
Volume 39
Issue 9
September 2012
ISSN 0305-0548
ELSEVIER
computers & operations research
Editor: Stefan Nickel
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Computers & Operations Research 39 (2012) 2232–2241

Contents lists available at SciVerse ScienceDirect

Computers & Operations Research

journal homepage: www.elsevier.com/locate/caor

A branch-and-cluster coordination scheme for selecting prison facility sites under uncertainty

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ARTICLE INFO

Available online 15 November 2011

Keywords:
Prison facility location
Stochastic integer programming
Multi-period scenario tree
Branch-and-fix coordination

ABSTRACT

A multi-period stochastic model and an algorithmic approach to location of prison facilities under uncertainty are presented and applied to the Chilean prison system. The problem consists of finding locations and sizes of a preset number of new jails and determining where and when to increase the capacity of both new and existing facilities over a time horizon, while minimizing the expected costs of the prison system. Constraints include maximum inmate transfer distances, upper and lower bounds for facility capacities, and scheduling of facility openings and expansion, among others. The uncertainty lies in the future demand for capacity, because of the long time horizon under study and because of the changes in criminal laws, which could strongly modify the historical tendencies of penal population growth. Uncertainty comes from the effects of penal reform in the capacity demand. It is represented in the model through probabilistic scenarios, and the large-scale model is solved via a heuristic mixture of branch-and-fix coordination and branch-and-bound schemes to satisfy the constraints in all scenarios, the so-called branch-and-cluster coordination scheme. We discuss computational experience and compare the results obtained for the minimum expected cost and average scenario strategies. Our results demonstrate that the minimum expected cost solution leads to better solutions than does the average scenario approach. Additionally, the results show that the stochastic algorithmic approach that we propose outperforms the plain use of a state-of-the-art optimization engine, at least for the three versions of the real-life case that have been tested by us.

1. Introduction

The prison location and sizing problem seeks to find optimal locations and sizes (in terms of number of inmates) for a given number of new prison facilities, and to decide where and when to increase their capacities, as well as the capacities of existing jails, so as to minimize the expected cost of a prison system that satisfies all the demands for space. As a pilot case we use the Chilean prison system. We must also solve the inmate-to-jail assignment problem. We consider the case of Chile, where the penal population is comprised of detainees who stay in the prison system for a few days at most; of persons under trial, who are held in the prison system as long as the trial is not finished; and of convicted inmates, whose sentences go from a few months to life. Inmates belonging to the first two categories must remain in prisons that are close to the courts, because of frequent travel between prison and court. Convicted inmates can be far from court, but they need to be within a preset distance from their family's residence, for humanitarian reasons. Although there are small confinement cells for some of the detainees, most of the penal population share the same prisons, with internal separation.

Recently, the authority in charge of the prison system overtook the task of planning its prison system for a time horizon that was set to 20 years. Apart from the inherent difficulties of the jail location, assignment and sizing problem, a critical aspect of planning for such a long time horizon is the uncertainty in the forecast of the population. In addition to this uncertainty, criminal law reforms recently implemented throughout Chile are bringing large changes to historical trends. The reforms have introduced deep modifications in sentencing policies and increase uncertainty regarding future inmate population. The percentage of convicted offenders who are imprisoned is now smaller, but

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0305-0548/$ - see front matter © 2011 Published by Elsevier Ltd.
doi:10.1016/j.cor.2011.11.006

changes in the perceived efficiency of law enforcement will also impact the quantity and frequency of legal action.

An additional difficulty comes from the fact that the prison system is currently overcrowded. As a consequence, inmates, especially sentenced inmates, instead of being kept close to their family's hometown, are assigned to jails that still have space. Thus, when forecasting, one should not use historical tendencies in the prison population without taking this into account. The full description of the planning problem is presented in [18], where a model is proposed for the prison location problem, in which expected inmate population over a 20 year horizon is predicted via a flow model together with historical information for each Chilean province during the period 1992–1998. The uncertain demand comes from the effects of the penal reform on the capacity demand. The demand estimates were used to generate eight independent scenarios and the problem was solved using a minimum regret procedure for choosing good solutions in a two-stage simple recourse environment. There were two groups of four scenarios: the first four were simple projections of the current population of the prison system, with different values for police effectiveness and different outcomes of the criminal laws reforms. The remaining four scenarios considered the same parameters, but the projections were corrected by taking into account the overcrowding effect. Because of the great uncertainty in the forecast, the different scenarios pose very different requirements as far as location and size of the facilities are concerned. The problem was addressed by stochastic optimization.

In contrast to the approach in [18], in this paper uncertainty will be represented via a multi-period scenario tree, and we present a compact full recourse model. A heuristic combination of the branch-and-fix coordination (BFC) algorithmic framework [2,9,10] and of a branch-and-bound scheme, so-called branch-and-cluster coordination (BCC) scheme, is developed to coordinate the management of feasible decisions under the various scenarios. The proposed minimum expected cost strategy compares favorably with the average scenario strategy. The stochastic algorithmic approach that we propose outperforms the plain use of a state-of-the-art optimization engine, at least for the three versions of the real-life case that we have tested. The results of the paper can be easily extended to other facility location and sizing problems under uncertainty of future demand. Other examples of this type of problem appear when locating any public (hospitals) or private (malls) facilities in areas of uncertain development.

The remainder of the paper is structured as follows. Section 2 introduces the background to the site selection problem and previous work on the subject. Section 3 describes the theoretical framework for the minimum expected cost strategy employed in the paper. Section 4 sets up the mixed integer deterministic equivalent model representing the siting problem. Section 5 presents the proposed solution algorithm. Section 6 reports the computational experience and, finally, Section 7 draws conclusions.

2. The prison system

Gendarmería de Chile is the Chilean government entity responsible for administering the country's correctional facilities. Some 65,000 persons are housed by the national prison system, falling into three categories, which depend on the status of the criminal proceedings against them. They are as follows:

1. Detainees: persons in custody awaiting formal charges.
2. Charged inmates: persons formally charged (arraigned) and awaiting trial or sentencing.
3. Convicted inmates: persons tried and sentenced, whether serving in open or in closed facilities.

The total prison population, made up of prisoners in all three categories, can also be classified according to the type of prison infrastructure. Institutional statistics reveal that about 31,000 persons were held in closed prisons (where inmates have to remain at all times, until their sentence expiry date), and 34,000 in the open or semi-open system (in which inmates can be released for some periods of time, as for example, during the day). The prison population is assigned to the various penal institutions based on criteria such as gender and type of sentence received. In this work, we restrict our analysis to inmates held in the closed prison system.

Each Chilean province consists of one or more court jurisdictions depending on its population. Each jurisdiction is located entirely within a single province, and all inmates in a jurisdiction are assigned to one of its facilities.

In order to ensure sufficient capacity for an uncertain inmate population, the system must decide which *existing* institutions should be expanded, when such expansions should be implemented and what the final capacities should be. The size, the timing of the opening and the location of *new* institutions must also be decided.

The problem considers the costs of opening and expanding facilities subject to a series of physical and legal constraints. The system must be designed so as to meet the uncertain demand for space over a 20 year time horizon which is decided by the authorities in charge.

The inmates are divided into two types: those detained for a few days or awaiting trial or sentencing, and those who have been sentenced. The first group is held at facilities that should be located as close as possible to their respective courts, given that inmates must travel frequently between the two locations. It is not mandatory, however that sentenced prisoners be imprisoned in the same city or geographical division as the court that sentenced them. But it is desirable that they be held in institutions located as close as possible to their families. Factors included in the problem are controlled use of prison overpopulation, limits on transfer distances for the two inmate groups, lower and upper bounds for facility capacities, scheduling of facility openings and expansions over the planning horizon, and solutions for various possible space demand scenarios.

In [18] uncertainty is handled with a *minimum regret* strategy in a two-stage environment. The prison-siting model solved there had the following characteristics:

1. Aggregate data for various scenarios are treated simultaneously in one common model via a compact representation. This means that a single model included all variables and constraints, which are repeated for each possible scenario.
2. There was only one set of site selections for all scenarios. The opening date decisions had to anticipate all possibilities, regardless of the uncertainty involved, i.e., a simple recourse environment was considered.
3. The worst (biggest) cost among the various scenarios was minimized.

A simulation model was defined in an earlier paper on modeling the future growth of the prison population [19]. It was based on a simplified diagram describing the flow of persons through the system. The flow included detainees, inmates under trial and convicted inmates.

However, the main characteristics of our current model are as follows:

1. The capacities of prison facilities in a given province should be sufficient to house all detainees and charged prisoners from

that province plus the convicted inmates assigned to it (from the same or any other province). Note that overpopulation is inevitable when the number of detainees plus prisoners under trial exceeds the province's prison capacity, because these categories cannot be transferred to facilities in other provinces. In such cases, convicted inmates assigned to these jails also contribute to the overpopulation. Our model deals with this phenomenon by allowing overpopulation, but penalizing it in the objective function. The overpopulation cost or penalty has a relation to facility opening costs, and was set in such a way as to force the model to open a new facility when there is enough overpopulation in its province for filling 1/3 of its opening size. Notice that the capacity of prison facilities is a soft constraint in the model.
2. No province capacity ever declines but rather grows in accordance with implemented expansions. No existing facility closes during the 20 year time horizon. Any new facility remains open for the duration of the horizon.
3. Existing facilities have a current capacity and a maximum expanded capacity determined by Gendarmería de Chile.
4. Convicts can be transferred between provinces with limits on transfer distances.
5. The decision to open a new facility must be taken one period in advance to allow sufficient time for construction given the variability in the evolution of the uncertain parameters.
6. Recourse is allowed along the time horizon, thus, the opening decisions do not need to anticipate all scenarios.

Additionally, the planning horizon is divided into six time periods to agree with the budgetary constraints of the agency in charge of the prison system. There are policy derived constraints on the number of facilities to be opened along the time horizon, such as (1) a given number of facilities must be opened in the first time period, (2) at most one new facility can be opened in each province, and (3) a fixed number of facilities must be opened along the time horizon.

3. Modeling under uncertainty

The problem as described in the previous section involves three sets of random parameters that will be treated by using a Stochastic Integer Programming (SIP) approach. These are convicted inmate-, charged inmate- and detainee-populations for each province and time period.

See [6,14,15], among others, for successful methods dealing with uncertainty in continuous problems. Also see [1,3,10,20,24,25], among others, for successful implementations of SIP.

Facility location problems with uncertainties have been addressed in different contexts. When different future scenarios are possible, the two preferred schemes are to minimize expected cost and to minimize regret. A review of both is offered in [26], where a procedure mixing both approaches is presented. From different points of view, a good review of general location problems has been provided in [5], where there is stochastic demand and congestion at the facilities. See in [18] an approach to locate facilities to maximize satisfied demand.

Now, let us consider the following deterministic problem:

$$\begin{aligned} \min \quad & ax+cy \\ \text{s.t.} \quad & Ax+By=b, \\ & x\in\{0,1\}^n, \quad y\ge 0, \end{aligned} \tag{1}$$

where m, n and n_c are the number of constraints, 0–1 variables and continuous variables, respectively, a and c are n- and n_c-dimensional objective function coefficient vectors, respectively; b is the m-dimensional right-hand side (rhs) of the constraint system; A and B are $m\times n$ and $m\times n_c$ constraint matrices, respectively; and x and y are an n-vector of 0–1 variables and an n_c-vector of continuous variables, respectively.

We use a scenario tree approach based on [1, 2, 3], in which uncertainty is modeled in terms of a set of scenarios. For this purpose we introduce the following definitions. A period of a time horizon is one or several years in which the random parameters are realized and decisions can be taken; a scenario is the realization of uncertain and deterministic parameters during all the periods of the time horizon; and a scenario group for a given time period is the group of scenarios with the same uncertain parameter realizations up to that period. (That is, a scenario group defines a group of partial scenarios).

To illustrate the multi-period scenario tree concept, let Fig. 1 be a scenario tree in which each branching level represents a time period in which a decision can be taken. In the figure, the time periods are $t=1$, 2, 3 and 4. Once a decision has been taken at some of these time periods, various possible situations may occur. In our example, after time period 1, there are two such situations in time period $t=2$, represented by nodes 2 and 3. This information is generally presented in the form of a tree in which each path from the root to a leaf represents a scenario and corresponds to the realization of the entire set of uncertain parameters. For example, path 1, 3, 6, 12 represents one scenario, and it is customary to call it scenario 12, i.e., denote the scenario using its last node. In what follows, we do not distinguish between a scenario (or a group) and the corresponding node on the tree (with the same number). Each node in the tree must be associated with a scenario group in such a manner that any two scenarios belong to the same group (i.e., they have the same partial scenario) in a given time period if they include the same occurrences of uncertain parameters up to that time period. For example, for time period 3, scenarios 12 and 13 belong to the same group associated with path $\{1,3,6\}$, i.e., with group $g=6$. Notice the difference between a scenario (a path from the root node to a leaf node) and a partial scenario (a path from the root to an intermediate node).

The notation for the scenario tree to be used in the paper is as follows:

$\mathcal{T}$	set of periods in the time horizon
$\mathcal{T}^+$	set of all time periods except the last one
Ω	set of scenarios
$\mathcal{G}$	set of scenario groups
$\mathcal{G}^t$	set of scenario groups in time period t ($\mathcal{G}^t \subseteq \mathcal{G}$), for $t\in\mathcal{T}$

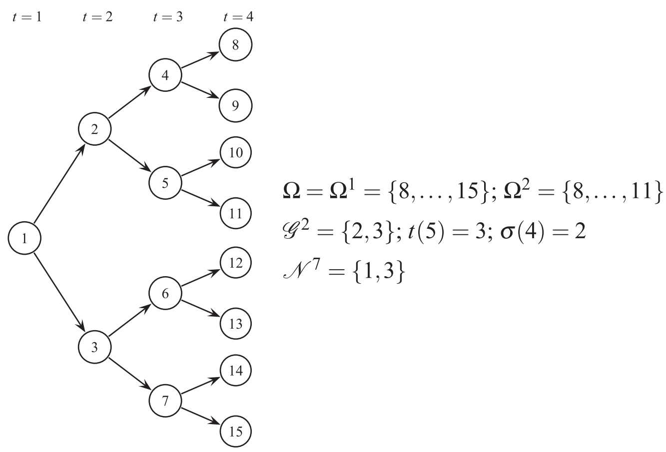

Fig. 1. Example of multi-period scenario tree.

$\mathcal{G}^-$ set of all scenario groups except the first one (that is associated with the root node)

Ω^g set of scenarios in group g ($\Omega^g \subseteq \Omega$), for $g \in \mathcal{G}$. The non-anticipativity principle, stated in [27] and restated in [6,21], among many others, requires that decisions pertaining to scenarios in a same group (i.e., partial scenarios with the same value in the parameters) be the same

$\sigma(g)$ immediate ancestor node of node g, for $g \in \mathcal{G}$

$\mathcal{N}^g$ set of nodes in the ancestor path from node g to the root node, for $g \in \mathcal{G}$. (*Note*: $\sigma^g \in \mathcal{N}^g$)

$t(g)$ time period of scenario group g.

If the parameters a, b and c in problem (1) are random parameters with a set of discrete occurrences, say, a^ω, b^ω and c^ω over the set Ω of scenarios $\omega \in \Omega$, we will model our problem as follows:

$$\begin{aligned} \min \quad & \sum_{\omega \in \Omega} w^\omega (a^\omega x^\omega + c^\omega y^\omega) \\ \text{s.t.} \quad & Ax^\omega + By^\omega = b^\omega \quad \forall \omega \in \Omega, \\ & (x,y) \in \mathcal{N}, \\ & x^\omega \in \{0,1\}^n, \quad y^\omega \geq 0 \quad \forall \omega \in \Omega, \end{aligned} \tag{2}$$

where w^ω is a positive weight assigned to scenario ω, for instance its probability; $\sum_{\omega \in \Omega} w^\omega = 1$; x^ω and y^ω represent the x and y variables for scenario ω, respectively; and the so-called *non-anticipativity* set is defined by

$$\mathcal{N} = \{v \,|\, v_t^\omega = v_t^{\omega'} \forall \omega, \omega' \in \Omega^g, g \in \mathcal{G}^t, t \in T\}, \tag{3}$$

where $v = (x,y)$ and v_t^ω is such that $v^\omega = (v_t^\omega, \forall t \in \mathcal{T})$.

The non-anticipativity principle ensures that the solution for time period t in the model does not depend on information that is yet unavailable. To model the constraints in (3), two different approaches can be used, namely, the compact representation and the splitting variable representation. The latter has two alternative formulations. The first one is known as a node-based (or scenario group-based) representation. It uses copies of the variables with nonzero coefficients in constraints belonging to different time periods. The second formulation, referred to as a scenario-based representation, requires that all variables be copied. In both formulations the non-anticipativity constraint (for short, NAC) must be explicitly incorporated into the model, but for the purpose of the paper the second formulation preserves the model structure better.

Upon incorporating the set (3) in model (2), we obtain the Deterministic Equivalent Model (DEM) for the splitting variable representation:

$$\begin{aligned} \min \quad & \sum_{\omega \in \Omega} w^\omega (a^\omega x^\omega + c^\omega y^\omega) \\ \text{s.t.} \quad & Ax^\omega + By^\omega = b^\omega \quad \forall \omega \in \Omega, \\ & x_t^\omega = x_t^{\omega'} \quad \forall t \in \mathcal{T}^+, \quad \forall g \in \mathcal{G}^t, \; \omega, \omega' \in \Omega^g, \; \omega \neq \omega', \\ & x^\omega \in \{0,1\}^n, y^\omega \geq 0 \quad \forall \omega \in \Omega. \end{aligned} \tag{4}$$

Notice that $x_t^\omega = x_t^{\omega'}$ implies that t is up to $t(g)$ as the parameters for each scenario group are identical.

We now modify the notation to consider scenario groups instead of single scenarios in DEM. The compact representation of the stochastic model (2) can be written

$$\begin{aligned} \min \quad & \sum_{g \in \mathcal{G}} w^g (a^g x^g + c^g y^g) \\ \text{s.t.} \quad & A' x^{\sigma(g)} + A x^g + B' y^{\sigma(g)} + B y^g = b^g \quad \forall g \in \mathcal{G}, \\ & x^g \in \{0,1\}^{n_t}, y^g \geq 0 \quad \forall g \in \mathcal{G}, \end{aligned} \tag{5}$$

where n_t is the number of the x variables at each time period t, and a^g, b^g and c^g are the counterparts of parameters a^ω, b^ω and c^ω related to scenario group g, for $g \in \mathcal{G}$, such that the values of the parameters for each scenario in the group are identical. Additionally, x^g and y^g represent the x and y variables for scenario group g, respectively, and A' and B' are the constraint matrices for the x and y variables related to the immediate ancestor of group g.

Different approaches exist for solving the type of problem under consideration here. Typically, Lagrangean relaxation schemes (see a survey in [12]) and Benders [4] decomposition schemes are used for stochastic continuous linear programming problems. For integer problems there are a few valid alternatives, including schemes such as Benders decomposition [6,7,17], Lagrangean decomposition [8,16,22,23], disjunctive decomposition [20], stochastic branch-and-cut [24], Benders decomposition based branch-and-bound [25] and the already cited branch-and-fix coordination (BFC) [2,9,10], among others. This last method is the basis for the approach utilized in our work.

If one ignores the NAC in model (4), the model decomposes into $|\Omega|$ submodels, and the model for scenario ω is as follows:

$$\begin{aligned} \min \quad & a^\omega x^\omega + c^\omega y^\omega \\ \text{s.t.} \quad & Ax^\omega + By^\omega = b^\omega, \\ & x^\omega \in \{0,1\}^n, \quad y^\omega \geq 0. \end{aligned} \tag{6}$$

The BFC scheme is specifically designed to coordinate the selection of the variable and node to be branched from in the branch-and-fix (BF) tree corresponding to each scenario so that the NAC, which have been relaxed from the model, are automatically satisfied when the appropriate variables are fixed at 0 or 1. The procedure also coordinates and reinforces active node pruning, variable fixing and objective function bounding for each node subproblem.

To gain computational efficiency it is not necessary to relax the NAC for all scenario pairs. The number of joint scenarios kept in a given model (i.e., scenarios whose NAC are explicitly included in the model) depends basically on the dimensions of the scenario model. A scenario cluster, then, is a set of scenarios whose mutual NAC are explicitly modeled. The criterion for forming the clusters, say $\Omega_1, \ldots, \Omega_q$, where q is the number of clusters, will be problem dependent. However, we favor the approach that shows higher scenario clustering for greater number of scenario groups in common. So, each scenario cluster has a related submodel. In any case, the clusters define a partition of Ω, that is, $\Omega_p \cap \Omega_{p'} = \emptyset$, $p,p' = 1, \ldots, q$, $p \neq p'$ and $\Omega = \bigcup_{p=1,\ldots,q} \Omega_p$. Thus, every scenario exclusively belongs to a cluster.

The model for each scenario cluster $p = 1, \ldots, q$ can be written using the compact representation, and it should include only the variables and constraints related to the scenarios in the cluster.

A heuristic procedure for obtaining good solutions for the pure 0–1 cluster model is given in [3]. For a heuristic approach to the mixed 0–1 case, see Section 5 below.

4. DEM for the selecting prison facility site problem

Let us introduce the following additional notation:

Sets

$\mathcal{J}$ set of provinces

$\mathcal{J}_i$ subset of provinces located close enough to province i, including itself, for $i \in \mathcal{J}$

Deterministic parameters

A "unit" cost is always a "per inmate" cost.

c_{jt} fixed cost of opening a new facility in province j at time period t, for $j \in \mathcal{J}, t \in \mathcal{T}^+$

$\hat{c}_{jt}$ unit cost of expanding a facility in province j at time period t. Note: Existing facilities cannot be expanded at time period $t=1$, nor can new facilities be available

k_{ij}^{t} unit cost of transferring convicted inmates between province (court) i and prison facilities in province j at time period t

s_{jt} unit cost of overpopulation in province j at time period t

$\overline{XE}_j$ maximum capacity of existing facilities in province j

$\overline{XN}_j$ maximum capacity of each new facility in province j

$\underline{XN}_j$ minimum capacity of each new facility in province j, if any

d^t discount factor used in the cost evaluation of time period t

$\overline{\delta}_j$ bound on the number of new prison facilities in province j

np number of new prisons that must be built along the time horizon

np^1 number of new prisons that must be built during time period 1

XE_j^1 total prison capacity (number of inmates) in currently existing facilities in province j at the beginning of the time horizon

w^g positive weight assigned to scenario group g, for $g \in \mathcal{G}$

Stochastic parameters for each scenario group g for $g \in \mathcal{G}$

D_j^{+g} demand for convicted inmate capacity in province j at time period $t(g)$

D_j^{-g} demand for charged inmate and detainee capacity in province j at time period $t(g)$

Variables for each scenario group g for $g \in \mathcal{G}$

C_{ij}^g number of convicted inmates originating in courts of province i assigned to facilities of province j at time period $t(g)$

δ_j^g 0–1 variable that is equal to 1 if a new prison is opened in province j at time period $t(g)$ and zero otherwise, for $t(g) \in \mathcal{T}^+$ because of construction delays

γ_j^g number of new facilities that are opened in province j up to time period $t(g)$ for $t(g) > 1$

XE_j^g total prison capacity (number of inmates) in currently existing facilities in province j at time period $t(g)$ for $t(g) > 1$ (such that XE_j^g is a parameter for $g=1$)

XN_j^g total prison capacity in new facilities in province j at time period $t(g)$ for $t(g) > 1$.

YE_j^g volume (number of inmates) of expansion of existing prison facilities in province j at time period $t(g)$, for $t(g) > 1$

YN_j^g volume of expansion of new prison facilities in province j at time period $t(g)$ for $t(g) > 1$

S_j^g volume of overpopulation in prison facilities in province j at time period $t(g)$.

DEM

$$\min \sum_{g \in \mathcal{G}} \sum_{j \in \mathcal{J}} w^g d^{t(g)} c_{j,t(g)} \delta_j^g \tag{7}$$

$$+ \sum_{g \in \mathcal{G}^-} \sum_{j \in \mathcal{J}} w^g d^{t(g)} \hat{c}_{j,t(g)} (YE_j^g + YN_j^g) \tag{8}$$

$$+ \sum_{g \in \mathcal{G}} \sum_{i \in \mathcal{J}} \sum_{j \in \mathcal{J}_i} w^g d^{t(g)} k_{ij}^{t(g)} C_{ij}^g \tag{9}$$

$$+ \sum_{g \in \mathcal{G}} \sum_{j \in \mathcal{J}} w^g d^{t(g)} s_{j,t(g)} S_j^g \tag{10}$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} \delta_j^1 = np^1, \tag{11}$$

$$\sum_{k \in \mathcal{N}^g} \sum_{j \in \mathcal{J}} \delta_j^k = np \quad \forall g \in \mathcal{G}^{|\mathcal{T}|}, \tag{12}$$

$$\sum_{k \in \mathcal{N}^g} \delta_j^k \le \overline{\delta}_j \quad \forall j \in \mathcal{J},\ g \in \mathcal{G}^{|\mathcal{T}|}, \tag{13}$$

$$\gamma_j^g = \gamma_j^{\sigma(g)} + \delta_j^{\sigma(g)} \quad \forall j \in \mathcal{J},\ g \in \mathcal{G}^-, \tag{14}$$

$$XE_j^g \le \overline{XE}_j \quad \forall j \in \mathcal{J},\ g \in \mathcal{G}^-, \tag{15}$$

$$\underline{XN}_j \gamma_j^g \le XN_j^g \le \overline{XN}_j \gamma_j^g \quad \forall j \in \mathcal{J},\ g \in \mathcal{G}^-, \tag{16}$$

$$XE_j^g = XE_j^{\sigma(g)} + YE_j^g \quad \forall j \in \mathcal{J},\ g \in \mathcal{G}^-, \tag{17}$$

$$XN_j^g = XN_j^{\sigma(g)} + YN_j^g \quad \forall j \in \mathcal{J},\ g \in \mathcal{G}^-, \tag{18}$$

$$\sum_{j \in \mathcal{J}_i} C_{ij}^g = D_i^{+g} \quad \forall i \in \mathcal{J},\ g \in \mathcal{G}, \tag{19}$$

$$D_j^{-g} + \sum_{i \in \mathcal{J}: j \in \mathcal{J}_i} C_{ij}^g \le XE_j^g + XN_j^g + S_j^g \quad \forall j \in \mathcal{J},\ g \in \mathcal{G}, \tag{20}$$

$$C_{ij}^g, S_j^g \ge 0 \quad \forall i \in \mathcal{J}_j,\ j \in \mathcal{J},\ g \in \mathcal{G}, \tag{21}$$

$$XE_j^g, XN_j^g, YE_j^g, YN_j^g, \gamma_j^g \ge 0 \quad \forall i \in \mathcal{J}_j,\ j \in \mathcal{J},\ g \in \mathcal{G}^-, \tag{22}$$

$$\delta_j^g \in \{0,1\} \quad \forall j \in \mathcal{J},\ g \in \mathcal{G}. \tag{23}$$

The objective function includes the cost of installing new facilities (7), the cost of expanding existing and new facilities (8), the cost of transferring convicted inmates (9) and the overpopulation cost (10). Notice that no facility expansion can be performed at time period t=1 and no new facility will be opened in the last period $|\mathcal{T}|$. Constraints (11) and (12) specify the total number of new prison facilities. Constraints (13) bound the number of new prison facilities per province. Constraints (14) give the equations for the facility opening update, such that a facility is available in the period immediately following the time period in which it is open. Constraints (15) and (16) bound the inmate capacity of existing and new facilities, respectively. Constraints (17) and (18) update the inmate capacity of existing and new facilities, respectively. Constraints (19) give the equation for the convicted prisoners assignment. Constraints (20) balance the inmate numbers. Notice that the integer constraint of the γ variables is implicitly satisfied, given the integer character of the δ variables.

5. Branch-and-cluster coordination algorithmic scheme

In the previous sections a mixed 0–1 DEM was presented for the prison facility siting problem. To deal with uncertainty in the real-life problem considered in this paper, 108 scenarios were generated based on the model for the prison population prediction given in [18]. The topology of the scenario tree is shown in Fig. 2. The dashed lines in the figure indicate that the structure for the other branches is repeated.

It was also decided for computational reasons that the problem would be treated using clusters of 36 scenarios, each containing compact representations of the submodels. Thus, the tree shown in Fig. 2 can be divided into three clusters ($q=3$).

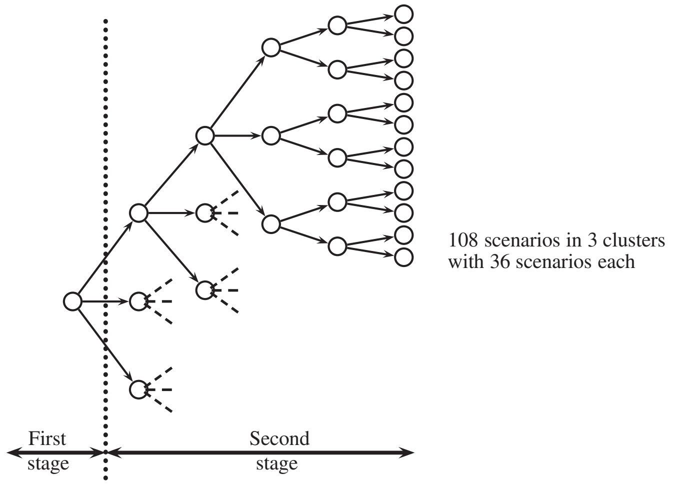

Fig. 2. Scenario tree for the prison facility sites selection problem.

Notice that the only decisions shared by all clusters are those related to the first time period. The related 0–1 variables are the prison opening decisions to be made in that period for implementation in time period 2, due to the time lag between making the decision and the actually opening of the facilities.

We present a heuristic specialization of the BFC-MSMIP approach given in [10] to handle the necessary coordination of the first period decisions. We call it branch-and-cluster coordination (BCC), it provides very good feasible solutions in affordable computing effort (in terms of memory and elapsed time), see Section 6. The basic steps of the algorithm are described below. For this purpose, we split the time horizon in two stages. The first stage only includes the first time period. The second stage includes all other time periods. In the first phase we solve model (7)–(23), where the integrality constraints on and the NAC of the second stage 0–1 variables are removed. In the second phase, after fixing the first stage variables to the values just obtained in the first phase, we solve the individual cluster submodels by using an MIP solver.

We now analyze the first phase of the algorithm, thus, in what follows, $p \in \{1,q\}$ denotes the index of a cluster. Variables related to that cluster are indexed with a superscript p. Let δ_j^{1p} denote the jth element of the variables vector δ^{1p}, where 1 refers to the first stage. Let also $\mathcal{R}^p$ denote the branch-and-fix (for short, BF) tree associated with cluster p (see below the definition of a BF tree as a special branch-and-bound tree, for short B&B), and $\mathcal{H}^p$ be the set of active nodes in $\mathcal{R}^p$. Any two active nodes, say, $h \in \mathcal{H}^p$ and $h' \in \mathcal{H}^{p'}$ with $p \neq p'$ are called twin nodes if the paths from the root node to each of them in their own BF trees, say $\mathcal{R}^p$ and $\mathcal{R}^{p'}$, have been branched on the same values of the δ variables. A set of twin nodes will be called a *Twin Node Family* (for short, TNF), say f, if any node is a twin node to all nodes in the family. Let $\mathcal{F}$ denote the set of families of twin nodes, so that one can say that the nodes h and h' are twins if $h,h' \in f, f \in \mathcal{F}$. A BF tree differs from a simple B&B tree associated with a scenario in that the fixing of a 0–1 variable in the B&B tree (in our case, BF tree) automatically produces the fixing of that variable to the same value in all related nodes in the TNF under consideration. As an illustration, let us consider the active node h in the BF tree $\mathcal{R}^p$ and assume a branching is required on the variable δ_j^{1p}; in that case, two new subproblems are created in the BF trees associated with the clusters, such that the new branches from each node h' in set f where $h \in f, f \in \mathcal{F}$ are as follows: $\delta_j^{1p} = \delta_j^{1p'} = 1$ on one descendant node from each node in set f and $\delta_j^{p} = \delta_j^{1p'} = 0$ on the other descendant node. So, the proposal is to execute in a coordinated way the q BF phases, one per cluster.

The main differences between the algorithms BCC and BFC-MSMIP are as follows: First, BCC splits the full DEM model into q scenario cluster submodels for the second phase, once the values of the first stage variables have been fixed. Second, the integrality constraints of the 0–1 variables are removed in the submodel to be solved by BCC at any branching TNF in the BF phase, so a simple LP model is solved as a provider of a lower bound of the solution value for the TNF submodel under consideration. And, third, BCC as a heuristic only looks for a good feasible solution, while BFC-MSMIP is an exact algorithm and, then, looks for the optimal solution of the original problem.

Roughly speaking, the BCC algorithm is as follows:

- *Step* 1: Solve the LP relaxation of the original problem by solving the corresponding q cluster related LP models. Each model will be the root node problem of $\mathcal{R}^p\ \forall p \in \{1,q\}$. If all NAC and integrality constraints are satisfied for all periods in all scenario clusters, then stop; the optimal solution to the original stochastic mixed 0–1 problem (2) has been obtained. Otherwise, a lower bound for the optimal solution has been found.
- *Step* 2: Following a criterion that has performed well in the past, see for example [2,25], the first stage TNF branching is done following the *depth first* strategy. The selection of the branching variable is done by using the highest expected minimum fractional value. The scheme is as follows: Let

$$\Delta_j = \min\left\{\sum_{p=1}^{q} w^p \delta_j^{*1p}, q - \sum_{p=1}^{q} w^p \delta_j^{*1p}\right\} \quad \forall j \in \overline{\mathcal{J}}^1,$$

  where δ_j^{*1p} is the current fractional value of the variable δ_j^{1p}, $\overline{\mathcal{J}}^1$ is the subset of 0–1 variables not yet branched on in the branching TNF, and w^p represents the weight assigned to cluster p. The branching variable to select, say β, is such that

$$\beta = argmax_{j \in \overline{\mathcal{J}}^1}\{\Delta_j\}.$$

  Branching on the β variable will create a new TNF, such that the strategy is "branching on the zero value" for

$$\sum_{p=1}^{q} w^p \delta_j^{*1p} \leq q - \sum_{p=1}^{q} w^p \delta_j^{*1p}$$

  and, otherwise, "branching on the one value".
- *Step* 3: The selected 0–1 variable is branched on the same 0–1 value at all the node members of the branching TNF.
- *Step* 4: Optimization of the q LP models attached to the newly created TNF after branching on the chosen first stage 0–1 β variable. If its related LP model is infeasible then go to Step 6.
- *Step* 5: If the solution that has been obtained in Step 4 has fractional values for the first stage 0–1 variables, then go to Step 2 to select another branching variable to continue the branching phase. Otherwise, a first stage integer solution for the branching TNF has been obtained. If all NAC and integrality constraints are satisfied for all periods in all scenario clusters, then stop; a feasible solution to the original stochastic mixed 0–1 problem (2) has been obtained. Otherwise, the δ_j^{1p} variables are fixed to their just obtained values. If the NAC for the first stage continuous variables are not satisfied, an LP relaxation of the original model (7)–(23) must be solved, where the first stage 0–1 variables are fixed to the just obtained values. If the integrality constraints of the 0–1 variables are satisfied for all periods in all scenarios in the just solved LP model, the feasible solution for the original model is found then stop. Otherwise, the first stage variables are fixed at their values and, as a consequence, the q mixed 0–1 scenario clusters submodels for the second stage become independent (i.e, there

is one model per cluster) and, then, they are optimized by using a MIP solver. If there is an infeasible solution for any of the cluster submodels then go to Step 6. Otherwise, the optimal solution of the cluster submodels gives a feasible solution to the original problem.

- *Step* 6: "Branch on the one value" if it has only been "branched on the zero value", and vice versa. In any case, goto Step 3, unless it has been branched on in both directions. In this latter case, goto Step 7.
- *Step* 7: (Both branches have already been used) Do back-tracking and go to Step 6.

6. Computational experience

The dimensions of the real-life problem that is used for obtaining the computational results reported in this section are: $|\mathcal{J}|=51$ provinces, 13 of them are eligible for being candidates to open prisons, $|\mathcal{T}|=6$ time periods, $|\Omega|=108$ scenarios and $|\mathcal{G}|=202$ scenario groups. The topology of the scenario tree is $1^1,3^3,2^2$, that is, there are (1), (3,9,27), (54,108) nodes in the time periods (1), (2,3,4), (5,6), respectively, see Fig. 2. The time periods 1 to 6 correspond to the six periods: year 1, years 2–3, 4–7, 8–11, 12–15 and 16–20, respectively.

For illustrative purposes, the demand for the scenarios is shown in Fig. 3. For confidentiality reasons, the demand is shown relative to the first time period's demand, which has been set to 100. For comparison purposes, the scenario tree has been developed based on the 1992–1998 data used in [18].

The scenario tree has been developed based on [18]. $np=10$ prison facilities must be open in the time horizon, $np^1=3$ of them must be open in time period 1. $\overline{\delta}_j=0$ for all provinces but the 13 given eligible provinces to open prisons, such that $\overline{\delta}_j=1$ for those candidates provinces (i.e., only a new prison facility per province is allowed, at most) and, so, the γ variables may only have 0–1 values.

The models and the BCC algorithm were written in GAMS v23.3 [11] using CPLEX v12.1 [13] as the MIP solver, and were executed on a 2.67 GHz Pentium IV PC with 12.0 GB of RAM running Windows 7 Ultimate.

It should be noted that all models have a similar basic structure. What changes from one BCC iteration to the next are the parameters for each scenario cluster and the variables to be fixed. This structural characteristic is reflected in the separation of the scheme into two parts. The first part defines the generic model structure while the second incorporates the variables to be fixed plus the data. In this manner the generic structure is generated only once, and is thereafter invoked each time the control routine is iterated.

One alternative strategy is analyzed in addition to the minimum expected cost strategy proposed above, namely, the average

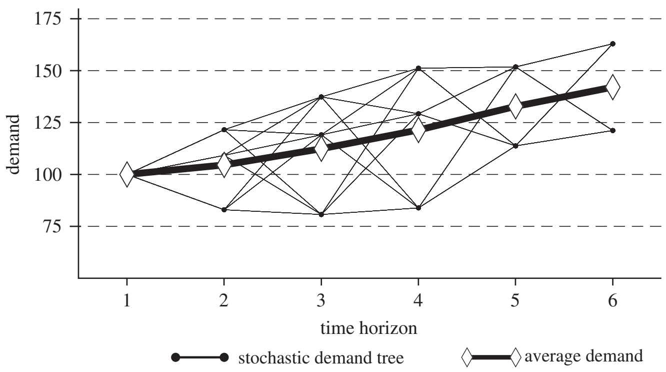

Fig. 3. Tree of the stochastic demand.

scenario strategy. This approach replaces the uncertain parameters by their average over all scenarios along the time horizon.

Table 1 shows for the real-life case the weights for the three clusters in each of the three versions of the case that are tested, the weights being the only difference among versions.

The scenarios of largest demand are in Cluster 1, those of intermediate demand are found in Cluster 2, and those with lowest demand are concentrated in Cluster 3. Thus, in Version 1 the clusters with the worst scenarios have the lowest probability, in Version 2 all clusters are equiprobable, and in Version 3 the clusters with the worst scenarios have the highest probability. Moreover, the scenarios within each cluster remain equiprobable in all versions.

Table 2 gives the dimensions of the scenario-related deterministic model. It also gives the dimensions of the compact representation of the deterministic equivalent model (DEM) to the stochastic problem (2). The notation is as follows: m, number of constraints; n_c, number of continuous variables; n, number of 0–1 variables; and *dens*, constraint matrix density. 13 0–1 variables out of 1222 belong to the first time period (that actually gives the set of decisions to be made). There are $13\times 5 = 65$ δ variables in the deterministic model and $13\times 94 = 1222$ δ variables in the DEM compact model. (Remember that only 13 given provinces out of 51 provinces are eligible to open new prisons, and no new prison facility can be built in the last time period.) As we will show below the DEM has been extremely difficult to solve, even the optimality of the incumbent solution could not be proven by CPLEX after spending much computing effort.

As previously noted, our algorithm divides its operation into two phases. The values of the variables in the first phase are determined via the BCC scheme and, once a first phase integer solution has been obtained, the algorithm proceeds to the second phase where the MP submodels are split by cluster and solved independently by CPLEX.

It is worth pointing out the small influence in the solution value that have the cost of facility expansion, convicted inmate transferring and facility overpopulation when it is compared with the much higher cost of new facilities opening. So, the NAC for the continuous variables have been relaxed for the cases that we have experimented with (and whose results are reported below). Additionally, the periods have from two up to five years and, then, decisions on the jail capacity can be very easily adjusted. So, the value of those variables have been averaged in the results presented below.

For Version 1 (cluster probabilities 0.15, 0.30, 0.55), the BCC algorithm obtains a feasible solution, whose value is 306.80, in 0.73 h of computing time. The plain use of CPLEX obtains the first

Table 1
Cluster probabilities for the versions of the real-life case.

	Version 1	Version 2	Version 3
Cluster 1	0.15	0.33	0.55
Cluster 2	0.30	0.33	0.30
Cluster 3	0.55	0.33	0.15

Table 2
Model dimensions.

	Deterministic model	DEM compact model
m	1615	57,907
n_c	2759	93,324
n	65	1222
dens(%)	0.396	0.017

(bad) solution value 2108.72, in 143 s, and the best feasible solution, whose value is 304.04, in 11.57 h of computing time with a 0.52% optimality gap and, then, the execution was stopped.

For Version 2 (cluster probabilities 0.33, 0.33, 0.33), the BCC algorithm obtains a feasible solution, whose value is 304.14, in 0.9 h of computing. The plain use of CPLEX proves in 6.7 h of computing that the solution value has a 0.81% optimality gap and, then, the execution was stopped. CPLEX has found the same solution after 2.6 h of computing.

For Version 3 (cluster probabilities 0.55, 0.30, 0.15), the BCC algorithm obtains a feasible solution, whose value is 316.46, in 0.84 h of computing. The plain use of CPLEX obtains the first (bad) solution value 2165.662, in 153 s, and the best feasible solution, whose value is 314.40, in 8.83 h of computing with a 0.56% optimality gap and, then, the execution was stopped.

For comparison purposes several computational experiments were carried out, producing jointly the *average scenario* solution for each time period. Given the decisions implied in that approach, the results of such decisions are evaluated for each scenario.

Figs. 4 and 5 show some results by using the stochastic and deterministic approaches. Fig. 4 shows the number of jails opened at each period along the time horizon for each of the scenario groups in the three versions of the case under consideration as the solutions provided by the stochastic approach. It also gives the solution provided by the deterministic approach.

Fig. 5 shows the differences in (a) number of new jails, (b) the total capacity of these jails, and (c) the average capacity of these jails in the stochastic and deterministic solutions for Version 1. The results are normalized such that they are relative to the solution provided by the deterministic approach, which has been set to 100. The shadowed zone represents the variation range of the solution provided by the stochastic approach over the 108 scenarios. The line with circle icons in the zone gives the average of the proposed solution. We can observe that the stochastic approach anticipates the jails opening with respect to the opening provided by the solution of the deterministic approach. It also gives a bigger total jails capacity while the average capacity varies with respect to the deterministic solution depending on the time period. The results are similar for Versions 2 and 3. Notice that the solution of the deterministic model proposes a smaller number of jails than the stochastic model, since it does not take into account that the demand can be small enough for some scenarios when comparing it with the average demand. On the other hand, the stochastic model, while proposing a higher number of jails, has more flexibility on matching capacity and demand and, then, the capacity of these jails can be adapted. As a result, the jails are smaller or larger than those provided by the deterministic model, depending on the demand for each scenario. In any case, it always offers more total capacity, and then, it reduces overpopulation cost. Additionally, it advances decisions to earlier time periods.

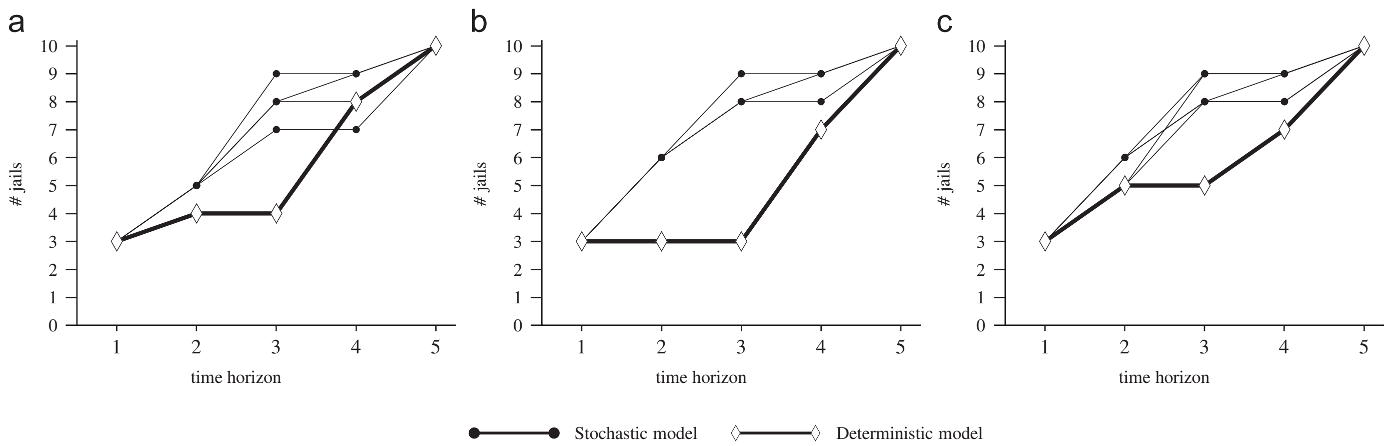

Fig. 4. Number of new jails opened. (a) Version 1. (b) Version 2. (c) Version 3.

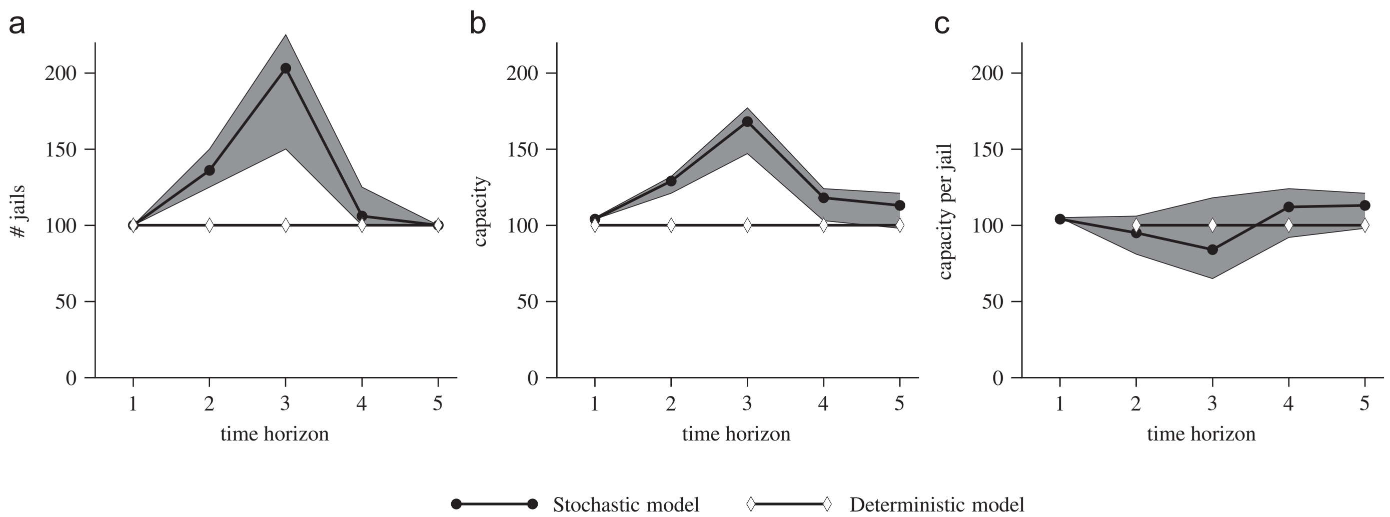

Fig. 5. Solution analysis. (a) Number of jails. (b) Total capacity. (c) Average capacity.

Table 3
Comparison of results.

Version	Strategy	Obj. fun. value	Diff. %
V1	Min expected cost	306.40	–
	Average scenario	365.07	19.15
V2	Min expected cost	304.14	–
	Average scenario	369.67	21.15
V3	Min expected cost	316.46	–
	Average scenario	361.24	14.15

While the deterministic model opens more jails in time period 4, the stochastic model concentrates the effort on opening jails in the first three time periods. There is not too-much work to be done in time period 4, and it simply satisfies in time period 5 the minimum jail opening requirements for the whole time horizon. It seems that both types of models could provide a better solution if the constraint forcing to open 10 jails along the time horizon is relaxed. This observation is clearer in the stochastic model since it opens many jails in the last available time period.

Table 3 summarizes the objective function values under the two strategies in each of the three cases that are studied. The *Diff.%* column gives the percentage difference between the minimum expected cost and the average scenario solution value over the min expected cost. Notice that in all the three versions of the real-life case, the minimum expected cost strategy obtains much better results than the other strategy.

Let us point out that when one uses the average scenario strategy and moves forward in the time horizon, the initial decisions about later time periods can be modified, while optimizing a new smaller average scenario problem each time. The time horizon gets reduced by one time period at each iteration, thus reducing the variance of the uncertain parameters. In other words, once one has obtained the solution for the first time period by solving the deterministic problem then, either after replacing the uncertainties by the average scenario or after observing the realization of the uncertain parameters, the average scenario approach fixes the solution for that time period. Next, it solves the average scenario problem for the remaining time periods. This procedure is repeated until there are no time periods left. A more accurate approach could be to consider a rolling horizon for a number of time periods, but this is beyond the scope of this work. This method, however, could show a bigger difference *Diff.%* with rolling horizon in favor of the minimum expected cost approach than the difference shown in Table 3.

7. Conclusions

The paper addresses a real-life problem of selecting prison facility sites under uncertainty through a stochastic integer programming approach, by modeling the uncertainty of convict-, untried prisoner- and detainee-populations using a scenario tree based approach. The model and solution approach can be easily adapted to other prison site selection problems and some other facility location problems. For example, the first stage in the BCC scheme could include more than one time period.

There are costs associated with enforcing the proximity of the inmates to either courts or their families. In the case of detainees and inmates under trial, in addition to the direct cost, there are penalties related to (i) the time it takes to transfer an inmate from the jail to the court and back; (ii) the use of guards, in a situation in which the prisons are understaffed; and (iii) even more important, a penalty related to the risk of escape. This risk is high and its impact is obviously very important, since most of the courts are located in urban areas. In the case of convicted inmates, the closeness to the families has to be maintained for humanitarian reasons; otherwise, the families would feel that they are also punished for the same crime. In addition, experience shows that closeness to families is helpful in inmate rehabilitation.

The results for the minimum expected cost strategy in the BCC scheme were compared with the average scenario approach. (Notice that the average scenario approach is the most frequently used approach today, even though it gives worse results than a stochastic approach.) In the present application, the clear and consistent superiority of the minimum expected cost approach over the other strategy demonstrates that, in the three versions of the real-life case we have experimented with, the latter is only optimal for the average scenario, which in fact may not even exist. Finally, the proposed stochastic algorithmic approach outperforms the plain use of a state-of-the-art optimization engine in computing effort (one order of magnitude), having a similar solution quality.

Acknowledgments

This research has been partially supported by Fondecyt under grant 1060807 (Chile, Andrés Weintraub) and Milenium Institute Complex Engineering Systems (Chile, Vladimir Marianov and Andrés Weintraub), Ministry of Science and Innovation under grants MTM2009-14039-C06-03 (Spain, Antonio Alonso-Ayuso), MTM2009-14087-C04-01 (Spain, Laureano F. Escudero), RIESGOS CM from Comunidad de Madrid (Spain, Antonio Alonso-Ayuso and Laureano F. Escudero), and National Science Foundation under grant DMI-0400155 (USA, Monique Guignard).

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