Scheduling on a machine with varying speed

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Problem Definition

Input

1. n jobs:
   - work volume v_j.
   - weight w_j.
2. Speed function $s : \mathbb{R}_+ \to \mathbb{R}_+$.

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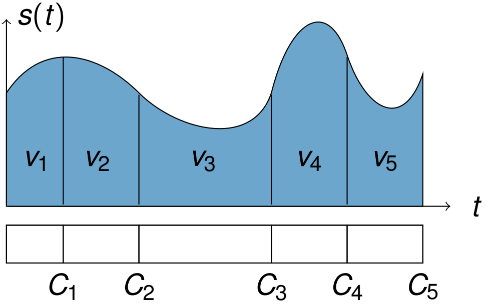

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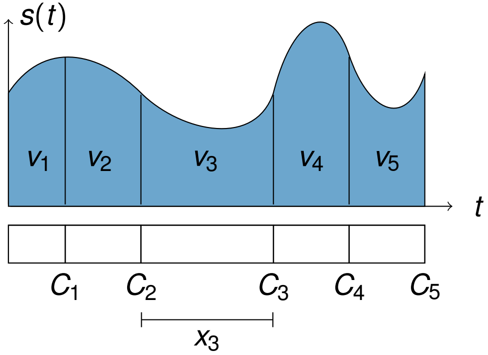

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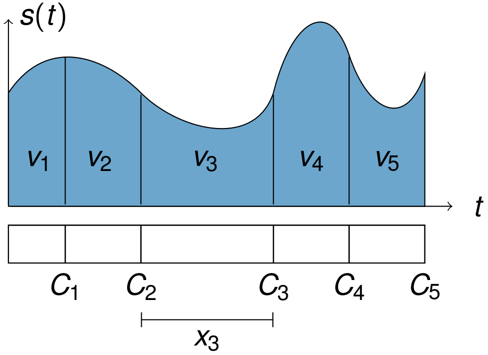

Objective

$$\min \sum_j w_j C_j$$

Problem Definition

Equivalent problem

1. Unit speed machine ($s \equiv 1$).
2. Different Objective:

$$\min \sum_j w_j f(C_j') = \sum_j w_j \underbrace{f(\sum_{k \le j} v_k)}_{=C_j},$$

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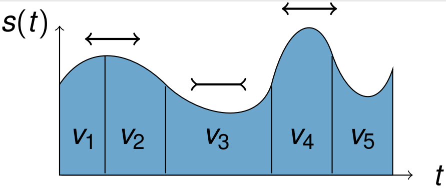
$$\text{where } f(t) := \inf\{b : \int_0^b s(\xi)d\xi \ge t\}$$



Approximation algorithms

Definition

For $\alpha \geq 1$, a solution S is α-approximate if

$$\mathit{cost}(S) \leq \alpha\, \mathit{cost}(S_{\mathrm{OPT}}).$$

Known Results

- 4-approx. (for all speeds functions simultaneously)

  Epstein et al. 2012 (SICOMP 2012)

- PTAS for $\sum_j w_j f(C_j)$ if f is concave (s non-decreasing).

  Stiller & Wiese (ISAAC 2010)

- $\sum_j w_j f(C_j)$ strongly NP-hard for piece-wise linear f (s piece-wise constant).

  Höhn & Jacobs (LATIN 2012)

- $O(1)$-approx. for $\min \sum_j f_j(C_j)$.

  Bansal & Pruhs (FOCS 2010)

- $(2+\varepsilon)$-approx. for $\sum_j f_j(C_j)$.

  Shmoys & Cheung (APPROX 2011)

Results Overview

Theorem

There exists a PTAS for any given function s, i.e., for any $\varepsilon > 0$ there exists a polynomial algorithm that returns a $(1+\varepsilon)$-approximate solution.

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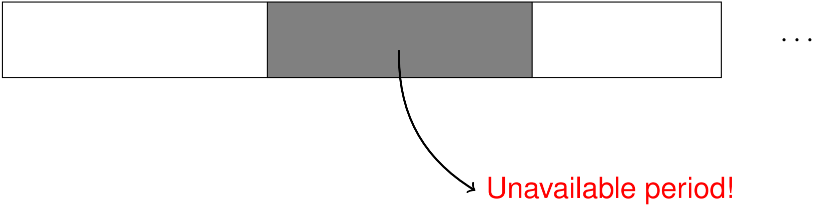
Energy

Several results for dynamic speed allocation.

Difficulty



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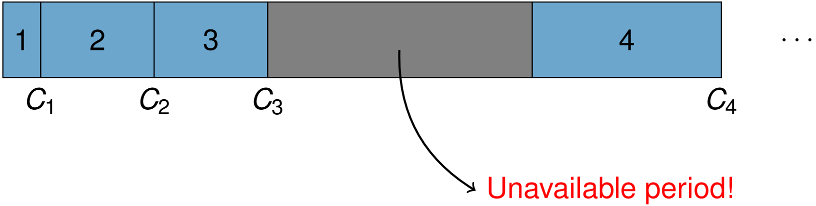

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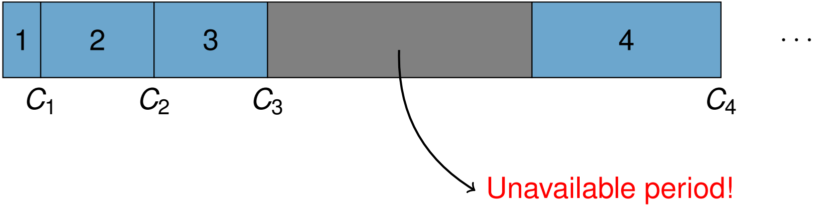
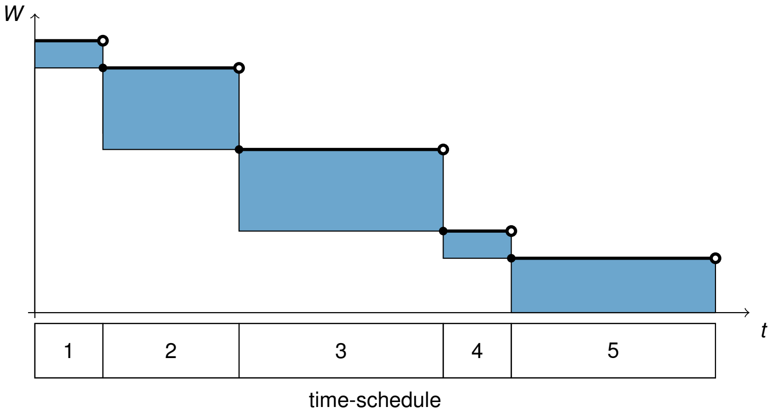


Observation

Rounding in the time axis might be problematic!

Dual Schedules

2D-Gantt Charts [Eastman et al. '64]



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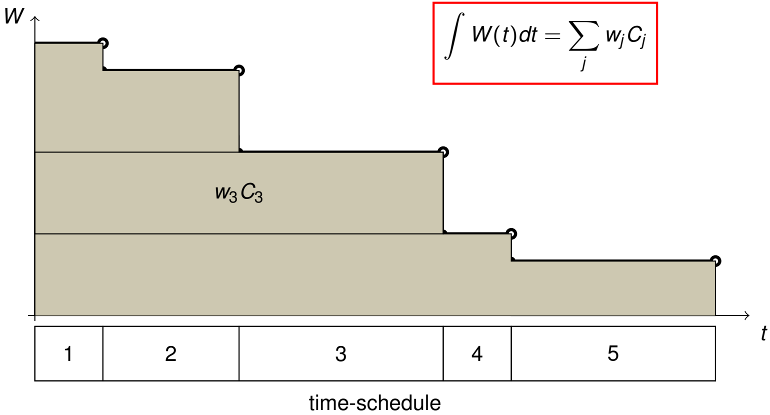

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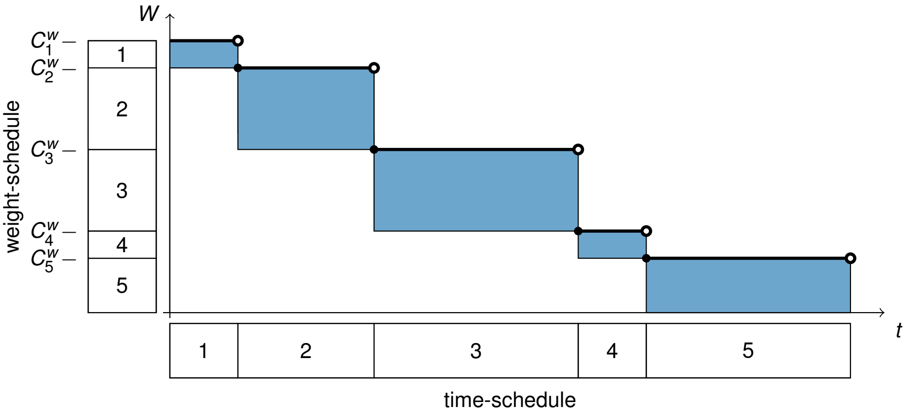

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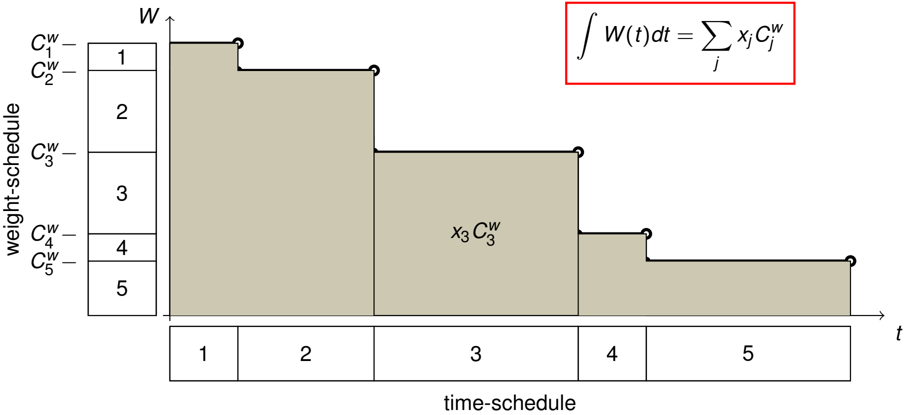

$$\boxed{\int W(t)dt = \sum_j x_j C_j^w}$$

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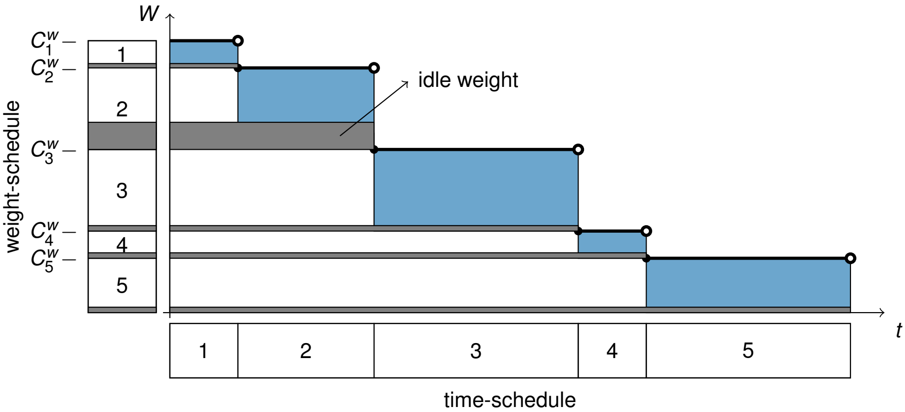

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Dynamic Program
(exponential time)

Basics

- Round $w_j := (1+\varepsilon)^k$ for $k \in \mathbb{Z}$.
- Weight intervals $I_u = ((1+\varepsilon)^{u-1}, (1+\varepsilon)^u]$.
- $\mathcal{F}_u :=$ collection of possible subsets of jobs to be processed before $(1+\varepsilon)^u$.

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DP Table

For each u and $S \in \mathcal{F}_u$:

$$T(u,S) := (1+\varepsilon)\text{-approximation of scheduling } S \text{ in } [0,(1+\varepsilon)^u]$$

$$= \min \left\{ T(u-1,S') + \sum_{j\in S\setminus S'} x_j(1+\varepsilon)^u \;:\; S' \in \mathcal{F}_{u-1}, S' \subseteq S \right\}.$$

Key Ideas

- Light jobs: $w_j \leq \varepsilon^2 S_j^w$,
  $\rightsquigarrow$ greedily order jobs by w_j/v_j.

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Reducing table's size

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Lemma

We can construct sets $\mathcal{F}_u$ of constant size.

Sets $\mathcal{F}_u$ are independent of the speed!

Main result

Theorem

There exists an efficient PTAS for minimizing $\sum_j w_j C_j$ on a machine with variable speed.

Theorem

There exists an efficient PTAS for minimizing $\sum_j w_j f(C_j)$ for any non-decreasing f on a unit speed machine.

Energy Constraint Scheduling

Model Definition

- Available set of speeds $\mathcal{S} \subseteq \mathbb{R}_+$.
- Speed $s \in \mathcal{S} \Rightarrow$ power $= s^\alpha$ ($\alpha = 2, 3$ usually).
- Total energy available E.
- Obj: $\min_j w_j C_j$.

Results

Theorem (Indep. by Vazquez '12 and Carrasco et al. '11)

If $\mathcal{S} = \mathbb{R}_+$ then the optimal value is

$$\min_{\textit{permutation } \pi} \frac{1}{E^{\frac{1}{\alpha-1}}} \cdot \left(\sum_{j=1}^{n} v_{\pi(j)} \left(\sum_{k \geq j} w_{\pi(k)} \right)^{\frac{\alpha-1}{\alpha}} \right)^{\frac{\alpha}{\alpha-1}} .$$

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Complexity Open!

Results
cont...

Theorem

- *If $|\mathcal{S}| = 2$ then the problem is NP-hard.*
- *There exists a PTAS if $\mathcal{S}$ is part of the input.*

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Similar ideas as previous PTAS for given speeds.