A Continuous-Time Model of Career Concerns and Human Capital Accumulation*

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Abstract

We develop a continuous-time model of career concerns that incorporates human capital accumulation throughout the working life. Workers are able to generate an output that follows a diffusion process with a drift that is the sum of effort and ability (“talent”). Talent corresponds to a hidden process that mean-reverts toward a trend that is endogenously changing over time as a consequence of on-the-job skills accumulation. We find that estimates of talent discount past output observations at excessively high rates, leading to broad inefficiencies in the standard setting of career concerns with exogenous talent specifications. Nevertheless, it is precisely this high-discounting which makes human capital accumulation an additional channel for extracting gains from belief-distortion. This in turn implies that effort levels are typically above the ones predicted by traditional models of career concerns, yet inefficiencies are far from being eliminated.

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1 Introduction

The importance of jobs as environments in which workers further develop their skills beyond the ones given by education has become key in explaining the dynamics of human capital accumulation and the properties of wages over the life-cycle (see for example Mincer (1974)). From this perspective, workers continue acquiring skills throughout their working life, and differences in experience accumulation have been established as a significant determinant of the heterogeneity observed in wages. Another important aspect of jobs is that they also provide workers with platforms for signaling to their employers how skilled or talented they are at their particular field of expertise. This dual role that jobs play is therefore crucial for shaping workers’ incentives to accumulate human capital and to generate good performance observations. This is particularly important in the early stages of the working life, a period in which both investing in human capital and in reputation formation generate the highest returns.

We develop a model of career concerns that incorporates human capital accumulation resulting from workers’ increased experience in their jobs. The main objective is to study the trade-offs that workers face in the presence of rich dynamics of skills and human capital throughout their career, and to determine how the workers’ incentives are constrained or limited by these forces. The outcome is a very flexible framework that provides strikingly clean results on how belief-distortion mechanisms operate and on how wages and effort levels evolve over time.

More specifically, in our model workers become more skilled as a consequence of repeated exposure to the tasks that they perform. The motivation for experience accumulation as a by-product of final-goods production is the following: sometimes the task itself provides the worker with the necessary training to improve performance, without the need to divert resources away from production in order to accomplish that goal. For example, PhD students, post-docs or assistant professors become more skilled at their fields of expertise as a consequence of permanent attempts to tackle similar problems or because of constant deep thinking about ideas. Young traders learn about the profitability of their trading strategies based on years of exposure to how markets react to different economic events or news. CEOs gain expertise on how to define their companies’ targets more effi-
ciently by learning from predecessors’ past performance and from their own experience in the position. Furthermore, even though years of work experience is a proxy for the human capital benefits obtained in any working environment, it is rather more sensible to think of how actively involved the worker was with his task as a more direct source of experience accumulation. Controlling for years of work experience, we would expect PhD students, traders, CEOs, etc. to exhibit higher productivity improvements the harder they work, the more frequently they face different challenges that demand creative solutions, and the more they try to understand what defines their tasks and what problems they may encounter.

In order to incorporate the above ideas in a unique framework, we build on Holmstrom’s (1982, 1999) paper to construct a continuous-time model of career concerns that includes a wider class of stochastic processes for skills and allows for human capital accumulation. In particular, we assume that talent mean-reverts towards a trend that, at any point in time, is endogenously determined by the worker’s experience up to that instant, where the latter is defined as the current history of effort choices. We interpret this mean-trend as the worker’s human capital stock, which is continuously changing over time depending on the degree of commitment that the worker has had towards production. In this sense, workers suffer productivity shocks that push current skills away from the amount of human capital so far acquired in life, but these deviations are not likely to be very persistent and talent tends to move back to its mean-trend. This specification tries to capture the idea that, even though workers suffer daily productivity shocks that may affect short-run performance, what really defines long-run productivity is the set of skills acquired and developed throughout life.

A standard insight from the career concerns literature is that, even when no output-based contracts can be written, incentives to exert effort can still be created if there are reputational concerns: whenever wages depend on estimates of talent, workers can use effort to generate good output observations and thus signal to their employers that that they are talented (see for example, Holmstrom (1982, 1999) and Dewatripont, Jewitt and Tirole (1999)). Starting from this premise, the purpose of this paper is two-fold. First, we attempt to give a clean and general treatment of the learning structure in the model: we want to understand in detail how beliefs evolve over time and how belief-distortion
operates in favor of the worker. In this aspect, a continuous-time formulation allow us to obtain simple insights on how short and long-run incentives are determined in a framework with more complex talent dynamics. Second, we want to study if experience accumulation constitutes an additional channel of belief-distortion on top of the one provided by the traditional career concerns model without human capital improvements. Given that jobs are environments in which individuals can improve their skills, we ask whether or not workers can take advantage of this and obtain further gains from belief-distortion even when human capital does not affect output directly.

All the results in this paper can be understood in a stationary-learning setting, which we assume for the moment in order to explain the main intuitions. As a benchmark, we study a career concerns model with mean-reverting talent but without human capital accumulation. We show that incentives are determined by two forces: the sensitivity of the principal’s beliefs to new information (surprises in output) and the rate at which the principal’s beliefs decay over time. Beliefs instantaneously react to the new information conveyed by changes in output, and the sensitivity parameter is what measures the intensity of these reactions. Yet, these distortions happen to decay over time precisely because of the “brownian” nature of the productivity shocks: past information becomes less of a good statistic for current talent when compared to most recent observations. As a consequence, when constructing estimates of talent, less weight is assigned to output observations back in the past. The rate at which the principal discounts past information is therefore what defines the persistence of belief-distortion over time and, in conjunction with the payoffs’ discount rate, it determines the long run benefits of belief-distortion.

The previous findings also hold when learning is non-stationary (both the sensitivity of beliefs and the rate at which the principal discounts past information will now evolve over time) and when talent is either mean-reverting or a standard Wiener process (random walk equivalent). In any of these talent specifications, distorting the principal’s beliefs comes at a expense: the principal will discount past information at a excessively high rates, reducing the long run benefits from belief-distortion. In fact, since the principal cannot distinguish between changes in output consequence of the signal’s noise or due to skills improvement, it will construct estimates of talent using the history of observations generated by the output component not explained by effort, which is a process
with non-zero expected increments. Therefore, anticipating the mentioned persistence, the market’s beliefs will indeed react to output changes that are the consequence of effort deviations, but these distortions will decay relatively fast over time. In the particular case of stationary-learning, this high-discounting generates effort levels that are uniformly bounded away from efficiency for all discount rates. This shows that Holmstrom’s (1982, 1999) asymptotic efficiency result in the random walk case is also particularly sensitive to the stochastic nature of talent: adding a little bit of mean-reversion eliminates any chance for efficiency.

We then add human capital accumulation to the model and we show that, for an extremely broad class of human capital accumulation technologies, the existence of a deterministic equilibrium is reduced to the existence of a solution to a particularly simple optimization problem that the worker must solve. In this problem, incentives arise in a remarkably clear way: the worker’s flow benefit at any point in time can be decomposed into the traditional career concerns component plus the monetary benefits that an additional unit of human capital stock at that particular instant generates. That is, even though human capital by itself does not affect output directly, it indeed constitutes an additional channel for belief-distortion. To understand why this result holds, observe that when the human capital stock increases the agent will become, on average, a more talented worker in the future. As a consequence, a marginal change in the worker’s human capital stock generates an additional output stream due to skills improvement. In fact, the effect that this human capital improvement has on future talent decays at the rate of mean-reversion. Nonetheless, because the principal’s beliefs decay at a rate faster than the one just mentioned, the principal will constantly underestimate the manager’s true talent over time. This implies that the principal will expect an additional output stream that is below the one actually realized and will acknowledge the difference between both flows as talent improvement. This unexplained output flow is therefore internalized by the worker in the form of an increase in wages. The most interesting finding is that, while a principal that heavily discounts past information introduces inefficiencies in the traditional career concerns setting, it allows the worker to obtain further gains of belief-distortion through the channel of human capital accumulation.

In order to fully characterize incentives in the presence of human capital accumulation,
we add more structure to the model. Under the assumption that the human capital dynamic is governed by an ordinary differential equation depending on current effort, we are able to treat the worker’s optimization problem as one of deterministic control and establish necessary conditions that any optimal deterministic strategy must satisfy. In particular, incentives are the sum of the standard career concerns component plus a new human capital term, where the latter is proportional to the shadow value of human capital. In fact, we are able to show that, under mild regularity conditions, the induced effort levels in this setting are permanently above the career concerns benchmark.

We conclude the paper by studying the special case when human capital is governed by a traditional physical-capital accumulation dynamic (effort plays the role of investment and human capital depreciates at a constant rate). Making full use of dynamic programming techniques we are able to show the existence of an optimal control and we provide analytic expressions for the optimal effort strategy and the for the worker’s value function. Regarding dynamics, this example predicts that both the human capital component and traditional career concerns component of effort should decrease over the life-cycle. It also predicts that individuals with lower human capital depreciation rates should invest more in education before entering the labor market.

Concerning the literature on career concerns, Holmstrom (1982, 1999) provided a formal framework for analyzing Fama’s (1980) conjecture that competitive markets are able to remove all moral hazard issues because workers use effort as a substitute for skills to signal to the market that they are highly talented. He analyzed two extreme (and exogenous) talent specifications: when skills are either fixed over time or when they evolve as a random walk. He finds that, because estimates of talent become infinitely precise in the former case, the optimal effort level decreases to zero. In contrast, when talent evolves as a random walk, the environment creates enough uncertainty for effort exertion to be profitable. In the special case of stationary learning, he shows that effort levels converge to efficiency when the discount factor tends to one. Nevertheless, inefficiencies are observed along the convergence path. Dewatripont, Jewitt and Tirole (1999) analyze the effects of additional tasks on incentives in a static setting, in the absence of market forces and under the existence of complementarities between effort and talent. In the particular case of additive technology, they find that the equilibrium level of effort decreases monotonically
to zero as the number of tasks is increased. Hence, they stress the importance that reducing the workers’ number of tasks has on incentives.

Some papers have analyzed the incentives that arise when dynamic contracts have a learning component. In the setting of Harris and Holmström (1982), firms and workers are imperfectly informed about workers’ abilities, the latter are risk averse and there is no moral hazard. Markets are competitive but long-term contracts can be signed, so firms can earn positive profits in the short run by paying wages below the estimated value of talent. They show that wages are downward rigid and that workers pay a risk premium in order to insure themselves from future low-outcome realizations. This premium decreases with the precision of talent’s estimate, so senior workers have higher salaries on average. Gibbons and Murphy (1992) develop a model to analyze the effects of short-term linear contracting on incentives, in the context of career concerns, moral hazard, competitive markets and risk aversion in the form of a exponential utility function of aggregate monetary benefits. They show that the optimal compensation contract takes into account explicit and implicit incentives and, since for young (old) workers career concerns are more (less) relevant, wages have a lower (higher) sensitivity to performance (measured by the slope of the contract). They also present empirical support for their theoretical findings using data on CEO compensation and stock market performance.

This paper is related to the literature of human capital accumulation through work experience. According to Rosen (1972), workers gain experience in their jobs, which allows them to improve their skills and thus to increase their productivity. In this line, wage differentials also reflect the various learning opportunities that different jobs offer and workers are willing to forgo part of their current skills’ market value in order to have access to the learning opportunities that jobs provide. In a similar vein, Jovanovic (1979) analyzes a matching model without learning, but including on-the-job search and training, where the latter provides the worker with firm-specific human capital. Among other things, he finds that the quality of the match and investing in human capital are complements: the higher the investment in human capital, the less likely the match will be dissolved and the lower the separation probability, the larger the investment will be. These two models assume that investing in human capital is costly in the sense that it diverts resources from final-good production. Our motivation is different: we do not want
to model the choice of a job seen as an investment opportunity in human capital, but to study how reputational incentives arise when this choice has already been made and human capital accumulation arises the a consequence of becoming more skilled at the task performed.

Finally, this paper is to some extent related to continuous-time techniques for addressing dynamic incentives problems. In particular, Sannikov (2008) developed a continuous-time framework to analyze a principal-agent interaction from a dynamic programming perspective. Using the agent’s continuation-value as a state variable, he is able to provide strikingly clean characterizations of incentives and of the optimal contract. In particular, the principal’s value function is simply characterized by an ordinary differential equation. DeMarzo and Sannikov (2008) is probably the most complete treatment involving learning in contracts since they also add intermediate consumption in a tractable way. In their paper, a principal and a manager learn about firm’s profitability through the observation of cash-flows and the manager can secretly divert effort and firm’s resources to his own private benefit at any point in time. They show that the optimal contract can be implemented by giving the agent a fraction of the firm’s equity and an optimal payout policy dependent on the level of firm’s cash reserves is determined.

In the next section we present the general model. In Section 3 we analyze as a benchmark the continuous-time career concerns model in the absence of human capital accumulation. The main intuitions regarding incentives are derived there. Section 4 adds human capital accumulation, Section 5 discusses some variations of the model and in Section 6 we conclude. All the proofs are relegated to the Appendix.

2 The Model

We build on Holmstrom’s model of career concerns. Consider a manager (worker or agent) that, upon working in a firm, is able to produce an output \( \xi := (\xi_t)_{t \geq 0} \) continuously over time. This performance measure is a public signal in the economy. More specifically, it obeys the following dynamic

\[
d\xi_t = (a_t + \theta_t) dt + \sigma \xi_t dZ_t^\xi, \quad t \in \mathbb{R}_+.
\]
Here, $Z^{\xi} := (Z^{\xi})_{t \geq 0}$ corresponds to a one-dimensional Brownian motion, $\sigma^{\xi} > 0$ is the signal’s volatility and $a_t$ is the manager’s effort choice at time $t$, which is subject to moral hazard. The variable $\theta_t$ represents the manager’s skills or talent level at time $t$. We assume that the process $\theta := (\theta_t)_{t \geq 0}$ is unobservable to all the agents in the economy and satisfies the stochastic differential equation (SDE)

$$d\theta_t = -\kappa(\theta_t - \bar{\theta}_t(a))dt + \sigma_{\theta}dZ^\theta_t, \ t \in \mathbb{R}_+,$$

where $Z^\theta := (Z^\theta)_{t \geq 0}$ is a one-dimensional Brownian motion independent of $Z^{\xi}$, $\kappa \geq 0$ is the mean-reversion coefficient and $\sigma_{\theta} > 0$ is the volatility of the Brownian shocks.

At any time $t$, we interpret the effort history $(a_s : 0 \leq s \leq t)$ as the manager’s experience at that instant, since it corresponds to a measure of how engaged in production the worker was up to that point. For example, we would expect workers that have spent more hours in production to develop a deeper understanding of their task and hence to be more productive. Yet different workers acquire skills in different ways and, therefore, the mapping between experience and human capital may vary across individuals. More specifically, we assume that this worker enters the market with a human capital stock of $\theta \in \mathbb{R}_+$, which may vary across workers in order to capture, for example, different schooling experiences. Then, for any time $t > 0$ there is a functional $\theta_t$ that takes the manager’s experience up to time $t$, $(a_s : 0 \leq s \leq t)$, and delivers a real number $\bar{\theta}_t(a)$, which we interpret as the manager’s human capital stock at time $t$.

The specification (2) has a very appealing interpretation. Consider first when there is no experience accumulation, i.e. $\bar{\theta}_t(\cdot) \equiv \bar{\theta} \in \mathbb{R}_+$ for all $t \geq 0$, case in which (2) becomes a standard mean-reverting process around $\bar{\theta}$. We can interpret the latter as an exogenously given human capital level: for instance, a long-tenured manager that has reached a fixed level of expertise. What we try to capture with this specification is a worker who, in the absence of productivity shocks, would be able to produce final goods by selling to a principal his labor and human capital. Nevertheless, since the presence of randomness affects the worker’s current skills, it is actually talent (human capital plus the history of productivity shocks) what, along with effort, determines production. While these shocks push talent away from the mean-trend $\bar{\theta}$, these deviations are not likely to last for long because of the mean-reverting force that constantly drives talent back to its long-run
average level. Finally, the presence of human capital accumulation implies that skills are now evolving around a mean-trend that is *endogenously* changing over time, reflecting the fact that workers do acquire skills from their working experience.

A model like the one above is quite general to fit several plausible ways in which a worker can experience human capital accumulation. We pay special attention to the case in which the stock of human capital evolves in a *continuous* way over time. In other words, we model human capital accumulation as a *slow* phenomenon (with potentially long-term effects) in which instantaneous jumps are not possible, but where discontinuities in its rate of change are permitted.\(^1\) From this perspective, given any strategy \( a := (a_t)_{t \geq 0} \), we model \((\bar{\theta}_t(a))_{t \geq 0}\) as the solution an ordinary differential equation (ODE) of the form

\[
d\bar{\theta}_t = f(\bar{\theta}_t, a_t)dt, \quad \bar{\theta}_0 = \bar{\theta} \geq 0.
\]

At any particular point in time \( t \), this solution will typically depend on some aggregate measure of the path \((a_s : 0 \leq s \leq t), \ t \geq 0\). An example of this sort is the standard “physical-capital accumulation” dynamic \( f(\bar{\theta}, a) = \alpha a - \lambda \bar{\theta} \), where \( \lambda \in (0, 1) \) is the depreciation rate and \( \alpha \) is the marginal benefit of an extra unit of effort on the speed at which human capital is accumulated. In such a specification, workers that learn fast from their work environment would exhibit higher values of \( \alpha \), and workers that are able to retain for longer periods what they have already learned would present low values of \( \lambda \).

The literature of career concerns has modeled talent in two different exogenous ways: when it corresponds to a random walk or when it is fixed over time. We are able to recover them as particular cases of our mean-reverting model in the absence of human capital accumulation: with an exogenously fixed mean-trend, the random walk specification is obtained when \( \kappa = 0 \) and the fixed-talent model when \( \kappa = \sigma_\theta = 0 \). We believe that both models, though interesting in their own, are probably too extreme to capture real-life features of how workers’ skills evolve over time. On the one hand, it is particularly unrealistic to model skills as exogenously fixed, not allowing for instance, the possibility that workers may suffer productivity shocks that influence short-run performance. But,

\(^1\)The model can be adjusted to the case in which the manager can directly affect the drift of the talent process (2). Over time, this may create discontinuities in the talent’s mean trend and the human capital stock interpretation of it is less appealing. However, a model like that still has a human capital accumulation interpretation, but now with less persistent effects on talent. See Section 5.
on the other hand, the random walk specification can generate extremely erratic path-realizations of talent with high probability. What we want is a model lying in between these two extremes, that is, allowing for short-term fluctuations, but in such a way that talent is always close a specific target representing what the worker has learned throughout experience. The nature of this target will depend of course on whether the working environment provides the worker with general skills that can be accumulated over time or not. We believe that the mean-reverting model with an exogenous or endogenous trend meets these requirements. As an illustration, the next figure shows how all the mentioned models behave under a particular realization of productivity shocks:

![Figure 1: Talent Models](image)

In the figure, the lowest path corresponds to a particular realization of a standard Wiener process of volatility $\sigma_\theta = 0.4$. In such a specification, the worker receives strong positive shocks in the beginning of his career, but then talent tends to decrease over time. When we add mean-reversion around zero ($\kappa = 3$), we observe that the same sequence of shocks generates a less erratic path that is permanently fluctuating around zero. Finally, if human capital is accumulated through a dynamic governed by $f(\bar{\theta}, a) = a - \lambda \bar{\theta}$, $\lambda = 0.3$, and the worker exerts one unit of effort all the time, we obtain the highest path observed.
in the graph. In this sense, long-run productivity performances are driven only by human
capital accumulation, while the Brownian shocks influence short-run productivity.

We now move on to the assumptions on preferences and market structure. The man-
ger is assumed to be risk neutral and his utility function is separable in consumption
and effort. Effort takes values on a set $A = [0, \ell]$ and the disutility of it is summarized in
a function $g : A \rightarrow \mathbb{R}_+$, which we assume to be increasing, strictly convex, and satisfying
$g(0) = 0$. As a consequence, if at time $t$ the manager is paid $w_t$ and exerts effort $a_t$, he
will get a utility flow of $w_t - g(a_t)$, $t \geq 0$.

We assume that firms are risk neutral and no output-based contracts can be written,
which allows us to isolate the career concerns motives. Also, the market for managers
is perfectly competitive, so firms earn zero expected discounted profits from production.
Moreover, we impose that they cannot commit to long-term contracting and hence firms
earn zero profits at any point in time.\footnote{For a model with competition but allowing for long-term contracting, see Harris and Holmstrom (1982), which lacks of moral hazard but allows for risk aversion on the manager’s side. In their model, learning helps to explain why senior workers are, on average, paid more: since they have more precise estimates of their own ability, they don’t need to insure themselves against low-output realization as much as young workers.} Since in this model it is the market that ultimately
sets wages, it corresponds to the “principal” in this market-agent interaction.

Because the market cannot observe the manager’s talent, it will create estimates of it
based on the public signal $\xi$. The zero-profit condition at every point in time implies that,
if the market expects the manager to follow the effort strategy $a^* := (a^*_t)_{t \geq 0}$, it will pay
him an amount $w_t$ over the interval $[t, t + dt)$, corresponding to the expected increment
in production during that period:

$$w_t := \mathbb{E}^{a^*}[d\xi_t | \mathcal{F}_t^\xi]/dt = \mathbb{E}^{a^*}[\theta_t | \mathcal{F}_t^\xi] + a^*_t,$$

where $\mathcal{F}_t^\xi$ denotes the public information at time $t$ (the information generated by the
family of random variables $(\xi_s : 0 \leq s \leq t)$), and $\mathbb{E}^{a^*}[\cdot | \mathcal{F}_t^\xi]$ is the market’s conditional
expectation operator under the assumption that the manager is following the strategy
$a^*$.\footnote{Different effort strategies generate different probability measures on the set of paths of $\xi$. Therefore, if the market conjectures that the manager is following $a^*$ when he has actually chosen the process $a$, their beliefs will differ. See footnote 5 for the formal connection between them.} Equivalently, at any particular point in time the manager’s wage will correspond to
the expected drift of the output process, conditional on the information publicly available at that instant. This shows that wages have a reputational component and an effort component, both depending on what the market conjectures is the strategy that the worker will follow.

The manager’s effort strategy needs to satisfy some measurability requirements with respect to the public information \( (\mathcal{F}_t^\xi)_{t \geq 0} \). Nevertheless, if effort were to depend explicitly on the output history, the contract \((w_t)_{t \geq 0}\) given by (4) would explicitly depend on performance, violating the assumption that no output-based contracts can be written. This would in turn raise the issue of why not implementing, for example, compensation packages that induce any desired behavior. Since what we actually want is to isolate the career concerns incentives from the ones that may arise under explicit methods, we will the class of equilibria to effort strategies that do not depend on output.

**Definition 1.** (Manager’s strategy) A feasible strategy for the manager corresponds to any Borel-measurable function of time that takes values in \( A \). Let \( A \) be the set of feasible strategies for the manager.

Of course, this definition is up to additional integrability conditions such that the filtering equations we use exist. Also, additional restrictions over the set of feasible strategies will be assumed once we have imposed more structure on the model.

With this in hand, we can state the equilibrium concept:

**Definition 2.** (Equilibrium) An equilibrium of this economy is a feasible strategy for the manager \( a^* := (a^*_t)_{t \geq 0} \) and a wage process \( w := (w_t)_{t \geq 0} \), such that:

\[(i) \text{ Given } a^*, \text{ the market sets a wage of the form } w_t = \mathbb{E}^{a^*}[\theta_t | \mathcal{F}_t^\xi] + a^*_t \text{ for all } t \geq 0; \]

\[(ii) a^* \text{ is optimal for the manager given the wage process in (i): } \]

\[
a^* \in \arg \max_{a \in A} \mathbb{E}^a \left[ \int_0^\infty e^{-rt}(w_t - g(a_t))dt \right] \]

\[s.t. \quad w_t = \mathbb{E}^{a^*}[\theta_t | \mathcal{F}_t^\xi] + a^*_t, \forall t \geq 0 \tag{5}\]

Note that in this definition we emphasize the fact that, by choosing an effort strategy \( a \neq a^* \), the manager induces a distribution over outcome paths that differs from the one anticipated by the market (\(\mathbb{E}^a[\cdot] \) operator). This in turn will affect the market’s beliefs.
about how talented the manager is. For instance, if the manager deviates from $a^*$ at some instant by exerting more effort, this will generate, in expectation, higher output observations and the market will revise its expectations upwards. As a consequence, an increase in effort today will generate, on average, a boost in the reputational component of future wages. Finally, the last part of the definition states that, in equilibrium, beliefs must coincide.

These are the basic ingredients of the model. In the next section we analyze the “pure career concerns model”, that is, the one without human capital accumulation. We treat it as a benchmark for three reasons. First, we want to connect the simplest version of our model with Holmstrom’s paper to see which new insights arise under mean-reversion. Second, it is on our purpose to give a detailed analysis of how learning takes place in an environment as basic as possible. Finally, we want to illustrate the basic forces that map belief-distortion into incentives. A number of important basic intuitions derived in the next section will facilitate the understanding of the more elaborated model with human capital accumulation.

3 Pure Career Concerns Model

Consider a setting in which there is no human capital accumulation, so talent evolves around a level $\bar{\theta} \geq 0$. Thus, the partially observed system (1)-(2) takes the form

$$d\xi_t = (a_t + \theta_t)dt + \sigma_\xi dZ^\xi_t$$
$$d\theta_t = -\kappa(\theta_t - \bar{\theta})dt + \sigma_\theta dZ^\theta_t$$

and suppose that the initial belief is such that $\theta_0|\mathcal{F}_0 \sim \mathcal{N}(m_0, \gamma_0)$. Then under suitable integrability conditions on the set of effort strategies, the conditional distribution of $\theta_t$ given the information $\mathcal{F}_t^\xi$ is also a Gaussian process for both parties (see Lipster and Shiryaev [12]).

In fact, suppose that the market conjectures that the manager follows $a^* \in \mathcal{A}$ and denote by $m_t^* := \mathbb{E}^{a^*}[\theta_t|\mathcal{F}_t^\xi]$ and $\gamma_t := \mathbb{E}^{a^*}[\theta_t - m_t^*]^2|\mathcal{F}_t^\xi]$ the market’s posterior estimates of the talent’s mean and variance using the information up to $t$, respectively. They satisfy
the equations

\[ dm_t^* = -\kappa(m_t^* - \bar{\theta})dt + \frac{\gamma_t}{\sigma_\xi} dZ_t^{a^*} \]  

(6)

\[ \dot{\gamma}_t = -2\kappa\gamma_t + \sigma_\theta^2 - \left( \frac{\gamma_t}{\sigma_\xi} \right)^2, \]

(7)

where the innovation process

\[ Z_t^{a^*} := \frac{1}{\sigma_\xi} \left( \xi_t - \int_0^t (a_s^* + m_s^*) ds \right), \quad t \geq 0, \]

is a \( \mathcal{F}_t^\xi \)-Brownian motion from the market’s perspective. Two interesting features of (6) and (7) are worth noting. First, the evolution of the posterior mean preserves the stochastic structure in which talent evolves. In particular, the posterior mean follows a mean-reverting process and its response to unexpected output observations (captured by the innovation process) increases with the size of the mean-square error and decreases with the signal’s volatility (\( \sigma_\xi \)). This implies that beliefs react more strongly in settings where either less information has been accumulated, or where signals are more accurate. Second, the mean-square error evolves faster for extreme values of it, reflecting the fact that fast learning occurs when there is very imprecise information.

Suppose for the moment that learning is constant. That is, the posterior variance is at the stationary level \( \gamma^* \) given by the solution to the equation \( 0 = -2\kappa\gamma^* + \sigma_\theta^2 - (\gamma^*/\sigma_\xi)^2 \). It is easy to check that

\[ \gamma^* = \frac{\sigma_\xi^2}{\sigma_\xi^2} \left( \sqrt{\kappa^2 + \sigma_\theta^2/\sigma_\xi^2} - \kappa \right) \]  

(8)

and we assume that the market’s initial belief is exactly at this level of variance. Replacing \( \gamma_t \) with \( \gamma^* \) in (6) we get:

\[ dm_t^* = -\kappa(m_t^* - \bar{\theta})dt + \frac{\gamma^*}{\sigma_\xi} \left[ \frac{d\xi_t}{\sigma_\xi} - \frac{(a_t^* + m_t^*) dt}{\sigma_\xi} \right] dZ_t^{a^*} \]  

(9)

All the insights regarding incentives that this paper offers are condensed in slight variations of (9). Observe first that because talent is not observable, the market cannot distinguish between output changes coming from the signal’s noise and output changes that are the consequence of variations in talent across time. Therefore, once the market has
conjectured that the manager will follow a particular strategy, say \( a^* \), the only information that it has available to construct statistics about the manager’s talent is the component of output not explained by effort, \( \xi_t - \int_0^t a^*_s ds \). Furthermore, since the manager controls the true distribution of output through his effort decision, the moral hazard present in the model prevents the market from having the same information that the manager possesses about output’s distribution. In fact, once a deviation has occurred, \( Z^a^* \) is not a Brownian motion from the manager’s perspective anymore, so the correct way to express (9) is as

\[
d m^*_t = -[(\beta + \kappa) m^*_t - \kappa \bar{\theta}] dt + \beta [d \xi_t - a^*_t dt],
\]

where \( \beta := \gamma^*/\sigma^2_\xi \). In other words, this is the way in which the manager evaluates the effect that his effort choice has on the market’s beliefs through controlling the output distribution.\(^4\) Two important concepts are derived here. First, \( \beta \) represents the sensitivity of market’s beliefs to new information: it measures how beliefs instantaneously react to increments in the component of output not explained by effort. The second important point is that the manager takes into account not only the initial reaction of the market’s beliefs to output changes, but also how these distortions persist over time. It is straightforward to check that the solution to (10) has the form

\[
m^*_t = m_0 e^{-\delta t} + \frac{\kappa(1 - e^{-\delta t})}{\delta} \bar{\theta} + \beta \int_0^t e^{-\delta(t-s)} [d \xi_s - a^*_s ds]
\]

where \( \delta = \kappa + \beta \). As a consequence, \( \delta \) represents the rate at which beliefs decay over time, or equivalently, the rate at which the market discounts past information. A high \( \delta \) implies that the market rapidly discounts past performance and, therefore, beliefs’ distortions will not last for long. Moreover, because \( \xi_t - \int_0^t a^*_s ds \) has a non-zero drift, its

\(^4\)Technically speaking, we need to write the market’s beliefs under the manager’s probability measure defined over the set of outcome paths of \( \xi \), which is our reference measure throughout the paper. Denote it by \( \mathbb{P}^a \) under the assumption that the manager actually follows \( a \in \mathcal{A} \). Thanks to Girsanov theorem, the market measures events according to an equivalent probability measure \( \mathbb{P}^{a^*} \) completely characterized by the relative density process \( \eta_t := \mathbb{E}^a[\mathbb{P}^{a^*}/d\mathbb{P}^a | \mathcal{F}^\xi_t] \). This process satisfies the SDE

\[
d \eta_t = -\eta_t \frac{(a_s + m_s) - (a^*_s + m^*_s)}{\sigma_\xi} dZ^*_t,
\]

where \( Z^*_t \) is a Brownian motion under \( \mathbb{P}^a \) and

\[
Z^*_t := Z^*_0 + \int_0^t \frac{[a_s + m_s] - [a^*_s + m^*_s]}{\sigma_\xi} ds
\]

is a Brownian motion under \( \mathbb{P}^{a^*} \).
increments have some persistence over time and, hence, are not truly “surprises” from the market’s perspective. For this reason, even though the market will react to the information conveyed in this process, these distortions will have a relatively low persistence over time.

To see this, observe that true talent takes the form

\[ \theta_t = \theta_0 e^{-\kappa t} + (1 - e^{-\kappa t}) \bar{\theta} + \sigma_\theta \int_0^t e^{-\kappa(t-s)} dZ^\theta_s, \quad t \geq 0 \]

so the effect that a particular productivity shock has on future talent decays at rate \( \kappa \), which is less than \( \delta \). This is why we say that the market discounts past information at excessively high rates.

We allow the manager to have access to additional sources of information than the one provided by output. This is a reasonable assumption given the fact that talent is an inherent characteristic of the manager himself. We do impose that the stochastic structure of the additional signals must lie within the Gaussian filtering framework. Letting \((F_t)_{t \geq 0}\) be the manager’s information structure (containing, in particular, \((F^\xi_t)_{t \geq 0}\)), we assume that his beliefs evolve as

\[ dm_t = -\kappa (m_t - \bar{\theta}) dt + \sigma_t dZ_t \]  \tag{12}

where \( Z := (Z_t)_{t \geq 0} \) is a \( F_t \)-Brownian motion and \( (\sigma_t)_{t \geq 0} \) is a non-negative process.\(^5\)

Suppose that the manager follows a strategy \( a \in A \), not necessarily equal to \( a^* \). Then, as before, the process

\[ Z^a_t := \frac{1}{\sigma_\xi} \left( \xi_t - \int_0^t (m_t + a_t) dt \right), \quad t \geq 0 \]  \tag{13}

is a Brownian motion from his perspective and it is correlated with \( Z \). Therefore, from the manager’s standpoint output can be written as \( d\xi_t = (m_t + a_t) dt + \sigma_\xi dZ^a_t \) (Girsanov

\(^5\)A process like this can be generated as follows: suppose the manager also observes \( d \) signals of the form \( d\xi^i = \theta_i dt + B^i_t dZ^i_t + C^i_t dZ^\theta_t \) where \( \{Z^i, Z : i = 1, \ldots, d\} \) is a family of independent one dimensional Brownian motions. Then, if the coefficients \{(\(B^i_t, C^i_t)_{t \geq 0} : i = 1, \ldots, n\)\} (which may depend on output) satisfy some measurability and integrability conditions, standard filtering techniques yield that the conditional mean \((m_t)_{t \geq 0}\) evolves as \( dm_t = -\kappa (m_t - \bar{\theta}) dt + \Sigma_t dZ^\theta_t \) where \( \Sigma_t \in \mathbb{R}^d \) and \( Z^\theta_t \) is a \( d \)-dimensional innovation process. Letting \( dZ_t := \Sigma_t dZ^\theta_t / ||\Sigma_t|| \) and \( \sigma_t := ||\Sigma_t|| \), we conclude.
and the market’s beliefs take the form:
\[
m^*_t = m_0e^{-\delta t} + \frac{\kappa(1-e^{-\delta t})}{\delta} + \beta \int_0^t e^{-\delta(t-s)}(a_s - a^*_s)ds + \beta \int_0^t e^{-\delta(t-s)}m_sds
\]
\[
+ \beta \sigma \xi \int_0^t e^{-\delta(t-s)}d\tilde{Z}_s, \ t \geq 0\tag{14}
\]
From (*) it is extremely clear how incentives are determined in this model: a marginal increase in effort over \([s, s+1]\) generates the additional wage flow \((e^{-\delta(t-s)}/\beta)\). In this sense, \(\beta\) is what really defines the short-term gains of belief-distortion (since for a short time horizon the discount plays a minor role) while \(\delta\) determines the long-run benefits from it.

With this in hand, we can state the first result regarding incentives. It is a particular case of a more general result presented in the next section. The interested reader may refer to the proof of Proposition 6 in the Appendix:

**Proposition 3.** Let \((a^*_t)_{t \geq 0}\) denote the equilibrium effort strategy in the career concerns model with mean-reverting talent, no human capital accumulation and under stationary learning. It corresponds to a constant effort strategy characterized by the condition
\[
g'(a^*) = \frac{\beta}{r + \delta} = \frac{\sqrt{\kappa^2 + \sigma^2 / \xi^2} - \kappa}{r + \sqrt{\kappa^2 + \sigma^2 / \xi^2}} < 1, \tag{15}
\]
where \(r \in (0, \infty)\) corresponds to the payoffs’ discount rate. Therefore, if \(\kappa > 0\), effort is uniformly bounded away from efficiency for all discount rates.

**Proof:** See the Appendix.

The intuition straightforward. In a stationary-variance setting the market learns at a constant rate and therefore the manager’s benefits coming from belief-distortion will remain fixed over time, leading to a constant effort level. On the other hand, from the market’s beliefs expression (14) and the fact that the manager discounts payoffs at a rate \(r\), we can see that a permanent marginal increase in effort generates an expected discounted benefit of size \(\beta / (r+\delta)\), the net present value of the dividend stream \(d_t = \int_0^t e^{-\delta(t-s)}ds, \ t \geq 0\), coming from permanently distorting the market’s beliefs.
The second equality in (15) is obtained when we replace the value of $\gamma^*$ in $\beta = \gamma^*/\sigma_\xi^2$ and $\delta = \beta + \kappa$. More interestingly, we recover Holmstrom’s asymptotic efficiency result for infinitely patient agents only if $\kappa = 0$.\(^6\) That is, just by adding a little bit of mean reversion, efficiency is no longer attained. This implies that Holmstrom’s asymptotic efficiency result is not only very sensitive to risk neutrality, discount rates and transient effects (which will be analyzed next), but also to the specific stochastic structure of talent.

The inefficiency result under mean-reversion is interesting on its own. When changing the drift of a diffusion (and Girsanov’s theorem applies) the paths that arise with positive are exactly the same, but their likelihood may vary. Mean-reversion provides the market with the additional information that paths close to $\bar{\theta}$ have now a higher probability to occur, and thus beliefs should evolve in a different way as in the random walk case. More specifically, the asymptotic inefficiency comes from the fact that, because of mean-reversion, the market discounts past information at a rate higher that the size at which beliefs react to new information: $\delta = \beta + \kappa > \beta$. The case $\kappa = 0$ is the unique mean-reverting rate at which both short and long-term forces have equal strength, but even in this case efficiency is achieved only in the limit when the payoffs’ discount rate goes to zero.

The next comparative statics will be useful to interpret how beliefs depend on the underlying parameters of the model:

**Corollary 4.** $\beta(\kappa, \sigma_\theta/\sigma_\xi) = \sqrt{\kappa^2 + \sigma_\theta^2/\sigma_\xi^2} - \kappa$, is decreasing in $\kappa$ and increasing in the ratio $\sigma_\theta/\sigma_\xi$. As a consequence, the rate at which beliefs decay over time, $\delta(\kappa, \sigma_\theta/\sigma_\xi) = \beta(\kappa, \sigma_\theta/\sigma_\xi) + \kappa$, is increasing in both $\kappa$ and $\sigma_\theta/\sigma_\xi$.

The Wiener process specification (equivalent to the traditional random walk case) presents the highest sensitivity of beliefs to new information: $\beta(0, \sigma_\theta/\sigma_\xi) > \beta(\kappa, \sigma_\theta/\sigma_\xi)$, $\kappa \geq 0$. In fact, the erratic behavior of talent in the random walk case increases the chance that output surprises are the consequence of talent improvement rather than the consequence of output’s noise, when compared to the mean-reverting specification. Moreover, as the rate of mean reversion increases, the market expects the manager’s talent to be close to $\bar{\theta}$ more frequently, and it will consider output surprises to be more likely the consequence of

\(^6\)It is direct to verify that, in this setting, efficiency requires a constant effort level satisfying $g'(a^*) = 1$. This comes from maximizing the drift of the surplus generated by the firm and the worker.
noise rather than the result of talent improvement. This explains why $\beta$ decreases with $\kappa$. Observe also that the rate at which the market discounts past information is minimized when $\kappa = 0$, so the market is giving relatively more importance to past information in the random walk than in the mean-reverting setting: because of the more erratic behavior of talent in the Wiener process specification, it is necessary to keep track of more past information in order to assess the talent’s today position. As a consequence, when $\kappa$ increases, recent observations become a better statistic of talent than past information, explaining why the market discounts past performance at rates that increase with the size of the mean-reversion coefficient. In the extreme case when $\kappa$ goes to infinity, the productivity shocks tend to disappear instantaneously and talent is virtually pegged to the level $\theta$. The market, anticipating this, will ascribe any change in output as the consequence of the signal’s noise ($\beta$ goes to zero), and thus no incentives to exert effort are generated. The comparative statics with respect to $\sigma_\theta/\sigma_\xi$ are straightforward. An increase in this ratio is either because talent becomes more volatile or output becomes less noisy. In the first case, output changes as the consequence of talent improvement are now more likely to occur, so $\beta$ increases. At the same time, past performance becomes less of a good predictor of current talent, so $\delta$ increases as well. Finally, when the public signal becomes more accurate, it is clear that present information improves its quality and becomes a better statistic of talent than past performance, explaining the increase in both $\beta$ and $\delta$.

Now that we have derived the main insights in terms of belief-distortion and incentives, we move on to characterize incentives away from the steady state $\gamma^*$. Recall that the mean-square error $\gamma_t$ evolves according to (7). We obtain the following generalization of the previous result, which is a particular case of the more general result presented in Proposition 6:

**Proposition 5.** Away from steady state, the equilibrium effort strategy $(a^*_t)_{t \geq 0}$ is characterized by the first order condition

$$g'(a^*_t) = \beta_t \int_t^\infty e^{-\int_s^t (r+\delta_u)du} ds, \ \forall t \geq 0 \quad (16)$$

where $\beta_t := \gamma_t/\sigma^2_\xi$ and $\delta_t := \beta_t + \kappa$, for all $t \geq 0$. Moreover,

$$\frac{da^*_t}{dt} \leq 0 \iff \frac{d\gamma_t}{dt} \leq 0.$$
Given the previous analysis the intuition for the result should be straightforward. An additional unit of effort at time $t$ implies that the market’s beliefs at $s \geq t$ increase by
\[
d_s := \exp\left(-\int_s^t \delta_u du\right) \beta_t.
\]
The only change with the stationary-learning setting is that now both the sensitivity of beliefs to new information and the rate at which the market’s beliefs decay change over time, as the market learns about the manager’s talent from new observations. Nonetheless, the structure of incentives is preserved. In this case, a marginal increase in effort generates an additional flow dividend $(d_s)_{s \geq t}$ which is in turn discounted at the rate $r > 0$.

The last part of the proposition states that, as information improves, the incentives to exert effort decay over time. This goes in line with the traditional idea that career concerns motives generate higher returns in environments with more uncertainty (within such environments there is more gap for belief-distortion). However, there are two opposing forces that make this conclusion non-trivial: when $\gamma_t$ decreases over time, both the sensitivity of beliefs to new information, $\beta_t$, and the rate at which the market’s beliefs decay, $\delta_t$, decrease. The first force reduces the short-term benefits from belief-distortion, while the second force makes any reputational gain more persistent over time. Therefore, the result says that the short-term losses from increased precision always outweigh the long-term benefits from more permanent distortions.

As we have seen, the continuous-time formulation of the pure career concerns model has proven itself to be a tractable benchmark to incorporate a more general, and certainly less extreme, specification for talent. It also allowed us to gain simple insights on how short and long-term incentives are shaped by the way beliefs evolve, and how the latter map into incentives. For the remainder of the paper it is necessary to remember only two things. First, that $\beta_t$ determines how beliefs instantaneously react to new information. Second, that because the market constructs statistics using the component of output not explained by effort, $\xi_t - \int_0^t a^*_s ds$, it discounts past information according to the rate process $\delta_t$, which is always higher than the rate at which true talent keeps track of the history of productivity shocks, $\kappa$. 

Proof: See the Appendix.
4 General Model

Suppose that output is given by (1), that is,

\[
d\xi_t = (a_t + \theta_t)dt + \sigma_\xi dZ_\xi^\xi, \ t \in \mathbb{R}_+,
\]

generating the public information \((\mathcal{F}_t^\xi)_{t \geq 0}\). Also, recall that talent evolves according to (2)

\[
d\theta_t = -\kappa(\theta_t - \bar{\theta}(a))dt + \sigma_\theta dZ_\theta^\theta, \ t \in \mathbb{R}_+.
\]

We impose that, for every feasible strategy \(a \in \mathcal{A}\), (i) \(\bar{\theta}_0(a) \equiv \bar{\theta} \in \mathbb{R}_+\), (ii) for each \(t > 0\), \(\bar{\theta}_t(a)\) is a deterministic functional of the history of effort chosen by the manager up to time \(t\), \((a_s : 0 \leq s \leq t)\). and (iii) the mapping \(t \mapsto \bar{\theta}_t(a)\) is Borel-measurable and satisfies standard integrability conditions that ensure that the filtering is well defined. At this point we do not necessarily assume that for each \(a \in \mathcal{A}\), \((\bar{\theta}_t(a))_{t \geq 0}\) is the solution to an ODE like (3).

Learning occurs as usual: if the market anticipates a deterministic effort strategy \(a^* \in \mathcal{A}\), then the trajectory \(t \mapsto \bar{\theta}_t(a^*)\) is completely anticipated at time zero and the market’s beliefs are as before:

\[
d m_t^* = -\kappa(m_t^* - \bar{\theta}_t(a^*))dt + \gamma_t \underbrace{\frac{d\xi_t - (m_t^* + a_t^*)dt}{\sigma_\xi}}_{dZ_\xi^*}, \quad (17)
\]

\[
\gamma_t = -2\kappa\gamma_t + \sigma_\theta^2 - \left(\frac{\gamma_t}{\sigma_\xi}\right)^2.
\]

where \(Z^{a^*}\) is a Brownian motion from the market’s perspective. In particular, observe that the mean-square error \(\gamma\) evolves in the same way as in the pure career concerns model.

As before, we allow the manager to have access to more information about his talent than the one provided by the public signal. The only restriction we impose is that his posterior beliefs lie within the Gaussian filtering framework. His posterior mean about his own talent will evolve in a similar way as in (12), but now with a trend depending on the effort strategy he will actually follows, say \(a \in \mathcal{A}\):

\[
d m_t = -\kappa(m_t - \bar{\theta}_t(a))dt + \sigma_t dZ_t, \ t \in \mathbb{R}_+ \quad (18)
\]
with \( Z := (Z_t)_{t \geq 0} \) a Brownian motion from his own perspective.

The manager then chooses \( a \in \mathcal{A} \) to maximize \( \mathbb{E}^a[\int_0^\infty e^{-rt}(w_t - g(a_t))dt] \) subject to the wages equation \( w_t = m^*_t + a^*_t \), the evolution of the market’s beliefs (17), the way that output evolves from his standpoint \( d\xi_t = (m_t + a_t)dt + \sigma_\xi dZ^a_t \), and finally, the evolution of his own posterior mean (18).

The main result of the paper is presented next. It shows that this seemingly complex problem that the manager faces can be reduced to a particularly clean deterministic optimization problem. Two features deserve attention. First, even though the talent’s dynamics are much richer than in the traditional career concerns benchmark, looking for deterministic effort strategies is logically correct: whenever the market conjectures that the manager will follow a deterministic effort strategy, the manager’s problem is indeed reduced to a deterministic optimization problem. Second, and most importantly, the result shows that human capital accumulation turns out to be an additional channel to distort beliefs. To understand how this channel operates, fix a particular strategy and suppose an exogenous marginal increase in the manager’s human capital stock at a specific point in time. The first thing to observe is that this change will make the manager, on average, more talented in the future. Specifically, because of mean-reversion, the effect that a marginal change in human capital has on future talent decays at a rate \( \kappa \). On the other hand, recall that the market’s beliefs decay according to the rate process \( (\delta_t)_{t \geq 0} \), which is always higher than \( \kappa \). Therefore, because of this excessive discounting, the market will constantly underestimate the manager’s talent after an exogenous change in the worker’s human capital has taken place. But since a more talented manager is able to generate, in expectation, higher output observations, the market will then underestimate the size of the additional output flow generated by this, on average, more skilled worker. The difference between the output stream actually observed versus the output stream anticipated by the market will constitute a “surprise” from the market’s perspective and, therefore, it will be ascribed to talent improvement and internalized by the manager in the form of an increase in future wages. It is the fact that the market discounts past information at relatively high rates what precisely enables the manager to extract reputational gains from human capital accumulation.

**Proposition 6.** In the career concerns model with human capital accumulation defined
by (1) and (2), a deterministic equilibrium effort strategy \( a := (a_t)_{t \geq 0} \) is an equilibrium if and only if it is the solution to the following deterministic problem

\[
\max_{a \in A} \int_0^\infty e^{-rt} (a_t \beta_t \lambda_t - g(a_t) + \rho_t \theta_t(a)) \, dt
\]  

where \( \rho_t = \frac{r}{r+\kappa} - \kappa \lambda_t \), \( \lambda_t := \int_t^\infty \exp \left( -\int_t^s (r + \delta_u) \, du \right) \, ds \), \( \beta_t = \gamma_t / \sigma_t^2 \) and \( \delta_t = \beta_t + \kappa \), \( t \geq 0 \), are all exogenous deterministic functions.

Proof: See the Appendix.

The above proposition shows that incentives in the model have a simple decomposition. Observe that the flow \( a_t \beta_t \lambda_t - g(a_t) \) at any point in time satisfies \( g'(a_t) = \beta_t \lambda_t \), the first order condition that we showed in Proposition 5. The term \( \rho_t = \frac{r}{r+\kappa} - \kappa \lambda_t \) is strictly larger than zero for all \( t \geq 0 \) and it represents the manager’s expected discounted gains from a marginal increase in his human capital stock at time \( t \). To understand this expression recall that talent takes the form

\[
\theta_t = \theta_0 e^{-\kappa t} + \kappa \int_0^t e^{-\kappa(t-s)} \overline{\theta}(a) \, ds + \sigma_\theta \int_0^t e^{-\kappa(t-s)} dZ_\theta
\]  

so a marginal increase of human capital at time \( s \) implies that the manager is, in expectation, more talented by an amount \( e^{-\kappa(t-s)} \kappa \) at time \( t \geq s \). This in turn translates into an additional output flow of net present value \( \frac{r}{r+\kappa} \) that a more skilled manager is able to produce. However, since the market’s beliefs decay at rates \( (\delta_t)_{t \geq 0} \), the market expects an additional output stream of value \( \kappa \lambda_t \), where \( \lambda_t := \int_t^\infty \exp \left( -\int_t^s (r + \delta_u) \, du \right) \, ds \), which is strictly below the one actually generated. As a consequence, the difference \( \rho_t \) is attributed to skills improvement and thus captured by the manager in the form of additional earnings. Finally, it is worth mentioning that, at any point in time, the manager’s incentives depend only on the publicly available information and not on private one that he may have. This is because there are no complementarities between effort and talent in the production function.

Fama’s (1980) conjecture that reputational concerns would remove all moral hazard issues within competitive markets motivated Holmstrom to build a formal framework to
study the validity of that claim. In fact, as we can see next, inefficiencies are a broad phenomena. The first best-solution maximizes the expected discounted surplus generated by the interaction between the firm and the manager:

\[
\max_{a \in A} \mathbb{E} \left[ \int_0^\infty e^{-rt}(d\xi_t - g(a_t)dt) \right] = \int_0^\infty e^{-rt}(a_t + \theta_t - g(a_t))dt
\]

\(\Leftrightarrow \max_{a \in A} \int_0^\infty e^{-rt} \left( a_t - g(a_t) + \frac{\kappa}{r + \kappa} \bar{\theta}_t(a) \right) dt \tag{21}\)

Comparing this problem with (19) we can see that, in addition to the traditional inefficiency arising from the pure career concerns model, there is an efficiency loss in what respect to human capital accumulation. The intuition is straightforward: as the consequence of an extra unit of human capital, the manager is able to generate an extra output flow of net present value \(\frac{\kappa}{r + \kappa}\), but he is able to internalize only \(\rho_t := \frac{\kappa}{r + \kappa} - \kappa \lambda_t\) of it in the form of additional wages.

The previous specification of human capital accumulation is quite general. In order to solve (19) we need to add more structure to the family of functionals \((\bar{\theta}_t(\cdot))_{t \geq 0}\) and explicitly define the class of deterministic strategies that we will look for. We do this in the next subsection.

\[\text{Definition 7. (Feasible Control)} \text{ We will say that a control is feasible if it corresponds}\]

\[\text{to a piecewise continuous}\] function of time \(a : \mathbb{R}_+ \to A := [0, \ell].\)

We will denote the set of feasible controls by \(A\) and observe that for any \(a \in A\), the human capital path generated by it, \((\bar{\theta}_t(a))_{t \geq 0}\), is piecewise continuously differentiable.

\[7\text{We say that } \phi : \mathbb{R} \to A \subseteq \mathbb{R} \text{ is piecewise continuous if for any interval } [a, b] \subset \mathbb{R} \text{ there exists a finite set of points } a = t_0 < t_1 < \ldots < t_n = b \text{ such that } \phi \text{ is continuous in } [t_0, t_1] \text{ and } (t_i, t_{i+1}), \text{ for } i = 1, \ldots, n - 1 \text{ and has a finite right hand limit for each } t_i, i = 1, \ldots, n. \text{ This definition follows from Halkin 1974.}\]
In order to capture the idea that the manager accumulates human capital as he becomes more experienced (as measured by how engaged in production the worker has been), we assume that $a \mapsto f(\bar{\theta}, a)$ is strictly increasing in for all $\bar{\theta} \geq 0$. More generally, we will ask this function to satisfy this very weak requirements:

**Assumption 8.** We assume that

(i) $f : \mathbb{R} \times A \rightarrow \mathbb{R}$ is such that $f(\cdot, a) \in C^1(\mathbb{R})$ for all $a \in A$ and $f(\bar{\theta}, \cdot)$ is differentiable for all $\bar{\theta} \in \mathbb{R}$.

(ii) For all $\bar{\theta} \geq 0$, the function $a \mapsto f(\bar{\theta}, a)$ is strictly increasing.

The following result gives necessary conditions that any optimal control must satisfy. It is an application of Pontryagin’s Maximum Principle for infinite horizon problems (see Halkin (1974)):

**Proposition 9.** Let $a^* \in A$ be an optimal control and suppose $a \neq 0, \ell$. Then, there exists a piecewise continuously differentiable function $q : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

(i) For almost every $t \in \mathbb{R}_+$

$$dq_t = \left\{ q_t \left[ r - \frac{\partial f}{\partial \bar{\theta}}(\bar{\theta}_t, a^*_t) \right] - \rho_t \right\} dt; \quad (22)$$

(ii) For every $t \geq 0$ such that $0 < a^*_t < \ell$, $a^*_t$ satisfies

$$g'(a^*_t) = \beta_t \lambda_t + q_t \frac{\partial f}{\partial a}(\bar{\theta}_t(a^*), a_t^*) \quad (23)$$

where $\bar{\theta}_t(a^*)$ is the solution to $d\bar{\theta}_t = f(\bar{\theta}_t, a^*_t)dt$, $\bar{\theta}_0 = \bar{\theta} \geq 0$. 

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The last proposition shows that the human capital accumulation model that we propose, though rich in complex dynamics, has a simple characterization. In any interior solution, incentives are decomposed into a standard career concerns component, $\beta_t \lambda_t$, and a human capital component, $q_t \frac{\partial f}{\partial a}(\theta, a^*)$. The latter corresponds to the benefits from an increase in the speed at which human capital is accumulated, adjusted by the shadow price of human capital at time $t$, $q_t$. As a consequence, effort will be typically different to the one in the career concerns benchmark, highlighting the role that experience accumulation has on incentives.

The question whether effort is above or below the pure career concerns model depends on the sign of the adjoint variable $q := (q_t)_{t \geq 0}$. In order to answer it, the first step would be to find all the solutions to the system

$$
\begin{align*}
    a(t, \theta, q) \in & \arg \max_{a \in A} \alpha \beta_t \lambda_t - g(a) + q \frac{\partial f}{\partial a}(\theta, a) \\
    d\theta_t &= f(\theta_t, a(t, \theta_t, q_t)) dt, \quad \theta_0 = \bar{\theta} \geq 0 \\
    dq_t &= \left\{ q_t \left[ r - \frac{\partial f}{\partial \theta}(\theta_t, a(t, \theta_t, q_t)) \right] - \rho_t \right\} dt
\end{align*}
$$

but such a system will typically have too many of them.\textsuperscript{8} Nevertheless, standard phase diagram analysis allow us to study the behavior of the system around a stationary solution. We illustrate some dynamics in the particular case of stationary-learning, quadratic effort cost and when the human capital dynamic is separable in both effort and in the stock of human capital.

**Example 10.** Consider $g(a) = \frac{1}{2} a^2$ and $f(a, \theta) = a - c(\theta)$, $c(0) = 0$ and $c' > 0$. In this case $c(\theta) dt$ represents the amount of human capital depreciated over the interval $[t, t + dt)$. Suppose that learning is stationary and thus $\beta_t = \beta := \gamma^* / \sigma^2$, $\delta_t = \delta := \beta + \kappa$, and $\rho_t = \rho := \kappa \left( \frac{1}{1 + \kappa} - \frac{1}{r + \delta} \right)$ for all $t \geq 0$. In any interior solution, effort takes the form $a_t^* = \frac{\beta}{r + \delta} + q_t$. Then, the set of points such that $\dot{q}_t = 0$ is represented by the function

\textsuperscript{8}In a finite horizon setting with no endpoint condition on the state, a terminal condition on the adjoint variable $q$ is part of the necessary conditions and hence the above system of ODEs will be uniquely determined (one forward ODE plus one backward ODE). In an infinite horizon setting, transversality conditions on $q$ do not necessarily apply. See Hallkin (1974) for examples on this.
$q_1(\theta) = \frac{\rho}{r + c'(\theta)}$. On the other hand, the set of points such that $\ddot{\theta}_t = 0$ corresponds to $q_2(\theta) = c(\theta) - \frac{\beta}{r+\delta}$ and, therefore, the steady state of the system is represented by the pair $(\theta^*, q^*)$ such that $q^* = q_1(\theta^*) = q_2(\theta^*)$. Depending on whether the depreciation function is convex, linear or concave, we illustrate the dynamics in a neighborhood of the steady-state equilibrium, emphasizing the corresponding convergence paths:

Figure 2: $c(\cdot)$ convex (top left), concave (top right) and linear (bottom)

In the above examples, the stationary-equilibrium value of the adjoint variable is strictly larger than zero. Hence, in a neighborhood of steady state, the optimal effort level is higher than the one found in the pure career concerns benchmark. In order to extract more qualitative properties of any solution away from the stationary solution, we change the approach.

In this section we follow a dynamic-programming approach to obtain further properties that any solution to $P(\bar{\theta})$, defined as

$$
\max_{a \in A} \int_0^\infty e^{-rt} \left( a_t \beta_t \lambda_t - g(a_t) + \rho_t \bar{\theta}_t(a_t) \right) dt \\
\text{s.t.} \quad d\bar{\theta}_t = f(\bar{\theta}_t, a_t)dt, \quad \bar{\theta}_0 = \bar{\theta} \geq 0,
$$

must satisfy.

For any $t \geq 0, \bar{\theta} \in \mathbb{R}$ and $a \in A$ we define

$$
V[t, \bar{\theta}; a] := \int_t^\infty e^{-rs} \left[ a_s \beta_s \lambda_s - g(a_s) + \rho_s \bar{\theta}_s^t(a) \right] ds
$$

(24)

where $(\bar{\theta}_s^t(a))_{s \geq t}$ is the solution of the ODE

$$
d\bar{\theta}_s = f(\bar{\theta}_s, a_s)ds, \quad s \geq t \\
\bar{\theta}_t = \bar{\theta}
$$

That is, $V[t, \bar{\theta}; a]$ is the manager’s continuation value at $t$ under the continuation strategy $(a_s)_{s \geq t}$, seen from a time-zero perspective. Also, the continuation-value function, $\mathcal{V}(t, \bar{\theta})$, is defined as

$$
\mathcal{V}(t, \bar{\theta}) := \sup_{a \in A} V[t, \bar{\theta}; a].
$$

(25)

Finally, we say that $a^* \in A$ is an optimal control for $P(\bar{\theta})$ if and only if

$$
V[0, \bar{\theta}; a^*] = \mathcal{V}(0, \bar{\theta})
$$

We say that $\mathcal{V}(0, \bar{\theta})$ is the value of $P(\bar{\theta})$.

The next result states that differentiability of the continuation-value function with respect to the state variable $\bar{\theta}$ ensures that effort is always above the career concerns benchmark:

**Proposition 11.** Fix $t \geq 0$ and suppose $\bar{\theta} \in \mathbb{R}$ is such that $\mathcal{V}(t, \bar{\theta}) < +\infty$. Let $(a_s^*)_{s \geq t}$ be the continuation strategy that attains this value. Then, the continuation-value function is increasing in the state variable. Moreover, if $\mathcal{V}(t, \cdot)$ is differentiable in a neighborhood $\Theta$ of $\bar{\theta}$, then for any $s \geq t$ such that $\bar{\theta}_s^t(a) \in \Theta$, $q_s = e^{rs} \frac{\partial \mathcal{V}}{\partial \bar{\theta}}(s, \bar{\theta}_s^t(a^*)) > 0$. 

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As a consequence, under enough regularity conditions in our model we can ensure that human capital accumulation indeed increases the manager’s career concerns motives. Of course, we expect this result to hold even when the continuation value is not differentiable, but an analysis of this kind is beyond the scope of this paper.

So far we have showed the necessary conditions that an optimal control must satisfy in case it exists, but nothing has been said concerning existence and uniqueness of a solution. The following result is an adaptation of finite-horizon dynamic-programming tools that gives a sufficient condition for the existence of an optimal control. Its proof can be found in the Appendix.

**Proposition 12.** Suppose there exists a function $\Phi \in C^1(\mathbb{R}_+ \times \mathbb{R})$ such that:

(i) It solves the Hamilton-Jacobi equation: for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R},$

$$\sup_{u \in A} \left\{ e^{-rt} \left[ \beta_t \lambda_t u - g(u) + \rho_t x \right] + \frac{\partial \Phi}{\partial t}(t, x) + \frac{\partial \Phi}{\partial x}(t, x) f(x, u) \right\} = 0 \quad (HJ);$$

(ii) For every $\theta \in \mathbb{R}_+$ and any control $a \in A$

$$\lim_{t \to \infty} \Phi(t, \theta_t(a; \theta)) = 0 \quad (26)$$

where $t \mapsto \theta_t(a; \theta)$ is the trajectory generated by the ODE $d \theta_t = f(\theta_t, a_t)dt$, starting from $\theta$ at $t = 0$.

Then, if $(t, x) \mapsto u^*(t, x)$ is the maximizer of the right-hand side in $(HJ)$, we have that

$$a^* : \mathbb{R}_+ \to A$$

$$t \mapsto a^*_t = u^*(t, \theta_t) \quad (27)$$

is an optimal control and $\Phi$ is the continuation-value function: $V(t, \theta) = \Phi(t, \theta)$ for all $(t, \theta) \in \mathbb{R}_+ \times \mathbb{R}$.

**Proof:** See the Appendix.
The previous proposition states that the existence of an optimal control can be “reduced” to the existence of a solution to a partial differential equation (PDE) satisfying a transversality condition. The time-dependence of this PDE comes from that fact that, away from steady state, the manager’s problem is non-stationary. The result also confirms the relationship between $q_t$ and $V(t, \bar{\theta}_t(a^*))$ stated in Proposition 11: just by comparing the first-order condition (ii) in Proposition 9 and the one delivered by the (HJ) equation, we have that $q_t = e^{rt} \partial V \partial \theta_t(t, \bar{\theta}_t(a^*))$. Therefore, whenever the previous result is to hold, the optimal effort level will be always above the career concerns benchmark.

Given the high degree of generality of the model, an analytic solution for the PDE generated by (HJ) is typically impossible to obtain in generic terms. Nevertheless, a lot can be said when human capital technology is additive in effort and accumulated talent. Suppose that the dynamic of human capital is governed by a function of the form

$$ f(a, \bar{\theta}) = \alpha a - \phi \theta, \quad \alpha > 0, \phi \in (0, 1). $$

In this case, $\alpha > 0$ measures how fast the manager acquires skills from his experience at work, while $\phi$ measures the rate at which acquired skills depreciate over time. Also, a dynamic of this form ensures that, whenever the stock of human capital starts from a non-negative value, the entire path of human capital will remain non-negative. The simple structure of the above specification allows us to obtain an extremely simple expression for the optimal effort strategy and for the partial differential equation induced by (HJ).

**Proposition 13.** In the career concerns model with human capital accumulation technology of the form $f(a, \bar{\theta}) = \alpha a - \phi \theta$, $\alpha > 0, \phi \in [0, 1]$, and $(t, \bar{\theta}) \in \mathbb{R}^2_+$, an optimal control exists. In any interior solution it satisfies the following first order condition:

$$ g'(a^*_t) = \beta_t \lambda_t + \alpha \mu_t $$

(28)

where $\mu_t := \int_t^\infty e^{-(r+\phi)(s-t)} \rho_s ds$ and $\rho_t = \frac{\alpha}{r+\phi} - \kappa \lambda_t$, $t \geq 0$. Also, the continuation-value function takes the form

$$ V(t, \bar{\theta}) = e^{-rt}[\eta_t + \mu_t \bar{\theta}], \quad (t, \bar{\theta}) \in \mathbb{R}^2_+, $$

(29)

where $\eta_t := \int_t^\infty e^{-r(s-t)}[g'(a^*_s)a^*_s - g(a^*_s)]ds$ for all $t \geq 0$.

**Proof:** See the Appendix.
The additivity of the model plus its Gaussian structure yield a clean separation between the two channels of belief-distortion that are available to the manager. Effort is strictly positive throughout the manager’s entire career and, at any point in time, it does not depend on the current stock of human capital: the separability of the human capital accumulation technology buys this result. Also, even though the manager’s problem is not stationary, the continuation-value function follows a very simple structure over time. Regarding the evolution of effort over the worker’s life, recall from Proposition 5 that incentives decrease in the pure career concerns model if and only if uncertainty gradually diminishes over time ($\gamma_t \downarrow \gamma^*$). The same occurs with the human capital component of incentives:

**Proposition 14.** $(\mu_t)_{t \geq 0}$ is strictly decreasing over time if and only if $\gamma_0 > \gamma^*$.

**Proof:** See the Appendix.

The result confirms the standard result from human capital theory that the returns from investing in human capital decrease over the workers’ life-cycle. In this particular case, the quality of the market’s information gradually improves over time and therefore, when constructing statistics, past performance is progressively given higher weights across time ($\delta_t \geq 0$ is decreasing). Hence, as the manager’s tenure increases, the market will expect more output benefits from a marginal increase in the stock of human capital of the former. This in turn reduces the fraction of additional output that manager internalizes as extra earnings, decreasing the incentives to exert effort.

We would like to conclude this particular example with two discussions: first, on the inefficiencies that arise in it, and second, on the the incentives that workers face before entering the labor market.

**Inefficiencies:** They appear naturally in the model because the manager is not able to internalize all the output benefits from belief-distortion. Recall that the efficient effort allocation solves (21), that is,

$$\max_{a \in A} \int_0^{\infty} e^{-rt} \left( a_t - g(a_t) + \frac{\kappa}{r + \kappa} \bar{u}_t(a) \right) dt$$

(30)
subject to the fact that \((\bar{\theta}_t(a))_{t \geq 0}\) is the solution to \(d\bar{\theta}_t = (\alpha a_t - \phi \bar{\theta}_t)dt\), \(\bar{\theta}_0 = \bar{\theta}\). This problem has the same structure as the one that the manager actually solves, \(P(\bar{\theta})\). Therefore, the first-best effort strategy is constant over time and satisfies the condition

\[
g'(a^*) = 1 + \frac{\alpha}{r + \phi} \frac{\kappa}{r + \kappa}.
\]

It is easy to see that the stationary-learning equilibrium effort strategy satisfies

\[
g'(a^*) = \frac{\beta}{r + \delta} + \frac{\alpha}{r + \phi} \left( \frac{\kappa}{r + \kappa} - \frac{\kappa}{r + \delta} \right) < g'(a^*)
\]

Nevertheless, depending on the degree of uncertainty in the early stages of the manager’s working life, it is plausible to see effort levels above efficiency when the agent starts his career.

**Schooling:** The incentives to acquire human capital before entering the job market are particularly simple in this example. The worker’s expected discounted benefits from entering the market with a human capital level of size \(\bar{\theta} > 0\) are given by

\[
\mathcal{V}(0, \bar{\theta}) = \eta_0(\alpha, \phi) + \mu_0(\phi)\bar{\theta}
\]

where we made explicit the dependence of the initial values \(\eta_0\) and \(\mu_0\) on \(\alpha\) and \(\phi\). Of course, the speed at which the workers acquires human capital from his working experience \((\alpha)\) plays no role in his marginal decision of education before entering the job market. However, given the long-term effects that it has on the worker’s skills, the human capital depreciation rate \(\phi\) indeed affects the manager’s schooling choice. Suppose that the “future manager” makes a static choice on how much education to acquire before entering the labor market. Education is costly according to an increasing and differentiable function \(c(e)\). Also, \(e\) units of education map into \(\bar{\theta}(e)\) units of human capital, where \(\bar{\theta}(\cdot)\) is increasing and concave. The first-order condition of this problem yields

\[
\mu_0(\phi)\bar{\theta}'(e^*(\phi)) - c'(e^*(\phi)) = 0,
\]

and since \(\mu_0(\phi)\) is decreasing in \(\phi\), \(e^*(\phi)\) will be decreasing. That is, individuals that suffer of less human capital depreciation would choose a higher level of education before entering the labor market.
5 Extensions

In this section we discuss some possible extensions of the model.

(1) Beliefs’ Sensitivity and Multitasking. We stressed the importance of the sensitivity process \((\beta_t)_{t\geq 0}\) in determining the short-term gains from belief-distortion, even though the analysis could have been made only in terms of the evolution of the mean-square error \((\gamma_t)_{t\geq 0}\) (recall that \(\beta_t := \gamma_t/\sigma^2_\xi\)). In more complex settings, the sensitivity process can take more elaborated forms and thus, talking about the mean-square error as the only measure of how beliefs instantaneously react to new information may be misleading. Because of the same reason, it is also wrong to rank the quality of different tasks as channels for belief distortion based only on how precise the signal’s volatility is. The following example illustrate both issues.

Example 15. (Correlation Among Noise Terms) Suppose that talent mean-reverts to a fixed trend \(\bar{\theta}\), but now there are 2 lines of production with the following structure:

\[
\begin{align*}
    d\xi^1_t &= (a^1_t + \theta_t)dt + \sigma_\xi dZ^\xi^1_t, \\
    d\xi^2_t &= (a^2_t + \theta_t)dt + \sigma_\xi \tau dZ^\theta_t + \sigma_\xi \sqrt{1 - \tau^2} dZ^\xi^2_t,
\end{align*}
\]

with \(\tau \in (0,1)\). Both lines of production have the same overall volatility \(\sigma_\xi\), however, the second task provide more accurate information regarding talent (due to the correlation with \(Z^\theta\)), so the market beliefs will react more strongly to innovations in \((\xi^2_t)_{t\geq 0}\). In a stationary learning setting, the filtering equation takes the form:

\[
\begin{align*}
    dm_t &= -\kappa(m_t - \bar{m})dt + \frac{\gamma^*}{(\sigma_\xi^2)^2} [d\xi^1_t - (a^1_t + m_t)dt] + \frac{\sigma_\theta \sigma_\xi \tau + \gamma^*}{(\sigma_\xi^2)^2} [d\xi^2_t - (a^2_t + m_t)dt] \\
    \beta^1 &= \left[ d\xi^1_t - (a^1_t + m_t)dt \right] + \sigma_\theta \sigma_\xi \tau + \gamma^* [d\xi^2_t - (a^2_t + m_t)dt]
\end{align*}
\]

implying \(\beta^2 > \beta^1\). As a consequence, if effort among different tasks were perfectly substitutable in the manager’s utility function, the manager would specialize in the second task.

(2) On-the-Job Training. Classic papers of human capital accumulation model the choice of human capital accumulation as a costly one, that is, it requires to move resources away from final-goods production. In the spirit of Jovanovic (1979), suppose for example that if the manager decides to allocate \(a\) units of effort to final goods-production, the
remaining \( \ell - a \) are devoted to human capital accumulation (in this case, effort could be interpreted as time). In order to model this decision, we can assume that the dynamics of human capital are according to a function \( \tilde{f}(\bar{\theta}, a) := f(\bar{\theta}, \ell - a) \) which is decreasing in the effort. The necessary conditions of Proposition 9 still hold (with the obvious modifications), since they only require the function \( a \mapsto f(\bar{\theta}, a) \) to be strictly monotone.

In any interior solution, effort would take the form

\[
g'(a^*_t) = \beta_t \lambda_t + q_t \frac{\partial \tilde{f}}{\partial a}(\bar{\theta}_t, a^*_t) \leq 0.
\]

Since the continuation-value function is still increasing in the state, \( q_t \) will be typically positive. Therefore, due to substitution effect present in this model, effort would be below that traditional career concerns benchmark.

(3) Discontinuous Changes in the Mean-Trend: Suppose that the manager is able to choose a pair of effort profiles \( a := (a^1_t, a^2_t)_{t \geq 0} \) where \( a^1 \) determines output and \( a^2 \) affects the drift of talent. More specifically suppose that \( \bar{\theta}_t(a) = \bar{\theta} + a^2_t \), so talent evolves as

\[
d\theta_t = -\kappa(\theta_t - (\bar{\theta} + a^2_t))dt + \sigma_{\theta} dZ^\theta_t, \quad t \in \mathbb{R}_+,
\]

i.e. the trend towards which talent mean-reverts may jump over time. Given this discontinuity, interpreting the current trend \( \bar{\theta} + a^2_t \) as the worker’s human capital stock is less appealing. Nevertheless, a model like this is still “smooth” enough to be interpreted as a one of human capital accumulation. The solution to the above SDE has the form

\[
\theta_t = \theta_0 e^{-\kappa t} + \bar{\theta}[1 - e^{-\kappa t}] + \int_0^t e^{-\kappa(t-s)}\kappa a^2_s ds + \sigma_{\theta} \int_0^t e^{-\kappa(t-s)} dZ^\theta_s
\]

and thus, \( L_t := \int_0^t e^{-\kappa(t-s)}\kappa a^2_s ds \) can be interpreted as a measure of the manager’s skills acquired over his working life. In contrast to the model analyzed in Section 4, how engaged is the worker in production today has now a less persistent effect on future talent. Suppose that effort is costly according to a function \( g : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \) that is strictly increasing in both arguments. Thanks to Proposition 19, the manager will solve

\[
\max_{a \in A} \int_0^\infty e^{-rt} \left( a^1_t \beta_t \lambda_t + \rho_t (\bar{\theta} + a^2_t) - g(a^1_t, a^2_t) \right) dt
\]

so the optimal effort strategy \( (a^1_t^*, a^2_t^*)_{t \geq 0} \) satisfies the first order condition \( \nabla g(a^1_t^*, a^2_t^*) = (\beta_t \lambda_t, \rho_t) \), \( t \geq 0 \). The usual interpretation in terms of marginal costs and marginal benefits applies.
6 Conclusion

We develop a continuous-time model of career concerns that includes a wide class of stochastic processes for talent and, at the same time, that incorporates human capital accumulation over the working life of individuals. The outcome is an extremely flexible framework that allows us to obtain clear insights on how belief-distortion mechanisms operate and on how incentives throughout the agents’ careers are shaped by the learning structure of the environment. Within this general setting, we are able to establish two important insights concerning incentives in career concerns models: First, that human capital accumulation is able to create long-term benefits from belief-distortion and, second, that inefficiencies are a robust phenomena in environments in which wages are based on estimates of workers’ abilities.

In the traditional career concerns setting, incentives are determined by two forces: how beliefs immediately react to new information and how persistent these distortions are over time. While a high sensitivity to changes in output favors the career concerns motives, beliefs decaying at fast rates reduce the long-term benefits from exerting effort. Because the non-observability of talent prevents the market distinguishing between changes in output due to the signal’s noise and output changes coming from talent improvement, the market will give too little weight to past output observations, reducing the manager’s future benefits from exerting effort and thus generating inefficiencies even in the long run.

The opportunity to accumulate human capital while at work enable workers to extract further gains from belief-distortion, without the need to assume that human capital is a direct input in production. In fact, because its beliefs decay too fast, the market will constantly underestimate the manager’s talent after an improvement in the latter’s human capital stock has taken place. As a consequence, the manager is able to generate an additional output stream that will constitute a surprise from the market’s perspective and, hence, it will be internalized by the former as an increase in the reputational component of future wages. Nevertheless, since the manager is able to obtain only a fraction of this additional output, inefficiencies are exacerbated when experience accumulation is incorporated.

In a example when human capital accumulates according to a standard physical-capital
dynamic, the model predicts effort levels that decay over time as the consequence of longer time-series of output that, in turn, yield more precise estimates of talent. It also predicts that individuals that suffer of less human capital depreciation would choose higher levels of education, because they face the highest returns from having more human capital upon entering the labor market.

To conclude, our model presents a general and clean treatment of the interaction between reputational concerns and human capital accumulation when jobs provide workers with general skills. The main message of the paper is that belief-distortion incentives appear in two channels: in daily observations through the direct effect of effort on output, and in long-term performance through the effect that human capital accumulation has on future skills.

References


Appendix

The proof of Propositions 3 and the first order condition in Proposition 5 are special cases of Proposition 6, which we prove next.

Proof of Proposition 6: Suppose that the market conjectures that the manager will follow a deterministic strategy \(a^* := (a_t^*)_{t \geq 0}\). Since wages take the form \(w_t = m_t^* + a_t^*\), only \((m_t^*)_{t \geq 0}\) matter for incentives.
Also, the fact that for each \( t \geq 0 \), \( \theta_t(\cdot) \) is a deterministic functional of paths of the form \((y_s : 0 \leq s \leq t)\), implies that the trajectory of human capital conjectured by the market, \( (\overline{\theta}_t(a^*)) \), is fixed at time zero and unaffected by the manager’s effort choice. The manager’s beliefs evolve according to

\[
dm_t = -\kappa(m_t^* - \overline{\theta}_t(a^*))dt + \frac{\gamma_t}{\sigma_{\xi}} \frac{d\xi_t - (m_t^* + a_t^*)dt}{dZ_t^*}
\]

where \( \gamma_t \) follows the dynamic (7) and \( Z^{a^*} \) is a Brownian motion from the market’s perspective. The solution to the above SDE is given by

\[
m_t^* = \exp \left( -\int_0^t \delta_s ds \right) m_0 + \int_0^t \exp \left( -\int_s^t \delta_u du \right) \left[ s\overline{\theta}_s(a^*)ds + \beta_s(d\xi_s - a_s^*ds) \right]
\]

where \( \beta_t := \frac{\gamma_t}{\sigma_{\xi}} \) and \( \delta_t := \beta_t + \kappa \) for all \( t \geq 0 \). Since from the manager’s perspective \((\overline{\theta}_t(a^*))_{t \geq 0}\) and \( a^* \) are exogenously given, incentives are determined only by

\[
G_t := \int_0^t \exp \left( -\int_s^t \delta_u du \right) \beta_s d\xi_s.
\]

\( G_t \) is only component of wages that can be distorted via output observations. Let \((m_t)_{t \geq 0}\) denote the manager’s posterior belief process of his own talent when he follows any strategy \( a := (a_t)_{t \geq 0} \). We assume that is evolves according to an SDE of the form

\[
dm_t = -\kappa(m_t - \overline{\theta}_t(a))dt + \sigma_t dZ_t
\]

where \( Z := (Z_t)_{t \geq 0} \) is a Brownian motion from the manager’s standpoint (in this specification we allow a manager with potentially more information than the market about himself). Moreover, the process \( Z^a := (Z^a_t)_{t \geq 0} \) defined by

\[
Z^a_t := \frac{1}{\sigma_{\xi}} \left( \xi_t - \int_0^t (a_s + m_s)ds \right), \ t \geq 0
\]

is also a Brownian motion from the manager’s perspective that is correlated to \( Z \). By Girsanov’s theorem we can write output from the manager’s perspective as

\[
d\xi_t = (m_t + a_t)dt + \sigma_{\xi} dZ^a_t, \ t \geq 0.
\]

Inserting this into the expression for \( G_t \) gives us how the manager evaluates belief-distortions on the market’s side,

\[
G_t := \int_0^t \exp \left( -\int_s^t \delta_u du \right) \beta_s \left[ (m_s + a_s)ds + \sigma_{\xi} dZ^a_s \right]
= \int_0^t \exp \left( -\int_s^t \delta_u du \right) \beta_s \left[ e^{-\kappa s}m_0 + \int_s^t e^{-\kappa(s-u)}(\kappa \overline{\theta}_u(a)du + dZ_u) + a_sds + \sigma_{\xi} dZ^a_s \right]
\]

where we used that \( m_s = e^{-\kappa s}m_0 + \int_0^s e^{-\kappa(s-u)}(\kappa \overline{\theta}_u(a)du + dZ_u), s \geq 0 \). The first term in \( G_t \) is unaffected by the effort decision, and therefore the manager’s optimization problem is reduced to

\[
\max_{a \in A} \int_0^\infty e^{-rt} \left( \int_0^t \exp \left( -\int_s^t \delta_u du \right) \beta_s \left[ \int_0^s e^{-\kappa(s-u)}(\kappa \overline{\theta}_u(a)du + dZ_u) + a_sds + \sigma_{\xi} dZ^a_s \right] - g(a_t) \right) dt
\]

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For any strategy that the manager follows, \( Z^a \) and \( Z \) will be a Brownian motions from the manager’s point of view. Moreover, since of any initial condition \( \gamma_0, (\beta_t)_{t \geq 0} \) and \( (\delta_t)_{t \geq 0} \) are uniformly bounded, all the stochastic integrals above will have zero expectation. As a consequence, the problem is reduced to

\[
\max_{a \in A} \mathbb{E}^\pi \left[ \int_0^\infty e^{-rt} \left( \int_0^t \exp \left( -\int_s^t \delta_u du \right) \beta_s \left( \int_s^t e^{-\kappa(s-u)} \bar{\theta}(a) du + a_s ds \right) \right. \right. 
\]

\[
- \left. \left. g(a_t) \right) dt \right] \quad (43)
\]

Integration by parts and the fact that \( \delta_t = \beta_t + \kappa \) yield

\[
\int_0^t \exp \left( -\int_s^t \delta_u du \right) \beta_s \int_s^t e^{-\kappa(s-u)} \bar{\theta}(a) du = \kappa e^{-\kappa t} \int_0^t e^{\kappa \tau} \bar{\theta}(a) d\tau 
\]

\[
- \kappa \exp \left( -\int_0^t \delta_u du \right) \int_0^t \exp \left( \int_0^s \delta_u du \right) \bar{\theta}(a) d\tau \quad (44)
\]

With this in hand, the manager’s objective function has 3 integrals of the form (up to multiplicative constants)

\[
I := \int_0^\infty e^{-rt} \left[ \exp \left( -\int_0^t \tau_s ds \right) \right. 
\]

\[
\int_0^t \exp \left( \int_0^s \tau_u du \right) \nu_s \right) dt \]

where \( \tau = \delta \) or \( \kappa \) and \( \nu = \alpha \) or \( \bar{\theta}(a) \). A direct application of Fubini’s theorem implies that

\[
I = \int_0^\infty \exp \left( \int_0^t \tau_s ds \right) \nu_t \int_t^\infty \exp \left( -\int_0^s (r + \tau_u) du \right) ds dt = \int_0^\infty e^{-rt} \int_t^\infty \exp \left( -\int_0^s (r + \tau_u) du \right) ds dt
\]

Defining \( \rho_t = \frac{\kappa}{r+\kappa} - \kappa \lambda_t \) and \( \lambda_t := \int_t^\infty \exp \left( -\int_t^s (r + \delta_u) du \right) ds \), the manager will solve

\[
\max_{a \in A} \int_0^\infty e^{-rt} \left( a_t \beta_t \lambda_t - g(a_t) + \rho_t \bar{\theta}(a_t) \right) dt 
\]

(45)

concluding the proof of Proposition 6.

Observe that when there is no human capital accumulation, \( \bar{\theta}(\cdot) \equiv \bar{\theta} \in \mathbb{R}_+ \), so the last term in the manager’s problem can be ignored. In this case, point-wise maximization implies that, in any interior equilibrium, the optimal strategy satisfies \( g'(a^*_t) = \beta \lambda_t \), \( t \geq 0 \). This proves the first order condition in Proposition 5. Furthermore, if in addition learning is stationary, \( \beta = \beta := \gamma^*/\sigma^2 \) and \( \delta_t = \delta := \beta + \kappa \), for all \( t \geq 0 \), where \( \gamma^* \) is the unique stationary solution of (7)

\[
\gamma^* = \sigma^2 \left( \sqrt{\kappa^2 + \sigma^2_a/\sigma^2_\xi} - \kappa \right).
\]

This yields an optimal effort strategy that is constant over time and characterized by the first order condition \( g'(a^*) = \frac{\beta}{\tau_\kappa} \). The rest of the statement in Proposition 3 are straightforward, concluding the proof.

\[\blacksquare\]

**Proof of Proposition 5.** We only need to show that \( l_t := \beta_t \lambda_t \) is decreases over time, where \( \beta_t = \gamma_t/\sigma^2_\xi \), \( \lambda_t = \int_t^\infty \exp \left( -\int_t^s (r + \delta_u) du \right) ds \) and \( \delta_t = \beta_t + \kappa \), \( t \geq 0 \). Observe that

\[
\frac{d \log(l(t))}{dt} = \frac{\gamma_t}{\gamma_t} + \frac{\gamma_t}{\sigma^2_\xi} + \kappa - \frac{1}{\int_t^\infty \exp \left( -\int_t^s (r + \kappa + \gamma_u/\sigma^2_\xi) du \right) ds} 
\]

(46)
Suppose $\gamma_t > \gamma^*$, which occurs if and only if $\dot{\gamma}_t \leq 0$. Then, $\lambda_t = \int_t^\infty \exp \left( -\int_t^s (r + \kappa + \gamma_u/\sigma_u^2)du \right) ds < 1$, implying that

$$\frac{d \log(l(t))}{dt} < \frac{\dot{\gamma}_t}{\gamma_t} + \frac{\gamma_t}{\sigma_t^2} - \frac{\gamma^*}{\sigma_t^2}$$

Finally, from the ODE that governs $\gamma_t$, we observe that $\dot{\gamma}_t/\gamma_t + \gamma_t/\sigma_t^2 = -2\kappa + \sigma_t^2/\gamma_t$, so

$$\frac{d \log(l(t))}{dt} < -2\kappa + \frac{\sigma_t^2}{\gamma_t} - \frac{\gamma^*}{\sigma_t^2} = 0$$

by definition of $\gamma^*$. When $\dot{\gamma}_t \geq 0$ (and so $\gamma_t < \gamma^*$ for all $t \geq 0$) an analogous argument shows that $l_t$ increases over time (the above inequalities just reverse). This concludes the proof.

Proof of Proposition 9: By the Pontryagin Maximum Principle for infinite horizon (Halkin 1974), if $a^* := (a^*_t)_{t \geq 0}$ is an optimal control then there exists $\mu \geq 0$ and a piecewise continuously differentiable function $^9 q : \mathbb{R}_+ \to \mathbb{R}$ s.t.

I. $[\mu, 0] \neq 0$;

II. $\dot{q}_t - rq_t = -\frac{\partial}{\partial x} \mathcal{H}(t, x, a^*_t, \mu, q_t) \big|_{x=\theta_t}$, a.s.

III. $\mathcal{H}(t, \theta_t, a^*_t, \mu, q_t) \geq \mathcal{H}(t, \theta_t, a^*_t, \mu, q_t)$, for all $t \geq 0$, $a \in A$.

where the Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H}(t, x, a, \mu, y) := \mu [a\beta_1 \lambda_t - g(a) + \rho t x] + y f(x, a).$$

with $\rho_t, \beta_t$ and $\lambda_t$ as in the Proposition.

Now we will prove that, under the hypothesis of the proposition, $\mu \neq 0$. Replacing the expression for the Hamiltonian in II yields the ODE

$$\dot{q}_t = q_t \left[ r - \frac{\partial f}{\partial x}(\theta_t(a^*_t), a^*_t) \right] - \mu \rho_t \tag{47}$$

Recall that the set of times where the last ODE does not hold is the set of points at which $a^*$ is discontinuous (at those points $q$ is not differentiable). By definition of piece-wise continuity, for any $T > 0$ there is only a finite number of times less than $T$ at which the optimal control is discontinuous. Therefore, II holds for intervals $[0, t_1], \{[t_i, t_{i+1}] \mid i \in \mathbb{N}\}$ such that their union is the real line. Moreover, since $q$ is continuous, it must that the solution of the above ODE at any subinterval $[t_i, t_{i+1}]$ must satisfy

$$q_{t_i^+} = q_{t_i}$$

where we understand that $q_{t_i^+}$ is the limit as $t$ decreases to $t_i$ of the solution to the ODE (47) in $[t_i, t_{i+1}]$ with final condition $q_{t_{i+1}}, i \geq 1$. The proof is based on the following

That is, a differentiable function which derivative is piecewise continuous

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Lemma 16. If $\mu = 0$, then either $a^* \equiv 0$ or $a^* \equiv \bar{a}$.

Proof of the Lemma:

Suppose $\mu = 0$. Then it must be that the following relationship holds for $t \in [0, t_1^*]$:

$$q_t = q_0 \exp \left( \int_0^t \left( r - \frac{\partial f}{\partial \theta} (\bar{\theta}_s(a^*), a^*_s) \right) ds \right)$$

where $s \mapsto \bar{\theta}_s(a^*)$ is the trajectory generated by $(a^*_s)_{s \in [0, t_1^*]}$. If $q_{t_1} = 0$, then $q_0 = 0$, contradicting I. Thus, $q_{t_1} \neq 0$ and therefore $q$ cannot vanish in $[0, t_1]$. Suppose that $q > 0$ over this set. Then, the maximum condition II implies that the optimal control must satisfy

$$a^*_s \in \arg \max_{a \in A} q_s f(\bar{\theta}_s(a^*), a), \forall s \in [0, t_1]$$

But $q_s f(\bar{\theta}_s(a^*), \cdot)$ is increasing, and thus $a^*_s \equiv \bar{a}$ for all $s \in [0, t_1]$. As a consequence, whenever $\mu = 0$, if $q_{t_1} > 0$ then the optimal control takes the maximum possible value in the first interval. If in turn, $q_{t_1} < 0$ the same reasoning shows that the optimal control will take the minimum value over the same set, this because $q_s f(\bar{\theta}_s, \cdot)$ would be decreasing for all $s \in [0, t_1]$. In the remainder of the proof, we assume without loss of generality that $q_{t_1} > 0$ (the other case is analogous). If $q_{t_2} \leq 0$, then (47) in $(t_1, t_2]$ implies that $q_s \leq 0$ in the same interval. Therefore

$$q_{t_1}^+ := \lim_{s \searrow t_1} q_s \leq 0 < q_{t_1}$$

contradicting the fact that $q$ is continuous. Hence, $q_{t_2} > 0$ implying that $q$ is strictly positive in $(t_1, t_2]$ and thus the optimal control must take value $\bar{a}$ over that interval. Proceeding inductively, if $q_{t_i} > 0$ then $q_{t_i} > 0$ for all $i = 0, 2, 3...$ and by the maximum condition $\bar{a}$ is the optimal control. The same reasoning allows us to conclude that when $q_{t_1} < 0$, $a^* \equiv 0$ must be optimal. This concludes the proof.

The previous Lemma shows that when an optimal control exists and is neither identically zero nor equal to $\bar{a}$, then $\mu > 0$. When this is the case it is clear that we can assume $\mu = 1$ (equivalently, redefine $q$ as $q/\mu$ and note that $q/\mu$ satisfies all the conditions of the theorem). This proves part (i) in the proposition. Finally part (ii) is simply the necessary condition that an unconstrained optimum must satisfy. This concludes the proof.

Proof of Proposition 11: If $V(t, \bar{\theta}_t) = \infty$ the first part of the Proposition is trivially true. Suppose that is finite. Let $a^* := (a^*_s)_{s \geq t}$ be the optimal control (a function of time) that attains value $V(t, \bar{\theta}_t)$ when starting from the level $\bar{\theta}_t \geq 0$ at time $t$. In the same vein let $(\bar{\theta}_s(a^*; \bar{\theta}))_{s \geq t}$ denote the path of human capital generated by the feasible control $a^*$ when starting from point $\bar{\theta} \in \mathbb{R}_+$ at time $t$, that is, the solution to

$$d\bar{\theta}_s = f(\bar{\theta}_s, a^*_s), \ s > t, \ \bar{\theta}_t = \bar{\theta}$$
Since the solutions of these two ordinary differential equations cannot cross (they differ only in the initial condition), it must be the that
\[ \bar{\theta}_s(a^*; \bar{\theta}) > \bar{\theta}_s(a^*; \bar{\theta}_t), \forall s \geq t \]
which implies that \( V[t, \bar{\theta}; a^*] > V[t, \bar{\theta}; a^*] = V(t, \bar{\theta}_t) \), so \( V(t, \cdot) \) is increasing. If, moreover, it turns out to be differentiable in a neighborhood \( \Theta \) of \( \bar{\theta}_t \) of human capital accumulation. Implicit in the formulation of Proposition 9 is the fact that \( \Phi(0, t, s) \) is the continuation-value function, concluding the proof.

\[ \blacksquare \]

**Proof of Proposition 12:** Let \( \Phi \) continuously differentiable satisfying (HJ) and the limit condition (ii) in the Proposition. For \( x \geq 0 \), denote by \( u^*(t, x) \) the solution to the problem

\[
\sup_{u \in A} \left\{ e^{-rt} \left[ \beta \lambda_t a - g(a) + \rho_t x \right] + \frac{\partial \Phi}{\partial t}(t, x) + \frac{\partial \Phi}{\partial x}(t, x) f(x, u) \right\} = 0
\]

(48)

Let \( a := (a_t)_{t \geq 0} \) be a feasible control and \( (\bar{\theta}_t(a; \bar{\theta}))_{t \geq 0} \) be the trajectory generated by \( a \) starting from \( \bar{\theta} \in \mathbb{R} \). Consider the function

\[ G_t[a] = \int_0^t e^{-rs} \left\{ \beta_s \lambda_s a_s - g(a_s) + \rho_s \bar{\theta}_s(a; \bar{\theta}) \right\} ds + \Phi(t, \bar{\theta}_t(a; \bar{\theta})) \]

Observe that for all \( t \geq 0 \)

\[
dG_t[a] = \left\{ e^{-rt} \left[ \beta_t \lambda_t a_t - g(a_t) + \rho_t \bar{\theta}_t(a; \bar{\theta}) \right] + \frac{\partial \Phi}{\partial t}(t, \bar{\theta}_t(a; \bar{\theta})) + \frac{\partial \Phi}{\partial x}(t, \bar{\theta}_t(a; \bar{\theta})) f(\bar{\theta}_t(a; \bar{\theta}), a_t) \right\} dt
\]

\[
\leq 0
\]

because \( \Phi \) solves (HJ). So, \( \lim_{t \to +\infty} G_t[a] \leq G_0[a] = \Phi(0, \bar{\theta}_0(a; \bar{\theta})) = \Phi(0, \bar{\theta}) \). Also, thanks to (ii)

\[ \lim_{t \to +\infty} G_t[a] = \int_0^\infty e^{-rs} \left\{ \beta_s \lambda_s a_s - g(a_s) + \rho_s \bar{\theta}_s(a; \bar{\theta}) \right\} ds =: V[0, \bar{\theta}; a] \]

(49)

Hence \( \Phi(0, \bar{\theta}) \) is an upper bound on the manager’s discounted utility. Finally, since \( (t, x) \mapsto u^*(t, x) \) is the maximizer of the right hand side in (HJ), we obtain \( G_t[a^*] = G_0 = \Phi(0, \bar{\theta}) \) for all \( t \geq 0 \). Again, the transversality condition (ii) allows us to conclude that

\[ V[0, \bar{\theta}; a^*] := \lim_{t \to +\infty} G_t[a^*] = G_0 = \Phi(0, \bar{\theta}) \]

(50)

so the upper bound is attained when choosing the feasible control \( a^* := (u^*(t, \bar{\theta}_t))_{t \geq 0} \). Thus \( a^* \) is optimal and \( \Phi(0, \bar{\theta}) \) is the value of \( \mathcal{P}(\bar{\theta}) \). The same analysis could have been done by instead starting from time \( t \) instead of time zero. This implies that \( \Phi(t, \bar{\theta}) \) is the continuation-value function, concluding the proof.

\[ \blacksquare \]
Proof of Proposition 13: We will find a solution of the form $\Phi(t, x) = b_t + c_t x$. Plugging this in (HJ) we get

$$\sup_{u \in A} \left\{ e^{-rt} [\beta_t \lambda_t u - g(u)] + \frac{db_t}{dt} + \frac{dc_t}{dt} x + c_t [\alpha u - \phi x] \right\} = 0$$  \hspace{1cm} (51)

Suppose that $(c_t)_{t \geq 0}$ satisfies the ODE $dc_t + (e^{-rt} \rho_t - \phi c_t) dt = 0$ with transversality condition $\lim_{t \to \infty} e^{-\phi t} c_t = 0$. Then,

$$c_t = e^{-rt} \int_0^\infty e^{-(r + \phi)(s-t)} \rho_s ds, \ t \in \mathbb{R}_+$$ \hspace{1cm} (52)

In fact, since $(\rho_t)_{t \geq 0}$ is bounded we get a stronger transversality condition: $c_t \to \infty$ as $t \to \infty$. The (HJ) equation then becomes $\sup_{u \in A} \left\{ e^{-rt} [\beta_t \lambda_t u - g(u)] + \frac{db_t}{dt} + c_t [\alpha u - \phi x] \right\} = 0$ which yields the first order condition

$$g'(u^*_t) = \beta_t \lambda_t + \alpha \mu_t, \ t \in \mathbb{R}_+$$

and that $b_t$ must satisfy $db_t = e^{-rt} [g(u^*_t) - g(u^*_\infty)] dt$. Observe that since $(u^*_t)_{t \geq 0}$ is bounded ($(\rho_t)_{t \geq 0}$ is bounded and $(\beta_t \lambda_t)_{t \geq 0}$ is decreasing and non-negative) the last condition has as a solution

$$b_t := e^{-rt} \int_0^\infty e^{-r(s-t)} [g'(u^*_s) u^*_s - g(u^*_s)] ds, \ t \in \mathbb{R}_+$$

and moreover, $b_t \to 0$ as $t \to \infty$. Therefore, we have found a function $\Phi(t, x) = b_t + c_t x$ of class $C^1 \mathbb{R}_+^d$ such that it satisfies (HJ). We now need to show that this satisfies (ii). To see this, fix an initial condition $\vec{\theta} \geq 0$. For any feasible control $a$ the path $t \mapsto \vec{b}_t(a; \vec{\theta})$ takes values in the interval $[0, \max \{\vec{\theta}, \ell/\phi\}]$, so it remains bounded all the time. Because $b_t, c_t \to 0$ as $t \to \infty$, we trivially conclude that $\lim_{t \to \infty} b_t + c_t \vec{b}_t(a; \vec{\theta}) = 0$. Applying Proposition 12 we conclude that $\Phi(t, x)$ is the continuation value and that $(u^*_t)_{t \geq 0}$ is an optimal control. This concludes the proof.

Proof of Proposition 14: Recall that $\rho_t = \frac{\kappa}{\kappa + \gamma} - \kappa \lambda_t$ where

$$\lambda_t := \int_t^\infty \exp \left( - \int_1^s (r + \delta u) du \right) ds, \ t \geq 0.$$ 

Direct calculations show that $\lambda_t$ satisfies the ODE $d\lambda_t = ([r + \delta_t] \lambda_t - 1) dt$. If $\gamma_t$ is decreasing over time $(\gamma_0 > \gamma^*)$, so will be $\delta_t = \gamma_t^* / \sigma_t^2 + \kappa$, which implies that $\lambda_t > \frac{1}{r + \delta_t}$ for all $t \geq 0$. As a consequence $\lambda_t$ is increasing and, furthermore, bounded above by $\frac{1}{r + \delta}$ (with $\delta := \gamma^* / \sigma_t^2 + \kappa$), so it converges. With this in hand, we conclude that the marginal benefit from an extra unit of human capital at time $t$, $\rho_t$, decreases over time and will also converge (it is bounded below by zero). Because of this,

$$\mu_t = \int_t^\infty e^{-(r + \phi)(s-t)} \rho_s ds < -\frac{\rho_t}{r + \phi}$$

for all $t \geq 0$. Finally, observing that $(e^{\gamma t} \mu_t)_{t \geq 0}$ satisfies the ordinary differential equation $d\mu_t = ([r + \phi] \mu_t - \mu_t) dt$, we conclude. The case $\gamma_0 < \gamma^*$ follows from an analogous argument.