Local Cuts for Mixed Integer Programming

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August 11, 2006

Outline

Introduction

- MIP modeling paradigm allows to model a wide range of problems.
- Advances in the last 20 years allow MIP to be used in practice (Bixby [Bix02]). Speed-up of up to six order of magnitude.
- Cutting plane techniques (Dantzig, Fulkerson and Johnson [DFJ54]) an important part of these developments.
- Many of the cutting planes exploit some specific structure of different problems.
- Relatively few cutting planes for general MIP's (MIR inequalities, Gomory cuts, cover inequalities, disjunctive cuts, lift and project, etc)
- Facet description of MIP is a hard problem.

Facet Description of MIP's

- Christof and Reinelt [CR01] computed all facets for several small MIP problems:
  - STSP on 10 cities, has 181,440 vertices, at least 51 billion facets and at least 15,379 classes of inequalities.
  - Linear Ordering on 8 elements has 40,320 vertices, at least 488 million facets and at least 12,231 classes of inequalities.
  - Cut polytope on graphs with 9 nodes has 256 vertices, at least 12 thousand billion facets and at least 164,506 classes of inequalities.
  - Some 0-1 polytopes with dimension 13 with only 254 vertices have as many as 17 million facets.
- Bushta et al [BMT85] showed that if we randomly chose $N = 2^{\alpha n}$ points on a sphere in $\mathbb{R}^n$, the expected number of facets bounded above and bellow by $\left(\frac{cn}{\log(n)}\right)^{n/2}$.

Local Cuts in the TSP

- Introduced by Applegate et al [ABCC03] as a technique to find *general* cuts for the TSP.
- Idea is to exploit their ability to solve small TSP problems very fast.
- Project the TSP into a small space.
- Separate in the small space.
- Get back a cut for the original problem.
- In theory the procedure can get any facet of the small space, not restricted to particular templates of cuts.

The results for the TSP

- Overall running time reduction.
  - 26% on medium instances.
  - 50% on large instances.
- Increase problems (TSPLIB) solved at root node from 42 to 66.
- Solve the (then) two largest TSP instances from MIPLIB, with 13,509 and 15,112 cities respectively.

What should be our target?

- Rely on *small* and *easy* problems.
- Be able to separate using optimization.
- Be able to obtain facets/high dimensional faces.
- Embed part of the original problem into these *easy* problems.
- Obtain guarantees about separability.

The Optimization Oracle

- Used to hide the (small) problem.
- Idea is that optimize small problem is fast.
- The description (facets) may be unknown, too long, or hard to store.

Definition (Optimization Oracle)

Given a cost vector c, the oracle return an optimal solution if one exists, or a *maximal* unbounded ray, or ensure that the problem is infeasible.

Separating through Optimization Oracles

An LP formulation

$$
\begin{array}{lrcll}
\max & & s & & \\
s.t. & sx^* - \sum_{i\in I_c} \lambda^c_i v_i - \sum_{i\in I_r} \lambda^r_i r_i + w & = 0 & (a) \\
 & -s + e^t \lambda^c & = 0 & (b) \\
 & \lambda^c, \lambda^r \geq 0, \quad -e \leq w \leq e. & &
\end{array}
\tag{1}
$$

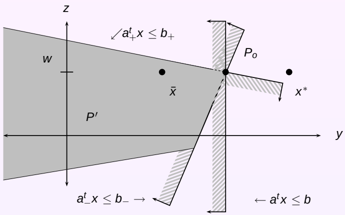
- x^* point that we want to separate from P.
- $\{v_i\}$ set containing all extreme points of P, $\{r_i\}$ set containing all extreme rays P.
- Can be solved by column generation.
- Always feasible, if unbounded, then $x^* \in P$.
- optimal dual solution give us a separating inequality.

The Separation Algorithm

1. Start with some set (may be empty) of points/rays.
2. Solve equation (1).
3. If unbounded, $x^* \in P$, stop.
4. let a, b be dual solution.
5. Maximize in P with objective a.
6. if $Z_{ip} > b$ add new point/ray, otherwise output $ax \leq b$.
7. go to step 2.

- If separation feasible, produces a face of P.
- Not necessarily a facet.
- Procedure is finite (is polynomial?).

From Faces to Facets



- Use tilting algorithm (Access P only through oracle).
- Tilting algorithm is correct and finite (no extra hypothesis required).
- At every iteration increase dimension of set of tight points or reduce dimension of overall space (implicit equalities).
- Produce faces of dimension $\geq k$ or facets.

The facet procedure, a graphical example

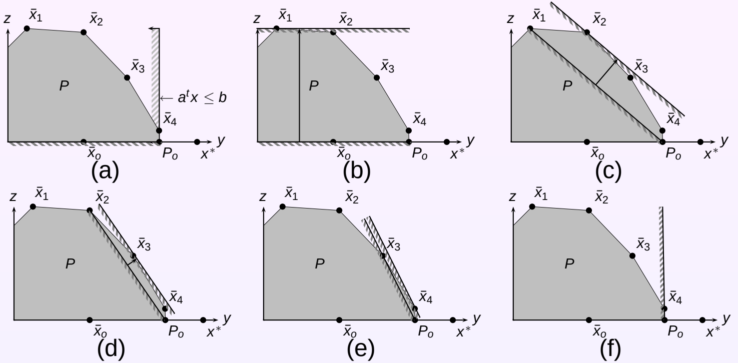


Figure: Example of a tilting operation.

The Mapping Problem

- We need simple problems.
- MIP with fixed number of integer variables is polynomial.
- Automatic transformation of generic problem to some known problem.
- Ability to translate cuts in the small problem to the original problem.
- Are conditions to ensure cutting? (like Gomory).

Definition (Valid Mapping)

We say that $P'_{ip} \subset \mathbb{R}^{n'}$ is a *valid mapping* for $P_{ip} \subset \mathbb{R}^{n}$, if there exists a function $\pi : \mathbb{R}^n \to \mathbb{R}^{n'}$ such that $\pi(P_{ip}) \subseteq P'_{ip}$. The function π is called the mapping function.

Linear (Pointed) Mappings

- Are linear affine transformations π.
- If P is the feasible set and P' our *small* problem, then $\pi(P) \subseteq (P')$.
- If $ax' \leq b$ is valid cut in P' then $a\pi(x) \leq b$ is valid cut in P.
- Implies $\pi(\mathrm{Conv\_hull}(P)) \subseteq \mathrm{Conv\_hull}(P')$.
- Can not assume that we know P, nor P'.
- Necessary to ensure $\pi(P_{lp}) \subseteq P'_{lp}$.

Definition (Pointed Mappings)

A valid mapping is pointed if and only if $\pi(P_{lp}) \subseteq P'_{lp}$, $\pi(x^*)$ is an extreme point of P'_{lp} and is fractional.

Sufficient Conditions

Theorem

(Sufficient Conditions) All Pointed Mappings produce a separating cutting plane for P_{ip}

- Note that the definition of a pointed mapping depend on the linear descriptions of P_{lp} and P'_{lp}, and also on selecting some extreme point in P'_{lp}.
- Are there pointed mappings?
- Are there conditions on the mapped spaces P'_{lp}?

Sufficient Conditions

Enough to consider :

- $P_{lp}: A_B x_B + A_N x_N = b,\ x \geq 0,\ x_B^* = A_B^{-1} b$ fractional.
- $\pi(x) = \begin{pmatrix} M_{BB'} & 0 \\ M_{BN'} & M_{NN'} + M_{BN'} A_B^{-1} A_N \end{pmatrix} \begin{pmatrix} x_B \\ x_N \end{pmatrix}$.
- $P'_{lp}: A'_{B'} y_{B'} + A'_{N'} y_{N'} = b',\ y \geq 0,\ y_B^* = M_{BB'} x_B^*$ fractional.

Then, π, P'_{lp} is a pointed mapping if the following holds:

- $M_{BB'} \geq 0$, $M_{NN'} \geq 0$ and $M_{BN'} x_B^* = 0$.
- $A'_{N'} M_{NN'} = A'_{B'} M_{BB'} A_B^{-1} A_N$.
- Some coefficient of $y_{B'}^*$ fractional.
- $\forall v \in \mathbb{R}^{m'} \exists q \in \mathbb{R}_+^{n'-m'}$ such that $A'_{N'} q = v$.

Final Thoughts on Local Cuts

- Computational experiments show that the technique may be applicable in practice.
- How should we design P' and π?
- Are there natural relaxations for MIP's?
- Can we manage the precision issues in practice?
- What are the interesting spaces P'_{ip} that we should look?
- Should we care about ensuring separation?

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