THE SIZE-RAMSEY NUMBER

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1. Introduction

The size-Ramsey number of a graph G is the smallest number of edges in a graph Γ with the Ramsey property for G, that is, with the property that any colouring of the edges of Γ with two colours (say) contains a monochromatic copy of G. The study of size-Ramsey numbers was proposed by Erdős, Faudree, Rousseau, and Schelp in 1978, when they investigated the size-Ramsey number of certain classes of graphs and, among others, raised some questions concerning the size-Ramsey number of paths. In this talk, we shall survey some results that have been discovered since, focusing on a couple of recent results obtained by the study of Ramsey properties of fairly sparse random graphs by means of the regularity lemma.

We give some details below.

2. The size-Ramsey number

Given an integer $q > 0$ and graphs Γ and H we write $\Gamma \to (H)_q$ if Γ contains a *monochromatic* copy of H in any *q-colouring* of the edges of Γ. That is, for any $\varphi\colon E(\Gamma) \to \{1, 2, \ldots, q\}$, there is a copy H' of H in Γ (that is, a subgraph of Γ isomorphic to H) such that φ is constant on $E(H')$. For simplicity, we shall always take $q = 2$ in what follows.

The *Ramsey number* $r(H)$ of a graph H is the smallest number of vertices in a graph Γ such that $\Gamma \to (H)_2$. In contrast, the *size-Ramsey number* $r_{\mathrm{e}}(H)$ of a graph H is the smallest number of edges in a graph Γ such that $\Gamma \to (H)_2$, that is,

$$r_{\mathrm{e}}(H) = \min\{|E(\Gamma)|\colon \Gamma \to (H)_2\}. \tag{1}$$

Note that, clearly, we have

$$r_{\mathrm{e}}(H) \le \binom{r(H)}{2}. \tag{2}$$

The study of size-Ramsey numbers was proposed by Erdős, Faudree, Rousseau and Schelp [9] in 1978. Those authors introduced the notion of *o-sequences*: sequences of graphs (H_n) for which we have

$$\lim_{n\to\infty} r_{\mathrm{e}}(H_n)\binom{r(H_n)}{2}^{-1} = 0, \tag{3}$$

that is, sequences of graphs for which the trivial upper bound in (2) may be substantially improved. Let P^n be the path on n vertices. The question whether (P^n) is a o-sequence was put forward in [9], and, in [8], Erdős stated the following version of this problem.

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Problem 1. *Is it true that*

$$r_{\mathrm{e}}(P^n)/n \to \infty \quad \textit{and} \quad r_{\mathrm{e}}(P^n)/n^2 \to 0? \tag{4}$$

Beck [4], using probabilistic methods, proved the surprising fact that $r_{\mathrm{e}}(P^n) \le cn$, where c is an absolute constant, that is, the size-Ramsey number of paths is 'linear'. Explicit examples of linear sized graphs that are Ramsey for P^n were given by Alon and Chung [2], that is, they showed how to construct explicitly graphs Γ with $O(n)$ edges such that $\Gamma \to (P^n)_q$.

The linearity of the size-Ramsey number of paths was generalized to bounded degree trees by Friedman and Pippenger [11] (see also [13, 18]). (See [15, 16, 17] for more on tree embeddings.) It was proved in [14] that cycles also have linear size-Ramsey numbers.

Beck [5] asked whether $r_{\mathrm{e}}(H)$ is always linear in the size of H for graphs H of bounded degree, and this was settled in the negative by Rödl and Szemerédi [25], who proved that there are graphs of order n, maximum degree 3, and size-Ramsey number $\Omega(n(\log n)^{1/60})$. It is conjectured in [25] that, for some $\varepsilon = \varepsilon(\Delta) > 0$, we have

$$n^{1+\varepsilon} \le r_{\mathrm{e}}(n, \Delta) \le n^{2-\varepsilon}, \tag{5}$$

where $r_{\mathrm{e}}(n, \Delta)$ is the maximum of $r_{\mathrm{e}}(H)$ over all graphs H on n vertices and of maximum degree at most Δ. The upper bound in (5) has been proved by Rödl, Schacht, Szemerédi, and the speaker [19]. For further results on size-Ramsey numbers, see [10, 22, 23, 24]. We emphasize that the the lower bound in (5) remains open.

Subdivision of graphs. Let I be a graph and h a positive integer. We denote by $I^{(h)}$ the *h-subdivision* of I, namely the graph $I^{(h)}$ is obtained by replacing each edge of I by a path with $h+1$ edges (so, for instance, $I^{(0)} = I$). The following result, which confirms a conjecture of Burr and Erdős [6], was proved by Alon [3].

Theorem 2 (Alon 1994)**.** *If an n-vertex graph H has no two vertices of degree at least 3 adjacent, then its Ramsey number is at most $12n$.*

Therefore, if we subdivide every edge of a graph I at least once, then we obtain a graph with linear Ramsey number. Thus, clearly, the size-Ramsey number of such a subdivision is at most quadratic. Pak [21] put forward the following conjecture.

Conjecture 3 (Pak 2002)**.** *There is an absolute constant c for which the following holds. For every integer D, there is a constant C_D such that if H is a graph with $\Delta(H) = D$ and h is an integer with $h > c \log N$, where*

$$N = |V(H^{(h)})| = |V(H)| + h|E(H)|, \tag{6}$$

then $r_{\mathrm{e}}(H^{(h)}) \le C_D N$.

Making use of results on mixing times of random walks on expanders, Pak [21] proved Conjecture 3 in a weaker form (he obtained the desired upper bound for $r_{\mathrm{e}}(H^{(h)})$ up to a polylogarithmic factor in N). Donadelli, Haxell, and the speaker [7] observed that Conjecture 3 holds in the case in which H is a fixed graph and $h \to \infty$. A recent result obtained together with Rödl and Tengan [20] addresses the case in which we subdivide every edge of a bounded degree graph a fixed, bounded number of times. Our proof makes use of random graphs, the regularity lemma, a path abundance result derived from results in [12], a result concerning embeddings of bounded degree graphs into almost complete graphs, and the Aharoni–Haxell generalization of Hall's matching theorem to hypergraphs [1].

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