widehat{V_5^U}(S)$ and $\widehat{V_5^L}(S)$ with $V_5(S)$. As can be seen, $\widehat{V_5^L}(S)$ provides a reasonable approximation of $V_5(S)$ for most values of S, and converges to it. $\widehat{V_5^U}$ also provides a reasonable approximation for low values for S but degrades as the market price increases.

Let $\mathcal{E}(\mathcal{V}_i)$ denote the average relative error of the approximation function $\mathcal{V}_i$ with respect to function V_i over the range $[S_{\min}, S_{\max}]$. That is,

$$\mathcal{E}(\mathcal{V}_i) \triangleq \frac{100}{S_{\max} - S_{\min}} \int_{S_{\min}}^{S_{\max}} \frac{|\mathcal{V}_i(S) - V_i(S)|}{V_i(S)} dS.$$

The average relative error of our approximations for the interval $[0, 100]$ is shown in Table 2. The expression V_5^{EL} is the value of V_5 under the optimal policy generated by V_5^L. These results suggest that $\widehat{V_i^L}$ and V_i^L are the best approximations of V_i. Note also that V_5^{EL} has an error close to 0, which highlights the quality of our methodology.

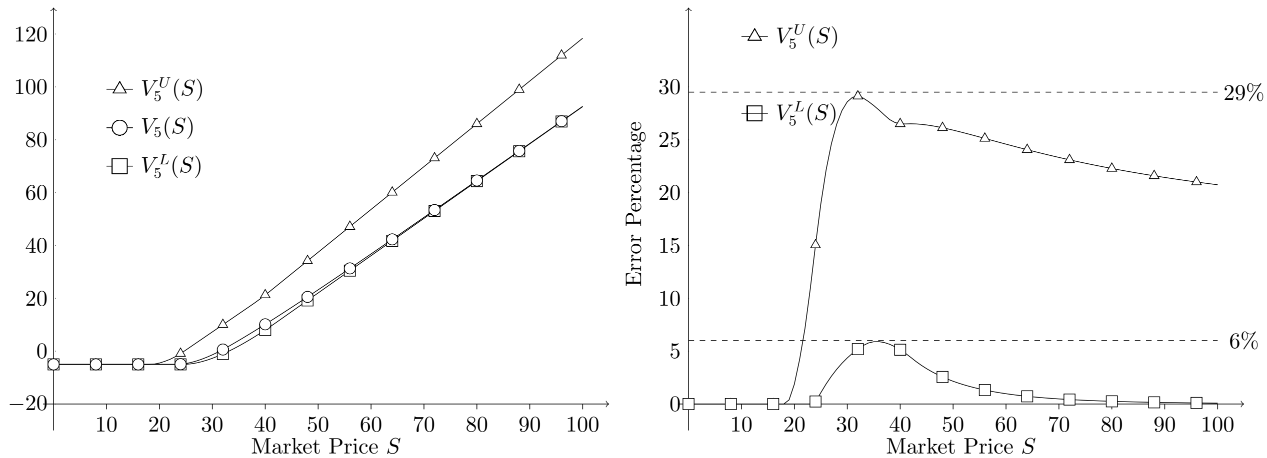

Figure 1 *Left panel:* Value of the functions V_5 (circle),V_5^U (triangle) and V_5^L (square) as the market price varies. *Right panel:* Relative error of V_5^U (triangle) and V_5^L (square) with respect to V_5. The dashed lines mark the maximum relative error of the two approximations.

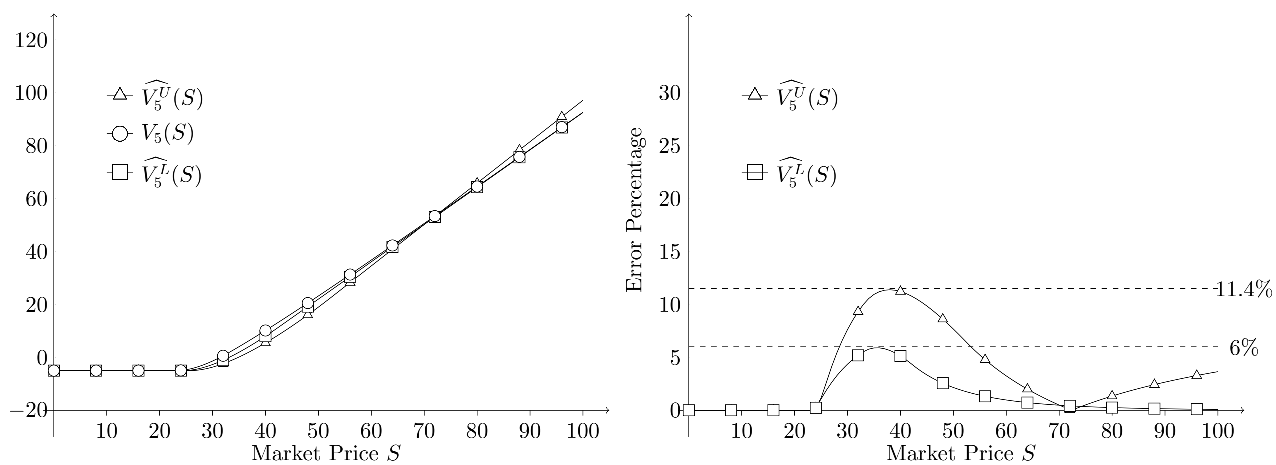

Figure 2 *Left panel:* Value of the functions V_5 (circle),$\widehat{V_5^U}$ (triangle) and $\widehat{V_5^L}$ (square) as the market price varies. *Right panel:* Relative error of $\widehat{V_5^U}$ (triangle) and $\widehat{V_5^L}$ (square) with respect to V_5. The dashed lines mark the maximum relative error of the two approximations.

5.2. Case Study: Mid-term Mine Planning

We apply the methodology developed so far to evaluate a real project at Chuquicamata, the iconic copper mine in northern Chile owned by Codelco—the world's largest copper producer. While still the biggest and longest-running open pit operation in the world, declining economic viability will force its closure towards the end of the present decade,

Approximation $\mathcal{V}$	$\mathcal{E}(\mathcal{V})$
V_5^L	1.33%
V_5^U	18.66%
$\widehat{V_5^L}$	1.34%
$\widehat{V_5^U}$	3.54%
V_5^{EL}	0.32%

Table 2 Mean relative error.

by which time a replacement underground mine will begin operating. According to plans for the massive new operation, the underground mine is to be divided into four sectors which are projected to yield up to 1,700 million tonnes of ore averaging 0.71% Cu. The undertaking represents an investment totalling about US$5 billion.

Instance setup. The instance we present is one of the four *sectors* of the new Chuquicamata mine, which we will call S1. It consists of 20 *phases* containing a total of 440 million tonnes of ore with an average grade of 0.68%. Its spatial distribution is shown in Figure 3.

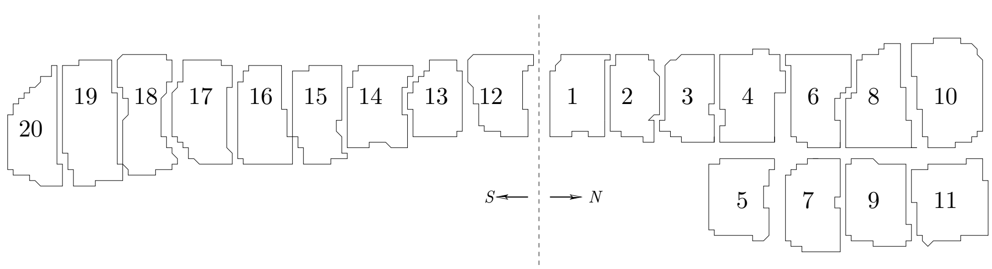

Figure 3 S1 spatial distribution.

In order to evaluate S1 using our methodology we first need to outline the key components of our dynamic programming formulation (see Section 2). Long-term planning of mining operations usually identifies a series of possible extraction sequences for the sector's phases. In this case, the phases in the north section of S1 can be processed (i.e. their ore extracted) in any of the 6 different sequences set out in Table 5.2.

Phases in the south section, on the other hand, can be processed in a unique sequence: 12, 13, 14, 15, 16, 17, 18, 19 and 20. We apply our methodology to find the economic value

Sequence	Phases order
N1	1-2-3-4-5-6-7-8-9-10-11
N2	1-2-3-5-4-6-7-9-8-10-11
N3	1-2-5-3-4-6-7-9-8-10-11
N4	1-2-5-3-4-6-7-8-9-10-11
N5	1-2-3-4-5-7-6-8-9-11-10
N6	1-2-5-3-4-6-7-9-8-11-10

Table 3 Extraction sequences, North section.

of a predetermined set of feasible extraction sequences. We let $\pi = (\pi_1, \ldots, \pi_N)$ denote one such a sequence (which corresponds to a permutation of $\{1, \ldots, N\}$), understanding that π_N is the first phase to be extracted in the sequence, and π_1 the last. In terms of our formulation, phases correspond to stages.

Next, for each phase, we show how to compute R_i, C_i and T_i. Phase i in the sector is characterized by a tonnage Q_i and an ore content L_i. The specific parameters for each phase are detailed in Table 4.

Phase	Q_i[Ton]	L_i[%]	Phase	Q_i[Ton]	L_i[%]
1	4333285	0.742	11	22640604	0.699
2	7646668	0.674	12	3708814	0.805
3	10451116	0.844	13	8882645	0.803
4	11801936	0.766	14	12883028	0.796
5	14495435	0.761	15	21390450	0.753
6	29220025	0.753	16	26742924	0.676
7	16695793	0.706	17	35356325	0.589
8	33268812	0.707	18	44094300	0.586
9	15297264	0.736	19	40080765	0.611
10	43069997	0.696	20	34275177	0.562

Table 4 S1's tonnage and ore content, by phase.

The marginal cost of processing a phase depends on which other phases have already been extracted, implying that each sequence has its own marginal cost. For a given sequence π

we assume this cost increases linearly with the distance from the first phase in the sequence (π_N) to the phase currently being extracted (π_i) according to

$$A_i^{\pi} = 9.514 + 0.0008 \cdot d_{\pi_N \pi_i},$$

where $d_{\pi_N \pi_i}$ is the distance between π_N and π_i. The parameters in the formula above represent the projected production costs of the underground mine. The distances between phases are displayed in Table 5.

The north and south sections of S1 were extracted simultaneously following the sequences indicated above. We consider that each section has an installed production capacity K of 7.3 [million Tonnes/year], a shutdown cost C_0 is 10 [million $] and a production capacity maintenance cost of M is 30 [million $/year]. In addition, we model the spot price of copper as a GBM with $\mu = r - \rho$, where $r = 12\%$ is the discount factor and $\rho = 6.3\%$ is the convenience yield[1] and $\sigma = 0.233$ is the price volatility. [2] From this, we can compute the

d_{ij}	1	2	3	4	5	6	7	8	9	10	11
1	0	159	318	476	595	633	731	791	870	949	1014
2	159	0	159	317	465	474	588	632	721	790	861
3	318	159	0	158	358	315	457	474	576	632	710
4	476	317	158	0	312	158	358	316	449	474	569
5	595	465	358	312	0	350	157	443	314	564	472
6	633	474	315	158	350	0	322	158	356	316	447
7	731	588	457	358	157	322	0	357	158	448	316
8	791	632	474	316	443	158	357	0	318	158	352
9	870	721	576	449	314	356	158	318	0	353	158
10	949	790	632	474	564	316	448	158	353	0	312
11	1014	861	710	569	472	447	316	352	158	312	0
d_{ij}	12	13	14	15	16	17	18	19	20	-	-
12	0	158	323	485	647	813	970	1129	1290	-	-

Table 5 Distance between S1's phases, in metres.

parameters R_i, C_i and T_i associated with each phase in S1 as follows

$$\begin{aligned}
R_i &= \mathbb{E}\Big[\int_0^{T_i} e^{-rt} L_i K S_{\tau_i+t} dt | \mathcal{F}_{\tau_i}\Big] / S_{\tau_i} = L_i K \left\{ \frac{1 - e^{-(r-\mu)T_i}}{r-\mu} \right\}, \\
C_i &= \int_0^{T_i} e^{-rt} A_i K dt = A_i K \left\{ \frac{1 - e^{-rT_i}}{r} \right\}, \\
T_i &= Q_i / K.
\end{aligned}$$

In computing the parameters above we assumed that ore is extracted from the phase at a constant rate, given by the production capacity K, and the ore percentage content L_i. Note that the expected present value (at the moment extraction begins) of the payoff from extraction is indeed linear in the prevailing market price. Similarly, we assumed that extraction costs are also incurred at a constant rate.

Discussion. The value functions $V_9^S(S)$ and $V_{11}^N(S)$ were used to represent the project value of the north and south sections, respectively. Total project value is given by $V_{20}(S) = V_9^S(S) + V_{11}^N(S)$. The results of the project evaluation for the six sequences considered (recall that the south section has only one possible sequence) when the initial price S_0 varies from 50[¢/lb] to 600[¢/lb] are summarized in Tables 6 and 7.

Price	$N1$			$N2$			$N3$		
S	V_{20}	$\widehat{V_{20}^L}$	Err(%)	V_{20}	$\widehat{V_{20}^L}$	Err(%)	V_{20}	$\widehat{V_{20}^L}$	Err(%)
50	207.38	189.45	8.65	207.17	189.32	8.62	207.37	189.57	8.58
100	1,401.65	1,393.39	0.59	1,401.34	1,393.15	0.58	1,401.82	1,393.79	0.57
150	2,679.38	2,688.49	0.34	2,679.15	2,688.24	0.34	2,679.84	2,689.18	0.35
200	3,960.11	3,983.58	0.59	3,959.96	3,983.34	0.59	3,960.86	3,984.57	0.60
250	5,240.84	5,278.68	0.72	5,240.77	5,278.43	0.72	5,241.87	5,279.96	0.73
300	6,521.57	6,573.77	0.80	6,521.58	6,573.53	0.80	6,522.88	6,575.35	0.80
350	7,802.29	7,868.87	0.85	7,802.39	7,868.63	0.85	7,803.9	7,870.74	0.86
400	9,083.02	9,163.97	0.89	9,083.19	9,163.72	0.89	9,084.91	9,166.13	0.89
450	10,363.75	10,459.06	0.92	10,364	10,458.82	0.91	10,365.92	10,461.52	0.92
500	1,1644.48	11,754.16	0.94	11,644.81	11,753.91	0.94	11,646.94	11,756.91	0.94
550	12,925.2	13,049.26	0.96	12,925.62	13,049.01	0.95	12,927.95	13,052.3	0.96
600	14,205.93	14,344.35	0.97	14,206.43	14,344.11	0.97	14,208.96	14,347.69	0.98

Table 6 Exact value of value function V_{20} in millions of US\$, approximate value of $\widehat{V_{20}^L}$ and relative error Err(%) for north section sequences $N1$, $N2$ and $N3$, for the price range 50 to 600 U.S. cents.

In Tables 6 and 7 we compare V_i with $\widehat{V_i^L}$. These results show that, as price S increases, value $V_i(S)$ converges to a linear function similar to the linear segment of $\widehat{V_i^L}$ and the relative error remains low. Moreover, for a plausible range of values of S (e.g., between 200[¢/lb] and 400[¢/lb]) $\widehat{V_i^L}$ produces a good-quality approximation across all execution sequences. Since $\widehat{V_i^L}$ and V_i^L behave similarly, we expect that V_i^L will also have low relative error. In turn, this suggests that we can use the execution policy derived from V_i^L to time the execution of $S1$ and achieve good performance.

Price	$N4$			$N5$			$N6$		
S	V_{20}	$\widehat{V_{20}^L}$	Err(%)	V_{20}	$\widehat{V_{20}^L}$	Err(%)	V_{20}	$\widehat{V_{20}^L}$	Err(%)
50	207.57	189.7	8.61	204.85	189.07	7.70	204.89	189.24	7.64
100	1,402.12	1,394.05	0.58	1,400.88	1,392.66	0.59	1,401.14	1,393.17	0.57
150	2,680.05	2,689.44	0.35	2,678.61	2,687.76	0.34	2,679.11	2,688.56	0.35
200	3,960.98	3,984.83	0.60	3,959.43	3,982.86	0.59	3,960.18	3,983.95	0.60
250	5,241.9	5,280.22	0.73	5,240.26	5,277.95	0.72	5,241.26	5,279.34	0.73
300	6,522.82	6,575.61	0.81	6,521.09	6,573.05	0.80	6,522.33	6,574.73	0.80
350	7,803.74	7,871	0.86	7,801.92	7,868.15	0.85	7,803.4	7,870.12	0.85
400	9,084.66	9,166.39	0.90	9,082.75	9,163.24	0.89	9,084.47	9,165.51	0.89
450	10,365.58	10,461.78	0.93	10,363.58	10,458.34	0.91	10,365.55	10,460.89	0.92
500	11,646.5	11,757.17	0.95	11,644.41	11,753.43	0.94	11,646.62	11,756.28	0.94
550	12,927.42	13,052.56	0.97	12,925.24	13,048.53	0.95	12,927.69	13,051.67	0.96
600	14,208.35	14,347.94	0.98	14,206.06	14,343.63	0.97	14,208.77	14,347.06	0.97

Table 7 Continuation of 6 for north section sequences $N4$, $N5$ and $N6$.

As a final remark we note that finding the optimal sequence of extraction can boost revenues. Observe that some of the analyzed sequences yield larger values for the project than others. In particular, $N3$ revenue dominates most of the sequences for intermediate values of S. In turn, a decision-maker may choose to execute the project in this order. We comment on this and the challenges it raises in the next section.

6. Conclusions and Future Research

In this paper, we present a solution approach to the problem of timing the execution of a project consisting of N stages with the objective of maximizing expected present value of the cumulative payoff, that depends on a market price which we assume follows a GBM. The proposed solution involves a dynamic program in which, at each stage, the decision-maker has the options of executing the next stage of the project, wait, or abandon the project altogether. Although the proposed approach does not solve the problem in closed-form, we have identified the value function's asymptotic behavior. Also, we have proven properties of the value function such as its convexity and growth, which enabled us to construct upper- or lower-bound functions of the value function. In addition, we have developed an efficient algorithm for computing these approximations.

Our results demonstrate that some of the proposed approximations have good performance (close to optimality) and have threshold-type structures. They also show that the best approximations are given by the lower bounds, a conclusion corroborated by the empirical evidence drawn from the application of the proposed approach to a real-world case of a major mining investment project.

As for extensions of our work, there are a number of interesting extensions that could be investigated both for their application to real problems and the theoretical challenges they pose. One example is the option of dynamically adjusting the execution sequence of the project's stages as new information on the asset price is obtained. We observed in our case study that several such sequences needed to be evaluated. Ideally, the algorithm should help find the best one, which we argue should be adaptive in nature. This adds a combinatorial aspect to the basic problem. For a given state $(S, \mathfrak{N})$ where $\mathfrak{N}$ is the set of stages already executed with $||\mathfrak{N}|| = M$, the expected optimal profit would have to be computed $\binom{N}{M}$ times, which in practice would be impossible for realistic values of M and N.

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Appendix A: Proof of main results

Proof of Proposition 1. We must prove both inequalities in the statement of the result. First, for the lower bound we demonstrate that

$$\sum_{k=1}^{i} e^{-rT^+_{k,i}} \mathbb{E}[W_k(S_{T^+_{k,i}})|S_0 = S] - e^{-rT^+_{0,i}} C_0 \leq V_i(S), \tag{A-1}$$

and then, noting that

$$\sum_{k=1}^{i} e^{-rT^+_{k,i}} \mathbb{E}[W_k(S_{T^+_{k,i}})|S_0 = S] - e^{-rT^+_{0,i}} C_0 = \mathcal{R}_i S - \mathcal{C}_i,$$

we have the desired inequality. Proceeding by induction on i, for i equal to 1 the result is immediate. If (A-1) is satisfied for $i-1$, we have

$$\begin{aligned}
V_i(S) &= \sup_{\tau \geq 0} \mathbb{E}[e^{-r\tau} \max\left\{W_i(S_\tau) + e^{-rT_i} \mathbb{E}^{S_\tau}[V_{i-1}(S_{T_i})], -C_0\right\} - \int_0^\tau e^{-rt} M dt | S_0 = S] \\
&\geq \max\left\{W_i(S) + e^{-rT_i} \mathbb{E}^S[V_{i-1}(S_{T_i})], -C_0\right\} \\
&\geq \max\left\{W_i(S) + e^{-rT_i} \mathbb{E}^S\Big[\sum_{k=1}^{i-1} e^{-rT^+_{k,i-1}} \mathbb{E}[W_k(S_{T^+_{k,i-1}})|S_0 = S_{T_i}]\Big] - e^{-rT^+_{0,i-1}} C_0, -C_0\right\} \\
&= \max\left\{\sum_{k=1}^{i} e^{-rT^+_{k,i}} \mathbb{E}^S[W_k(S_{T^+_{k,i}})] - e^{-rT^+_{0,i}} C_0, -C_0\right\} \\
&\geq \sum_{k=1}^{i} e^{-rT^+_{k,i}} \mathbb{E}^S[W_k(S_{T^+_{k,i}})] - e^{-rT^+_{0,i}} C_0,
\end{aligned}$$

where in the first equality we use the strong Markov property (Karatzas and Shreve 1991, Theorem 5.4.20), in the first inequality we define $\tau \equiv 0$ and in the second inequality we apply the inductive hypothesis.

For the upper bound we consider a modified price process $\mathcal{Z}_t$ defined by

$$\mathcal{Z}_t \triangleq S_t + \sum_{k: T^+_{k,i} \leq t} (S^b_k - \mathcal{Z}_{T^+_{k,i}-})^+, \qquad \mathcal{Z}_{0-} = S_0.$$

The idea behind the use of this process is that the moment stage i is completed, process S_t moves to the price S^b_{i-1}, which imply that the next stage should start its execution. In this way, the entire sequence of stages can be processed consecutively with no suspensions.

Let $\mathcal{H}_i(S)$ be the expected value of a project with i stages under process $\mathcal{Z}_t$. Since $S_t \le \mathcal{Z}_t$ for every realization of these processes, it is clear that $V_i(S) \le \mathcal{H}_i(S)$. Therefore,

$$\begin{aligned}
V_i(S) \le \mathcal{H}_i(S) &= \sum_{k=1}^{i} e^{-rT^+_{k,i}} \mathbb{E}[W_k(\mathcal{Z}_{T^+_{k,i}})|S_0 = S] - e^{-rT^+_{0,i}} C_0 \\
&= \sum_{k=1}^{i} e^{-rT^+_{k,i}} (R_k \mathbb{E}[\mathcal{Z}_{T^+_{k,i}}|S_0 = S] - C_k) - e^{-rT^+_{0,i}} C_0 \\
&= \sum_{k=1}^{i} e^{-rT^+_{k,i}} R_k \mathbb{E}[\mathcal{Z}_{T^+_{k,i}} - S_{T^+_{k,i}}|S_0 = S] + \sum_{k=1}^{i} e^{-rT^+_{k,i}} (R_k \mathbb{E}[S_{T^+_{k,i}}|S_0 = S] - C_k) - e^{-rT^+_{0,i}} C_0 \\
&= \sum_{k=1}^{i} e^{-rT^+_{k,i}} R_k \mathbb{E}[\mathcal{Z}_{T^+_{k,i}} - S_{T^+_{k,i}}|S_0 = S] + (\mathcal{R}_i S - \mathcal{C}_i). \qquad \text{(A-2)}
\end{aligned}$$

We now bound the term $\mathbb{E}[\mathcal{Z}_{T^+_{k,i}} - S_{T^+_{k,i}}|S_0 = S]$ on the right-hand side of the last expression. Since $S_t = S_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t)$, we have

$$\mathbb{P}(\{S^b_k \ge S_{T^+_{k,i}}\}|S_0 = S) = \mathbb{P}(\frac{1}{\sigma}\Big[\ln\Big(\frac{S}{S^*_k}\Big) + \{\mu + \frac{\sigma^2}{2}\}T^+_{k,i}\Big] \le B_{T^+_{k,i}}). \qquad \text{(A-3)}$$

By a theorem of (Asmussen 2003, Theorem XIII-2.1), for $S > S^m_i \cdot \exp(-\min_{1\le k\le i}\{(\mu + \frac{\sigma^2}{2})T^+_{k,i}\})$ we can bound the left-hand side of (A-3) by an exponential term as follows:

$$\mathbb{P}(\frac{1}{\sigma}\Big[\ln\Big(\frac{S}{S^*_k}\Big) + \{\mu + \frac{\sigma^2}{2}\}T^+_{k,i}\Big] \le \tilde{B}_{T^+_{k,i}}) \le \exp\left(-\frac{1}{T^+_{k,i}\sigma^2}\Big[\ln\Big(\frac{S}{S^*_k}\Big) + \{\mu + \frac{\sigma^2}{2}\}T^+_{k,i}\Big]^2\right).$$

Putting it all together, we get

$$\begin{aligned}
\mathbb{E}[\mathcal{Z}_{T^+_{k,i}} - S_{T^+_{k,i}}|S_0 = S] &= \sum_{n=k}^{i} \mathbb{E}[(S^b_k - \mathcal{Z}_{T^+_{k,i}-})\mathbf{1}_{\{\{S^b_k \ge \mathcal{Z}_{T^+_{k,i}-}\}\}}|S_0 = S] \\
&\le \sum_{n=k}^{i} S^b_k \mathbb{P}(\{S^b_k \ge \mathcal{Z}_{T^+_{k,i}-}\}|S_0 = S) \\
&\le \sum_{n=k}^{i} S^b_k \mathbb{P}(\{S^b_k \ge S_{T^+_{k,i}}\}|S_0 = S) \\
&= \sum_{n=k}^{i} S^b_k \mathbb{P}(\frac{1}{\sigma}\Big[\ln\Big(\frac{S}{S^b_k}\Big) + \{\mu + \frac{\sigma^2}{2}\}T^+_{k,i}\Big] \le B_{T^+_{k,i}}) \\
&\le (i + 1 - k) \cdot S^m_i \cdot \exp\left(-\frac{1}{\sigma^2} \min_{1\le k\le i} \frac{1}{T^+_{k,i}}\Big[\ln\Big(\frac{S}{S^m_i}\Big) + \{\mu + \frac{\sigma^2}{2}\}T^+_{k,i}\Big]^2\right).
\end{aligned}$$

Finally, using the bound in (A-2), we deduce that

$$V_i(S) \le \frac{i(i+1)}{2} \cdot S_i^m \cdot R_i^m \cdot \exp\left(-\frac{1}{\sigma^2} \min_{1\le k\le i} \frac{1}{T_{k,i}^+} \left[\ln\left(\frac{S}{S_i^m}\right) + \{\mu + \frac{\sigma^2}{2}\} T_{k,i}^+ \right]^2 \right) + (\mathcal{R}_i S - \mathcal{C}_i).$$

■

Proof of Proposition 2. We first prove the convexity result. Define $X_t \triangleq \exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t)$ so S_t is equal to $S_0 X_t$ and, therefore, we can write $V_1(S)$ as

$$V_1(S) = \sup_{\tau\ge 0} \mathbb{E}\left[e^{-r\tau} \max\{W_1(SX_\tau) - e^{-rT_1} C_0, -C_0\} - \int_0^\tau e^{-rt} M\,dt \right].$$

Because the maximum of convex functions is itself a convex function, so is $V_1(S)$. The result follows immediately for $i > 1$ by induction. The monotonicity of $V_i(S)$ simply follows from the monotonicity of $W_1(S)$ and induction. ■

Proof of Proposition 3. We prove the results by induction. Let us begin with the lower bound result. For i equal to 1, the lower bound is satisfied because $V_1^L(S)$ is equal to $V_1(S)$ by definition. If the lower bound is valid for $i-1$, then

$$\begin{aligned}
V_i^L(S) &= \sup_{\tau\ge 0} \mathbb{E}\Big[e^{-r\tau} \max\big\{ W_i(S_\tau) + e^{-rT_i} V_{i-1}^L(e^{\mu T_i} S_\tau), -C_0 \big\} - \int_0^\tau e^{-rt} M\,dt \Big| S_0 = S \Big] \\
&\overset{\text{Ind. Hyp}}{\le} \sup_{\tau\ge 0} \mathbb{E}\Big[e^{-r\tau} \max\{ W_i(S_\tau) + e^{-rT_i} V_{i-1}(e^{\mu T_i} S_\tau), -C_0 \} - \int_0^\tau e^{-rt} M\,dt \Big| S_0 = S \Big] \\
&\overset{(a)}{=} \sup_{\tau\ge 0} \mathbb{E}\Big[e^{-r\tau} \max\{ W_i(S_\tau) + e^{-rT_i} V_{i-1}(\mathbb{E}[S_{\tau+T_i} | \mathcal{F}_\tau]), -C_0 \} - \int_0^\tau e^{-rt} M\,dt \Big| S_0 = S \Big] \\
&\overset{\text{Jensen}}{\le} \sup_{\tau\ge 0} \mathbb{E}\Big[e^{-r\tau} \max\big\{ W_i(S_\tau) + e^{-rT_i} \mathbb{E}[V_{i-1}(S_{\tau+T_i}) | \mathcal{F}_\tau], -C_0 \big\} - \int_0^\tau e^{-rt} M\,dt \Big| S_0 = S \Big] \\
&= V_i(S).
\end{aligned}$$

In (a) we have made use of the fact that $e^{-\mu t} S_t$ is a $\mathbb{P}$-martingale. Note that we can make use of Jensen's inequality since by Proposition 2, $V_i(S)$ is a convex function.

Let us now prove the upper bound result. First, note that the validity of the upper bound for i equal to 1 immediately follows from the condition that $C_0 < \frac{M}{r}$. To prove this bound is valid for $i > 1$ we will use the following lemma, whose proof can be found at the end of this appendix.

Lemma 1. *For all $i \geq 1$ and $t \geq 0$, $V_i(\cdot)$ satisfies the following inequality:*

$$e^{-rT_{i+1}} V_i(S_{\tau+T_{i+1}}(\omega)) - \int_0^{T_{i+1}} e^{-rs} M ds \leq V_i(S_\tau(\omega)), \quad \textit{for all } \omega \in \Omega.$$

We note that Lemma 1 simply estates that, in any situation, waiting for a period of time T_{i+1} as part of an strategy is no better than the optimal strategy.

Let us now consider the case of $i > 1$, so assume that the upper bound result is valid for $i-1$. Then,

$$\begin{aligned}
V_i(S) &= \sup_{\tau \geq 0} \mathbb{E}\Big[e^{-r\tau} \max\big\{ W_i(S_\tau) + e^{-rT_i} \mathbb{E}[V_{i-1}(S_{\tau+T_i}) | \mathcal{F}_\tau], -C_0 \big\} - \int_0^\tau e^{-rt} M dt \Big| S_0 = S \Big] \\
&\overset{(a)}{\leq} \sup_{\tau \geq 0} \mathbb{E}\Big[e^{-r\tau} \max\{ W_i(S_\tau) + e^{-rT_i}(e^{rT_i} V_{i-1}(S_\tau) + e^{rT_i} \int_0^{T_i} e^{-rt} M dt), -C_0 \} \\
&\quad - \int_0^\tau e^{-rt} M dt \Big| S_0 = S \Big] \\
&\overset{\text{Ind. Hyp}}{\leq} \sup_{\tau \geq 0} \mathbb{E}\Big[e^{-r\tau} \max\{ W_i(S_\tau) + V_{i-1}^U(S_\tau) + \int_0^{T_i} e^{-rt} M dt, -C_0 \} - \int_0^\tau e^{-rt} M dt \Big| S_0 = S \Big] \\
&= V_i^U(S),
\end{aligned}$$

where the inequality proved in Lemma 1 is used in (a). ■

Proof of Theorem 1. This proof is divided into two parts. In the first part we find a solution for the system of equations (9) for the case $i = 1$. Then, in the second part, we use this solution and the condition on the statement of the result to check the hypotheses in (Øksendal 2007, Theorem 10.4.1), from which we can conclude the desired result. Throughout the proof we assume that $(C_1 - \gamma_1 + \alpha_1 C_0) \leq M/r$. The the case where $(C_1 - \gamma_1 + \alpha_1 C_0) > M/r$ is analogous and we omit it. In order to avoid degenerate cases in which the thresholds collapse or the abandonment option is never taken, throughout the proof we assume that $C_0(1-\alpha_1) < C_1 - \gamma_1$.

To simplify notation we let

$$x_1 \triangleq \mathcal{S}_1^a \quad \text{and} \quad x_2 \triangleq \mathcal{S}_1^b.$$

System (9) with i equals to 1 then becomes

$$\mathcal{B}_1 x_1^{\lambda_1} + \mathcal{D}_1 x_1^{\lambda_2} - M/r = -C_0, \tag{A-4}$$

$$\mathcal{B}_1 x_2^{\lambda_1} + \mathcal{D}_1 x_2^{\lambda_2} - M/r = R_1 x_2 - (C_1 - \gamma_1) - \alpha_1 C_0, \tag{A-5}$$

$$\lambda_1 \mathcal{B}_1 x_1^{\lambda_1 - 1} + \lambda_2 \mathcal{D}_1 x_1^{\lambda_2 - 1} = 0, \tag{A-6}$$

$$\lambda_1 \mathcal{B}_1 x_2^{\lambda_1 - 1} + \lambda_2 \mathcal{D}_1 x_2^{\lambda_2 - 1} = R_1. \tag{A-7}$$

Because $M/r > C_0$, by combining (A-4) and (A-6) it can be seen that it cannot be the case that $\mathcal{D}_1$ equals 0. Also, since $\lambda_2 < 0$ it must be the case that $x_1 > 0$. By a similar argument we conclude that $\mathcal{B}_1$ is not equal to 0. If we combine equations (A-4) and (A-6), we can solve for $\mathcal{B}_1$ and $\mathcal{D}_1$ as functions of x_1, thereby obtaining

$$\mathcal{B}_1 = -\frac{(M/r - C_0)\lambda_2}{(\lambda_1 - \lambda_2)x_1^{\lambda_1}}, \qquad \mathcal{D}_1 = \frac{(M/r - C_0)\lambda_1}{(\lambda_1 - \lambda_2)x_1^{\lambda_2}}.$$

Then, exploiting the fact that $M/r - C_0 > 0$ and $\lambda_2 < 0 < 1 < \lambda_1$, we may conclude that $\mathcal{B}_1 > 0$ and $\mathcal{D}_1 > 0$. All that remains is to show that $x_1 < x_2$ and that a solution of the system (A-4) -(A-7) exists. From equations (A-5) and (A-7), we can also solve for $\mathcal{B}_1, \mathcal{D}_1$ but this time as functions of x_2. Therefore,

$$\mathcal{B}_1 = \frac{(C_1 - \gamma_1 + \alpha_1 C_0 - M/r)\lambda_2 - R_1 x_2(\lambda_2 - 1)}{(\lambda_1 - \lambda_2)x_2^{\lambda_1}}, \qquad \mathcal{D}_1 = -\frac{(C_1 - \gamma_1 + \alpha_1 C_0 - M/r)\lambda_1 - R_1 x_2(\lambda_1 - 1)}{(\lambda_1 - \lambda_2)x_2^{\lambda_2}}.$$

At this point it is convenient to define the following functions for $x > 0$:

$$\begin{aligned}
I_1(x) &\triangleq \frac{\lambda_2(C_0 - M/r)}{x^{\lambda_1}}, \\
I_2(x) &\triangleq \frac{\lambda_1(C_0 - M/r)}{x^{\lambda_2}}, \\
J_1(x) &\triangleq \frac{(C_1 - \gamma_1 + \alpha_1 C_0 - M/r)\lambda_2 - R_1 x(\lambda_2 - 1)}{x^{\lambda_1}}, \\
J_2(x) &\triangleq \frac{(C_1 - \gamma_1 + \alpha_1 C_0 - M/r)\lambda_1 - R_1 x(\lambda_1 - 1)}{x^{\lambda_2}}.
\end{aligned}$$

The derivatives of these functions are:

$$\begin{aligned}
I_1'(x) &= \frac{-\lambda_1\lambda_2(C_0 - M/r)}{x^{\lambda_1+1}}, \\
I_2'(x) &= -\frac{\lambda_1\lambda_2(C_0 - M/r)}{x^{\lambda_2+1}}, \\
J_1'(x) &= \frac{R_1 x(\lambda_1 - 1)(\lambda_2 - 1) - (C_1 - \gamma_1 + \alpha_1 C_0 - M/r)\lambda_1\lambda_2}{x^{\lambda_1+1}}, \\
J_2'(x) &= \frac{R_1 x(\lambda_1 - 1)(\lambda_2 - 1) - (C_1 - \gamma_1 + \alpha_1 C_0 - M/r)\lambda_1\lambda_2}{x^{\lambda_2+1}}.
\end{aligned}$$

With these definitions, what we have to prove is the existence of a pair (x_1, x_2), with $x_1 < x_2$ such that

$$I_1(x_1) = J_1(x_2) \quad \text{y} \quad I_2(x_1) = J_2(x_2). \tag{A-8}$$

Note that $I_1(x)$ is positive and decreasing over its domain, with $I_1(x) \overset{x\to\infty}{\to} 0$ and $I_1(x) \overset{x\to 0^+}{\to} \infty$, while $I_2(x)$ is negative and decreasing over its domain with $I_2(x) \overset{x\to\infty}{\to} -\infty$ and $I_2(x) \overset{x\to 0^+}{\to} 0$. Also, $J_1(x)$

takes positive values and is decreasing over its entire domain, with $J_1(x) \overset{x\to\infty}{\to} 0$ and $J_1(x) \overset{x\to 0^+}{\to} \infty$, while $J_2(x)$ is negative and decreasing over its domain with $J_2(x) \overset{x\to\infty}{\to} -\infty$ and $J_2(x) \overset{x\to 0^+}{\to} 0$. Note that at the points

$$\underline{x} = \frac{(C_1 - \gamma_1 + \alpha_1 C_0 - C_0)\lambda_2}{R_1(\lambda_2 - 1)}, \qquad \overline{x} = \frac{(C_1 - \gamma_1 + \alpha_1 C_0 - C_0)\lambda_1}{R_1(\lambda_1 - 1)},$$

it is the case that $I_1(\underline{x}) = J_1(\underline{x})$ and $I_2(\overline{x}) = J_2(\overline{x})$, and $\underline{x} < \overline{x}$.

From the analysis above we can conclude as follows regarding the solution of (A-8). First, $x_1, x_2 \in (\underline{x}, \overline{x})$ and $x_1 < x_2$. This is so because for $x \geq \overline{x}$, it is the case that $I_2(x) \geq J_2(x)$ and therefore all pairs of points (x_1, x_2) greater than $\overline{x}$ such that $I_2(x_1) = J_2(x_2)$ satisfy $x_1 \geq x_2$. At the same time, since $J_1(x) > I_1(x)$ for $x \geq \overline{x}$, all pairs of points (x_1, x_2) greater than $\overline{x}$ such that $I_1(x_1) = J_1(x_2)$ satisfy $x_1 < x_2$. Therefore, it must also be the case that $x_1, x_2 < \overline{x}$. Similarly, for $x \leq \underline{x}$ it is the case that $I_1(x) \geq J_1(x)$ and therefore the pairs of points (x_1, x_2) less than $\underline{x}$ such that $I_1(x_1) = J_1(x_2)$ satisfy $x_1 \geq x_2$. At the same time, since $J_2(x) > I_2(x)$ for $x \leq \underline{x}$, all pairs of points (x_1, x_2) less than $\underline{x}$ such that $I_2(x_1) = J_2(x_2)$ satisfy $x_1 < x_2$. From the foregoing we may conclude that $x_1, x_2 > \underline{x}$ and therefore that $x_1, x_2 \in (\underline{x}, \overline{x})$. Also, on this interval all possible solutions of (A-8) satisfy $x_1 < x_2$.

Second, consider $x_2 \in (\underline{x}, \overline{x})$ and let $x_1^1(x_2)$ be the only solution of $I_1(x_1^1) = J_1(x_2)$ with $x_1^1 < x_2$, and $x_1^2(x_2)$ the only solution of $I_2(x_1^2) = J_2(x_2)$ with $x_1^2 < x_2$. Clearly, $x_1^1(x_2)$ and $x_1^2(x_2)$ are continuous functions defined in $(\underline{x}, \overline{x})$. We must prove that there exists an $x_2 \in (\underline{x}, \overline{x})$ such that $x_1^1(x_2) = x_1^2(x_2)$. For x_2 close to $\underline{x}$, $x_2 - x_1^2(x_2) > 0$, but since $x_2 - x_1^1(x_2) \approx 0$ there must exist an x_2 such that $x_2 - x_1^1(x_2) = x_2 - x_1^2(x_2)$, that is, $x_1^1(x_2) = x_1^2(x_2)$.

A graphical description of the above points is given in Figure 4. Briefly, it was demonstrated that assuming $M/r > (C_1 - \gamma_1 + \alpha_1 C_0) > C_0$, system (A-4)-(A-7) has a solution in the variables $x_1, x_2, \mathcal{B}_1, \mathcal{D}_1$.

To conclude our proof we continue with the verification of the conditions required by (Øksendal 2007, Theorem 10.4.1), which allow us to establish that our optimal stopping problem has a solution. We propose the following continuation region

$$D = \{S : \mathcal{S}_k^a < S < \mathcal{S}_k^b\},$$

and the following candidate solution

$$\phi(S) = \begin{cases} -C_0 & \text{if } S \leq \mathcal{S}_1^a \\ \mathcal{B}_1 S^{\lambda_1} + \mathcal{D}_1 S^{\lambda_2} - M/r & \text{if } \mathcal{S}_1^a < S < \mathcal{S}_1^b \\ R_1 S - (C_1 - \gamma_1 + \alpha_1 C_0) & \text{if } \mathcal{S}_1^b \leq S, \end{cases}$$

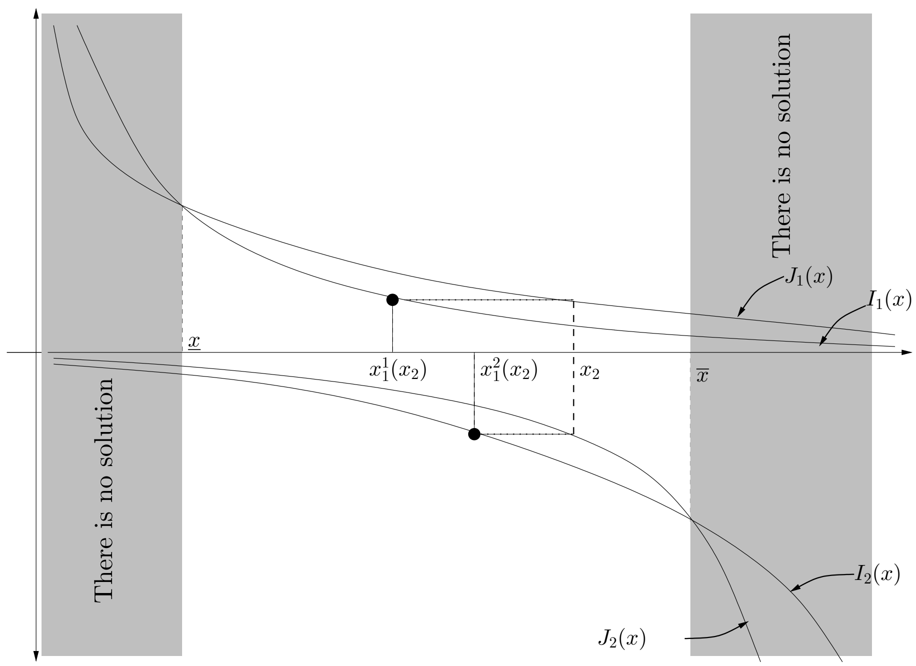

Figure 4 The curves are the functions I_1, I_2, J_1 and J_2 defined in Theorem 1. The last point in the demonstration of this proposition is that close to $\underline{x}$, the value of $x_2 - x_1^1(x_2)$ is close to 0 while the value of $x_2 - x_1^2(x_2)$ is greater than 0. Therefore, as x_2 grows towards $\overline{x}$, the value of $x_2 - x_1^1(x_2)$ rises while the value of $x_2 - x_1^2(x_2)$ falls to 0. This implies that there must exist some $x_2 \in (\underline{x}, \overline{x})$ such that $x_2 - x_1^1(x_2) = x_2 - x_1^2(x_2)$, that is, $x_1^1(x_2) = x_1^2(x_2)$.

where $\mathcal{S}_1^a, \mathcal{S}_1^b$ are found by imposing the value matching and smooth pasting conditions. Conditions $iii), iv), v), viii)$ and $ix)$ of (Øksendal 2007, Theorem 10.4.1) are easily verified. Because we have already demonstrated that (9) has a solution, condition $i)$ of the theorem is proved and condition $vii)$ is verified by construction of $\phi(S)$ (see Equation (4) and the related discussion). This leaves only conditions $ii)$ and $vi)$ to be verified. In the case of $ii)$, it must be proven that $\phi(S) \geq \max\{R_1 S - (C_1 - \gamma_1 + \alpha_1 C_0), -C_0\}$ for all $S \geq 0$. This holds because the definition of $\psi(S)$ and system (A-4)-(A-7) imply that $\phi(S)$ is convex. As for $vi)$, we must show that

$$-r\phi(S) + \mu S\phi'(S) + \frac{1}{2}\sigma^2 S^2 \phi''(S) \leq M, \quad \text{in } D^c.$$

For $S \leq \mathcal{S}_1^a$, since $\phi(S)$ is equal to $-C_0$ the inequality is equivalent to $C_0 \leq M/r$, which also holds true. For $S \geq \mathcal{S}_1^b$, $\phi(S)$ is equal to $R_1 S - (C_1 - \gamma_1 + \alpha_1 C_0)$ and the inequality is equivalent to

$$\frac{r(C_1 - \gamma_1 + \alpha_1 C_0) - M}{(r-\mu)R_1} \leq S.$$

Given that $S \geq \mathcal{S}_1^b$ and that the numerator on the left-hand side of the above inequality is negative, the inequality is true (note that $r > \mu$). This proves the result. ■

Proof of Proposition 4. Let I_j^i be the jth piece defining $\mathcal{V}_i$. By (8) we know that $\mathcal{V}_i$ always has two invariable pieces, $I_1^i = (-\infty, \mathcal{S}_i^a]$ and $I_2^i = (\mathcal{S}_i^a, \mathcal{S}_i^b)$. The third and last piece $I_3^i = [\mathcal{S}_i^b, \infty)$ divides depending on the quantity m_{i-1} of pieces in $\mathcal{V}_{i-1}$. Because we only need to know $\mathcal{V}_{i-1}$ for the values of S such that $\eta_i S > \eta_i \mathcal{S}_i^b$, and $\eta_i \mathcal{S}_i^b \in (a^*, b^*) \triangleq I_{l_{i-1}}^{i-1}$, then for $S > \mathcal{S}_i^b$, $\mathcal{V}_i(S)$ will have a different definition for each interval: $(\mathcal{S}_i^b, b^*/\eta_i), I_{l_{i-1}+1}^{i-1}/\eta_i, \ldots, I_{m_{i-1}}^{i-1}/\eta_i$. Therefore,

$$m_i = 2 + (m_{i-1} - l_{i-1} + 1). \tag{A-9}$$

If m_{i-1} equals l_{i-1} then m_i is at least 3, and by summing both sides of (A-9) from 2 to i and setting l_{i-1} equal to 1 we get that m_i is at most $2i+1$. ■

Proof of Corollary 1. When i equals 1, only one system needs to be solved and therefore only one iteration is required. We now show by induction that the result is also true for $i \geq 2$. Let us define ν_j, where $j \in \{1, \ldots, i-1\}$, as the number of iterations executed by the algorithm to calculate Y^j, not counting the iterations used to calculate Y^k for $k \in \{1, \ldots, j-1\}$. Also, let m^j be the number of rows in matrix Y^j. Calculating Y^{i-1} will involve a total of $\sum_{j=1}^{i-1} \nu_i$ iterations, while calculating Y^i may entail from 1 to m^{i-1} iterations, then it follows that the number of iterations for the algorithm to finish in the worst case is given by $\sum_{j=1}^{i-1} \nu_j + m^{i-1}$. For example, in the case illustrated by the dotted line in Figure 5, Y^3 is calculated in $\nu_1 = 1, \nu_2 = 1$ iterations; ν_3 can therefore take values between 1 and 5, the worst case thus being $\nu_1 + \nu_2 + m^2 = 7$.

Continuing now with the induction on i, note that by Proposition 4 we have

$$m^{i-1} = 2 + (m^{i-2} - \nu_{i-1} + 1). \tag{A-10}$$

Assume that the result is true for $i-1$. Then,

$$\begin{aligned}
\sum_{j=1}^{i-1} \nu_j + m^{i-1} &= \sum_{j=1}^{i-2} \nu_j + \nu_{i-1} + m^{i-1} \\
&\leq (3(i-1) - 2 - m^{i-2}) + \nu_{i-1} + m^{i-1} \\
&= 3i - 2 + \big(m^{i-1} - (2 + (m^{i-2} - \nu_{i-1} + 1))\big) \\
&= 3i - 2,
\end{aligned}$$

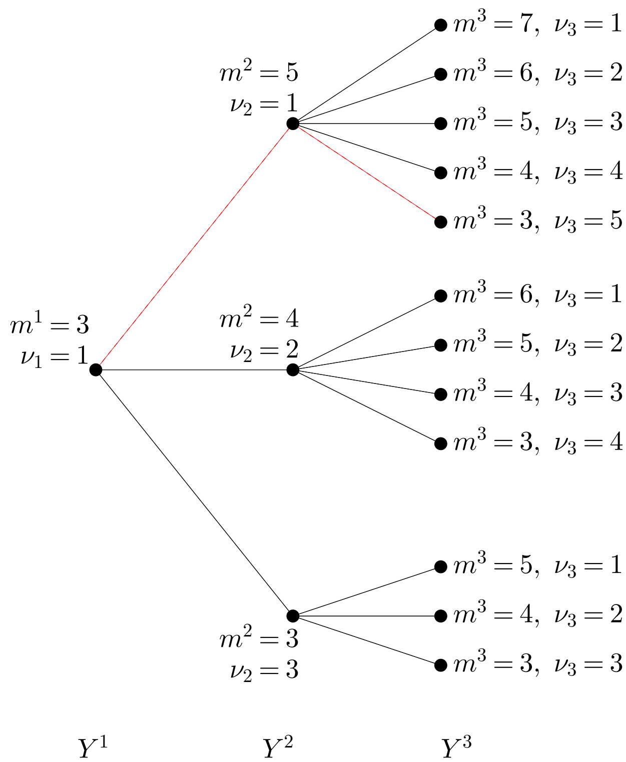

Figure 5 Possible number of rows and iterations for calculating Y^3.

Were the inequality follows from the induction hypothesis and the last equality follows from equation (A-10). Since the ν_j were chosen arbitrarily, the result is proved. ■

Proof of Lemma 1. Let τ be an $\mathcal{F}_t$ - stopping time. Then $V_i(S_{\tau+T_{i+1}})$ is equal to

$$\sup_{\xi\geq 0}\mathbb{E}\Big[e^{-r\xi}\max\big\{W_i(S_\xi)+e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\xi+T_i})|\mathcal{F}_\xi],-C_0\big\}-\int_0^\xi e^{-rt}Mdt|S_0=S_{\tau+T_{i+1}}\Big]. \quad \text{(A-11)}$$

Using the change of variable $\xi=\lambda-T_{i+1}$, the fact that Brownian motion is a Lévy process and recalling that S_t is equal to $S_0\exp((\mu-\frac{\sigma^2}{2})t+\sigma B_t)$, it is the case that (A-11) equals

$$\sup_{\lambda\geq T_{i+1}}\mathbb{E}\Big[e^{-r(\lambda-T_{i+1})}\max\big\{W_i(S_\lambda)+e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\lambda+T_i})|\mathcal{F}_\lambda],-C_0\big\}-\int_0^{\lambda-T_{i+1}}e^{-rt}Mdt|S_0=S_\tau\Big].$$

Rearranging this last expression yields

$$\sup_{\lambda\geq T_{i+1}}\mathbb{E}\Big[e^{-r(\lambda-T_{i+1})}\max\big\{W_i(S_\lambda)+e^{-rT_i}\mathbb{E}[V_{i-1}(S_{\lambda+T_i})|\mathcal{F}_\lambda],-C_0\big\}-\int_0^{\lambda}e^{-r(u-T_{i+1})}Mdu|S_0=S_\tau\Big]$$
$$+\int_0^{T_{i+1}}e^{-r(u-T_{i+1})}Mdu,$$

which is less than or equal to

$$e^{rT_{i+1}} \sup_{\lambda>0} \mathbb{E}\Big[e^{-r\lambda} \max\big\{W_i(S_\lambda) + e^{-rT_i} V_{i-1}(S_{\lambda+T_i}), -C_0\big\} - \int_0^\lambda e^{-ru} M du | S_0 = S_\tau\Big] + e^{rT_{i+1}} \int_0^{T_{i+1}} e^{-ru} M du.$$

We therefore conclude that

$$V_i(S_{\tau+T_{i+1}}) \leq e^{rT_{i+1}} V_i(S_\tau) + e^{rT_{i+1}} \int_0^{T_{i+1}} e^{-ru} M du,$$

as required. ■